Topological Quantum Field Theories for Character Varieties

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Representation varieties

Representation variety

G a complex algebraic group and Γ a finitely generated group.

$$\mathfrak{X}_G(\Gamma) = \mathrm{Hom}\,(\Gamma, G)$$

Algebraic structure: $\Gamma = \langle \gamma_1, \ldots, \gamma_s \,|\, R_\alpha(\gamma_1, \ldots, \gamma_s) = 1 \rangle$.
We have an identification

$$\begin{array}{cccc} \psi : & \mathrm{Hom}\,(\Gamma, G) & \longrightarrow & G^s \\ & \rho & \mapsto & (\rho(\gamma_1), \ldots, \rho(\gamma_s)) \end{array}$$

with the algebraic set

$$\mathrm{Im}\ \psi = \left\{ (g_1, \ldots, g_s) \in G^s \,|\, R_\alpha(g_1, \ldots, g_s) = 1 \right\}.$$

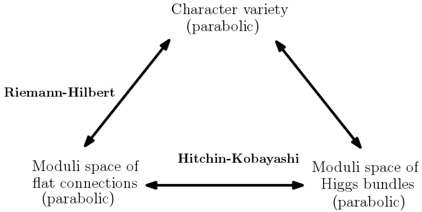
Accross the non-abelian Hodge theory

Character variety

With respect to the action of G by conjugation

$$\mathcal{R}_G(\Gamma) = \mathfrak{X}_G(\Gamma) \mathbin{/\!\!/} G$$

Non-abelian Hodge theory. For $\Gamma = \pi_1(\Sigma_g)$.



Arithmetic method

Problem

Compute the Hodge structure (or the E-polynomial) on the cohomology of $\mathcal{R}_G(\Sigma_g)$ for Σ_g the genus g surface.

Based on Katz' theorem of polynomial counting.

- Hausel and Rodriguez-Villegas (2008). $G = \mathrm{GL}_n(\mathbb{C})$, arbitrary g. Twisted.
- Hausel, Letellier and Rodrigez-Villegas (2011). $G = \mathrm{GL}_n(\mathbb{C})$, arbitrary g. Generic semi-simple marked points.
- Mereb (2015). $G = \mathrm{SL}_n(\mathbb{C})$, arbitrary g. Twisted.

In terms of generating functions of Macdonalds symmetric polynomials.

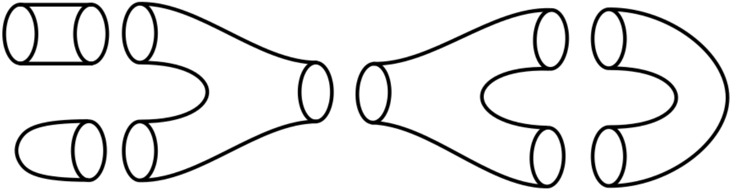
Geometric method

Based on Hodge monodromy representation.

- Logares, Muñoz and Newstead (2013). $G = \mathrm{SL}_2(\mathbb{C})$, $g = 1, 2$. At most 1 marked point.
- Logares and Muñoz (2014). $G = \mathrm{SL}_2(\mathbb{C})$, $g = 1$. At most 2 marked points.
- Martínez and Muñoz (2016). $G = \mathrm{SL}_2(\mathbb{C})$, arbitrary g.
- Martínez (2017). $G = \mathrm{PGL}_2(\mathbb{C})$, arbitrary g.
- Baraglia and Hekmati (2017). $G = \mathrm{GL}_2(\mathbb{C}), \mathrm{GL}_3(\mathbb{C})$, $\mathrm{SL}_2(\mathbb{C}), \mathrm{SL}_3(\mathbb{C})$, arbitrary g.

Explicit expressions but very involved calculations.

Topological Quantum Field Theories



Topological Quantum Field Theories

Topological Quantum Field Theory

A **TQFT** is a monoidal symmetric functor

$$Z : \mathbf{Bd}_n \to R\text{-}\mathbf{Mod}.$$

Let $n \geq 1$. The category $\mathbf{Bd}_n$ is:

- **Objects**: $(n-1)$-dimensional closed manifolds (maybe empty).
- **Morphisms**: $W : M_1 \to M_2$ is a n-dimensional compact manifold W such that $\partial W = M_1 \sqcup M_2$, up to boundary preserving diffeomorphism (**bordism**).
- **Composition**: Gluing of bordisms.

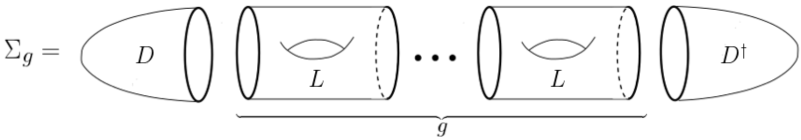
It is a symmetric monoidal category with disjoint union.

Algebraic topology via TQFTs

Goal: Given a closed manifold W, we want to compute an algebraic invariant $\chi(W) \in R$.

Key idea

Construct a TQFT, $Z : \mathbf{Bd}_n \to R\text{-}\mathbf{Mod}$, such that for all $W : \emptyset \to \emptyset$ closed, $Z(W)(1) = \chi(W)$.



$$\chi(W) = Z(\Sigma_g)(1) = Z(D^\dagger) \circ Z(L)^g \circ Z(D)(1)$$

Lax is good enough

Not so easy...

- Monoidality $\Rightarrow Z(M)$ is a finitely generated R-module.
- Natural constructions have $Z(M)$ infinitely generated.

A **lax monoidal TQFT** is a symmetric lax monoidal functor

$$Z : \mathbf{Bd}_n \to R\text{-}\mathbf{Mod}.$$

Lax monoidality

The map

$$Z(M_1) \otimes Z(M_2) \to Z(M_1 \sqcup M_2)$$

exists but it is no longer an isomorphism.

