Advanced Partial Differential Equations: Exercises

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1. *Given a bounded open set* $\Omega \subset \mathbf{R}^n$*, we consider the problem: Find* $u > 0$ *such that*

$$\begin{aligned} -\Delta u = u^p \quad & \mathbf{x} \in \Omega, \\ u = 0 \quad & \mathbf{x} \in \partial\Omega, \\ u > 0 \quad & \mathbf{x} \in \Omega. \end{aligned}$$

Prove that there is a solution when $1 < p+1 < p^*$*, where* $p^* = \infty$ *if* $n \leq 2$ *and* $p^* < \frac{2n}{n-2}$ *when* $n > 2$*.*

Consider the minimization problem

$$I = \mathrm{Min}_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, d\mathbf{x}}{\int_\Omega |u|^{p+1} \, d\mathbf{x}} = \mathrm{Min}_{u \in H_0^1(\Omega)} J(u).$$

The functional $J(u)$ to be minimized is positive, thus, bounded from below. Consider a minimizing sequence $u_n \in H_0^1(\Omega)$, such that $J(u_n) \to I$ as $n \to \infty$. The sequence $v_n = \frac{u_n}{\| u_n \|_{L^{p+1}}}$ is a minimizing sequence satisfying also $\|v_n\|_{L^{p+1}} = 1$. Then, $\int_\Omega |\nabla v_n|^2 d\mathbf{x} \to I$ implies that v_n is bounded in $H_0^1(\Omega)$ and v_n tends weakly in H_0^1 to a limit $v \in H_0^1(\Omega)$. By Sobolev injections, v_n is compact in L^{p+1}, $p+1 < p^*$, thus $v \in L^{p+1}(\Omega)$ and $\|v_n\|_{L^{p+1}} = 1 \to \|v\|_{L^{p+1}} = 1$. By lower semicontinuity of weak convergence, we have $J(v) \leq \lim_{n \to \infty} J(v_n) = I$. Since $v \in H_0^1(\Omega)$, we have $I \leq J(v)$. Therefore, $I = J(v)$ and the minimum is attained at v. Moreover, we can replace v by $|v|$ and $J(|v|) \leq I(v)$, so that $w = |v| \geq 0$ is a minimizer too and $I = J(w)$. $w \neq 0$ because $\|w\|_{L^{p+1}} = 1$.

Now, $J(w) \leq J(w + tr)$, $r \in H_0^1(\Omega)$ for real t. An asymptotic expansion first for $t > 0$ then for $t < 0$ leads to

$$\int_\Omega \nabla w \nabla r \, d\mathbf{x} = c \int_\Omega w^p r \, d\mathbf{x}$$

for all $r \in H_0^1(\Omega)$ and some $c > 0$. This implies $-\Delta w = c\, w^p$. Setting $u = c^{-1/(p-1)} w$, we get $-\Delta u = u^p$ and $u \geq 0$, $u \neq 0$. By the strong maximum principle, $u > 0$.

If $p+1 = p^* = \frac{2n}{n-2}$ and $n > 2$ existence depends on the geometry of Ω, see [1].

2. *Given a solution* $u \in W^{1,\infty}_{\rm loc}(\mathbf{R}^+, H^1_0(\Omega)) \cap W^{2,\infty}_{\rm loc}(\mathbf{R}^+, L^2(\Omega))$ *of*

$$u_{tt} - \Delta u + \alpha |u_t|^{p-1} u_t = 0 \quad \text{in } L^\infty(\mathbf{R}^+, H^{-1}(\Omega))$$

with $\alpha > 0$, $1 < p$ *and* $p+1 < p^*$, *we set*

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u(\mathbf{x},t)|^2 \, d\mathbf{x} + \frac{1}{2}\int_\Omega |u_t(\mathbf{x},t)|^2 \, d\mathbf{x}.$$

Then, for some positive constant $C(E(0))$, *we have*

$$E(t) \le C(E(0)) t^{-2/(p-1)}, \quad t > 0.$$

Proof taken from [2]. We set $\phi(t) = E^{(p-1)/2} \int_\Omega u u_t \, d\mathbf{x}$. Next, we differentiate with respect to t to get

$$\begin{aligned} E'(t) &= -\alpha \int_\Omega |u_t|^{p+1} d\mathbf{x} \le 0, \\ \phi'(t) &= E(t)^{(p-1)/2} \left(\int_\Omega |u_t|^2 \, d\mathbf{x} - \int_\Omega |\nabla u|^2 \, d\mathbf{x} - \alpha \int_\Omega |u_t|^{p-1} u_t u d\mathbf{x} \right) \\ &\quad + \frac{p-1}{2} E(t)^{(p-3)/2} E'(t) \int_\Omega u u_t \, d\mathbf{x} \end{aligned}$$

First, notice that $E(t) \le E(0)$ and $-\int_\Omega |\nabla u|^2 \, d\mathbf{x} = -2E(t) + \int_\Omega |u_t|^2 \, d\mathbf{x}$. Moreover,

$$E(t)^{-1} \left| \int_\Omega u u_t \, d\mathbf{x} \right| \le E(t)^{-1} \left(\frac{1}{2}\int_\Omega |u|^2 \, d\mathbf{x} + \frac{1}{2}\int_\Omega |u_t|^2 \, d\mathbf{x} \right) \le C(\Omega)$$

for some positive constant $C(\Omega)$ because Poincaré's inequality implies $\frac{1}{2}\int_\Omega |u|^2 \, d\mathbf{x} \le \frac{\lambda(\Omega)}{2} \int_\Omega |\nabla u|^2 \, d\mathbf{x}$. As a consequence, we get

$$\begin{aligned} \phi'(t) &\le 2E(t)^{(p-1)/2} \int_\Omega |u_t|^2 \, d\mathbf{x} - \alpha E(t)^{(p-1)/2} \int_\Omega |u_t|^{p-1} u_t u d\mathbf{x} \\ &\quad -2E(t)^{(p+1)/2} - \frac{p-1}{2} C(\Omega) E(0)^{(p-1)/2} E'(t). \end{aligned}$$

