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Deligne-Hodge Polynomials of Character Varieties of Doubly Periodic Instantons

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“Let’s go by pieces.”

Jack the Ripper

ABSTRACT

Deligne-Hodge Polynomials of Character Varieties of Doubly Periodic Instantons

by J. Ángel González

En este trabajo, estudiaremos un tipo especial de variedades algebraicas, las variedades de caracteres, y calcularemos para ellas un invariante algebro-geométrico conocido como polinomio de Deligne-Hodge o E-polinomio. Estas variedades de caracteres aparecen como espacios de móduli de representaciones del grupo fundamental de una superficie de Riemann compacta con algunos puntos eliminados, sobre un grupo algebraico reductivo.

En concreto, nos centraremos en el caso representaciones del grupo fundamental de una superficie de Riemann compacta de género 1, a saber, una curva elíptica, con uno o dos puntos marcados, sobre $SL(2, \mathbb{C})$. Calcularemos los polinomios de Deligne-Hodge para ambos casos e, incluso, para el caso de un punto marcado, podremos ir un paso adelante y calcular sus números de Hodge mixtos. Hasta el momento actual, esta información algebraica era desconocida para el caso de holonomía fijada en una clase de conjugación de tipo Jordan. Para llevar a cabo este propósito, usaremos una técnica recientemente desarrollada, basada en la estratificación de la variedad de caracteres y el subsecuente análisis de piezas más simples.

Finalmente, explicaremos la relación de estas variedades de caracteres con otros espacios de móduli que surgen de la física matemática. En particular, estudiaremos los fundamentos de teorías gauge y teoría de Yang-Mills, culminando en el estudio de los fibrados de Higgs.

Palabras clave: *Espacios de móduli, variedades de caracteres, polinomio de Deligne-Hodge.*

In this work, we shall study a special kind of algebraic varieties, the character varieties, and we will compute an algebro-geometric invariant of this varieties, known as the Deligne-Hodge polynomial or E-polynomial. This character varieties arise as moduli spaces of representations of the fundamental group of a compact Riemann surface with some removed points into a reductive algebraic group.

In particular, we focus on the case of representations of the fundamental group of a compact Riemann surface of genus 1, that is, an elliptic curve, with one or two marked points, into $SL(2, \mathbb{C})$. We compute the Deligne-Hodge polynomials in both cases and, for one marked point, we also compute their mixed Hodge numbers for all the cases. Until present, this algebraic information was unknown for the case of fixed holonomy in a conjugacy class of Jordan type. For this purpose, we use a recently developed technique based on stratifications of character varieties and subsequent analysis of simpler pieces.

Finally, we also put into context this character varieties, explaining their relation with other moduli spaces that arise in mathematical physics. In particular, we study the fundaments of gauge theory and Yang-Mills theory, reaching the vast area of Higgs bundles.

Key words: *Moduli spaces, character varieties, Deligne-Hodge polynomial.*

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Contents

Para Laura,
porque es mi fuente de vida.

Introduction

Given a topological space, X, with finitely generated fundamental group, $\pi_1(X)$, and a complex reductive algebraic group G, the character variety of representations of $\pi_1(X)$ into G, is, as explained in section 2.2, the algebraic variety

$$R_G(X) := Hom(\pi_1(X), G) /\!\!/ G$$

The quotient denoted by $/\!\!/$ is an special kind of quotient for the action of G on $Hom(\pi_1(X), G)$ by conjugation, known as the Geometric Invariant Theory quotient (usually shortened as GIT quotient). This quotient confers good properties to the orbit space, making it a algebraic variety. It is treated in section 2.2.4.

In particular, we will take $G = SL(2,\mathbb{C})$ and $X = \Sigma_g - \{p_1, \ldots, p_s\}$, a compact Riemann surface of genus g with s removed points, known as the marked points. We denote this character variety as $\mathcal{M}_s^g := R_{SL(2,\mathbb{C})}(X)$, that is

$$\mathcal{M}_s^g = \left\{ (A_1, \ldots, A_g, B_1, \ldots, B_g, C_1, \ldots, C_s) \in SL(2,\mathbb{C})^{2g+s} \mid \prod_{k=1}^{g} [A_k, B_k] \prod_{l=1}^{s} C_l = Id \right\} /\!\!/ SL(2,\mathbb{C})$$

with $SL(2,\mathbb{C})$ acting by simultaneous conjugation. Moreover, we will focus on the case of an elliptic curve, i.e. a compact Riemann surface of genus $g = 1$, and we will force the loops around the marked points to live in some fixed conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_s \subseteq SL(2,\mathbb{C})$. The resulting variety is called the parabolic $SL(2,\mathbb{C})$-character variety of and elliptic curve with s marked points

$$\mathcal{M}_{\mathcal{C}_1,\ldots,\mathcal{C}_s} = \left\{ (A, B, C_1, \ldots, C_s) \in SL(2,\mathbb{C})^{2+s} \,\middle|\, \begin{array}{c} [A,B] \prod_l C_l = Id \\ C_1 \in \mathcal{C}_1, \ldots, C_s \in \mathcal{C}_s \end{array} \right\} /\!\!/ SL(2,\mathbb{C})$$

This special kind of varieties are related with an important area of mathematical physics known as gauge theory, since they are homeomorphic to the space of some special solutions of this theory, the moduli space of Higgs bundles.

Gauge theory arose as a physical theory to explain the electromagnetic phenomena, but, in the present, is used in nuclear and high energy physics, as quantum electrodynamics and the standard model of particle physics. We will devote the chapter 1 of this work to explain the mathematical fundaments of gauge theory.

In gauge theory, one of the main concerns are the Yang-Mills equations and its solutions, the Yang-Mills connections. Since these equations are highly non-linear, a good aproach is to consider only special kind of solutions, as instantons of self-dual connections. In this philosophy, when considering self dual solutions of the Yang-Mills equations on the space-time $\mathbb{R}^4$ which are invariant in two directions, we can perform a technique, used in physics, known as dimensional reduction (explained in section 1.6.2), to restate these equations in $\mathbb{R}^2$ in a special way known as the Hitchin's self-duality equations

$$\begin{cases} F_A + [\Phi, \Phi^*] = 0 \\ \quad \overline{\partial}_A \Phi = 0 \end{cases}$$

where Φ is a field called the Higgs field. Moreover, since these equations are conformally invariant, we can consider solutions on any compact Riemannian surface. In this context, a solution of the self duality equations is known as a Higgs bundle.

In order to study the possible Higgs bundles on a compact Riemann surface X, we consider the moduli space of Higgs bundles on X, $\mathcal{M}_{Dol}(X)$. This a variety whose points are Higgs bundles on X and whose geometry reflects some notion of closeness of the solutions. The first part of chapter 2 is devoted to explain the concept of moduli space.

Now, by a general theory known as non-abelian Hodge theory with tame singularities (briefly sketched in section 2.3), we find that, for a special kind of Higgs bundles, called parabolic Higgs bundles, the moduli space of traceless parabolic Higgs bundles of parabolic degree 0 and s marked points is homeomorphic to the character variety $\mathcal{M}_{\mathcal{C}_1,\ldots,\mathcal{C}_s}$, where $\mathcal{C}_1, \ldots, \mathcal{C}_s \subseteq SL(2,\mathbb{C})$ are conjugacy classes of semisimple elements. Moreover, via a physics-inspired mechanism, the Nahm transform, it is obtained an homeomorphism between the moduli space of doubly periodic instantons and the parabolic character variety with two marked points, $\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2}$, with $\mathcal{C}_1, \mathcal{C}_2 \subseteq SL(2,\mathbb{C})$ conjugacy classes of semisimple elements.

For this reason, the study of topological and algebraic invariants of character varieties and moduli spaces of Higgs bundles, such as their Betti numbers and mixed Hodge numbers, becomes a subject of high importance in mathematical physics. However, this is not an easy work and there is not a general solution for this problem.

Nonetheless, there are several situations explored to the moment. For example, the Betti numbers of the moduli space of $SL(2,\mathbb{C})$-Higgs bundles was computed by Hitchin in [35] and for $SL(3,\mathbb{C})$ by Gothen in [27]. In the case of non-complex Lie groups, Gothen in [28] computed the Betti numbers fo the moduli space of $U(2,1)$-Higgs bundles. For the case of punctured Riemann surfaces, Betti numbers were computed by García-Prada, Gothen and Muñoz in [61] for $SL(3,\mathbb{C})$-parabolic Higgs bundles, and by Logares in [45] for $U(2,1)$-parabolic Higgs bundles. To this point, the technique used in these computations was Morse theory. Others techniques have also been introduced by García-Prada, Heinloth and Schmitt in [25] in order to compute the Betti numbers for some cases of moduli spaces of Higgs bundles of rank 4.

Another approach to the problem of finding algebro-geometric information of these moduli spaces is to consider a new invariant, called the mixed Hodge numbers. This invariant is computed via an algebraic structure attached to the cohomology ring of the variety, called the mixed Hodge structure. In contrast to other purely topological information, such as Betti numbers, the mixed Hodge numbers depends on the algebraic structure of the variety. In particular, though the moduli space of (parabolic)

Higgs bundles and the corresponding character varieties are homeomorphic, they are not algebraically isomorphic, so its mixed Hodge numbers do not agree. However, since this mixed Hodge numbers can be added for obtaining the Betti numbers, some special sums do agree.

Until now, only this alternate sums of mixed Hodge numbers are known for the moduli space of Higgs bundles, as computed in [33] by Hausel and Thaddeus. In this work, they discovered the first non-trivial example of Strominguer-Yau-Zaslow Mirror Symmetry using the so called Hitchin system for computing the stringy cohomology for the moduli space of $SL(3,\mathbb{C})$ and $PGL(3,\mathbb{C})$ Higgs bundles.

In the aim of finding another examples of non-trivial Mirror Symmety, Hausel introduced the study, on character varieties, of an alternate sum of mixed Hodge numbers, collected in a polynomial known as the Deligne-Hodge polynomial. For this purpose, Hausel and Rodríguez-Villegas introduced, in [32], a new arithmetic technique based on the Weil conjectures. With this technique, they computed the Deligne-Hodge polynomial of the G-character variety without marked points for $G = GL(n,\mathbb{C}), SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$ in terms of generating functions.

In order to study this Deligne-Hodge polynomial for character varieties, in [47], Logares, Muñoz and Newstead introduced a new geometric technique based on the stratification of these spaces. The key idea is that the Deligne-Hodge polynomial of a variety X, $e(X)$, has very important properties similar to the one of the Euler characteristic. In particular, if our space decomposes as $X = Y \sqcup Z$, then $e(X) = e(Y)e(Z)$ and, for some special kind of fibrations, which we call E-fibrations, $F \to X \to B$, we have $e(X) = e(F)e(B)$. In section 3.3.3, we explain these Deligne-Hodge polynomials and their properties.

Therefore, we can compute the Deligne-Hodge polynomial of a character variety by chopping it in simpler pieces that can be analyzed separately and, finally, adding up its correspondent Deligne-Hodge polynomials. Using this idea, in chapter 4 of this work, we implement this technique for studying parabolic $SL(2,\mathbb{C})$-character varieties with one and two marked points. This work is based on the paper [47] written by Logares, Muñoz and Newstead and [46] of Logares and Muñoz.

For the case of one marked point, we compute the Deligne-Hodge polynomials of all the possible parabolic character varieties, and, using this information, we recover its mixed Hodge numbers. Recall that $SL(2,\mathbb{C})$ has five different types of conjugacy classes, determined by the Jordan canonical forms

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad -Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for $\lambda \in \mathbb{C}^* - \{\pm 1\}$

The results can be summarized as follows.