TQFTs as lagrangian field theories

Let $\mathcal{C}$ be a category with final object $\star$ and pullbacks (category of fields).

$$Z : \mathbf{Bd}_n \xrightarrow{\text{Field theory}} \mathrm{Span}(\mathcal{C}) \xrightarrow{\text{Quantization}} R\text{-}\mathbf{Mod}$$

Field Theory

Let $\mathcal{G} : \mathbf{Emb}_c \to \mathcal{C}$ be a monoidal contravariant functor with the Seifert-van Kampen property. It induces a functor

$$\mathcal{F}_{\mathcal{G}} : \mathbf{Bd}_n \longrightarrow \mathrm{Span}(\mathcal{C}).$$

- **Objects:** $\mathcal{F}_{\mathcal{G}}(M) = \mathcal{G}(M)$.
- **Morphisms:** Given $W : M_1 \to M_2$, its image is the span

$$\mathcal{G}(M_1) \xleftarrow{\mathcal{G}(i_1)} \mathcal{G}(W) \xrightarrow{\mathcal{G}(i_2)} \mathcal{G}(M_2).$$

$\mathcal{C}$-Algebra

A $\mathcal{C}$-**algebra** $\mathcal{A}$ is a pair of functors:

$$A : \mathcal{C}^{op} \to \mathbf{Ring} \qquad\qquad B : \mathcal{C} \to A(\star)\text{-}\mathbf{Mod}$$

- They agree on objects: $A(c) = B(c)$ for all $c \in \mathcal{C}$.
- Beck-Chevaley condition: For a pullback diagram

$$\begin{tikzcd} d \arrow[r, "g'"] \arrow[d, "f'"'] & c_1 \arrow[d, "f"] \\ c_2 \arrow[r, "g"'] & c \end{tikzcd} \qquad A(g)\circ B(f) = B(f')\circ A(g')$$

- B preserves external product.

Grothendieck's six operators

$$\mathcal{A}_c = A(c) \in \mathbf{Ring} \qquad f^* = A(f) \qquad f_! = B(f)$$

Quantization

Given a $\mathcal{C}$-algebra $\mathcal{A}$, we define

$$\mathcal{Q}_{\mathcal{A}} : \mathrm{Span}(\mathcal{C}) \to \mathcal{A}_\star\text{-}\mathbf{Mod}.$$

- **Objects:** $\mathcal{Q}_{\mathcal{A}}(c) = \mathcal{A}_c$ for $c \in \mathcal{C}$.
- **Morphisms:** Given a span $S : c_1 \xleftarrow{f} d \xrightarrow{g} c_2$, we define

$$\mathcal{Q}_{\mathcal{A}}(S) = g_! \circ f^* : \mathcal{A}_{c_1} \xrightarrow{f^*} \mathcal{A}_d \xrightarrow{g_!} \mathcal{A}_{c_2}$$

Construction of TQFT

$$Z : \mathbf{Bd}_n \xrightarrow{\mathcal{F}_{\mathcal{G}}} \mathrm{Span}(\mathcal{C}) \xrightarrow{\mathcal{Q}_{\mathcal{A}}} \mathcal{A}_\star\text{-}\mathbf{Mod}$$

Introducing sheaves

Sheaves à la Galatius-Tillmann-Madsen-Weiss

A sheaf is a contravariant functor $\mathcal{S} : \mathbf{Emb}_c \to \mathbf{Cat}$ with 'good gluing properties'.

$$\mathcal{S}(M) = \{\text{Extra structures on } M\}$$

Examples

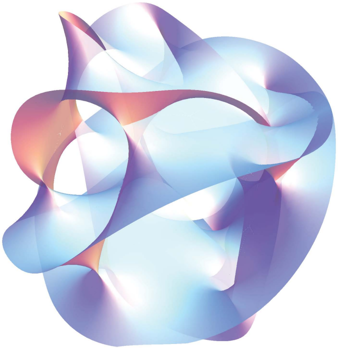
- $\mathcal{S}_{or}(M) = \{\text{Orientations on } M\}$.
- $\mathcal{S}_p(M) = \{A \subseteq M \text{ finite}\}$.
- $\mathcal{S}_\Lambda(M) = \{\text{Parabolic structures with datum in } \Lambda\}$.

TQFTs over a sheaf

$$\mathbf{Emb}_c \Rightarrow \mathbf{Emb}_c^{\mathcal{S}} \qquad \mathbf{Bd}_n \Rightarrow \mathbf{Bd}_n^{\mathcal{S}}$$

Notation: $\mathbf{Bdp}_n(\Lambda) = \mathbf{Bd}_n^{\mathcal{S}_p \times \mathcal{S}_\Lambda}$.

Hodge theory



Hodge structures

Let H be a $\mathbb{Q}$-vector space. A **(mixed) Hodge structure** is a double finite filtration

$$0 \subseteq \ldots \subseteq W^k H \subseteq W^{k+1} H \subseteq \ldots \subseteq H$$

$$h0 \supseteq \ldots \supseteq F_p H_{\mathbb{C}} \supseteq F_{p+1} H_{\mathbb{C}} \supseteq \ldots \supseteq H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C}$$

such that $F_p \operatorname{Gr}_W^k H \oplus \overline{F_{k-p+1} \operatorname{Gr}_W^k H} = H_{\mathbb{C}}$ for all k, p (**pure Hodge structure**).

Deligne's theorem

$$\begin{tikzcd}
\mathbf{Var}_{\mathbb{C}} \arrow[rr, "H_c^k(-;\mathbb{Q})"] \arrow[dr] & & \mathbb{Q}\text{-}\mathbf{Vect} \\
& \mathbf{HS} \arrow[ur] &
\end{tikzcd}$$

Deligne-Hodge polynomial

$$H^{p,q} = Gr_p^F \, Gr_W^{p+q} \, H \qquad h^{p,q} = \dim H^{p,q}$$

We have a morphism

$$e : \mathrm{K}\mathbf{HS} \to \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$$

such that $e(H) = \sum h^{p,q} u^p v^q$.

Deligne-Hodge polynomial

$$e(X) = \sum_k \sum_{p,q} (-1)^k h_c^{k;p,q}(X) \, u^p v^q \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}].$$

where $h_c^{k;p,q}(X) = h^{p,q}(H_c^k(X;\mathbb{Q}))$.