Now we set $\psi_\varepsilon(t) = (1+K_1\varepsilon)E(t) + \varepsilon\phi(t)$ with $K_1 = \frac{p-1}{2} C(\Omega) E(0)^{(p-1)/2}$. We get

$$\begin{aligned} \psi'_\varepsilon(t) &\le 2\varepsilon E(t)^{(p-1)/2} \int_\Omega |u_t|^2 \, d\mathbf{x} - \alpha\varepsilon E(t)^{(p-1)/2} \int_\Omega |u_t|^{p+1} d\mathbf{x} \\ &\quad -2\varepsilon E(t)^{(p+1)/2} - \alpha \int_\Omega |u_t|^{p+1} d\mathbf{x} \end{aligned}$$

Notice that $\|u_t\|_{L^2}^2 \leq \text{meas}(\Omega)^{(p-1)/(p+1)}(\int_\Omega |u_t|^{p+1})^{2/(p+1)}$. By Young's inequality

$$2\varepsilon E(t)^{\frac{(p-1)}{2}} \int_\Omega |u_t|^2\, d\mathbf{x} \leq 2\varepsilon \ \text{meas}(\Omega)^{\frac{p-1}{p+1}} E(t)^{\frac{p-1}{2}} \left(\int_\Omega |u_t|^{p+1}\right)^{\frac{2}{p+1}}$$
$$\leq \varepsilon E(t)^{\frac{p+1}{2}} + \varepsilon\delta \int_\Omega |u_t|^{p+1}$$

for some positive δ depending on Ω.

Using Sobolev injections for $p+1 < p^*$ we find

$$\int_\Omega |u_t|^{p-1} u_t u\, d\mathbf{x} \leq \left(\int_\Omega |u_t|^{p+1}\, d\mathbf{x}\right)^{\frac{p}{p+1}} \|u\|_{L^{p+1}} \leq S(\Omega)\|u_t\|_{L^{p+1}}^p \|\nabla u\|_{L^2}.$$

Notice that $\|\nabla u\|_{L^2} \leq 2E(t)$. By Young's inequality again

$$\varepsilon\alpha E(t)^{(p-1)/2} \int_\Omega |u_t|^{p-1} u_t u\, d\mathbf{x} \leq \varepsilon\alpha E(t)^{(p-1)/2} S(\Omega)\|u_t\|_{L^{p+1}}^p \|\nabla u\|_{L^2}$$
$$\leq \frac{\alpha}{2} \int_\Omega |u_t|^{p+1} + \varepsilon\eta(\varepsilon)E(t)^{(p+1)/2}$$

where $\eta > 0$ depends on $E(0)$, Ω, α and ε, and tends to zero as ε tends to zero. Adding up, we get

$$\psi_\varepsilon'(t) \leq (-\frac{\alpha}{2} + \varepsilon\delta) \int_\Omega |u_t|^{p+1} + \varepsilon(-1 + \eta(\varepsilon))E(t)^{(p+1)/2}.$$

On the other hand, for ε small enough,

$$\frac{1}{\varepsilon}E(t) \leq (1 - K_2\varepsilon)E(t) \leq \psi_\varepsilon(t) \leq (1 + K_2\varepsilon) \leq 2E(t).$$

Choosing ε small enough, we find

$$\psi_\varepsilon'(t) \leq -\frac{\varepsilon}{4}E^{(p+1)/2} \leq -\frac{\varepsilon K_3}{4}\psi_\varepsilon(t)^{(p+1)/2}.$$

Integrating the inequality we find $E(t) \leq C(E(0))t^{-2/(p-1)}$ for $t > 0$.

3. *Prove that the function* $v(\mathbf{x}, t) = |t|^{\frac{p}{p-1}}\phi(\mathbf{x})$, $1 < p < p^* - 1$, *where*

$$-\Delta\phi = \left(\frac{p}{p-1}\right)^p |\phi|^{p-1}\phi \quad \mathbf{x} \in \Omega,$$
$$\phi = 0 \quad \mathbf{x} \in \partial\Omega,$$

is a solution of the backward parabolic problem

$$-\Delta v + |v_t|^{p-1} v_t = 0 \quad \mathbf{x} \in \Omega \times (-\infty, 0],$$
$$v = 0 \quad \mathbf{x} \in \partial\Omega \times (-\infty, 0].$$

Proof taken from [3, 8]. We see that

$$v_t = -\frac{p}{p-1}|t|^{\frac{1}{p-1}}\phi(\mathbf{x}),$$

$$|v_t|^{p-1}v_t = -\left(\frac{p}{p-1}\right)^p |t|^{\frac{p}{p-1}}|\phi(\mathbf{x})|^{p-1}\phi(\mathbf{x}),$$

$$-\Delta v = -|t|^{\frac{p}{p-1}}\Delta\phi(\mathbf{x}) = |t|^{\frac{p}{p-1}}\left(\frac{p}{p-1}\right)^p |\phi(\mathbf{x})|^{p-1}\phi(\mathbf{x}),$$

so that the equation is fulfilled. Existence of ϕ follows from critical point theory.