Theorem 0.0.1. *Let $\mathcal{M}_\mathcal{C}$ be the parabolic $SL(2,\mathbb{C})$-character variety of an elliptic curve with one marked point and holonomy in the conjugacy class $\mathcal{C} \subseteq SL(2,\mathbb{C})$. The mixed Hodge information of these varieties is:*

- $e\left(\mathcal{M}_{Id}\right) = q^2 + 1$. *Moreover, its non-vanishing mixed Hodge numbers are*

$$h^{2;0,0}(\mathcal{M}_{Id}) = 1 \qquad h^{4;2,2}(\mathcal{M}_{Id}) = 1$$

- $e\left(\mathcal{M}_{-Id}\right) = 1$. *Moreover, its only non-vanishing mixed Hodge numbers is*

$$h^{0;0,0}(\mathcal{M}_{-Id}) = 1$$

- $e\left(\mathcal{M}_{J_+}\right) = q^2 - 2q - 3$. *Moreover, its non-vanishing mixed Hodge numbers are*

$$h^{1;0,0}(\mathcal{M}_{J_+}) = 4 \qquad h^{2;0,0}(\mathcal{M}_{J_+}) = 1 \qquad h^{3;1,1}(\mathcal{M}_{J_+}) = 2 \qquad h^{4;2,2}(\mathcal{M}_{J_+}) = 1$$

- $e\left(\mathcal{M}_{J_-}\right) = q^2 + 3q$. *Moreover, its non-vanishing mixed Hodge numbers are*

$$h^{1;0,0}(\mathcal{M}_{J_-}) = 1 \quad h^{2;0,0}(\mathcal{M}_{J_-}) = 1 \quad h^{2;1,1}(\mathcal{M}_{J_-}) = 4 \quad h^{3;1,1}(\mathcal{M}_{J_-}) = 1 \quad h^{4;2,2}(\mathcal{M}_{J_-}) = 1$$

- $e\left(\mathcal{M}_{D_\lambda}\right) = q^2 + 4q + 1$. *Moreover, its non-vanishing mixed Hodge numbers are*

$$h^{2;0,0}(\mathcal{M}_{D_\lambda}) = 1 \qquad h^{2;1,1}(\mathcal{M}_{D_\lambda}) = 4 \qquad h^{4;2,2}(\mathcal{M}_{D_\lambda}) = 1$$

Observe that, in particular, the mixed Hodge numbers of $\mathcal{M}_{J_-}$ were unknown and they are first computed in this work.

For the case of two marked points, the analysis is more difficult and, in most cases, reduces to the understanding of pieces that arises in the study of one marked point. In this case, we obtain the following Deligne-Hodge polynomials.

Theorem 0.0.2. *Let $\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2}$ be the parabolic $SL(2,\mathbb{C})$-character variety of an elliptic curve with two marked points and holonomy in the conjugacy classes $\mathcal{C}_1, \mathcal{C}_2 \subseteq SL(2,\mathbb{C})$. The Deligne-Hodge polynomials of these varieties is:*

- $e\left(\mathcal{M}_{[J_+],[J_-]}\right) = q^4 - 3q^2 + 6q$.
- $e\left(\mathcal{M}_{[J_+],[J_+]}\right) = q^4 + q^3 - q + 7$.
- $e\left(\mathcal{M}_{[J_-],[J_-]}\right) = q^4 + q^3 - q + 7$.
- $e\left(\mathcal{M}_{[J_+],[D_\lambda]}\right) = q^4 + q^3 + q^2 - 3q$.
- $e\left(\mathcal{M}_{[J_-],[D_\lambda]}\right) = q^4 + q^3 + q^2 - 3q$.
- $e\left(\mathcal{M}_{[D_\lambda],[D_\mu]}\right) = q^4 + 2q^3 + 6q^2 + 2q + 1$ *for* $\lambda \neq \mu, \mu^{-1}$.
- $e\left(\mathcal{M}_{[D_\lambda],[D_\lambda]}\right) = q^4 + q^3 + 7q^2 + 3q$.

Observe that the polynomials $e\left(\mathcal{M}_{[J_+],[D_\lambda]}\right)$ and $e\left(\mathcal{M}_{[J_-],[D_\lambda]}\right)$ do not agree with those computed in [46], due to an erratum found in section 4.1 of that paper.

Beyond this point, the problem becomes more difficult, since the computational cost of doing this kind of calculations grows combinatorially with genus and number of marked points. However, recently, in [50] and [52], the Deligne-Hodge polynomial of the character varieties of any genus has been computed recursively. From this point, future work should be devoted to compute the Deligne-Hodge polynomial of $SL(2,\mathbb{C})$ character varieties with arbitrary large number of marked points and developing a Topological Field Theory to compute them as reassembled pieces. In addition, analogous computations for $G = SL(3,\mathbb{C})$ or, more general, $G = SL(n,\mathbb{C})$ can be performed in the future following these ideas.

Chapter 1

Gauge Theory

1.1 Electromagnetism

One of the most important discoveries in physics was that electrodynamics can be completely captured by a system of partial differential equations. All the electric and magnetic phenomena were be explaned by a small set of laws. This laws are the well known **Maxwell's equations**, stated between 1861 and 1862 by James Clerk Maxwell.

In this framework, there exist two 3-dimensional vector fields, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ which mutually interact each other creating the electromagnetic phenomena. The rules of variations of this fields, in terms of the others, is determined by four equations, the Maxwell's equations, that, in appropiated units[1], are

$\nabla \cdot \mathbf{E} = \rho$	Gauss's Law	$\nabla \cdot \mathbf{B} = 0$	Magnetic Gauss's Law
$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	Faraday equation	$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}$	Ampère Equation

where the time-dependent scalar ρ is the charge density and the time-depentent vector field $\mathbf{j}$ is the electric current density.

We can restate this theory in a geometric context, that allow us to generalize it to a more abstract frame, the gauge theories. To this end, let us take, the space-time $\mathbb{R}^4$ as a differentiable manifold. In this manifold, let us define the 2-form F given by

$$F = \sum_{i=1}^{3} E_i \, dx^i \wedge dt + \sum_{0 \leq i < j \leq 3} \epsilon_{ijk} \, B_i dx^j \wedge dx^k$$

where ϵ_{ijk} is the sign of the permutation $(i\, j\, k)$, E_1, E_2, E_3 are the components of the electric field $\mathbf{E}$ and B_1, B_2, B_3 are the components of the magnetic field $\mathbf{B}$. Then, observe that, computing the

[1] Units in which the dielectic constant ϵ_0 and the magnetic constant μ_0 are both 1.

exterior differential of F

$$\begin{aligned} dF &= \left(\frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} + \frac{dB_1}{dt}\right) dx^2 \wedge dx^3 \wedge dt + \left(\frac{\partial E_3}{\partial x^1} - \frac{\partial E_1}{\partial x^3} + \frac{dB_2}{dt}\right) dx^1 \wedge dx^3 \wedge dt \\ &+ \left(\frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} + \frac{dB_3}{dt}\right) dx^1 \wedge dx^2 \wedge dt + \left(\frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

So, equating component by component, we observe that $dF = 0$ is equivalent to $\nabla \times \mathbf{E} = 0$, the Faraday equation and $\nabla \cdot \mathbf{B} = 0$, the magnetic Gauss's law.

In order to restate the other two equations, the Gauss's law and the Ampère equation, we need to introduce more structure in the space-time $\mathbb{R}^4$. It can be seen that electromagnetism is, in fact, a relativistic phenomenon, not a classical one. Hence, it is natural to introduce, in $\mathbb{R}^4$, the Minkowski metric of signature $(1,3)$ which is strongly linked with special relativity. In terms of coordinates (t, x^1, x^2, x^3) of $\mathbb{R}^4$, this metric, $g_{1,3}$, is given by

$$g_{1,3} = -dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$$

Let us denote this space $\mathbb{R}^{1,3} = (\mathbb{R}^4, g_{1,3})$. Let $\star : \Omega^*(\mathbb{R}^{1,3}) \to \Omega^{4-*}(\mathbb{R}^{1,3})$ be the Hodge star operator over $\mathbb{R}^{1,3}$.[2] In this particular case, over the basis $\{dt, dx^1, dx^2, dx^3\}$, with semi-riemannian volume form $\Omega = -dt \wedge dx^1 \wedge dx^2 \wedge dx^3$, the Hodge star operator is the linear operator given on the basis by

$$\begin{array}{rcc rcc} \star 1 & = & -dt \wedge dx^1 \wedge dx^2 \wedge dx^3 & \star\, dt \wedge dx^1 \wedge dx^2 \wedge dx^3 & = & 1 \\ \star\, dx^i & = & dx^j \wedge dx^k \wedge dt & \star\, dt & = & dx^1 \wedge dx^2 \wedge dx^3 \\ \star\, dx^i \wedge dx^j & = & dx^k \wedge dt & \star\, dx^i \wedge dt & = & -dx^j \wedge dx^k \\ \star\, dx^1 \wedge dx^2 \wedge dx^3 & = & dt & \star\, dt \wedge dx^i \wedge dx^j & = & dx^k \end{array}$$

where $(i\,j\,k)$ is an even permutation of $(1\,2\,3)$. With this operator, if we compute $d \star F$ we obtain

$$\begin{aligned} d \star F &= d\left(-\sum_{i,j,k} \epsilon_{ijk} E_i dx^j \wedge dx^k + \sum_{i=1}^{3} B_i\, dx^i \wedge dt\right) = -\left(\frac{\partial E_1}{\partial x^1} + \frac{\partial E_2}{\partial x^2} + \frac{\partial E_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 \\ &+ \left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3} - \frac{dE_1}{dt}\right) dx^2 \wedge dx^3 \wedge dt + \left(\frac{\partial B_3}{\partial x^1} - \frac{\partial B_1}{\partial x^3} + \frac{dE_2}{dt}\right) dx^1 \wedge dx^3 \wedge dt \\ &+ \left(\frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2} - \frac{dE_3}{dt}\right) dx^1 \wedge dx^2 \wedge dt \end{aligned}$$

So we have that

$$\begin{aligned} \star d \star F &= \left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3} - \frac{dE_1}{dt}\right) dx^1 + \left(\frac{\partial B_1}{\partial x^3} - \frac{\partial B_3}{\partial x^1} - \frac{dE_2}{dt}\right) dx^2 + \left(\frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2} - \frac{dE_3}{dt}\right) dx^3 \\ &- \nabla \cdot \mathbf{E}\, dt \end{aligned}$$

[2]See chapter 3.1.1 for the general definition of the Hodge star operator.

Let us define the **current form** $J = \sum_{i=1}^{3} j_i dx^i - \rho dt$, where j_1, j_2, j_3 are the components of the electric current density, Hence, by the previous computation, we have that the Gauss's law and the Ampère equations are equivalent to $\star d \star F = J$. Therefore, in this setting, the Maxwell's equations are equivalent to

$$\boxed{dF = 0 \qquad \star d \star F = J}$$

In particular, if we are looking for equations in vacuum, $J = 0$ and the Maxwell equations can be restated as

$$\boxed{dF = 0 \qquad d \star F = 0}$$

As we will see, this is the starting point of a more general equation, the Yang-Mills equation, that is in the core of a very more abstract theory, known as gauge theory. This theory, that can be used to model high-energy physics, is, connects mathematics and physics in such a powerful way that, even now, it is under very intense research.

1.2 Review of Lie Group Theory

First of all, let us note that along this section and the subsequents of this chapter, we will work exclusively in the smooth category, that is, all manifolds and maps will be C^∞ differentiable.