Mixed Hodge modules

Saito's mixed Hodge modules

Complex algebraic variety X $\Longrightarrow$ Abelian monoidal category $\mathcal{M}_X$ $\Longrightarrow$ Ring $\mathrm{K}\mathcal{M}_X$

- $\mathcal{M}_X$ contains variations of Hodge structures on X.
- $\mathcal{M}_\star = \mathbf{HS}$.

$$\begin{tikzcd}
\text{Local system with stalk } H \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \text{Locally constant sheaf} \arrow[dl, Leftrightarrow] \\
\rho : \pi_1(X) \to \mathrm{GL}(H) &
\end{tikzcd}$$

$\rho : \pi_1(X) \to \mathrm{GL}(H)$ preserving Hodge structure $\Rightarrow \rho \in \mathrm{K}\mathcal{M}_X$

Mixed Hodge modules as $\mathrm{Var}_{\mathbb{C}}$-algebra

Properties

- Every $\mathrm{K}\mathcal{M}_X$ has a natural $\mathrm{K}\mathbf{HS}$-module structure.
- For $f : X \to Y$ regular we have $\mathrm{K}\mathbf{HS}$-module morphisms

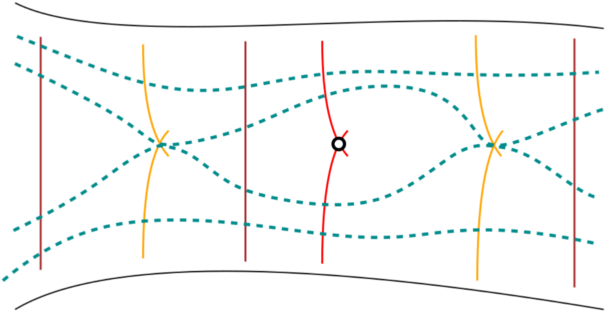
$$f_! : \mathrm{K}\mathcal{M}_X \to \mathrm{K}\mathcal{M}_Y, \qquad f^* : \mathrm{K}\mathcal{M}_Y \to \mathrm{K}\mathcal{M}_X.$$

  f^* is also a ring homomorphism.
- Beck-Chevalley condition for f^* and $f_!$.
- For the projection $c_X : X \to \star$

$$(c_X)_!(1) = [H_c^\bullet(X; \mathbb{Q})].$$

$\mathrm{K}\mathcal{M} = (f^*, f_!)$ is a $\mathbf{Var}_{\mathbb{C}}$-algebra with $\mathrm{K}\mathcal{M}_\star = \mathrm{K}\mathbf{HS}$.

TQFTs for representation varieties

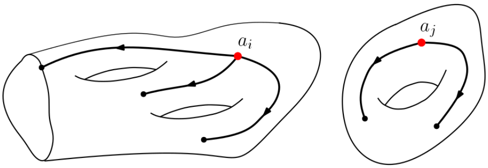


Generalized representation varieties

Let M be a compact manifold and $A \subseteq M$ finite.

$$\mathfrak{X}_G(M, A) = \mathrm{Hom}_{\mathbf{Grpd}}(\underbrace{\Pi(M, A)}_{\text{Fund. groupoid}}, G).$$

Algebraic structure



$$\mathfrak{X}_G(M, A) = G^{|A| - \mathrm{rk}\, H_0(M)} \times \prod_{a_i} \mathfrak{X}_G(\pi_1(M, a_i)).$$

Field theory

$$\mathfrak{X}_G : \mathbf{Embp}_c \to \mathbf{Var}_{\mathbb{C}}$$

- Objects. $(M, A) \Rightarrow \mathfrak{X}_G(M, A)$.
- Morphisms.
  $f : (M, A) \to (M', A') \rightsquigarrow f_* : \Pi(M, A) \to \Pi(M', A')$
  $\rightsquigarrow \mathfrak{X}_G(f) : \mathfrak{X}_G(M', A') \to \mathfrak{X}_G(M, A)$.

Quantisation

$$\mathrm{K}\mathcal{M} \text{ as } \mathbf{Var}_{\mathbb{C}}\text{-algebra with } \mathrm{K}\mathcal{M}_\star = \mathrm{K}\mathbf{HS}$$

$$Z_G : \mathbf{Bdp}_n \xrightarrow{\mathcal{F}_{\mathfrak{X}_G}} \mathrm{Span}(\mathbf{Var}_{\mathbb{C}}) \xrightarrow{\mathcal{Q}_{\mathrm{K}\mathcal{M}}} \mathrm{K}\mathbf{HS}\textbf{-Mod}$$

TQFT for representation varieties

Theorem. (G-P, Logares, Muñoz)

There exists a lax monoidal TQFT

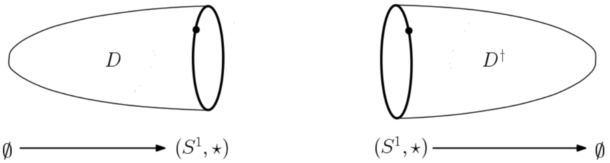
$$Z_G = \mathcal{Q}_{\mathrm{K}\mathcal{M}} \circ \mathcal{F}_{\mathfrak{X}_G} : \mathbf{Bdp}_n(\Lambda) \to \mathrm{K}\mathbf{HS\text{-}Mod},$$

computing the Hodge structure on representation varieties.

- $Z_G(\emptyset) = \mathrm{K}\mathcal{M}_\star = \mathrm{K}\mathbf{HS}$.
- $(M, A) \in \mathbf{Bdp}_{\Rightarrow} Z_G(M, A) = \mathrm{K}\mathcal{M}_{\mathfrak{X}_G(M,A)}$.
- $(W, A) : \emptyset \to \emptyset$ closed connected.

$$Z_G(W, A)(1) = [H_c^\bullet(\mathfrak{X}_G(W))] \otimes [H_c^\bullet(G)]^{|A|-1}.$$

TQFT in surfaces ($n = 2$)



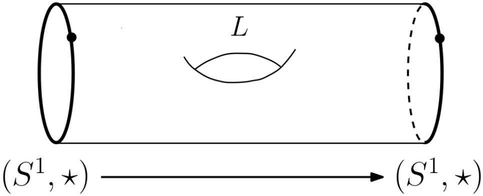
$$\mathfrak{X}_G(\emptyset) = 1 \qquad \mathfrak{X}_G(S^1, \star) = G$$

$$\mathcal{F}_{\mathfrak{X}_G}(D) = \left[1 \longleftarrow 1 \xrightarrow{i} G \right] \qquad \mathcal{F}_{\mathfrak{X}_G}(D^\dagger) = \left[G \xleftarrow{i} 1 \longrightarrow 1 \right]$$