4. *Consider the vorticity equation in two dimensions. Let* $v = \operatorname{curl}\mathbf{u} \in C((0,\infty); W^{1,p}(\mathbf{R}^2))$, $1 \le p \le \infty$, *be the solution of*

$$v_t - \Delta v + \mathbf{u}\cdot\nabla v = 0, \qquad \mathbf{x} \in \mathbf{R}^2 \times \mathbf{R}^+$$
$$v(\mathbf{x},0) = v_0, \qquad \mathbf{x} \in \mathbf{R}^2,$$

for a divergence free velocity field u *and an initial datum* $v_0 \in L^1(\mathbf{R}^2)$. *Prove 1) that the mass* $\int_{\mathbf{R}^2} v_0\, d\mathbf{x}$ *does not change with time and 2) that* $\|v(t)\|_{L^p(\mathbf{R}^2)} \le Ct^{-1+\frac{1}{p}}$ *for* $t > 0$.

Proof taken from [4, 5]. Notice that $\mathbf{u}\cdot\nabla v = \operatorname{div}(\mathbf{u}v) = 0$. Integrating the equation, using the divergence theorem, and the fact that v vanishes at infinity we get

$$\frac{d}{dt}\int_{\mathbf{R}^2} v_0\, d\mathbf{x} = 0.$$

The velocity vector is given by

$$\mathbf{u}(\mathbf{x},t) = K * v(\mathbf{x},t) = \frac{1}{2\pi}\int_{\mathbf{R}^2}\frac{(-y_2,y_1)}{|\mathbf{y}|^2}v(\mathbf{x}-\mathbf{y},t)d\mathbf{y}$$

where the kernel $K \in L^{2,\infty}$ and $\|K * v\|_{L^r} \le \|K\,\|_{L^{2,\infty}}\|v\|_{L^p}$ for $r > 2$, $1 < p < 2$, $1/r = 1/p - 1/2$.

Writing down the integral expression for the solution

$$v(t) = G(t) * v_0 + \int_0^t \nabla G(t-s) * [v(s)\,\mathbf{K} * v(s)]ds,$$

where $G(t)$ stands for the heat kernel, and taking norms we find

$$\|v(t)\|_{L^p} = \|G(t) * v_0\|_{L^p} + \int_0^t \|\nabla G(t-s) * [v(s)\,\mathbf{K} * v(s)]\|_{L^p} ds.$$

The integral terms decays faster than the rest, therefore

$$\|v(t)\|_{L^p} \sim \|G(t) * v_0\|_{L^p} \le Ct^{-1+\frac{1}{p}}.$$

Recall that $G(t) * v_0$ is a solution of the heat equation with datum v_0 and it belongs to L^p for all $1 \leq p \leq \infty$ for any $t > 0$ if $v_0 \in L^1$. Moreover, $\|G(t) * v_0\|_{L^p} \leq \|G(t)\|_{L^p} \|v_0\|_{L^1}$ and $\|G(t)\|_{L^p} = Ct^{-1+\frac{1}{p}}$.

5. *Let* $\mathbf{u}$ *be a solution of the incompressible Navier-Stokes equations in two dimensions with initial datum* $\mathbf{u}_0 \in L^1 \cap L^2(\mathbf{R}^2)$ *such that* $\operatorname{div}(\mathbf{u}_0) = 0$. *Then* $\mathbf{u}(t) \in L^p(\mathbf{R}^2)$ *for* $1 \leq p \leq 2$ *and* $t > 0$.

   Proof taken from [6, 10]. The theory of classical solutions with L^2 data, that is, $\mathbf{u}_0 \in L^2(\mathbf{R}^2)$ guarantees that $\mathbf{u}(t) \in L^\infty([0,\infty); L^2(\mathbf{R}^2))$ and is bounded by $\|\mathbf{u}_0\|_{L^2}$. By taking the divergence of Navier-Stokes equations

$$\mathbf{u}_t - \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nabla p, \quad \operatorname{div}(\mathbf{u}) = 0,$$

we get an equation for the pressure

$$-\Delta p = \operatorname{div}(\mathbf{u} \cdot \nabla \mathbf{u}).$$

The pressure is then the convolution $p = E_2 * \operatorname{div}(\mathbf{u} \cdot \nabla \mathbf{u})$, where E_2 is the fundamental solution of $-\Delta$ in $\mathbf{R}^2$, up to a function of time. Then $\mathbf{u}$ satisfies the integral equation

$$\mathbf{u}(t) = G(t) * \mathbf{u}_0 + \int_0^t \partial_i G(t-s) * u_i \mathbf{u}(s) ds + \int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * u_i u_j(s) ds,$$

where ∂_i denotes partial derivative with respect to x_i, u_i are components of $\mathbf{u}$ and summation with respect to repeated indices is intended. Since $u \in L^1$, $G(t) * u_0 \in L^q$ for all $q > 1$ and $t > 0$. On the other hand, $u(s) \in L^2$ implies that $u_i u_j(s) \in L^1$. Moreover,

$$\| \int_0^t \partial_i G(t-s) * u_i u_j(s) ds \|_{L^q} \leq C \int_0^t (t-s)^{-1+\frac{1}{q}-\frac{1}{2}} \| \mathbf{u}\|_{L^2}^2 ds \leq Ct^{\frac{1}{q}-\frac{1}{2}}$$

for $1 \leq q < 2$. Thus, the first integral belongs to L^q for $1 \leq q < 2$. Let us consider now the second integral. Since $\partial_i G(t)$ belongs to the Hardy space $\mathcal{H}^1(\mathbf{R}^2)$ and $\partial_j \nabla E_2$ is a Calderon-Zygmund kernel, we conclude that $\partial_i G(t-s) * \partial_j \nabla E_2 \in L^1$ and

$$\|\partial_i G(t-s) * \partial_j \nabla E_2\|_{L^1} \leq C \|\partial_i G(t-s)_{\mathcal{H}^1} < C(t-s)^{\frac{-1}{2}}.$$