Recall that a **Lie group** G is a manifold that also has structure of group, and such that the group operations are C^∞ maps. Intrinsically associated to the a real Lie group G is its **Lie algebra** $\mathfrak{g}$. By definition, a Lie algebra $\mathfrak{g}$ is a $\mathbb{R}$-vector space with a anticommutative linear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satifies the Jacobi identity. The Lie algebra of G is the left-invariant vector fields with the Lie bracket or, equivalently, the tangent space at the identity with the Lie bracket of their left-invariant extensions. We can go back and forth between both worlds using the exponential map $exp : \mathfrak{g} \to G$ (or, at least, defined on an neighbourhood of $0 \in \mathfrak{g}$). We will denote by $L_g : G \to G$ and $R_g : G \to G$ the *left-product* and *right-product* automorphisms, respectively, that is $L_g(h) = gh$ and $R_g(h) = hg$.

Given $g \in G$, we can define the conjugation map $c_g : G \to G$ given by $c_g(h) = ghg^{-1}$. Hence, we define the **adjoint representation** of G, $Ad : G \to GL(\mathfrak{g})$ as $Ad(g) = (c_g)_{*e}$.[3] We will denote $Ad_g := Ad(g) \in GL(\mathfrak{g})$.

Suppose now that G is a matrix group, that is $G \subseteq GL(V)$, for some vector space V. In this case, the adjoint representation is simple conjugation. Indeed, if $\xi \in \mathfrak{g}$ and $A \in G$, we have that

$$Ad_A(\xi) = \left.\frac{d}{dt}\right|_{t=0} c_A(exp(t\xi)) = \left.\frac{d}{dt}\right|_{t=0} A\, exp(t\xi)\, A^{-1} = A \left(\left.\frac{d}{dt}\right|_{t=0} exp(t\xi) \right) A^{-1} = A\xi A^{-1}$$

[3]Furthermore, diferentiating one more time, we have the adjoint representation of $\mathfrak{g}$, $ad = (Ad_*)_e : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. It holds $ad(\xi)(\chi) = [\xi, \chi]$ for all $\xi, \chi \in \mathfrak{g}$.

Moreover, for a differentiable manifold M and a Lie algebra $\mathfrak{g}$, we can extend the notion of k-forms to the space of **k-differential forms with values in $\mathfrak{g}$**, $\Omega^k(M,\mathfrak{g}) := \mathfrak{g} \otimes \Omega^k(M)$. In this space, we can extend several operations:

- *Exterior derivative*: Let us fix a basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$. Then, using this basis, every $\omega \in \Omega^k(M,\mathfrak{g})$ can be written in an unique way
$$\omega = \sum_{i=1}^{n} \xi_i \otimes \omega_i$$
where $\omega_i \in \Omega^k(M)$. Then, we define $d\omega \in \Omega^{k+1}(M,\mathfrak{g})$ by
$$d\omega = \sum_{i=1}^{n} \xi_i \otimes d\omega_i$$
This definition does not deppend on the basis choosen (essentially, because any change of matrix is constant along M) so $d\omega$ is well defined.

- *Graded Lie bracket*: Let $\omega \in \Omega^p(M,\mathfrak{g})$ and let $\eta \in \Omega^q(M,\mathfrak{g})$. Then, we define the graded Lie bracket of ω and η, $[\omega,\eta] \in \Omega^{p+q}(M,\mathfrak{g})$, by
$$[\omega,\eta](X_1,\dots,X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} sign(\sigma)[\omega(X_{\sigma(1)},\dots,X_{\sigma(p)}), \eta(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]$$
for all vector fields $X_1,\dots,X_{p+q}$ on M. It satisfies the following identities
  - *Graded commutation*: For $\omega \in \Omega^p(M,\mathfrak{g})$ and $\eta \in \Omega^q(M,\mathfrak{g})$
$$[\omega,\eta] = (-1)^{pq+1}[\eta,\omega]$$
  - *Graded Jacobi identity*: For $\omega \in \Omega^p(M,\mathfrak{g})$, $\eta \in \Omega^q(M,\mathfrak{g})$ and $\delta \in \Omega^r(M,\mathfrak{g})$
$$(-1)^{pr}[[\omega,\eta],\delta] + (-1)^{pq}[[\eta,\delta],\omega] + (-1)^{qr}[[\delta,\omega],\eta] = 0$$

Remark 1.2.1. If $\omega = \eta$ is a 1-form, we have
$$[\omega,\omega](X_1,X_2) = [\omega(X_1),\omega(X_2)] - [\omega(X_2),\omega(X_1)] = 2[\omega(X_1),\omega(X_2)]$$

- *Wedge product*: Let us fix a basis $\xi_1,\dots,\xi_n$ of $\mathfrak{g}$. Let $\omega \in \Omega^p(M,\mathfrak{g})$, $\eta \in \Omega^q(M,\mathfrak{g})$, written in this basis as $\omega = \sum \omega_i \otimes \xi_i$ and $\eta = \sum \eta_j \otimes \xi_j$. Then, we define $\omega \wedge \eta \in \Omega^{p+q}(M,\mathfrak{g})$ as
$$\omega \wedge \eta = \sum_{i,j=1}^{n} [\xi_i,\xi_j] \otimes (\omega_i \wedge \eta_j)$$

Again, this definition does not depend on the basis choosen. Furthermore, it holds

$$[\omega, \eta] = \omega \wedge \eta + (-1)^{pq} \eta \wedge \omega \qquad d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$$

For computational purposes, if G is a matrix group (so $\mathfrak{g}$ is a vector subspace of the space of matrices $\mathfrak{gl}(n, \mathbb{R})$ for some $n > 0$ and the Lie bracket is the commutator) we can describe $\omega \wedge \eta$ in a more manegable way. Let us write ω and η as matrices of p-forms and q-forms, respectively. Then $\omega \wedge \eta$ is the matrix obtained by matrix product of ω and η where the entries (elements of $\Omega^*(M)$) are multiplied by wedge product.

1.3 Connections on Vector Bundles

Recall that a **vector bundle** with base manifold M and fiber a vector space V is a fiber bundle $E \xrightarrow{\pi} M$ such that each fiber has a vector space structure isomorphic to V and such that, for all trivilizing neighbourhood $U \subseteq M$ and diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times V$ the map $\varphi_U|_{E_x} : E_x \to V$ is a linear isomorphism. Equivalently, the transition functions between trivializing neighbourhoods U_α and U_β, $g_{\alpha\beta}$, defined by $\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)(v))$ are linear automorphisms for all $x \in U_\alpha \cap U_\beta$, that is $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$. We will call the **rank** of E to the dimension of V.

In this context, we define a connection on a vector bundle as generalization of an affine connection on the tangent bundle of a smooth manifold, as used in riemannian geometry. Good references for the topic are [4], [42], [11] or [43].

Definition 1.3.1. Let $E \xrightarrow{\pi} M$ a vector bundle over a differentiable manifold M. An **affine connection** (also known simply as **connection** or **covariant derivative**) is a map $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Omega^1(M)$ such that

- *Additivity*: $\nabla(\sigma_1 + \sigma_2) = \nabla(\sigma_1) + \nabla(\sigma_2)$ for $\sigma_1, \sigma_2 \in \Gamma(E)$.
- *Leibniz rule*: $\nabla(f\,\sigma) = f\,\nabla(\sigma) + \sigma \otimes df$ for all $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$.

We will denote the set of all affine connections over E by $\mathcal{A}_E$. Given a vector field X over M, we will denote by $\nabla_X : \Gamma(E) \to \Gamma(E)$ the endomorphism defined as $\nabla_X(\sigma) = \nabla(\sigma)(X)$ for all $\sigma \in \Gamma(E)$.

Example 1.3.2. The exterior diferential d is an affine connection on $\Omega^k(M)$. In particular, d is an afine connection on $C^\infty(M)$, seen as $\mathbb{R}$-vector bundle.

Example 1.3.3. Every connection ∇, in the sense of riemannian geometry, induces an affine connection on TM.

Example 1.3.4. Let us take a trivial vector bundle $E = M \times V \xrightarrow{\pi} M$. Let us take a basis of V, $v_1, \ldots, v_n$ and we define $e_1, \ldots, e_n : M \to M \times V$ by $e_1(x) = (x, v_1), \ldots, e_n(x) = (x, v_n)$. Observe that $e_1(x),\dots,e_n(x)$ is a basis of E_x for all $x\in M$ so, for every section $\sigma\in\Gamma(E)$ there exist unique $\sigma^1,\dots,\sigma^n\in C^\infty(M)$ such that $\sigma(x)=\sum\sigma^i(x)e_i(x)$ for all $x\in M$.

In this case, we have that the map $d:\Gamma(E)\to\Gamma(E)\otimes\Omega^1(M)$ given by

$$d(\sigma)=d\left(\sum_i\sigma^ie_i\right)=\sum_i e_i\otimes d\sigma^i$$

is an affine connection. Furthermore, if we take 1-forms $A_i^j\in\Omega^1(M)$ for $i,j=1,\dots n$, then we can define the affine connection $d+A:\Gamma(E)\to\Gamma(E)\otimes\Omega^1(X)$ given by

$$(d+A)(\sigma)=(d+A)\left(\sum_i\sigma^ie_i\right)=d\left(\sum_i\sigma^ie_i\right)+A\left(\sum_i\sigma^ie_i\right)=\sum_i e_i\otimes d\sigma_i+\sum_{i,j}e_j\otimes\sigma^iA_i^j$$

Equivalently, gathering together the A_j^i in a matrix of 1-forms $A=\left(A_i^j\right)_{i,j=1}^n\in\Omega^1(M,End(V))=\Omega^1(M,\mathfrak{gl}(V))$ we have just define

$$(d+A)(\sigma)=d(\sigma)+A(\cdot)(\sigma)$$

Observe that $A(\cdot)(\sigma)\in\Omega^1(M)\otimes\Gamma(E)$. However, the key point is that all the connection on $E=M\times V$ are of this form. Indeed, if ∇ is any affine connection on E, it should satisfy

$$\nabla(\sigma)=\nabla\left(\sum_i\sigma^ie_i\right)=\sum_i e_i\otimes d\sigma^i+\sum_i\sigma^i\nabla(e_i)=\sum_i e_i\otimes d\sigma^i+\sum_{i,j}e_j\otimes\sigma^iA_i^j=(d+A)(\sigma)$$

where $\nabla(e_i)=\sum_j e_j\otimes A_i^j$.[4] Therefore, we have just prove that, in a trivial model, all the connections are of the form $d+A$ for some $A\in\Omega^1(M,\mathfrak{gl}(V))$. In this case, A is called the **gauge potential** of ∇.

In the general case, let $E\xrightarrow{\pi}M$ be any vector bundle of fiber V, and let $\varphi_U:\pi^{-1}(U)\to U\times V$ be a trivialization of the bundle. Choosing a basis of V $v_1,\dots,v_n$, we can take a basis of sections $e_1,\dots,e_n:U\to\pi^{-1}(U)\subseteq E$ by $e_i(x)=\phi_U^{-1}(x,v_i)$ for $i=1,\dots,n$. Then, using the previous example, we can locally write, with respect to this trivialization and basis, $\nabla|_{\pi^{-1}(U)}=d+A_U$ with $A_U\in\Omega^1(U,\mathfrak{gl}(V))$. Therefore, locally, all the connections are of the form $d+A$ for some $A\in\Omega^1(U,\mathfrak{gl}(V))$.