Morphisms of the TQFT

$$Z_G(D) = i_! : \mathrm{K}\mathbf{HS} \to \mathrm{K}\mathcal{M}_G \qquad Z_G(D^\dagger) = i^* : \mathrm{K}\mathcal{M}_G \to \mathrm{K}\mathbf{HS}$$

TQFT in surfaces ($n = 2$)



$$\mathcal{F}\mathfrak{X}_G(L) = \begin{bmatrix} G & \xleftarrow{p} & G^4 & \xrightarrow{q} & G \\ g & \leftarrow\!\shortmid & (g, g_1, g_2, h) & \mapsto & hg[g_1, g_2]h^{-1} \end{bmatrix}$$

Morphisms of the TQFT

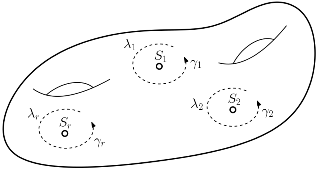
$$Z_G(L) = q_! p^* : \mathrm{K}\mathcal{M}_G \to \mathrm{K}\mathcal{M}_G$$

Considering parabolic structures

Set Λ a collection of subvarieties of G stable under conjugation. Given $Q = \{(S_1, \lambda_1), \cdots, (S_r, \lambda_r)\}$ we define

$$\mathfrak{X}_G(M, A, Q) = \{\, \rho : \Pi(M - \cup S_i, A) \to G \mid \rho(\gamma_i) \in \lambda_i \}\,,$$

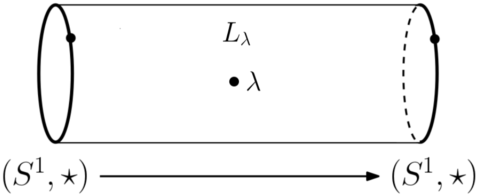
where γ_i is a loop 'around' S_i.



Extension of the results

$$Z_G : \mathbf{Bdp}_n(\Lambda) \to \mathrm{K}\mathbf{HS\text{-}Mod}.$$

TQFT in surfaces ($n = 2$)



$$\mathcal{F}\mathfrak{X}_G(L_\lambda) = \left[\begin{array}{ccccc} G & \xleftarrow{r} & G^2 \times \lambda & \xrightarrow{s} & G \\ g & \leftarrow\!\shortmid & (g, h, \xi) & \mapsto & hg\xi h^{-1} \end{array}\right]$$

Morphisms of the TQFT

$$Z_G(L) = s_! r^* : \mathrm{K}\mathcal{M}_G \to \mathrm{K}\mathcal{M}_G$$

Reduction of the TQFT

Piecewise algebraic varieties, $\mathbf{PVar}_k$

- Objects: Grothendieck semi-ring of algebraic varieties.
- Morphisms: Piecewise regular maps.

Ex: If G acts on X, then the orbit space $[X/G] \in \mathbf{PVar}_k$.

$$\begin{tikzcd} \mathbf{Embp}_c \arrow[r, "\mathfrak{X}_G"] \arrow[rr, bend left, dashed] & \operatorname{Im}\mathfrak{X}_G \subseteq \mathbf{Var}_{\mathbb{C}} \arrow[r, "\tau"] & \mathbf{PVar}_{\mathbb{C}} \end{tikzcd}$$

$$\mathfrak{X}_G(M, A) \longmapsto [\mathfrak{X}_G(M, A)/G]$$

$$Z_G^{gm} = \mathcal{Q}_{\mathrm{K}\mathcal{M}} \circ \mathcal{F}_{\tau \circ \mathfrak{X}_G} : \mathbf{Bdp}_n \to \mathrm{K}\mathbf{HS\text{-}Mod}$$

$\boxed{\mathbf{n = 2}}$ $\quad Z_G^{gm}(S^1, \star) = \mathrm{K}\mathcal{M}_{[G/G]} \subsetneq \mathrm{K}\mathcal{M}_G.$

Reduction of the TQFT

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$\boxed{\mathbf{n = 2}}$ $$Z_G^{gm}(S^1, \star) = \mathrm{K}\mathcal{M}_{[G/G]} \subsetneq \mathrm{K}\mathcal{M}_G.$$

New problem

Z_G^{gm} is no longer a functor.

Theorem

Let $\eta = \tau_! \circ \tau^*$. If $\eta_{(M,A)}$ is an isomorphism, then

$$\mathcal{Z}_G^{gm} = Z_G^{gm} \circ \eta^{-1}$$

is a lax monoidal TQFT computing the same invariant as Z_G.

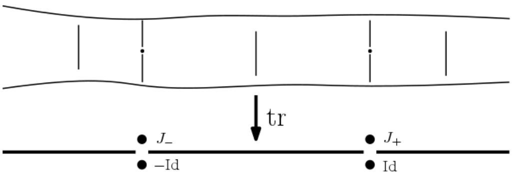
$$\begin{tikzcd}
Z_G(M_1, A_1) \arrow[r, "Z_G(W,A)"] \arrow[d, "\tau_!"'] & Z_G(M_2, A_2) \arrow[d, "\tau_!"] \\
Z_G^{gm}(M_1, A_1) \arrow[r, dashed, "\mathcal{Z}_G^{gm}(W,A)"'] & Z_G^{gm}(M_2, A_2)
\end{tikzcd}$$

Parabolic $\mathrm{SL}_2(\mathbb{C})$-representation varieties

Conjugacy classes in $\mathrm{SL}_2(\mathbb{C})$

$$\pm\mathrm{Id} \qquad \boxed{J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}} \qquad D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda \in \mathbb{C} - \{0, \pm 1\}$.

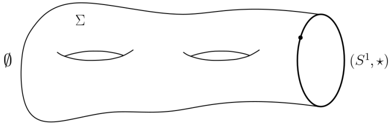


$$Z_G^{gm}(S^1, \star) = \mathrm{K}\mathcal{M}_1 \oplus \mathrm{K}\mathcal{M}_{-1} \oplus \mathrm{K}\mathcal{M}_{J_+} \oplus \mathrm{K}\mathcal{M}_{J_-} \oplus \mathrm{K}\mathcal{M}_{\mathbb{C}-\{\pm 2\}}$$

Question

Is $\eta : \mathrm{K}\mathcal{M}_{[\mathrm{SL}_2(\mathbb{C})/\mathrm{SL}_2(\mathbb{C})]} \to \mathrm{K}\mathcal{M}_{[\mathrm{SL}_2(\mathbb{C})/\mathrm{SL}_2(\mathbb{C})]}$ invertible?