Thus,

$$\| \int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * u_i u_j(s) ds \|_{L^1} \leq \int_0^t C(t-s)^{\frac{-1}{2}} \|\mathbf{u}(s)\|_{L^2}^2 ds \leq Ct^{\frac{1}{2}}.$$

In an analogous way, since $\partial_j \nabla E_2$ is a Calderon-Zygmund kernel, we conclude that $\partial_i G(t-s) * \partial_j \nabla E_2 \in L^q$, $1 < q < \infty$ and

$$\|\partial_i G(t-s) * \partial_j \nabla E_2\|_{L^q} \leq C \|\partial_i G(t-s)\|_{L^q} < C(t-s)^{-1+\frac{1}{q}-\frac{1}{2}}.$$

Thus,

$$\| \int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * u_i u_j(s) ds \|_{L^q} \le \int_0^t C(t-s)^{-1+\frac{1}{q}-\frac{1}{2}} \|\mathbf{u}(s)\|_{L^2}^2 ds$$
$$\le C t^{\frac{1}{q}-\frac{1}{2}}$$

for $1 < q \le 2$.

6. *Consider the convection diffusion equation*

$$u_t - \Delta u + \partial_y(|u|^{q-1}u) = 0$$

set in $\mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}^+$, *with* $\mathbf{x} = (x_1, \dots, x_{n-1}, y)$. *Assume that* V *is a solution with initial datum* $V_0 \in (L^1 \cap L^\infty)(\mathbf{R}^n)$ *and* v *is a solution with initial datum* $v_0 \in (L^1 \cap L^\infty)(\mathbf{R}^n)$. *Assume that*

$$v, V \in C^1([0,T]; L^2(\mathbf{R}^2)) \cap L^\infty([0,T]; H^2(\mathbf{R}^2)) \cap L^\infty((0,T) \times \mathbf{R}^2)$$

for every $T > 0$. *Then,* $v \le V$.

Proof taken from [7, 9]. The function $w = v - V$ satisfies

$$w_t - \Delta w + \partial_y(|v|^{q-1}v) - \partial_y(|V|^{q-1}V) \le 0$$

and $w(0) \le 0$. Multiplying the inequality by w^+ and integrating by parts, we obtain

$$\frac{d}{dt}\int \frac{|w^+(t)|^2}{2} d\mathbf{x} + \int |\nabla w^+(t)|^2 d\mathbf{x} \le \int a w^+(t) \partial_y w^+(t) d\mathbf{x}$$

where $a(\mathbf{x}, t) = \frac{|v|^{q-1}v - |V|^{q-1}V}{v - V}$ is a bounded function. Integrating in t and applying Young's inequality we get

$$\frac{\|w^+(t)\|_2^2}{2} + \int_0^t \|\nabla w^+(s)\|_2^2 ds \le K_1 \int_0^t \|w^+(s)\|_2^2 ds + \varepsilon \int_0^t \|\nabla w^+(s)\|_2^2 ds$$

for ε as small as needed. Notice that $w^+(0) = 0$. Gronwall's inequality for

$$\|w^+(t)\|_2^2 \le 2K_1 \int_0^t \|w^+(s)\|_2^2 ds$$

implies $w^+(t) = 0$.

7. *A line vortex lying along a curve* Γ *in an incompressible inviscid and irrotational fluid is a solution of the following equations*

$$\mathrm{div}(\mathbf{u}) = 0, \quad \mathrm{curl}(\mathbf{u}) = \omega_0 \delta_\Gamma(\mathbf{x}),$$

where $\mathbf{u}$ *is the fluid velocity,* $\omega_0 = 2\pi\gamma$ *is the circulation around the vortex and* γ *is the vortex strength.* δ_Γ *is a Dirac function supported at the curve* Γ. *Express this solution in terms of a vector stream function.*

Taken from [11]. We define a vector stream function $\mathbf{U}$ in $\mathbf{R}^3$ as the solution of $\mathrm{div}(\mathbf{U}) = 0$, $\mathrm{curl}(\mathbf{U}) = \mathbf{u}$. Then $-\Delta \mathbf{U} = \omega_0 \delta_\Gamma(\mathbf{x})$. Using the Green function for the Laplacian in $\mathbf{R}^3$ we get $\mathbf{U} = \frac{\omega_0}{4\pi} \int_\Gamma \frac{1}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$.