Let us study how the gauge potential 1-forms change when passing from a trivializing neighbourhood to another. Suppose that $U_\alpha,U_\beta\subseteq M$ are two trivializing neighbourhoods of E with $U_\alpha\cap U_\beta\neq\emptyset$. Then, on U_α, $\nabla=d+A_\alpha$ and, on U_β, $\nabla=d+A_\beta$. Let us take bases of sections $e_1^\alpha,\dots,e_n^\alpha$ on U_α and $e_1^\beta,\dots,e_n^\beta$ on U_β. Using Einstein summation convention,[5] observe that $e_i^\beta={g_{\alpha\beta}}_i^je_j^\alpha$ where $g_{\alpha\beta}:U_\alpha\cap U_\beta\to GL(V)$ are the transition functions. Then, on $U_\alpha\cap U_\beta$ we have

$$\nabla(e_i^\beta)=\nabla({g_{\alpha\beta}}_i^je_j^\alpha)=e_k^\alpha\otimes d\left({g_{\alpha\beta}}_i^k\right)+{g_{\alpha\beta}}_i^j\nabla(e_j^\alpha)=e_k^\alpha\otimes d\left({g_{\alpha\beta}}_i^j\right)+{g_{\alpha\beta}}_i^je_k^\alpha\otimes {A_\alpha}_j^k$$

[4]In the riemannian jargon, $\Gamma_{ik}^j:=A_i^j\left(\frac{\partial}{\partial x^k}\right)$ are the Christoffel symbols where ∇ acts on TM.

[5]That is, a repeated index up and down means an elipsed sum in that index.

and, since $\nabla(e_i^\beta) = e_j^\beta \otimes A_{\beta}{}_i^j = g_{\alpha\beta}{}_j^k e_k^\alpha \otimes A_{\beta}{}_i^j$. Therefore, using matrix product and $dg_{\alpha\beta}$ as a matrix of 1-forms we have that

$$g_{\alpha\beta} A_\beta = dg_{\alpha\beta} + A_\alpha g_{\alpha\beta}$$

or, equivalently

$$\boxed{A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}}$$

Remark 1.3.5. We can write this formula in the invariant form

$$\boxed{A_\beta(X_x) = \left(L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta}\right)_* (X_x) + Ad_{g_{\alpha\beta}^{-1}(x)}(A_\alpha(X_x))}$$

Indeed, let $\gamma : (-\epsilon, \epsilon) \to U$ be a curve with $\gamma'(0) = X_x$. Then, we have that

$$\begin{aligned} L_{g_{\alpha\beta}^{-1}(x)}{}_*(g_{\alpha\beta}{}_*(X_x)) &= L_{g_{\alpha\beta}^{-1}(x)}{}_*\left(\left.\frac{d}{dt}\right|_{t=0} g_{\alpha\beta}(\gamma(t))\right) = \left.\frac{d}{dt}\right|_{t=0} L_{g_{\alpha\beta}^{-1}(x)}\left(g_{\alpha\beta}(\gamma(t))\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} g_{\alpha\beta}^{-1}(x) g_{\alpha\beta}(\gamma(t)) = g_{\alpha\beta}^{-1}(x) \left.\frac{d}{dt}\right|_{t=0} g_{\alpha\beta}(\gamma(t)) = g_{\alpha\beta}^{-1}(x)\, dg_{\alpha\beta}(X_x) \end{aligned}$$

and, for the other term, we have only to remember that *Ad* in a matrix group is simple conjugation, so

$$Ad_{g_{\alpha\beta}^{-1}(x)}(A_\alpha(X_x)) = g_{\alpha\beta}^{-1}(x) A_\alpha(X_x) g_{\alpha\beta}(x)$$

In fact, this introduces the idea of what is the structure of the space of affine connections on a vector bundle.

Proposition 1.3.6. *Given a vector bundle $E \xrightarrow{\pi} M$, the set of affine connections over E, $\mathcal{A}_E$, is an affine space with underlying vector space $\Omega^1(M, End(E))$.*

Proof. First of all, let ∇ be an affine connection over E and $A \in \Omega^1(M, End(E))$. Let us define $\nabla' = \nabla + A : \Gamma(E) \to \Gamma(E) \times \Omega^1(M)$ by

$$\nabla'_X(\sigma) = \nabla_X(\sigma) + A(X)(\sigma)$$

where $\sigma \in \Gamma(E)$ and X is a vector field on M. Observe that ∇' is an affine connection, because it has the additivity property (since ∇ and A have it) and, for the Leibniz rule

$$\begin{aligned} \nabla'_X(f\sigma) &= \nabla_X(f\sigma) + A(X)(f\sigma) = X(f)\sigma + f\nabla_X(\sigma) + fA(X)(\sigma) \\ &= X(f)\sigma + f\left(\nabla_X(\sigma) + A(X)(\sigma)\right) = X(f)\sigma + f\nabla'_X(\sigma) \end{aligned}$$

so, ∇' is an affine connection on E.

Now, recall that an affine space is a space with a transitive free action of a vector space. So, let us define the action (on the right) of $\Omega^1(M, End(E))$ on $\mathcal{A}_E$ by $\nabla \cdot A = \nabla + A$, for $\nabla \in \mathcal{A}_E$ and $A \in \Omega^1(M, End(E))$. We just have checked that $\cdot$ is an action on $\mathcal{A}_E$, and it is free.

Therefore, it is enough to prove that this action is transitive or, equivalently, that, given two affine connections ∇_1, ∇_2 on E, $\nabla_1 - \nabla_2$ lives in $\Omega^1(M, End(E))$. Moreover, it is enough to check that ∇_1, ∇_2 is a homomorphism of $C^\infty(M)$-modules $\nabla_1 - \nabla_2 : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E)$ since, in that case, we will be a tensor so we will have that

$$\nabla_1 - \nabla_2 \in \Omega^1(M) \otimes \Gamma(E) \otimes \Gamma(E^*) \cong \Gamma(T^*M \otimes E \otimes E^*) \cong \Gamma(T^*M \otimes End(E)) \cong \Omega^1(M, End(E))$$

Of course, additivity is easy, so the only non-trivial fact is the $C^\infty(M)$-lineality. For this, observe that, given $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$ we have

$$(\nabla_1 - \nabla_2)(f\sigma) = \nabla_1(f\sigma) - \nabla_2(f\sigma) = (\sigma \otimes df + f\nabla_1(\sigma)) - (\sigma \otimes df + f\nabla_2(\sigma)) = f\,(\nabla_1 - \nabla_2)\,(\sigma)$$

as we wanted to prove. ■

1.3.1 Covariant Exterior Derivative

Hereupon, let us fix a vector bundle $E \xrightarrow{\pi} M$ with fiber V. We can enlarge this vector bundle and define the **space of k-differential forms with values in** E

$$\Omega^k(E) := \Gamma(E) \otimes \Omega^k(M) = \Gamma(E \otimes \Lambda^k M)$$

Recall that a tensor product inherits the module structure of any of its factor. For example, since $\Omega^*(M)$ is a ring with the wedge product, we can define, for $\sigma \in \Gamma(E)$ and $\omega, \eta \in \Omega^*(M)$

$$(\sigma \otimes \omega) \wedge \eta := \sigma \otimes (\omega \wedge \eta)$$

Therefore, given an affine connection ∇ over E we can use this information to create a generalization of the exterior derivative, called the **covariant exterior derivative**. This is defined, for all $k \geq 0$ as the map $d_\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$ given on basic elements by

$$d_\nabla(\sigma \otimes \omega) := \nabla(\sigma) \wedge \omega + \sigma \otimes d\omega$$

Remark 1.3.7. If we take $k = 0$, then, in this level, the covariant exterior derivative $d_\nabla : \Omega^0(E) = \Gamma(E) \to \Omega^1(E) = \Gamma(E) \otimes \Omega^1(M)$ reduces to $d_\nabla = \nabla$. If we take $E = M \times \mathbb{R}$, the trivial vector bundle of rank 1, then $\Gamma(E) = C^\infty(M)$ so $\Omega^k(E) = \Omega^k(M)$. Furthermore, E admits the connection $\nabla = d : \Omega^0(E) = C^\infty(M) \to \Omega^1(M)$, the usual exterior derivative in functions, so the covariant exterior derivative associated to this connection is the usual exterior derivative in forms $d_\nabla = d : \Omega^k(M) \to \Omega^{k+1}(M)$.

Remark 1.3.8. The previous definition of covariant exterior derivative comes from a general construction. Suppose that we have two vector bundles $E_1 \overset{\pi_1}{\to} M$ and $E_2 \overset{\pi_2}{\to} M$ with affine connections ∇_1, ∇_2. We can define a conection ∇ on $E_1 \otimes E_2$ given by

$$\nabla(\sigma_1 \otimes \sigma_2) = \nabla_1(\sigma_1) \otimes \sigma_2 + \sigma_1 \otimes \nabla_2(\sigma_2)$$

for $\sigma_1 \in \Gamma(E_1)$, $\sigma_2 \in \Gamma(E_2)$. Observe that we have defined $(\sigma_1 \otimes \omega) \otimes \sigma_2 := (\sigma_1 \otimes \sigma_2) \otimes \omega$. Moreover, ∇ induces a conection on E^*, ∇^*, given by

$$d(\mu(\sigma)) = \nabla^*(\mu)(\sigma) + \mu(\nabla(\sigma))$$

for all $\sigma \in \Gamma(E)$ and $\mu \in \Gamma(E^*)$.

1.3.1.1 Connections and holomorphicity

Finally, let us do a brief digresion for the relation between connections and holomorphicity. Let us suppose that M is a complex manifold and $E \to M$ is a C^∞-complex vector bundle. In that case, the almost complex structure on M induces a decomposition at the level of forms

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

and, defining $\Omega^{p,q}(E) := \Gamma(E) \otimes \Omega^{p,q}(M)$, this bigrading can be extended to a bigrading

$$\Omega^k_{\mathbb{C}}(E) = \bigoplus_{p+q=k} \Omega^{p,q}(E)$$

Moreover, since the almost complex structure of M integrates, we have a decomposition of the exterior derivative on M in terms of the Dolbeault and anti-Dolbeault operators

$$d = \partial + \overline{\partial}$$

with $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. Hence, since d_∇ is a combination of ∇ and d, we also have a decomposition

$$d_\nabla = \partial_\nabla + \overline{\partial}_\nabla$$

with $\partial_\nabla : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E)$ and $\overline{\partial}_\nabla : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$, respectively called the anti-Dolbeault and Dolbeault covariant exterior derivatives of ∇.

On the other hand, let us suppose that $E \to M$ is, not only a $\mathbb{C}^\infty$-complex vector bundle, but a holomorphic vector bundle, i.e., E is a complex manifold and $\pi : E \to M$ is holomorphic. In that case, we can define an operator

$$\overline{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$$

be decreeing that

- $\overline{\partial}_E(\sigma_1 + \sigma_2) = \overline{\partial}_E(\sigma_1) + \overline{\partial}_E(\sigma_2)$ for $\sigma_1, \sigma_2 \in \Gamma(E)$.
- $\overline{\partial}_E(f\sigma) = \overline{\partial} f\sigma + f\overline{\partial}_E(\sigma)$ for $f \in \mathbb{C}^\infty(M)$ and $\sigma \in \Gamma(E)$.
- $\overline{\partial}_E(\sigma) = 0$ on an open set $U \subseteq M$ if and only if σ is an holomorphic section $\sigma : \pi^{-1}(U) \to U$.

Remark 1.3.9. This operator $\overline{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ can be extended to and operator $\overline{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q}(E)$ for any $p, q \geq 0$ by $\overline{\partial}_E(\sigma \otimes \omega) = \overline{\partial}_E(\sigma) \otimes \omega + \sigma\overline{\partial}(\omega)$, for $\sigma \in \Gamma(E)$ and $\omega \in \Omega^{p,q}(M)$. Furthermore, this operator coincides with a natural one in a special case. Suppose than the complex dimension of M is n, and let $K_M = \Omega^{n,0}(M)$ be the canonical bundle of M, with its natural holomorphic structure. Then, it can be proven that, in the $(n, 0)$-bigrading, the operator

$$\overline{\partial}_E : \Omega^{n,0}(E) = \Gamma(E \otimes K_M) \to \Omega^{n,1}(E) = \Gamma(E \otimes K_M) \otimes \Omega^1(M)$$

coincides with the operator $\overline{\partial}_{E\times K_M}$ for the holomorphic structure on $E \times K_M$. In particular, given $\Phi \in \Omega^{n,0}(E)$, we have that $\overline{\partial}_E\Phi = 0$ if and only if Φ is holomorphic in $E \times K_M$.