Idea: Restrict to a ‘core submodule’ $\mathcal{W}$.



$$\mathcal{W} = \langle Z_G^{gm}(\Sigma)(1) \rangle \subseteq \mathrm{K}\mathcal{M}_{[G/G]}$$

Lemma

For $G = \mathrm{SL}_2(\mathbb{C})$ we have

$$\mathcal{W} = \langle \underbrace{T_1}_{\mathrm{K}\mathcal{M}_1}, \underbrace{T_{-1}}_{\mathrm{K}\mathcal{M}_{-1}}, \underbrace{T_+}_{\mathrm{K}\mathcal{M}_{J_+}}, \underbrace{T_-}_{\mathrm{K}\mathcal{M}_{J_-}}, \underbrace{T_{\mathbb{C}-\{\pm 2\}}, S_2, S_{-2}, S_2 \otimes S_{-2}}_{\mathrm{K}\mathcal{M}_{\mathbb{C}-\{\pm 2\}}} \rangle.$$

Moreover, on $\mathcal{W}$, the morphism $\eta = \mathrm{tr}_! \circ \mathrm{tr}^*$ is invertible.

In the previous set of generators of $\mathcal{W}$

Denote $q = [H_c^\bullet(\mathbb{C})]$.

$$\mathcal{Z}^{gm}_{\mathrm{SL}_2}(D) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathcal{Z}^{gm}_{\mathrm{SL}_2}(D^\dagger) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{Z}^{gm}_{\mathrm{SL}_2}(L_{[J_+]}) = (q^3 - q) \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ q^2-1 & 0 & q-2 & q & (q-1)^2 & -q+1 & -q+1 & -2q+2 \\ 0 & q^2-1 & q & q-2 & (q-1)^2 & -q+1 & -q+1 & -2q+2 \\ 0 & 0 & q & q & q^2-2q & -q+1 & -q+1 & -q+2 \\ 0 & 0 & 0 & 0 & 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & -1 \end{pmatrix}$$

$$\mathcal{Z}^{gm}_{\mathrm{SL}_2}(L_{[J_-]}) = (q^3 - q) \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^2-1 & q & q-2 & (q-1)^2 & -q+1 & -q+1 & -2q+2 \\ q^2-1 & 0 & q-2 & q & (q-1)^2 & -q+1 & -q+1 & -2q+2 \\ 0 & 0 & q & q & q^2-2q & -q+1 & -q+1 & -q+2 \\ 0 & 0 & 0 & 0 & 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & -1 \end{pmatrix}$$

Results

Theorem (G-P)

Let Q be a parabolic structure with r_+ classes in J_+ and r_- classes in J_-. Denote $r = r_+ + r_-$.

- If r_- is even

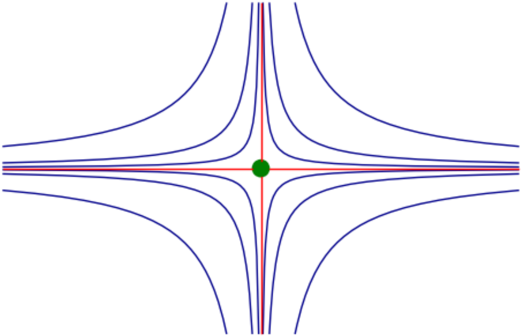
$$[H_c^\bullet(\mathfrak{X}(\Sigma_g, Q))] = \left(q^2-1\right)^{2g+r-1} q^{2g-1} + \frac{1}{2}(q-1)^{2g+r-1} q^{2g-1}(q+1)\left(2^{2g}+q-3\right)$$
$$+ \frac{(-1)^r}{2}(q+1)^{2g+r-1} q^{2g-1}(q-1)\left(2^{2g}+q-1\right).$$

- If r_- is odd

$$[H_c^\bullet(\mathfrak{X}(\Sigma_g, Q))] = (q-1)^{2g+r-1}(q+1)q^{2g-1}\left((q+1)^{2\,g+r-2}+2^{2g-1}-1\right)$$
$$+ (-1)^{r+1}\, 2^{2g-1}(q+1)^{2g+r-1}(q-1)q^{2g-1}.$$

New result for parabolic structures on high genus surfaces.

Topological Geometric Invariant Theory

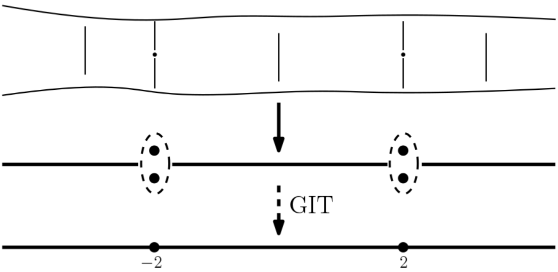


Geometric Invariant Theory

Let X be an affine variety with an action of G algebraic group.

The problem of quotients

Orbit space X/G might not be an algebraic variety.



Let $\pi : X \to X /\!\!/ G$ be a G-invariant regular morphism.

- **Categorical.**

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & Z \\ X /\!\!/ G \arrow[ur, dashed] & \end{tikzcd}$$

$$X = \operatorname{Spec} R \Longrightarrow X /\!\!/ G = \operatorname{Spec} R^G$$

- **Good.** If π is surjective and
  - (Topology) $W_1, W_2 \subseteq X$ closed G-stable.
  $$\overline{\pi(W_1)} \cap \overline{\pi(W_2)} \neq \emptyset \Leftrightarrow W_1 \cap W_2 \neq \emptyset.$$
  - (Algebra) $\mathcal{O}_{X /\!\!/ G}(U) \cong \mathcal{O}_X(\pi^{-1} U)^G$ for all $U \subseteq X /\!\!/ G$ open.
- **Geometric.** Good quotient $+$ orbit space.