8. *We know that the problem*

$$g_t - \Delta_v g + \mathbf{v} \cdot \nabla_x g + \mathbf{E}(\mathbf{x}, t) \cdot \nabla_v g = 0, \qquad \mathbf{x} \in \mathbf{R}^3, \mathbf{v} \in \mathbf{R}^3, t \in \mathbf{R}^+,$$
$$g(\mathbf{x}, \mathbf{v}, 0) = g_0(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in \mathbf{R}^3, \mathbf{v} \in \mathbf{R}^3,$$

with $g_0 \in L^1(\mathbf{R}^3 \times \mathbf{R}^3)$ and bounded and Lipschitz $\mathbf{E}$ admits fundamental solutions $\Gamma_{\mathbf{E}}$. The solution of the initial value problem can be expressed as

$$g(\mathbf{x}, \mathbf{v}, t) = \int \Gamma_{\mathbf{E}}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', 0) d\mathbf{x}' d\mathbf{v}'$$

and $\Gamma_{\mathbf{E}}$ satisfies the estimates

$$|\Gamma_{\mathbf{E}}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t')| \leq C(\|\mathbf{E}\|_{L^\infty_{\mathbf{x},t}}, T)\, G(\mathbf{x}/2, \mathbf{v}/2, t; \mathbf{x}'/2, \mathbf{v}'/2, t'),$$
$$|\partial_{v_i} \Gamma_{\mathbf{E}}(\mathbf{x}, \mathbf{v}, t; \mathbf{x}', \mathbf{v}', t')| \leq C(\|\mathbf{E}\|_{L^\infty_{\mathbf{x},t}}, T)\, \frac{G(\mathbf{x}/2, \mathbf{v}/2, t; \mathbf{x}'/2, \mathbf{v}'/2, t')}{(t - t')^{1/2}},$$

where G is the fundamental solution for the problem with $\mathbf{E} = 0$. Extend these results to problems for which $\mathbf{E}$ is just bounded.

Taken from [12]. We regularize $\mathbf{E}$ by convolution and consider $\mathbf{E}_\delta = \mathbf{E} * \eta_\delta$ where η_δ is a mollifying family of functions. Then $\mathbf{E}_\delta$ are bounded and Lipschitz, so for each of them we can construct solutions g_δ of the initial value problem and have estimates on the fundamental solutions Γ_δ. Moreover, $\|\mathbf{E}_\delta\|_{L^\infty_{x,t}} \leq \|\mathbf{E}\|_{L^\infty_{x,t}}$ and $\mathbf{E}_\delta \to \mathbf{E}$ as $\delta \to 0$.

Since Γ_δ is bounded (locally in t) in any L^p_{xvt} space, a subsequence converges weakly (locally in t) in any L^p_{xvt} (weakly * if $p = \infty$) to a function $\Gamma_{\mathbf{E}}$ and we can pass to the limit in the right-hand side of the integral expressions for the solutions g_δ in terms of Γ_δ.

Moreover, the integral expressions imply that g_δ are uniformly bounded in any space L^p_{xvt} with respect to δ and locally in t. Therefore, g_δ converges weakly (locally in t) in any L^p_{xvt} space to a function g and their derivatives also converge in the sense of distributions.

In the distribution sense, the derivatives of Γ_δ with respect to $\mathbf{v}$ converge weakly to the derivatives of $\Gamma_{\mathbf{E}}$. We can also pass to the limit in the inequalities satisfied by Γ_δ and establish similar inequalities for $\Gamma_{\mathbf{E}}$ because $\|\mathbf{E}_\delta\|_{L^\infty_{x,t}} \leq \|\mathbf{E}\|_{L^\infty_{x,t}}$.

Now, multiplying the differential equation satisfied by g_δ by g_δ we get a uniform L^2_{xvt} bound on $\nabla_v g_\delta$. If we multiply the equation by $|\mathbf{v}|^2$ we get a uniform L^1_{xvt} bound on $|\mathbf{v}|^2 g_\delta$.

Multiplying the differential equations satisfied by g_δ by test functions, we can pass to the limit in all the terms of the weak formulation of the equation except in $\mathbf{E}_\delta \nabla_v g_\delta$ with the convergences already established. The passage to the limit in this term is technical, see details in [12]. Finally, g is a solution for the initial value problem with bounded $\mathbf{E}$ and $\Gamma_{\mathbf{E}}$ an associated fundamental solution.

9. *Prove that the solution of*

$$z_t - \Delta z = \mathbf{d} \cdot \nabla(G^q), \quad z(0) = 0$$

can be calculated in terms of heat kernels.

Taken from [19]. Set $z = \mathbf{d} \cdot \nabla g$ where $g_t - \Delta g = G^q$, $g(0) = 0$, that is,

$$g(t) = \int_0^t G(t-s) * G^q(s) ds.$$

10. *Prove that the solution* Φ *of the equation*

$$-\frac{d^2}{dx^2}\Phi(x) = n_D(x) - \int_{\mathbf{R}} \frac{dk}{1 + \exp(\epsilon(k) - \Phi(x))}$$

with $\int_{\mathbf{R}^2} \frac{dk dx}{1+\exp(\epsilon(k)-\Phi(x))} = a$ *fixed and* $\frac{d\Phi}{dx} \in L^2$ *is unique.*

Taken from [21]. Assume that there are two solutions Φ_1 and Φ_2 satisfying such conditions. Set $U = \Phi_1 - \Phi_2$. Then, $\frac{dU}{dx} \in L^2$ and

$$\frac{d^2 U}{dx^2} = \int_{\mathbf{R}} \frac{dk}{1 + \exp(\epsilon(k) - \Phi_1(x))} - \int_{\mathbf{R}} \frac{dk}{1 + \exp(\epsilon(k) - \Phi_2(x))}.$$

Let us assume first that $U(x) > 0$ everywhere. Then

$$a = \int_{\mathbf{R}^2} \frac{dk dx}{1 + \exp(\epsilon(k) - \Phi_1(x))} > \int_{\mathbf{R}^2} \frac{dk dx}{1 + \exp(\epsilon(k) - \Phi_2(x))} = a,$$

which is impossible.