In some sense, this two concepts are equivalent, as explained in the following theorem, whose proof can be found in [44].

Theorem 1.3.10. *Let M be a complex manifold.*

- *Let $E \to M$ be a C^∞-complex vector bundle and let ∇ be a connection on E. If we have $\overline{\partial}_\nabla^2 = \overline{\partial}_\nabla \circ \overline{\partial}_\nabla = 0$, then there exists an unique complex structure on E such that $E \to M$ is a holomorphic vector bundle and $\overline{\partial}_E = \overline{\partial}_\nabla$.*
- *Let $E \to M$ be an holomorphic vector bundle. There exists an unique connection ∇ on E such that $\overline{\partial}_E = \overline{\partial}_\nabla$.*

1.3.2 Curvature on Vector Bundles

Let E be a trivial vector bundle with $d_\nabla = d$. In that case, we automaticaly have $d \circ d = 0$. However, in general it is not true that $d_\nabla \circ d_\nabla = 0$, that is, the complex

$$\Omega^0(E) \xrightarrow{d_\nabla} \Omega^1(E) \xrightarrow{d_\nabla} \Omega^2(E) \xrightarrow{d_\nabla} \dots$$

is not necessary an exact sequence. The curvature mesures how far a connection is of having that complex exact.

Definition 1.3.11. Let $E \xrightarrow{\pi} M$ be a vector bundle with fiber V and structure group $G \subseteq GL(V)$ and let ∇ be an affine connection on E. The **curvature** of ∇, F_∇ is the map

$$F_\nabla := d_\nabla \circ d_\nabla : \Omega^0(E) \to \Omega^2(E)$$

A connection ∇ is called **flat** if $F_\nabla = 0$.

Remark 1.3.12. In a trivial vector bundle with covariant exterior derivative $d_\nabla = d$ we have $F_\nabla = d \circ d = 0$, so, ∇ is flat.

Proposition 1.3.13. *F_∇ is a tensor, that is, it is $C^\infty(M)$-linear.*

Proof. It is simply a long computation. Let $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$ and let us write $\nabla\sigma = \sigma^i \otimes \omega_i$ for some $\sigma^i \in \Gamma(E)$ and $\omega_i \in \Omega^1(M)$. Then we have

$$\begin{aligned}
F_\nabla(f\sigma) &= d_\nabla(d_\nabla(f\sigma)) = d_\nabla(\nabla(f\sigma)) = d_\nabla(f\nabla\sigma + \sigma \otimes df) = d_\nabla(f\nabla\sigma) + \nabla\sigma \wedge df + \sigma \otimes d^2 f \\
&= d_\nabla(f\sigma^i \otimes \omega_i) + \nabla\sigma \wedge df = \nabla(f\sigma^i) \wedge \omega_i + f\sigma^i \otimes d\omega_i + \nabla\sigma \wedge df \\
&= \big(\sigma^i \otimes df + f\nabla(\sigma^i)\big) \wedge \omega_i + f\sigma^i \otimes d\omega_i + \nabla\sigma \wedge df \\
&= \sigma^i \otimes df \wedge \omega_i + f\nabla(\sigma^i) \wedge \omega_i + f\sigma^i \otimes d\omega_i + \sigma^i \otimes \omega^i \wedge df \\
&= f\nabla(\sigma^i) \wedge \omega_i + f\sigma^i \otimes d\omega_i = f\, d_\nabla(\sigma^i \otimes \omega_i) = f d_\nabla(\nabla(\sigma)) = f F_\nabla(\sigma)
\end{aligned}$$

as we wanted to prove. ∎

As consequence of this proposition, since F_∇ is a tensor, it can be seen as living in

$$(\Omega^0(E))^* \otimes \Omega^2(E) = \Gamma(E)^* \otimes \Gamma(E) \otimes \Omega^2(M) \cong \Gamma(E \otimes E^* \otimes \Lambda^2 T^* M) = \Omega^2(End(E))$$

and, abusing of notation, we will also denote this tensor by $F_\nabla \in \Omega^2(End(E))$.

1.3.2.1 Curvature in the trivial model

In this section, let us suppose that our vector bundle $E \to M$ is trivial, that is $E = M \times V$ for some finite dimensional $\mathbb{R}$-vector space V. Also, we will fix a basis $v_1, \dots, v_n$ of V and an affine connection ∇ on E. This will allow us to do some explicit computations on E that will be easily translated to the general case.

The first consequence of the triviality of the vector bundle is that the endomorphism bundle $End(E)$ is also trivial, that is $End(E) = M \times End(V) = M \times \mathfrak{gl}(V)$. Therefore, the space $\Omega^*(M, End(E))$ is canonically isomorphic to $\Omega^*(M, \mathfrak{gl}(V))$. The second consequence of this reductions is that the connection ∇ has the special form of

$$\nabla = d + A$$

for some $A \in \Omega^1(M, End(E)) = \Omega^1(M, \mathfrak{gl}(V))$.

This space $\Omega^*(M, End(E)) = \Omega^1(M, \mathfrak{gl}(V))$ will play a very important role in our computations. First of all, we can extend the wedge product to a map $\wedge : \Omega^p(M, End(E)) \times \Omega^q(M, E) \to \Omega^{p+q}(M, E)$. This map, for basic elements $T \otimes \omega$ and $\sigma \otimes \eta$ for $T \in \Gamma(End(E))$, $\sigma \in \Gamma(E)$, $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$, is defined as

$$(T \otimes \omega) \wedge (\sigma \otimes \eta) := T(\sigma) \otimes (\omega \wedge \eta)$$

Remark 1.3.14. If $\sigma \in \Gamma(E) = \Omega^0(M, E)$, then $\Omega \wedge \sigma = \Omega(\sigma)$ for all $\Omega \in \Omega^*(M, End(E))$.

Using this wedge product, we can rewrite the covariant exterior derivative in a more explicit way.

Proposition 1.3.15. *Let $E = M \times V$ be a trivial vector bundle with an affine connection $\nabla = d + A$, then*

$$d_\nabla(\Omega) = d\Omega + A \wedge \Omega$$

for all $\Omega \in \Omega^*(M, E)$

Proof. The proof is only a computation. Without lost of generality, we can suppose that Ω is a basic element, let us say $\Omega = \sigma \otimes \omega$ for $\sigma \in \Gamma(E)$ and $\omega \in \Omega^*(M)$. Then, we have

$$\begin{aligned} d_\nabla(\sigma \otimes \omega) &= \nabla\sigma \wedge \omega + \sigma \otimes d\omega = ((d + A)\sigma) \wedge \omega + \sigma \otimes d\omega \\ &= d\sigma \wedge \omega + A\sigma \wedge \omega + \sigma \otimes d\omega \end{aligned}$$

While, for the other term, we have

$$d(\sigma \otimes \omega) + A \wedge (\sigma \otimes \omega) = d\sigma \wedge \omega + \sigma \otimes d\omega + A \wedge (\sigma \otimes \omega)$$

Therefore, it is enough to prove that $A \wedge (\sigma \otimes \omega) = A\sigma \wedge \omega$. Let us write $A = T^i_j \otimes A^j_i$, where $T^i_j \in \mathfrak{gl}(V)$ does $T^i_j(e_i) = e_j$ and $T^i_j(e_k) = 0$ for $k \neq i$, and $\sigma = \sigma^k e_k$. Hence, we have

$$\begin{aligned} A \wedge (\sigma \otimes \omega) &= \left(T^i_j \otimes A^j_i\right) \wedge (\sigma^k e_k \otimes \omega) = \sigma^k T^i_j(e_k) \otimes A^j_i \wedge \omega = \sigma^i e_j \otimes A^j_i \wedge \omega \\ A\sigma \wedge \omega &= (T^i_j \otimes A^j_i)(\sigma^k e_k) \wedge \omega = \sigma^k T^i_j(e_k) \otimes A^j_i \wedge \omega = \sigma^i e_j \otimes A^j_i \wedge \omega \end{aligned}$$

as we wanted. ■

With this explicit form of the covariant exterior derivative, we can compute F_∇ explicitly, usually known as the structure equation.

Corollary 1.3.16 (Structure equation)**.** *Let $E = M \times V$ be a trivial vector bundle with an affine connection $\nabla = d + A$. Then, it holds*

$$F_\nabla = dA + A \wedge A = dA + \frac{1}{2}[A, A]$$

Proof. Again,

$$\begin{aligned} F_\nabla(\sigma) &= d_\nabla(d_\nabla(\sigma)) = (d + A\wedge)\left((d+A)(\sigma)\right) = (d + A\wedge)\left(d\sigma + A\wedge\sigma\right) \\ &= d^2\sigma + d\left(A\wedge\sigma\right) + A\wedge d\sigma + A\wedge A\wedge\sigma = dA\wedge\sigma - A\wedge d\sigma + A\wedge d\sigma + A\wedge A\wedge\sigma \\ &= (dA + A\wedge A)(\sigma) \end{aligned}$$

where we have used that $d(A\wedge\sigma) = dA\wedge\sigma + (-1)^p A\wedge d\sigma$ for $A\in\Omega^p(M, End(E))$ and $\sigma\in\Omega^q(M,E)$ ■

Overlooking following questions, we need to examine the case of the endomorphism bundle $End(E)$. Recall that, given a affine connection ∇ on E, we have automatically defined a connection on $End(E)$, also denoted by ∇, with the requirement that

$$\nabla(B(\sigma)) = (\nabla B)(\sigma) + B(\nabla\sigma)$$

for $B\in\Gamma(End(E))$ and $\sigma\in\Gamma(E)$.

Proposition 1.3.17. *Let $E = M\times V$ be a trivial vector bundle with an affine connection $\nabla = d + A$ and let $End(E) = M\times\mathfrak{gl}(V)$ be its endomorphism bundle. Then, the induced connection ∇ on $End(E)$ operates by means of the Lie bracket in the following way.*

$$\nabla B = dB + [A, B]$$

for all $B\in\Gamma(End(E))$.