Geometric $\Rightarrow$ Good $\Rightarrow$ Categorical $\Rightarrow$ Unique

The problem of stratifications

$$X = U \sqcup Z \Longrightarrow X /\!\!/ G \stackrel{?}{=} (U /\!\!/ G) \sqcup (Z /\!\!/ G)$$

Pseudo-quotients

A pseudo-quotient is a surjective regular morphism $\pi : X \to Y$ such that, for all $W_1, W_2 \subseteq X$ closed G-stable

$$\overline{\pi(W_1)} \cap \overline{\pi(W_2)} \neq \emptyset \Leftrightarrow W_1 \cap W_2 \neq \emptyset.$$

Properties

- No uniqueness. Ex: $\mathbb{A}^1$ and cuspidal curve.
- Uniqueness up to K-theory.
- Well-behaved for stratifications.

Stratification of $\mathrm{SL}_n(\mathbb{C})$-representation varieties

$$\mathfrak{X}(\Sigma_g, Q) = \underbrace{\mathfrak{X}(\Sigma_g, Q)^{red}}_{\to\text{Diagonal elements}} \sqcup \underbrace{\mathfrak{X}(\Sigma_g, Q)^{irr}}_{\text{Free action of } \mathrm{PGL}_n(\mathbb{C})}$$

$$\begin{aligned}[\mathfrak{X}(\Sigma_g, Q) /\!\!/ \mathrm{SL}_n(\mathbb{C})] &= [Diag/S_n] + \frac{[\mathfrak{X}(\Sigma_g, Q)^{irr}]}{[\mathrm{PGL}_n(\mathbb{C})]} \\ &= \underbrace{[Diag/S_n]}_{\text{Equivariant theory}} + \underbrace{\frac{[\mathfrak{X}(\Sigma_g, Q)] - [\mathfrak{X}(\Sigma_g, Q)^{red}]}{[\mathrm{PGL}_n(\mathbb{C})]}}_{\text{Explicit } \mathfrak{X}(\Sigma_g, Q)^{red}}.\end{aligned}$$

And finally...

Theorem (G-P)

Let Q be a parabolic structure with r_+ classes in J_+ and r_- classes in J_-. Denote $r = r_+ + r_-$.

- If $\sigma = 1$, then

$$
\begin{aligned}
e\left(\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(\Sigma_g, Q)\right) = & (-1)^r\, 2^{2g}(q-1)q^{2g-2}\left(1-(1-q)^{r-1}\right) \\
& + \frac{1}{2}(q-1)^{2g+r-2}q^{2g-2}\left(2^{2g}+q-3\right) \\
& + \frac{1}{2}(q+1)^{2g+r-2}q^{2g-2}\left(2^{2g}+q-1\right) + \left(q^2-1\right)^{2g+r-2}q^{2g-2}.
\end{aligned}
$$

- If $\sigma = -1$, then

$$
\begin{aligned}
e\left(\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(\Sigma_g, Q)\right) = & (-1)^{r-1}\, 2^{2g-1}(q+1)^{2g+r-2}q^{2g-2} \\
& + (q-1)^{2g+r-2}q^{2g-2}\left((q+1)^{2g+r-2} + 2^{2g-1} - 1\right).
\end{aligned}
$$

Future work

- Case of semi-simple punctures.
- Higher rank case.
- Extension for Grothendieck ring of varieties.
- Can we remove lax monoidality? Extended version?
- TQFTs along the non-abelian Hodge theory.
- Hausel et al. conjectures on E-polynomials.
- New perspectives for mirror symmetry for character varieties.

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Thank you

for your attention

Classical Hodge theory

Hodge's theorem

If X is a smooth projective variety

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

$$0 = W_{k-1} \subseteq W_k = H^k(X;\mathbb{Q}) \qquad F^p = \bigoplus_{s\geq p} H^{s,k-s}(X)$$

$$H_c^{k;p,q}(X) = H^{p,q}(X) \qquad e(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X)\, u^p v^q$$

- $e(\mathbb{C}P^n) = 1 + uv + \ldots + u^n v^n \Rightarrow e(\mathbb{C}^n) = u^n v^n$.
- $e(\text{Proj curve}) = 1 - gu - gv + uv$.

Soft TQFTs

Suppose that we only have 'half of the $\mathcal{C}$-algebra'

$$A : \mathcal{C}^{op} \to \mathbf{Ring}.$$

$$\mathcal{Q}_A^0 : \mathrm{Span}(\mathcal{C}) \to R\text{-}\mathbf{Bim}$$

- **Objects:** $\mathcal{Q}_A^0(c) = A(c)$ for $c \in \mathcal{C}$.
- **Morphisms:** Given a span $S : c_1 \stackrel{f}{\leftarrow} d \stackrel{g}{\rightarrow} c_2$, we define

$$\mathcal{Q}_A^0(S) = {}_{A(c_1)}[A(d)]_{A(c_2)}$$

Soft TQFT

$$\mathcal{Z} : \mathbf{Bd}_n \xrightarrow{\mathcal{F}_{\mathcal{G}}} \mathrm{Span}(\mathcal{C}) \xrightarrow{\mathcal{Q}_A^0} R\text{-}\mathbf{Bim}.$$

The morphism η

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^2-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^2-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^2 & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 0 & q^2 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & q & q^2 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & q^2 \end{pmatrix}$$