Let us assume now that there is a unique point x_0 at which $U(x_0) = 0$. We take $U(x) < 0$ for $x < x_0$ and $U(x) > 0$ for $x > x_0$. Thus, $\frac{d^2U}{dx^2} < 0$ if $x < x_0$ and $\frac{d^2U}{dx^2} < 0$ if $x > x_0$. Then, $\frac{dU}{dx}$ is decreasing if $x < x_0$ and $\frac{dU}{dx}$ is increasing if $x > x_0$. On the other hand,

$$\int_{\mathbf{R}} \left(\frac{dU}{dx}\right)^2 dx = \int_{-\infty}^{x^*} \left(\frac{dU}{dx}\right)^2 dx + \int_{x^*}^{\infty} \left(\frac{dU}{dx}\right)^2 dx$$

is finite. If there exists x^* such that $\frac{dU(x^*)}{dx} > 0$ and $x^* < x_0$ then $\int_{-\infty}^{x^*} \left(\frac{dU}{dx}\right)^2 dx > \left(\frac{dU(x^*)}{dx}\right)^2 \int_{-\infty}^{x^*} dx = \infty$. This is impossible, so that $\frac{dU}{dx} \leq 0$ for all x and U is decreasing. This contradicts our assumption on x_0. Therefore, we should have at least to points x_0 and x_1 at which U vanishes.

Let x_0 and x_1 be such that $U(x_0) = U(x_1) = 0$. If x_M is such that $U(x_M) = \max\{U(x), x_0 \leq x \leq x_1\} > 0$, then $\frac{d^2U(x_M)}{dx^2} \leq 0$ because the maximum is attained at an interior point. However,

$$0 \geq \frac{d^2U(x_M)}{dx^2} = \int_{\mathbf{R}} \frac{dk}{1+\exp(\epsilon(k)-\Phi_1(x_M))} - \int_{\mathbf{R}} \frac{dk}{1+\exp(\epsilon(k)-\Phi_2(x_M))} > 0,$$

since $U(x_M) > 0$. Hence, $\max\{U(x),\, x_0 \le x \le x_1\} = 0$. In an analogous way, we conclude that $U(x_m) = \min\{U(x),\, x_0 \le x \le x_1\} = 0$. Therefore, $U = 0$ on $[x_0, x_1]$.

Now we set $x_0 = \min\{x \,|\, U(x) = 0\}$ and $x_1 = \max\{x \,|\, U(x) = 0\}$. Then, either $U(x) < 0$ for $x < x_0$ and $U(x) > 0$ for $x > x_1$. Repeating the above arguments, we would obtain $x' \notin [x_0, x_1]$ such that $U(x') = 0$. This contradicts the definition of x_0 and x_1. Therefore, $U = 0$ everywhere and $\Phi_1 = \Phi_2$.

11. *Consider the hyperbolic problem*

$$\begin{aligned}
\frac{\partial^2 E}{\partial x \partial t} + A\frac{\partial E}{\partial t} + B\frac{\partial E}{\partial x} + C\frac{\partial J}{\partial t} + D = 0, &\qquad x \in (0, L),\ t > 0,\\
E(x, 0) = 0, &\qquad x \in (0, L),\\
E(0, t) = \rho J(t), &\qquad t \ge 0,\\
\int_0^L E(x, t)dx = \phi, &\qquad t \ge 0,
\end{aligned}$$

where ρ, ϕ, L are positive and A, B, C, D are bounded functions, A and B positive, while C is negative. What would be an adequate numerical scheme to solve this problem?

Hyperbolic problems are typically discretized in explicit ways. However, in this case i) we have an integral constraint which couples all the values at each time level, ii) the hyperbolic operator is given in non characteristic form. We use forward finite differences of first order for first order time derivatives of E and J. A second order approximation for the space derivative of E because the use of central differences leads to instabilities. The second order derivarive E_{xt} is approximated combining the space and time derivative approximation just described. At $x = 1$ we use for the first order spatial derivative of E a first order backward difference formula. The integral constraint is discretized by means of a composite trapezoidal rule. For a proof of the convergence and stability properties of the scheme see [16].

12. *Construct solutions of the scalar conservation law $w_t + (c(x)w)_0 = 0$ with $w(0) = w_0$.*

Taken from [17]. We set $v = cw$. Then, $v_t + cv_x = 0$. Thus, v is constant along the characteristic curves $x(t)$ solution of $x'(t) = c(x(t))$, $x(0) = x_0$, because

$$\frac{d}{dt}v(x(t), t) = v_x(x(t), t)x'(t) + v_t(x(t), t) = 0.$$

Given (x, t) we may be able to calculate $x_0(x, t)$ such that the characteristic curve with initial value $x_0(x, t)$ satisfies $x(t) = x$. Then $v(x, t) = v(x(t), t) = v_0(x_0(x, t))$ and $w(x, t) = \frac{v_0(x_0(x,t))}{c(x_0(x,t))}$. The feasibility of this procedure will depend on the function c.