Proof. Let us write $A = T^i_j\otimes A^j_i$. Then, for $\sigma = \sigma^i e_i\in\Gamma(E)$ and $B = B^j_i T^i_j\in\Gamma(End(E))$ we have $Be_i = B^j_i e_j$ so

$$\begin{aligned} \nabla(B(\sigma)) = \nabla(B(\sigma^i e_i)) = \nabla(\sigma^i B^j_i e_j) &= \sigma^i e_j\otimes dB^j_i + B^j_i\nabla(\sigma^i e_j) \\ &= \sigma^i e_j\otimes dB^j_i + B^j_i e_j\otimes d\sigma^i + B^j_i\sigma^i e_k\otimes A^k_j \end{aligned}$$

while, for the other term, we have

$$B(\nabla\sigma) = B(\nabla(\sigma^i e_i)) = B(e_i\otimes d\sigma^i + \sigma^i e_j\otimes A^j_i) = B^j_i e_j\otimes d\sigma^i + \sigma^i B^k_j e_k\otimes A^j_i$$

so, putting all together, we have

$$(\nabla B)(\sigma) = \nabla(B(\sigma)) - B(\nabla\sigma) = \sigma^i e_j\otimes dB^j_i + B^j_i\sigma^i e_k\otimes A^k_j - \sigma^i B^k_j e_k\otimes A^j_i = dB(\sigma) + [A,B]\sigma$$

as we wanted to prove. ■

With this computation in hand, we can prove

Corollary 1.3.18 (Bianchi identity)**.** *Let $E = M \times V$ be a trivial vector bundle with a connection ∇, covariant exterior derivative d_∇ and curvature F_∇. Then, it holds*

$$d_\nabla F_\nabla = 0$$

Proof. This computation uses the structure equation and the explicit formula for the connection on $End(E)$. Then, using that $[A, A \wedge A] = [A, 1/2[A, A]] = 0$ by the graded Jacobi identity, $d(\omega \wedge \eta) = d\omega \wedge \eta(-1)^{pq+1}\omega \wedge d\eta$ and $[\omega, \eta] = -(-1)^{pq}[\eta, \omega]$ for $\omega \in \Omega^p(M, End(E))$ and $\eta \in \Omega^q(M, End(E))$ we have

$$\begin{aligned} d_\nabla F_\nabla &= dF_\nabla + [A, F_\nabla] = d(dA + A \wedge A) + [A, dA + A \wedge A] = d(A \wedge A) + [A, dA] + [A, A \wedge A] \\ &= dA \wedge A - A \wedge dA + [A, dA] = [dA, A] + [A, dA] = 0 \end{aligned}$$

as we wanted to prove. ■

1.3.2.2 Return to the general case

Now, let us take a general vector bundle $E \to M$ with fiber V, not necessarelly trivial, with an affine connection ∇ on it. Let us fix a basis of V an let us consider $\{U_\alpha\}_{\alpha \in \Lambda}$ a covering of M of trivializing open sets. Then, for every $\alpha \in \Lambda$, we can consider $\nabla_\alpha := \nabla|_{\pi^{-1}(U_\alpha)}$. Since $\pi^{-1}(U_\alpha)$ is isomorphic to a trivial vector bundle, then we have that there exists $A \in \Omega^1(U_\alpha, End(E)) = \Omega^1(U_\alpha, \mathfrak{gl}(V))$ such that $\nabla_\alpha = d + A_\alpha$.

Let us write $F_{\nabla_\alpha} \in \Omega^2(U_\alpha, End(E)) = \Omega^2(U_\alpha, \mathfrak{gl}(V))$ for the pullback of the curvature to this trivializing open set $U_\alpha \subseteq M$. Then, since the connection only depends on a small neighbourhood around the considered point and the curvature is tensorial, all the identities valid in a trivial model can be translated to identities for F_{∇_α}. In particular, we have the structure equation.

Proposition 1.3.19 (Structure equation)**.** *Let $E \to M$ be a vector bundle with a connection ∇ and curvature F_∇. Let $U_\alpha \subseteq M$ be a trivializing open set and let F_{∇_α} be the pullback of the curvature to this open set. Then, if $\nabla_\alpha = d + A_\alpha$ we have*

$$\boxed{F_{\nabla_\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha]}$$

Furthermore, since the Bianchi identity can be stated pointwise, working locally in a trivial model and using the Bianchi identity for the trivial model, we have

Corollary 1.3.20 (Bianchi identity)**.** *Let $E \to M$ be a vector bundle with a connection ∇, covariant exterior derivative d_∇ and curvature F_∇. Then, it holds*

$$\boxed{d_\nabla F_\nabla = 0}$$

Remark 1.3.21. Using the structure equations, we can also recover the classical notion of curvature in riemannian geometry. Indeed, let us take a vector bundle $E \to M$ with a connection ∇ and let $n = \dim M$. By the structure equations, in a trivializing open set $U \subseteq M$, we have, locally

$$F_\nabla = dA + \frac{1}{2}[A, A]$$

so, in particular, for the coordinate vector basis $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$, writing $A_i = A\left(\frac{\partial}{\partial x^i}\right)$, we have

$$\begin{aligned}
F_{\nabla ij} = F_\nabla\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= dA\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + \frac{1}{2}[A, A]\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= \frac{\partial}{\partial x^i}A_j - \frac{\partial}{\partial x^j}A_i - A\left(\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]\right) + [A_i, A_j] \\
&= \frac{\partial}{\partial x^i}A_j - \frac{\partial}{\partial x^j}A_i + [A_i, A_j]
\end{aligned}$$

However, this formula can be recognized as a more familiar formula. In fact, let us define the Riemann curvature tensor $R : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(End(E))$ by

$$R_\nabla(X, Y)\,\sigma := \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma$$

it is a straighforward computation (see [4]) that, locally, the components of R are

$$R_{\nabla ij} = \frac{\partial}{\partial x^i}A_j - \frac{\partial}{\partial x^j}A_i + [A_i, A_j]$$

Therefore, both tensor agree locally so, by tensoriality, they agree locally. That is, seen F_∇ as map $F_\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(End(E))$ we have

$$\boxed{F_\nabla(X, Y) = R_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}}$$

which agrees with the usual definition of curvature in riemannian geometry.

Finally, in order to complete this section, let us examine how the local expresion of F_∇ varies from a chart to another. Suppose that we have two trivializing open sets $U_\alpha, U_\beta \subseteq M$ with $U_\alpha \cap U_\beta \neq \emptyset$ and let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ be their transition function. Let us write $\nabla_\alpha = d + A_\alpha$ and $\nabla_\beta = d + A_\beta$

Recall that, under this change of coordinates, the change in the gauge potential A is

$$A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}$$

So, let us compute. For the exterior differential we have

$$dA_\beta = d\left(g_{\alpha\beta}^{-1} dg_{\alpha\beta}\right) + d\left(g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}\right) = dg_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + dg_{\alpha\beta}^{-1} \wedge A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} - g_{\alpha\beta}^{-1} A_\alpha \wedge dg_{\alpha\beta}$$

and, for the wedge product

$$A_\beta \wedge A_\beta = \left(g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}, g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}\right) = g_{\alpha\beta}^{-1} dg_{\alpha\beta} \wedge g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$
$$+ g_{\alpha\beta}^{-1} dg_{\alpha\beta} \wedge g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha \wedge dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha \wedge A_\alpha g_{\alpha\beta}$$

Hence, we have

$$\begin{aligned} F_{\nabla\beta} = dA_\beta + A_\beta \wedge A_\beta &= \left(dg_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + dg_{\alpha\beta}^{-1} \wedge A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}\right) \\ &+ \left(g_{\alpha\beta}^{-1} dg_{\alpha\beta} \wedge g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} \wedge g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha \wedge A_\alpha g_{\alpha\beta}\right) \\ &= g_{\alpha\beta}^{-1} F_{\nabla\alpha} g_{\alpha\beta} + \left(dg_{\alpha\beta}^{-1} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} g_{\alpha\beta}^{-1}\right) \wedge dg_{\alpha\beta} + \left(dg_{\alpha\beta}^{-1} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} g_{\alpha\beta}^{-1}\right) \wedge A_\alpha g_{\alpha\beta} \\ &= g_{\alpha\beta}^{-1} F_{\nabla\alpha} g_{\alpha\beta} \end{aligned}$$

where the last equality follows from

$$0 = d(g_{\alpha\beta}^{-1} g_{\alpha\beta}) = dg_{\alpha\beta}^{-1} g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} \Rightarrow dg_{\alpha\beta}^{-1} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} g_{\alpha\beta}^{-1} = 0$$

Therefore, we have just computed its change of coordinates rule

$$\boxed{F_{\nabla\beta} = g_{\alpha\beta}^{-1} F_{\nabla\alpha} g_{\alpha\beta}}$$

1.3.3 Compatible Connections

In general, we do not want that the transtion functions of our vector bundles were completely free. Indeed, we will impose a very specific way of acting on it, given by a group G. As we will see, this is very important to achive a satisfactory framework to work with.

Let $E \xrightarrow{\pi} M$ be a vector bundle with fiber V. We will say that E has **structure group**[6] $G \subseteq GL(V)$ (or, more generaly, with respect to a faithful representation $\rho : G \to GL(V)$) if there exists a trivializing covering of M, $\{U_\alpha\}_{\alpha\in\Lambda}$, such that all the transition functions between them lie in G, that is $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. These kind of trivializing coverings are called compatibles with the G-structure.

The vector bundle E having structure group $G \subseteq GL(V)$ corresponds to the idea that E has a structure on it preserved by G. For example, if $G = GL^+(V)$, then E is a $GL^+(V)$-vector bundle if and only if it is orientable. Moreover, if $G = SL(V)$, then it corresponds to choosing a volume form on E and all the transition functions preserving this volume form. In addition, if $G = O(V)$, then E being a $O(V)$-vector bundle corresponds to having a bundle metric an all the transition functions beeing linear isometries with respect to this metric. Finally, if $G = GL(n, \mathbb{C}) \subseteq GL(2n, \mathbb{R})$, then E is a $GL(n, \mathbb{C})$-vector bundle if and only if E admits an almost-complex structure.

[6]Gauge group for physicists, but we reserve this word for another notion.

Now, suppose that our vector bundle E has structure group $G \subseteq GL(V)$. Then, we can restrict our attention only to connections that, in some sense, are compatible with the G-structure.

Definition 1.3.22. Let $E \xrightarrow{\pi} M$ be a vector bundle with fiber V (with a fixed basis) and structure group $G \subseteq GL(V)$. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a trivializing covering of M whose transition functions lie on G, that is $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. A connection $\nabla \in \mathcal{A}_E$ is called a **affine G-connection** or **compatible with the G-structure** if, for all $\alpha \in \Lambda$, writting $\nabla|_{\pi^{-1}(U_\alpha)} = d + A_\alpha$, we have $A_\alpha \in \Omega^1(M, \mathfrak{g}) \subseteq \Omega^1(M, \mathfrak{gl}(V))$. We will denote the set of G-connections as $\mathcal{A}_E^G \subseteq \mathcal{A}_E$.

Remark 1.3.23. Recall that, from 1.3.5, we have the invariant form of the change of coordinates formula

$$A_\beta(X_x) = \left(L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta}\right)_*(X_x) + Ad_{g_{\alpha\beta}^{-1}(x)}(A_\alpha(X_x))$$

that is also valid when we consider a matrix group $G \subseteq GL(V)$. In particular, if $U_\alpha, U_\beta \subseteq M$ are compatibles with the G-structure of E, then $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Hence, since

$$Ad_{g_{\alpha\beta}^{-1}(x)} : \mathfrak{g} \to \mathfrak{g} \qquad \left(L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta}\right)_* : TM \to \mathfrak{g}$$

we have that if $A_\alpha \in \Omega^1(M, \mathfrak{g})$ then $A_\beta \in \Omega^1(M, \mathfrak{g})$. Thus, the definition of a G-connection does not depend on the trivializing covering and the basis of V used to compute de 1-form A_α choosen.