The morphism $Z^{gm}_{\mathrm{SL}_2(\mathbb{C})}(L)$

$$\begin{pmatrix}
q+4 & 1 & q^2-2q-3 & q^2+3q & \begin{smallmatrix}q^3-2q^2\\-3q-2\end{smallmatrix} & -q^2-4q-1 & \begin{smallmatrix}2q^2-7q\\-1\end{smallmatrix} & -5q-1 \\
1 & q+4 & q^2+3q & \begin{smallmatrix}q^2-2q\\-3\end{smallmatrix} & \begin{smallmatrix}q^3-2q^2\\-3q-2\end{smallmatrix} & \begin{smallmatrix}2q^2-7q\\-1\end{smallmatrix} & \begin{smallmatrix}-q^2-4q\\-1\end{smallmatrix} & -5q-1 \\
\begin{smallmatrix}q^2-2q\\-3\end{smallmatrix} & q^2+3q & \begin{smallmatrix}q^4+q^3\\+3q+3\end{smallmatrix} & q^4-3q^2-6q & \begin{smallmatrix}q^5-2q^4-3q^3\\+q^2+3q\end{smallmatrix} & \begin{smallmatrix}-q^4+2q^3\\-4q^2+3q\end{smallmatrix} & \begin{smallmatrix}-q^4-q^3\\-4q^2+6q\end{smallmatrix} & \begin{smallmatrix}-2q^3-q^2\\+3q\end{smallmatrix} \\
q^2+3q & \begin{smallmatrix}q^2-2q\\-3\end{smallmatrix} & \begin{smallmatrix}q^4-3q^2\\-6q\end{smallmatrix} & \begin{smallmatrix}q^4+q^3\\+3q+3\end{smallmatrix} & \begin{smallmatrix}q^5-2q^4-3q^3\\+q^2+3q\end{smallmatrix} & \begin{smallmatrix}-q^4-q^3\\-4q^2+6q\end{smallmatrix} & \begin{smallmatrix}-q^4+2q^3\\-4q^2+3\end{smallmatrix}q & \begin{smallmatrix}-2q^3-q^2\\+3q\end{smallmatrix} \\
q^2+1 & q^2+1 & q^4-2q^2 & q^4-2q^2 & \begin{smallmatrix}q^5-2q^4-q^3\\+2q^2-2\end{smallmatrix} & \begin{smallmatrix}-q^4-q^3\\+q^2-q-1\end{smallmatrix} & \begin{smallmatrix}-q^4-q^3\\+q^2-q-1\end{smallmatrix} & \begin{smallmatrix}-2q^3\\+q^2-2q-1\end{smallmatrix} \\
0 & 3q & 3q^2 & -3q & -3q^2 & 4q^3-6q^2 & -4q^2 & -3q^2 \\
3q & 0 & -3q & 3q^2 & -3q^2 & -4q^2 & 4q^3-6q^2 & -3q^2 \\
q & q & q^3 & q^3 & \begin{smallmatrix}q^4-2q^3\\-q^2-2q\end{smallmatrix} & \begin{smallmatrix}-q^3-q^2\\-q\end{smallmatrix} & \begin{smallmatrix}-q^3-q^2\\-q\end{smallmatrix} & \begin{smallmatrix}q^3-2q^2\\-q\end{smallmatrix}
\end{pmatrix}$$

The problem with semi-simple points

Pick $t_0 \neq \pm 2$ and let $\lambda = \{A \in \mathrm{SL}_2(\mathbb{C}) \mid \operatorname{tr} A = t_0\}$. That is, $\lambda = [D_\mu]$ with $\mu + \mu^{-1} = t_0$.

New special fibers appear...

$$Z^{gm}_{\mathrm{SL}_2(\mathbb{C})}(L_\lambda)(\mathcal{W}) \subsetneq \mathcal{W}$$

$$\text{but} \qquad Z^{gm}_{\mathrm{SL}_2(\mathbb{C})}(L_\lambda)(\mathcal{W}) \subseteq \langle \mathcal{W}, T_{t_0}, T_{-t_0}, T_{2-t_0^2}, T_{t_0^2-2} \rangle.$$

Key idea

For a surface with a finite number of marked points, computations can be done in a finite dimensional module.

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Key idea

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Example of computation of the TQFT

Consider the affine group of the complex line

$$G = \mathrm{AGL}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{C}^*, b \in \mathbb{C} \right\}.$$

We have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{AGL}(\mathbb{C})^3 \arrow[dl, "\varpi"'] \arrow[d] \arrow[r] & 1 \arrow[d, "i"] \\
\mathrm{AGL}(\mathbb{C}) & \mathrm{AGL}(\mathbb{C})^4 \arrow[l, "q"] \arrow[r, "p"'] & \mathrm{AGL}(\mathbb{C})
\end{tikzcd}$$

$$\varpi\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & (a_1-1)b_2x-(a_2-1)b_1x \\ 0 & 1 \end{pmatrix}.$$

$$\begin{aligned}\text{General fiber} &= \{(a_1-1)b_2x-(a_2-1)b_1x=\alpha\}\\ &\cong \left[(\mathbb{C}-\{0,1\})\times(\mathbb{C}^*)^2\times\mathbb{C}^2\right]\sqcup\left[\mathbb{C}-\{0,1\}\times\mathbb{C}^*\times\mathbb{C}^2\right].\end{aligned}$$

$$\begin{aligned}\varpi^{-1}(I) &= \{(a_1-1)b_2=(a_2-1)b_1\}\\ &\cong \left[(\mathbb{C}-\{0,1\})\times(\mathbb{C}^*)^2\times\mathbb{C}^2\right]\sqcup\left[\mathbb{C}^*\times\mathbb{C}^3\right]\\ &\sqcup\left[\mathbb{C}-\{0,1\}\times\mathbb{C}^*\times\mathbb{C}^2\right].\end{aligned}$$

$$\begin{aligned}Z_G(L)\circ Z_G(D)(1) &= q(q-1)(q^3-q^2)\,T_1\\ &\quad + q(q-1)(q^3-2q^2)\,T_{\mathrm{ASO}(\mathbb{C})^*}.\end{aligned}$$

$$\begin{tikzcd}
& \mathrm{ASO}(\mathbb{C})^* \times \mathrm{AGL}(\mathbb{C})^3 \arrow[r] \arrow[dl, "\vartheta"'] \arrow[d] & \mathrm{ASO}(\mathbb{C})^* \arrow[d, "j"] \\
\mathrm{AGL}(\mathbb{C}) & \mathrm{AGL}(\mathbb{C})^4 \arrow[l, "q"] \arrow[r, "p"'] & \mathrm{AGL}(\mathbb{C})
\end{tikzcd}$$

$$\vartheta\left(\begin{pmatrix}1 & \beta \\ 0 & 1\end{pmatrix}, \begin{pmatrix}a_1 & b_1 \\ 0 & 1\end{pmatrix}, \begin{pmatrix}a_2 & b_2 \\ 0 & 1\end{pmatrix}, \begin{pmatrix}x & y \\ 0 & 1\end{pmatrix}\right) = \begin{pmatrix}1 & (a_1-1)b_2x - (a_2-1)b_1x + \beta x \\ 0 & 1\end{pmatrix}.$$

$$\begin{aligned}\text{General fiber} &= \{(a_1-1)b_2x - (a_2-1)b_1x + \beta = \alpha\} \\ &= \left[(\mathbb{C}^*)^3 \times \mathbb{C}^3\right] - \text{General fiber of } \varpi.\end{aligned}$$

$$\begin{aligned}\vartheta^{-1}(I) &= \{(a_2-1)b_1x - (a_1-1)b_2x = \beta\} \\ &= \left[(\mathbb{C}^*)^3 \times \mathbb{C}^3\right] - \varpi^{-1}(I).\end{aligned}$$

$$Z_G(L)\left(T_{\mathrm{ASO}(\mathbb{C})^*}\right) = q(q-1)(q^4-3q^3+2q^2)\,T_1 + q(q-1)(q^4-3q^3+3q^2)\,T_{\mathrm{ASO}(\mathbb{C})^*}.$$