13. *Consider the differential difference equation $u_n'(t) = u_{n+1} - 2u_n + u_{n-1} - A\sin(u_n)$, where A is a positive parameter. Prove that there is a monotone solution such that $u_{-\infty} = 0$ and $u_\infty = 2\pi$ with $u_0 = \pi$ and $u_n - \pi = \pi - u_{-n}$ for all n.*

Taken from [14]. We set $u_0 = \pi$ and vary u_1 in the interval $(\pi, 2\pi)$ to find the desired solution. The condition $u_0 = \pi$ ensures that $u_n - \pi$ is an odd function of n. We first choose $\epsilon > 0$ so that $-A\sin(u) > \epsilon(u - \pi)$ for $\pi < u \le \frac{3}{2}\pi$. Then, we choose N large so that $\epsilon(N-1) > 1$. Next, we choose $u_1 - \pi$ small so that $u_j \le \frac{3}{2}\pi$ for $1 \le j \le N$. We wish to show that under these conditions, the finite sequence $\{u_1, \ldots, u_N\}$ is not monotone increasing. It is convenient to let $U_n = u_n - \pi$. If $\{U_1, \ldots, U_N\}$ is monotone increasing, then $2 \le j \le N$ and $U_j \le (2-\epsilon)U_{j-1} - U_{j-2}$. Adding these inequalities results in $U_N - U_{N-1} \le \epsilon \sum_{i=2}^{-N-1} U_i + (1-\varepsilon)U_1$. Since we assumed that $U_i \ge U_1$ for $2 \le i \le N$, our lower bound on N then shows that $U_N < U_{N-1}$, a contradiction. Therefore, we have shown that for sufficiently small U_1, the sequence starts to decrease before crossing π. On the other hand, we have simply to choose $U_1 > \pi$ to have the sequence cross π before decreasing. Note that if the sequence increases until some first N such that $U_N = \pi$, then $U_{N+1} > \pi$. If, finally, there is an N such that the sequence increases up to $n = N$, with $U_N < \pi$, and $U_N = U_{N+1}$, then $U_{N+2} < U_{N+1}$ so that the sequence decreases before reaching π.

14. *Let $U_i(t)$ and $L_i(t)$, $i \in \mathbf{Z}$ be differentiable sequences such that*

$$U_i'(t) - d_1(U_i)(U_{i+1} - U_i) - d_2(U_i)(U_{i-1} - U_i) - f(U_i) \ge \\ L_i'(t) - d_1(L_i)(L_{i+1} - L_i) - d_2(L_i)(L_{i-1} - L_i) - f(L_i)$$

and $U_i(0) < L_i(0)$ for all i, where f, $d_1 > 0$ and $d_2 > 0$ are Lipschitz continuous functions. Then, $U_i(t) > L_i(t)$ for all $t > 0$ and $i \in \mathbf{Z}$.

Taken from [15]. By contradiction, set $W_i(t) = U_i(t) - L_i(t)$. At $t = 0$, $W_i(0) > 0$ for all i. Let us assume that W_i changes sign after a certain minimum time $t_1 > 0$, at some value of i, $i = k$. Thus $W_k(t_1) = 0$ and $W_k'(t) \le 0$, as $t \to t_1$. We shall show that this is contradictory. At $t = t_1$, there must be an index m (equal or different from k) such that $W_m(t_1) = 0$, while its next neighbor $W_{m+j}(t_1) > 0$ (j is either 1 or -1), and $W_i(t_1) = 0$ for all indices between k and m. For otherwise W_k should be identically 0 for all k. The differential inequality implies

$$W_m'(t_1) \ge d_1(U_m(t_1))W_{m+1}(t_1) + d_2(U_m(t_1))W_{m-1}(t_1) > 0.$$

This contradicts the fact that $W_m'(t)$ should have been nonpositive as $t \to t_1$, for $W_m(t_1)$ to have become zero in the first place.

15. *Consider the equation*

$$U'(t) = z_1(F/A) + z_3(F/A) - 2U(t) - A\sin(U(t)) + F,$$

for $|F| < A$, $A >> 1$ where $z_1(F/A) < z_2(F/A) < z_3(F/A)$ are three consecutive solutions of the equation $\sin(U(t)) = F/A$ in one period. Prove that there is a critical value F_c such that this equation has three stable constant solutions if $0 \leq F < F_c$ but one if $F > F_c$. Characterize F_c.

Taken from [18]. When $F = 0$, $z_1(0) = 0$, $z_2(0) = \pi$ and $z_3(0) = 2\pi$. We need to solve

$$2z + A\sin(z) = F + 2\mathrm{arcsin}(F/A) + 2\pi.$$

As we increase F from 0, we keep on finding three solutions $z_1(F/A) < z_2(F/A) < z_3(F/A)$ continuing these branches until $F + 2\mathrm{arcsin}(F/A) + 2\pi$ hits the first local maximum of $2z + A\sin(z)$ (remember that A is large). The value F_c at which this happens is characterized by the existence of a double zero, a value u_0 such that $2 + A\cos(u_0) = 0$ and $2u_0 + A\sin(u_0) = F_c + 2\mathrm{arcsin}(F_c/A) + 2\pi$. Then, $u_0 = \arccos(-2/A)$ and F_c is the solution of $2u_0(A) + A\sin(u_0(A)) = F_c + 2\mathrm{arcsin}(F_c/A) + 2\pi$. Below F_c we have three zeroes, at F_c two collapse, above F_c the collapsing ones, $z_1(F/A)$ and $z_2(F/A)$ are lost.

$z_1(F/A)$ and $z_3(F/A)$ are stable while they exist. This picture corresponds to a saddle node bifurcation in the system, see [18].

16. *The system of equations*

$$\frac{dE_i}{dt} + \frac{v(E_i)}{\nu}(E_i - E_{i-1}) - \frac{D(E_i)}{\nu}(E_{i+1} - 2E_i + E_{i-1}) = J - v(E_i),$$

for $i \in \mathbf{Z}$ admits traveling wave solutions of the form $E_i(t) = E(i - ct)$ propagating at constant velocity c when the parameter J is large enough. Here, v, D are positive functions and $\nu > 0$ is large. v is a cubic, it grows from 0 to a local maximum, decreases to a positive minimum, and increases to infinity later. Justify that the wavefront velocity scales as $(J - J_c)^{1/2}$ where J_c is the threshold for existence of travelling waves.