Example 1.3.24. Intuitively, as in the case of the structure group, ∇ beeing a G-connection corresponds to the idea that ∇ preserves the structure induced on E by G. Let us ilustrate this idea with and example. Suppose that our vector bundle E has a bundle metric $\langle \cdot, \cdot \rangle$ [7]. Then, a connection ∇ is said to be **metric** if, for all $\sigma_1, \sigma_2 \in \Gamma(E)$ we have

$$d\langle \sigma_1, \sigma_2 \rangle = \langle \nabla(\sigma_1), \sigma_2 \rangle + \langle \sigma_1, \nabla(\sigma_2) \rangle$$

or, equivalently, for all vector field X on M

$$X\langle \sigma_1, \sigma_2 \rangle = \langle \nabla_X(\sigma_1), \sigma_2 \rangle + \langle \sigma_1, \nabla_X(\sigma_2) \rangle$$

Suppose that we have a metric connection ∇ on a vector bundle E of rank n. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a covering of M of trivializing neighbourhoods that induces a $O(n)$-structure group reduction on E, that is, such that $g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n)$ ($g_{\alpha\beta}(x)$ is an isometry of E_x for all $x \in U_\alpha \cap U_\beta$). Let us take $U_\alpha \subseteq M$ for some $\alpha \in \Lambda$. Using the trivial model, we can find sections $e_1, \ldots, e_n : U_\alpha \to \pi^{-1}(U_\alpha) \subseteq E$ such that $e_1(x), \ldots, e_n(x)$ is an orthonormal basis of E_x for all $x \in U_\alpha$. Now, let us write $\nabla|_{\pi^{-1}(U_\alpha)} = d + A$ for some $A \in \Omega^1(M, \mathfrak{gl}(V))$ with respect to this basis of sections $e_1, \ldots, e_n$. Then, since they are orthogonal of each $x \in U_\alpha$, we have

$$\langle e_i, e_j \rangle = \delta_{ij}$$

[7] That is, $\langle \cdot, \cdot \rangle \in \Gamma(E^* \otimes E^*)$ which, in each fiber E_x, $\langle \cdot, \cdot \rangle : E_x \times E_x \to \mathbb{R}$ is a inner product (bilinear, simmetric and definite-positive). For example, if M is a riemannian manifold, the riemannian metric is a bundle metric of TM.

that is a constant function. Moreover, since in the trivialized model the e_i have constant coefficients, we have that $de_i = 0$. Hence, using that ∇ is a bundle metric, for all vector field X on M we have

$$\begin{aligned} 0 = X\langle e_i, e_j\rangle &= \langle \nabla_X(e_i), e_j\rangle + \langle e_i, \nabla_X(e_j)\rangle = \langle A_X(e_i), e_j\rangle + \langle e_i, A_X(e_j)\rangle \\ &= \langle A_i^k(X)e_k, e_j\rangle + \langle e_i, A_j^k(X)e_k\rangle = A_i^k(X)\delta_{kj} + A_j^k(X)\delta_{ik} = A_i^j(X) + A_j^i(X) \end{aligned}$$

so $A_i^j(X) = -A_j^i(X)$, that is, $A(X)$ is skew-symmetric, or, equivalently, $A(X) \in \mathfrak{o}(n)$. Hence, we have that $A \in \Omega^1(M, \mathfrak{o}(n))$. Therefore, we have that, given a bundle metric, an $O(n)$-connection is the same as a metric connection, that is, a connection that preserves the structure induced by $O(n)$.

It is very remarkable that the definition of a compatible G-connection ∇ can be given in a slightly different way, maybe more intrinsic, in a more general framework. Let us fix $x \in M$ and $T_x : T_xE \to T_xE$ an endomorphism of T_xE. Let us take also a trivializing neighbourhood $U_\alpha \subseteq M$ of x compatible with the G-structure and diffeomorphism $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$, and let us define $\tilde{T}_x^\alpha := \varphi_\alpha|_{E_x} \circ T_x \circ \varphi_\alpha^{-1}|_{E_x} : V \to V$. We will say that T_x **lives in** G if $\tilde{T}_x^\alpha$ is of the form

$$\tilde{T}_x^\alpha(v) = \rho(g_\alpha)(v)$$

for some $g_\alpha \in G$. Analogously, $T_x : E_x \to E_x$ **lives in** $\mathfrak{g}$ if $\tilde{T}_x^\alpha$ is of the form

$$\tilde{T}_x^\alpha(v) = \rho_*(\xi_\alpha)(v)$$

for some $\xi_\alpha \in \mathfrak{g}$, where $\rho_* : \mathfrak{g} = T_eG \to T_e\, GL(V) = End(V)$.

It can be checked that this definition does not depend on the trivialization $(U_\alpha, \varphi_\alpha)$. Specifically, it can be shown that if U_β is another G-compatible trivializing neighbourhood of x, then $g_\beta = g_{\alpha\beta}(x)g_\alpha g_{\alpha\beta}^{-1}(x) \in G$ in the case living in G and $\xi_\beta = g_{\alpha\beta}(x)g_\beta g_{\alpha\beta}^{-1}(x) = Ad_{g_{\alpha\beta}(x)}(\xi) \in Ad(\mathfrak{g})$ in the case living in $\mathfrak{g}$. However, as we have shown, $g_\alpha \in G$ and $\xi_\alpha \in \mathfrak{g}$ do depend on the trivialization choosen, so we can say that T_x lives in G (or $\mathfrak{g}$), but we can not assert which $g \in G$ (or $\xi \in \mathfrak{g}$) is.

In this context, a map $T \in End(E)$ is said to live in G (resp. in $\mathfrak{g}$) if $T_x : E_x \to E_x$ lives in G (resp. in $\mathfrak{g}$) for all $x \in M$. Therefore, a connection $\nabla \in \mathcal{A}_E$ is said to be compatible with the G-structure if, for all trivializing $U_\alpha \subseteq M$ compatible with the G-structure, the associated gauge potential $(A_\alpha)_X : End(\pi^{-1}(U_\alpha)) \to End(\pi^{-1}(U_\alpha))$ lives in $\mathfrak{g}$ for every vector field X on U_α.

In fact, from this definition, we can identify the fiber bundle structure of the space of endomorphisms of E that live in $\mathfrak{g}$, let us call it $End(E)_\mathfrak{g}$. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a covering of M of trivializing open sets for E compatible with the G-structure of E. Then, for every U_α let us define the map $\phi_\alpha : End(\pi^{-1}(U_\alpha))_\mathfrak{g} \to U_\alpha \times \mathfrak{g}$ by $\phi_\alpha(T_x) = (x, \xi_\alpha^{T_x})$, where $\tilde{T}_x^\alpha(v) = \rho_*(\xi_\alpha^{T_x})(v)$ in the trivial model defined by U_α. Using charts it can be shown that ϕ_α is a diffeomorphism that commutes with the proyection to M. Moreover, they are related by $\tilde{g}_{\alpha\beta}(x) = \phi_\beta|_{End(E)_x} \circ \phi_\alpha^{-1}|_{End(E)_x} : \mathfrak{g} \to \mathfrak{g}$, $\tilde{g}_{\alpha\beta}(x) = Ad_{g_{\alpha\beta}(x)}$.

By the identification of fiber bundles with their transition functions, we have that $End(E)_{\mathfrak{g}}$ is isomorphic to the vector bundle over M with fiber $\mathfrak{g}$, structure group G and transition functions $\tilde{g}_{\alpha\beta} = Ad \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to Ad(G) \subseteq GL(\mathfrak{g})$. We will call this vector bundle $\mathfrak{g}_E$.

Example 1.3.25. In particular, if $G = GL(V)$, then we have that $\mathfrak{gl}(V)_E = End(E) = E \otimes E^*$.

Putting all this information together and using proposition 1.3.6, we have just proven

Proposition 1.3.26. *Given a G-vector bundle $E \xrightarrow{\pi} M$, the set of affine connections over E compatible with the G-structure, $\mathcal{A}_E^G$, is an affine space with underlying vector space $\Omega^1(M, \mathfrak{g}_E)$.*

1.3.3.1 Curvature of a G-connection

Let us suppose that $E \to M$ is a vector bundle of fiber V and structure group G and ∇ is a G-compatible affine connection on E. Recall that, from this connection, we can define its curvature $F_\nabla \in \Omega^2(M, End(E))$. However, observe that, in a trivializing open set $U_\alpha \subset M$ where $\nabla|_{\pi^{-1}(U_\alpha)}$ we have, by the structure equation, that

$$F_{\nabla\alpha} = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha]$$

so, since $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$, we can see $F_{\nabla\alpha} \in \Omega^2(U_\alpha, \mathfrak{g})$. Therefore, $F_\nabla \in \Omega^2(M, \mathfrak{g}_E)$ that is, F_∇ is a 2-form with values in the bundle of Lie algebras $\mathfrak{g}_E$, seen as a subbundle of $End(E)$.

Recall that the change of coordinates rule for F_∇ for trivializing open sets $U_\alpha, U_\beta \subseteq M$ is that $F_{\nabla\alpha} \in \Omega^2(U_\alpha, \mathfrak{g})$ and $F_{\nabla\beta} \in \Omega^2(U_\beta, \mathfrak{g})$ are related by[8]

$$F_{\nabla\beta} = g_{\alpha\beta}^{-1} F_{\nabla\alpha} g_{\alpha\beta} = Ad_{g_{\alpha\beta}^{-1}} \circ F_\alpha$$

Observe that, in general, $\mathfrak{g}_E$ is not trivial, $\mathfrak{g}_E \ncong M \times \mathfrak{g}$ so we can not say that $F_\nabla \in \Omega^2(M, \mathfrak{g})$. However, if G is abelian (for example if $G = U(1)$) then $F_{\nabla\beta} = F_{\nabla\alpha}$. Therefore, the $F_{\nabla\alpha}$ can be reensambled to form a global form $F_\nabla \in \Omega^2(M, \mathfrak{g})$. This is a key point that makes the difference between abelian gauge theories (with G abelian) and non-abelian ones.

1.3.4 Gauge Transformations on Vector Bundles

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M$ be two vector bundles with fiber V. A map $f : E \to E'$ is said to be a **vector bundle map** if $f|_{E_x} : E_x \to E'_x$ is a linear map for all $x \in M$ and the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

[8]Observe that, since $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G \subseteq GL(V)$, the change of coordinates preserves the living in $\mathfrak{g}$.

Observe that, in this case, given $x \in M$ we can find a trivializing neighbourhood of x, $U \subseteq M$, for both vector bundles, that is $\varphi_U : \pi^{-1}(U) \subseteq E \to U \times V$ and $\varphi'_U : \pi'^{-1}(U) \subseteq E' \to U \times V$. Then since $\pi' \circ f = \pi$, we have that $\varphi'_U \circ f \circ \varphi_U^{-1} : U \times V \to U \times V$ is of the form

$$\varphi'_U \circ f \circ \varphi_U^{-1}(x, v) = (x, \tilde{f}_U(x)(v))$$

for some $\tilde{f}_U : U \to GL(V)$. Therefore, if E and E' has structure group $G \subseteq GL(V)$, we will say that $f : E \to E'$ is a **G-vector bundle map** if, for all $x \in M$ there is a common trivializing neighbourhood of x, $x \in U \subseteq M$ such that $\tilde{f}_U$ lies in G, that is, $\tilde{f}_U : U \to G$.

Definition 1.3.27. Given $E \xrightarrow{\pi} M$ a vector bundle with fiber V and structure group $G \subseteq GL(V)$, a **gauge transformation** is a diffeomorphism $f : E \to E$ which is a G-vector bundle map too. The set of gauge transformation of E form a group, called the **gauge group** of E, denoted by $\mathcal{G}_E$.

The gauge group also acts on the set of affine connections.

Definition 1.3.28. Let $E \xrightarrow{\pi} M$ be a vector bundle with fiber V and structure group $G \subseteq GL(V)$, let $\mathcal{G}_E$ be its gauge group and let $\mathcal{A}_E$ be it set of affine connections. Then $\mathcal{G}_E$ acts on $\mathcal{A}_E$ by $f \cdot \nabla = f^{-1} \circ \nabla \circ f$ for $f \in \mathcal{G}_E$ and $\nabla \in \mathcal{A}_E^G$, that is

$$\left(f \cdot \nabla\right)_X (\sigma) = f^{-1} \circ \nabla_X (f \circ \sigma)$$

for any vector field X on M and $\sigma \in \Gamma(E)$.