In this way, $\mathcal{W} = \langle T_1, T_{\mathrm{ASO}(\mathbb{C})^*}\rangle$ and

$$Z_G(L) = q(q-1)\begin{pmatrix} q^3-q^2 & q^4-3q^3+2q^2 \\ q^3-2q^2 & q^4-3q^3+3q^2 \end{pmatrix}$$

Final result

$$[H_c^\bullet(\mathfrak{X}_{\mathrm{AGL}(\mathbb{C})}(\Sigma_g))] = q^{2g-1}\left((q-1)^{2g}+q-1\right)$$

Example of computation of GIT quotient

Consider $\Gamma = F_n$ and $G = \mathrm{SL}_2(\mathbb{C})$.

Lemma

If an element of $\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(\Gamma)$ has non trivial isotropy for the conjugacy action, then it is conjugated to

$$\left(\begin{pmatrix} \pm 1 & \alpha_1 \\ 0 & \pm 1 \end{pmatrix}, \ldots, \begin{pmatrix} \pm 1 & \alpha_n \\ 0 & \pm 1 \end{pmatrix}\right) \text{ or } \left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix}\right)$$

for $\alpha_i \in \mathbb{C}$ and $\lambda_i \in \mathbb{C}^*$.

$$\mathfrak{X}_{\mathrm{SL}_2(\mathbb{C})}(F_n) = \underbrace{\mathfrak{X}^{\upsilon} \sqcup \mathfrak{X}^{\delta} \sqcup \mathfrak{X}^{\iota} \sqcup \mathfrak{X}^{\rho}}_{\mathfrak{X}^{red}} \sqcup \mathfrak{X}^{irr}$$

- $\mathfrak{X}^\upsilon$. Stab $= \mathbb{C}$.

$$\mathbb{C}^* \longrightarrow \mathrm{SL}_2(\mathbb{C})/\mathbb{C} \times \left[\{\pm 1\}^n \times \left(\mathbb{C}^n - \{0\}\right)\right] \longrightarrow \mathfrak{X}^\upsilon$$

$$\Rightarrow [H_c^\bullet(\mathfrak{X}^\upsilon)] = 2^n(q^2-1)\frac{q^n-1}{q-1}.$$

- $\mathfrak{X}^\delta$. Stab $= \mathbb{C}^*$.

$$\mathfrak{X}^\delta = \frac{\mathrm{SL}_2(\mathbb{C})/\mathbb{C}^* \times [(\mathbb{C}^*)^n - \{(\pm 1, \ldots, \pm 1)\}]}{\mathbb{Z}_2}$$

$$\Rightarrow [H_c^\bullet(\mathfrak{X}^\delta)] = \frac{q^3-q}{2}\left((q-1)^{n-1} + (q+1)^{n-1}\right) - 2^n q^2.$$

- $\mathfrak{X}^\iota$. Stab $= \mathrm{SL}_2(\mathbb{C})$.

$$\Rightarrow [H_c^\bullet(\mathfrak{X}^\iota)] = 2^n.$$

- $\mathfrak{X}^\rho$. $\mathrm{Stab} = \{\pm \mathrm{Id}\}$.

$$\mathbb{C}^* \times \mathbb{C} \longrightarrow \mathrm{PGL}_2 \times \Omega \longrightarrow \mathfrak{X}^\rho$$

where $\Omega = \left((\mathbb{C}^*)^n - \{\pm 1, \ldots, \pm 1\}\right) \times \left(\mathbb{C}^n - \mathbb{C}\right)$ is the set of allowed values for the eigenvalues and the antidiagonal component.

$$\Rightarrow [H_c^\bullet(\mathfrak{X}^\rho)] = \frac{q^3 - q}{(q-1)q}\left((q-1)^n - 2^n\right)\left(q^n - q\right).$$

Final result

$$
\begin{aligned}
[H_c^\bullet(\mathfrak{X}(F_n) /\!\!/ \mathrm{SL}_2(\mathbb{C}))] &= [H_c^\bullet(Diag/\mathbb{Z}_2)] + \frac{[H_c^\bullet(\mathfrak{X}(F_n))] - [H_c^\bullet(\mathfrak{X}(F_n)^{red})]}{[H_c^\bullet(\mathrm{PGL}_n(\mathbb{C}))]} \\
&= [H_c^\bullet((\mathbb{C}^*)^n/\mathbb{Z}_2)] + \frac{[H_c^\bullet(\mathrm{SL}_2(\mathbb{C}))]^n - [H_c^\bullet(\mathfrak{X}^\upsilon \sqcup \mathfrak{X}^\delta \sqcup \mathfrak{X}^\iota \sqcup \mathfrak{X}^\rho)]}{[H_c^\bullet(\mathrm{PGL}_n(\mathbb{C}))]} \\
&= \frac{1}{2}(q+1)^{n-1}q + \frac{1}{2}(q-1)^{n-1}q - (q-1)^{n-1}q^{n-1} + (q^3-q)^{n-1}.
\end{aligned}
$$

Result in agreement with the computations of Lawton-Munoz and Cavazos-Lawton.

Publications

Included in the PhD Thesis

- *(with M. Logares and V. Muñoz)*. A lax monoidal Topological Quantum Field Theory for representation varieties. arXiv:1709.05724. 2017.
- Hodge theory of representation varieties via Topological Quantum Field Theories. arXiv:1810.09714. 2018.
- Stratification of algebraic quotients and character varieties. arXiv:1807.08540. 2018.

In preparation

- E-polynomials of $\mathrm{SL}_2(\mathbb{C})$-character varieties with semisimple punctures.
- *(with M. Logares)*. Topological Quantum Field Theories for character varieties over nodal curves.