Taken from [20]. For ν large, we can construct stationary solutions, which can be approximated by

$$E_i \sim z_1(J) \quad i < 0, \quad E_i \sim z_3(J) \quad i > 0,$$

for $|J| < J_c$, while E_0 solves

$$J - v(E_0) - \frac{v(E_0)}{\nu}(E_0 - z_1(J)) + \frac{D(E_0)}{\mu}(z_3(J) - 2E_0 + z_1(J)) = 0,$$

where $z_1(J) < z_2(J) < z_3(J)$ are solutions of $J = v(z)$. At a value J_c, $z_1(J_c) = z_2(J_c)$ and these roots are lost for $J > J_c$, only $z_3(J)$ remains. The reduced equation

$$\frac{dE_0}{dt} = J - v(E_0) - \frac{v(E_0)}{\nu}(E_0 - z_1(J)) + \frac{D(E_0)}{\nu}(z_3(J) - 2E_0 + z_1(J)),$$

for the middle point undergoes a saddle node bifurcation at J_c with normal form

$$\phi' = \alpha(J_c)(J - J_c) + \beta(J_c)\phi^2,$$

which has solutions of the form $\sqrt{\frac{\alpha}{\beta}(J - J_c)}\tan(\sqrt{\alpha\beta(J - J_c)}(t - t_0))$, blowing up when the argument of the tangent approaches $\pm\pi/2$, over a time $t - t_0 \sim \pi/\sqrt{\alpha\beta(J - J_c)}$.

Now, for $J > J_c$ but close to J_c, simulations show staircase like wave profiles, in which a point stays near the vanished equilibrium $E_0(J_c)$ until it moves following the tangent path given by the normal form and is replaced at position $E_0(J_c)$ by a neighbouring one, once and again. The wave velocity is the reciprocal of the time this transition takes $c(J, \nu) \sim \frac{\sqrt{\alpha\beta(J-J_c)}}{\pi}$, see [20] for details.

17. *We consider a problem with noise*

$$\frac{du_i}{dt} = u_{i+1} - 2u_i + u_{i-1} + F - A\sin(u_i) + \gamma\xi_i,$$

where $A > 0$ is large and $\gamma > 0$ characterizes the disorder strength and ξ_i is a zero mean random variable taking values on an interval $(-1, 1)$ with equal probability. Show that the speed of the wavefronts for F larger than the critical value F_c^* scales as $(F - F_c^*)^{3/2}$.

Taken from [22]. Setting $\gamma = 0$, we can repeat with this equation the study done in the previous exercise and obtain a velocity that scales like $(F - F_c)^{1/2}$. However, when we add noise, for each realization of the noise, the threshold F_c is shifted slightly up or down by the noise. The observed velocity will be the average of the velocities observed for a large number of realizations. If

$$|c_R| \sim \frac{1}{\pi}\sqrt{\alpha(F_c)\beta(F_c)(F - F_c) + \gamma\beta(F_c)\xi_0}$$

the average

$$\bar{c} = \frac{1}{N}\sum_{R=1}^{N}|c_R| = \frac{1}{2\pi}\int_{-1}^{1}(\alpha\beta(F - F_c) + \gamma\beta\xi)^{1/2}d\xi \sim (F - F_c^*)^{3/2}$$

where the new critical field is $F_c^* = F_c - \frac{\gamma}{\alpha}$.

18. *Let $u_{i,j}(t)$ be a solution to*

$$\frac{\partial u_{i,j}}{\partial t} = u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + A(\sin(u_{i,j-1} - u_{i,j})\sin(u_{i,j+1} - u_{i,j}))$$

for $i, j \in \mathbf{Z}$ and $u_{i,j}(0) = \alpha_{i,j}$ satisfying $\alpha_{i+1,j} - 2\alpha_{i,j} + \alpha_{i-1,j} \in l^2$, $\sin(\alpha_{i,j-1} - \alpha_{i,j})\sin(\alpha_{i,j+1} - \alpha_{i,j}) \in l^2$ and $\alpha_{i,j} \in l^\infty_{\mathrm{loc}}$. If $(u_{i,j+1} - u_{i,j})(t) \in$

$\cap_{n\in\mathbf{Z}}\left[-\frac{\pi}{2}+2n\pi, \frac{\pi}{2}+2n\pi\right]$ *holds for all* i, j, t, *then* $u_{i,j}(t)$ *tends to a limit* $s_{i,j}$ *as* $t \to 0$ *which is a stationary solution of the problem.*

Taken from [23]. Define $w_{i,j}(t) = u_{i,j}(t+\tau) - u_{i,j}(t)$ for some $\tau > 0$. Then

$$\frac{d}{dt}\left(\frac{1}{2}\sum_{i,j}|w_{i,j}(t)|^2\right) = -\sum_{i,j}((w_{i+1,j} - w_{i,j})(t))^2 - \sum_{i,j}(\sin((u_{i,j+1} - u_{i,j})(t+\tau)) - \sin((u_{i,j+1} - u_{i,j})(t)))((u_{i,j+1} - u_{i,j})(t+\tau) - (u_{i,j+1} - u_{i,j})(t)) \leq 0.$$

This implies $w_{i,j}(t) \to 0$ as $t \to \infty$ for every i, j. In conclusion, $u_{i,j}(t)$ tends to a limit $s_{i,j}$ which is a stationary solution of the problem.

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