Let $f \in \mathcal{G}_E$ and $\nabla \in \mathcal{A}_E^G$. Let us take $x \in M$ and let $U \subseteq M$ be a trivializing neighbourhood of x. Then, locally in U, the connection ∇ can be written as $d + A$ and the connection $f \cdot \nabla$ as $d + A'$. Thus, following the proof of the transformation rule of connections under change of trivialization map, we get that

$$\boxed{A' = \tilde{f}^{-1} d\tilde{f} + \tilde{f}^{-1} A \tilde{f}}$$

Furthermore, this later formula can be rewritten in a more invariant way, as in 1.3.5 so we have

$$A'(X_x) = \left(L_{\tilde{f}^{-1}(x)} \circ \tilde{f}^{-1}\right)_* (X_x) + Ad_{\tilde{f}^{-1}(x)}(A_\alpha(X_x))$$

for any $x \in U$ and $X_x \in T_x M$. Thus, if $A \in \Omega^1(U, \mathfrak{g})$, then $A' \in \Omega^1(U, \mathfrak{g})$. Therefore, if ∇ is a G-connection, then $f \cdot \nabla$ is also a G-connection and, restricting, it defines an action of $\mathcal{G}_E$ on $\mathcal{A}_E^G$.

Moreover, analogously to the case of a change of coordinates, if F_∇ is the curvature of the G-connection ∇ and $F_{f \cdot \nabla}$ is the curvature of the G-connection $f \cdot \nabla$, then in a local trivialization $U_\alpha \subseteq M$ they are related by

$$F_{f \cdot \nabla_\alpha} = \tilde{f}^{-1} F_\nabla \tilde{f} = Ad_{\tilde{f}^{-1}} \circ F_{\nabla_\alpha}$$

so, returning to the global form, we have that

$$F_{f\cdot\nabla}(X,Y)(\sigma) = f^{-1} \circ (F_\nabla(X,Y))\,(f \circ \sigma)$$

for all $\sigma \in \Gamma(E)$ and X, Y vector fields on M.

1.4 Connections on Principal Bundles

As we have seen, connections and their curvature on vector bundles are a very powerful tool. However, its understanding requires a large knowledge of differential geometry and it is plenty of tricks. This difficulties can be overcomed using a different framework, but equivalent, via principal bundles. This point of view allow us to have a sometimes easier sometimes complementary framework to cope with connections, but loosing the geometrical intuition behind the affine connecions, very closed to riemannian connections. Therefore, in this section, will show how to restate this formalism in terms of principal bundles and, in the next one, how to translate between them.

Recall that, given a Lie group G and a base manifold M a **principal G-bundle** over M is a fiber bundle $P \xrightarrow{\pi} M$ with a right free action[9] of G such that $\pi(p \cdot g) = \pi(p)$ for all $p \in P$, the action is transitive in each fiber (hence, the fibers are exactly the G-orbits of P) and such that the trivilizing functions are G-equivariant. This last hypotesis means that, if $\varphi_U : \pi^{-1}(U) \to U \times G$ is a trivializing function for some $U \subseteq M$, then for all $g \in G$ and $p \in P$, $\varphi_U(p \cdot g) = \varphi_U(p) \cdot g$, where the action of G in $U \times G$ is given by right product on the second coordinate (i.e. $(x, h) \cdot g := (x, hg)$). We will denote the transition functions of the principal bundle between U_α and U_β by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, acting on the left and defined by $\varphi_\alpha \circ \varphi_\beta^{-1}(x, g) = (x, g_{\alpha\beta}(x)g)$. Moreover, we will abuse the notation and, if P is a principal G-bundle, we will denote $R_g : P \to P$ the right action of g on P, that is $R_g(p) = p \cdot g$.

For our computations, we introduce the notion of fundamental field.

Definition 1.4.1. Let $P \xrightarrow{\pi} M$ be a principal G-bundle and let $\xi \in \mathfrak{g}$. We define the **fundamental field** associated to ξ, X_ξ as the vector field over P given by

$$(X_\xi)_p := \left.\frac{d}{dt}\right|_{t=0} p \cdot exp(t\xi)$$

for all $p \in P$.

[9] The fact that the action of G was on the right is fundamental and it is linked with how the transition functions act on G. Usually, the transition functions $g_{\alpha\beta}$ are defined as $\varphi_\alpha \circ \varphi_\beta^{-1}(x, g) = (x, g_{\alpha\beta}(x)g)$, In this case, if we want that the action of G on P commutes with the transition function we need that G acts on the right. Indeed, in the local trivialized model, we have

$$R_h \circ L_{g_{\alpha\beta}}(x, g) = (g_{\alpha\beta} \cdot (x, g)) \cdot h = (x, g_{\alpha\beta}gh) = g_{\alpha\beta} \cdot ((x, g) \cdot h) = L_{g_{\alpha\beta}} \circ R_h(x, g)$$

and it does not hold if G acts on the left. Furthermore, as our computations will show, the right action of G is needed in order to be able to translate connections on P to translations on the associated vector bundle.

Remark 1.4.2. Given $p \in P$, a vector $X_p \in Ker\,\pi_{*p}$ if and only if there exists $\xi \in \mathfrak{g}$ such that $(X_\xi)_p = X_p$.

Definition 1.4.3. Given a G-principal bundle $P \xrightarrow{\pi} M$, a **connection** or **connection 1-form** is a 1-form over P with values on $\mathfrak{g}$, $A \in \Omega^1(P, \mathfrak{g})$, such that

- $A(X_\xi) = \xi$ for all $\xi \in \mathfrak{g}$.
- $A_{g \cdot p}(R_{g_*} X_p) = Ad_{g^{-1}}(A_p(X_p))$ for all $g \in G$, $p \in P$ and $X_p \in T_pP$. In a more compressed form, $R_g^* A = Ad_{g^{-1}} \circ A$

We will denote the set of connection 1-forms on P as $\mathcal{A}_P$.

Moreover, we can use the local formulation of the previous definition.

Definition 1.4.4. Let $P \xrightarrow{\pi} M$ be a G-principal bundle. A **connection system** is, for every local trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$, a 1-form over $U_\alpha \subseteq M$ with values in $\mathfrak{g}$, call it $A_\alpha \in \Omega^1(U, \mathfrak{g})$. This set of 1-forms should satisfy the compatibility condition that, if U_α, U_β are two trivializing neighbourhoods with $U_\alpha \cap U_\beta \neq \emptyset$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is its transition function, then, over $U_\alpha \cap U_\beta$

$$A_\beta(X_x) = \left(L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta}\right)_* (X_x) + Ad_{g_{\alpha\beta}^{-1}(x)}(A_\alpha(X_x))$$

for all $x \in U_\alpha \cap U_\beta$, and $X_x \in T_xM$.

Remark 1.4.5. Recall that, from remark 1.3.5, if G is a matrix group the change of coordinates formula for a connection system can be rewritten[10] as

$$\boxed{A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}}$$

where $dg_{\alpha\beta}$ is the matrix of same size as $g_{\alpha\beta}$ whose entries are d of the original entry.

Proposition 1.4.6. *Definitions 1.4.3 and 1.4.4 are equivalent. More precisely, locally a connection 1-form forms a connection system and given a connection system, they can be glue together to build a connection 1-form.*

Proof. 1.4.3 $\Rightarrow$ 1.4.4. To be more precise, given a local trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$, we can define the local section $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha) \subseteq P$ by $\sigma_\alpha(x) := \varphi_\alpha^{-1}(x, e)$. Then we define $A_\alpha := \sigma_\alpha^* A$. Observe that given $U_\alpha, U_\beta \subseteq M$ we have that $\sigma_\beta(x) = \sigma_\alpha(x) \cdot g_{\alpha\beta}(x)$. Hence, if $X_x \in T_xM$ and

[10] As the sharp-sighted reader could have observed, this formula is the same as the one of the change of coordinates of the 1-form A of a connection on a vector bundle (see 1.3). Of course, this is not a coincidence, but it will allow us to translate between both worlds. However, if we had used use a left action of G on P instead of the usual right action, the change of coordinates formula would not be this one, not allowing the translation.

$\gamma : (-\epsilon, \epsilon) \to M$ is a curve with $\gamma'(0) = X_x$ we have

$$\begin{aligned}
\sigma_{\beta_*}(X_x) &= \frac{d}{dt}\Big|_{t=0} \sigma_\beta(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} \sigma_\alpha(\gamma(t)) \cdot g_{\alpha\beta}(\gamma(t)) \\
&= \sigma_\alpha(x) \cdot \left(\frac{d}{dt}\Big|_{t=0} g_{\alpha\beta}(\gamma(t)) \right) + \left(\frac{d}{dt}\Big|_{t=0} \sigma_\alpha(\gamma(t)) \right) \cdot g_{\alpha\beta}(x) \\
&= \sigma_\beta(x) \left(\frac{d}{dt}\Big|_{t=0} g_{\alpha\beta}^{-1}(x)\, g_{\alpha\beta}(\gamma(t)) \right) + R_{g_{\alpha\beta}(x)_*}(\sigma_{\alpha *}(X_x)) \\
&= (X_\xi)_{\sigma_\beta(x)} + R_{g_{\alpha\beta}(x)_*}(\sigma_{\alpha *}(X_x))
\end{aligned}$$

where $\xi = (L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta})_*(X_x) \in T_eG = \mathfrak{g}$. In this way, using the properties of the connection 1-form, we have the rule of chart changing

$$\begin{aligned}
A_\beta(X_x) = A(\sigma_{\beta_*} X_x) &= A\left((X_\xi)_{\sigma_\beta(x)} \right) + A\left(R_{g_{\alpha\beta}(x)_*}(\sigma_{\alpha *}(X_x)) \right) = \xi + Ad_{g_{\alpha\beta}^{-1}(x)}(A(\sigma_{\alpha *} X_x)) \\
&= (L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta})_*(X_x) + Ad_{g_{\alpha\beta}^{-1}(x)}(A_\alpha(X_x))
\end{aligned}$$

as expected.

1.4.3 $\Leftarrow$ 1.4.4. Suppose that we have a connection system $\{(U_\alpha, A_\alpha)\}_{\alpha \in \Lambda}$ for some atlas of M of trivializing open sets. Let us fix some U_α and let $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha) \subseteq P$ be the local section as before. Observe that σ_α induces an horizontal distribution onto its image, that is, for all $x \in M$ we have the decomposition

$$T_{\sigma_\alpha(x)}P = \sigma_{\alpha *}(T_xM) \oplus V_{\sigma_\alpha(x)}$$

Hence, we can use the 1-form A_α on α to define a 1-form with values in $\mathfrak{g}$ on $\sigma(U_\alpha)$, $\tilde{A}^\alpha$, by

$$\tilde{A}^\alpha_{\sigma_\alpha(x)}(\sigma_{\alpha *}(X_x) + X_\xi) = A_\alpha(X_x) + \xi$$

Furthermore, we can extend $\tilde{A}^\alpha$ to the whole $\pi^{-1}(U_\alpha)$ by

$$\tilde{A}^\alpha_{\sigma_\alpha(x)\cdot g} = Ad_{g^{-1}} \circ \left(R^*_{g^{-1}} \tilde{A}^\alpha \right)_{\sigma_\alpha(x)\cdot g}$$

Observe that, by construction, $\tilde{A}^\alpha$ is right-invariant and if $p = \sigma_U(x)$ for some $x \in U$ we have that $\tilde{A}^\alpha_{\sigma_\alpha(x)}(X_\xi) = \xi$ for all fundamental field X_ξ with $\xi \in \mathfrak{g}$. For the general case, if $p = \sigma_\alpha(x) \cdot g$ then

$$\begin{aligned}
\tilde{A}^\alpha_p(X_{\xi_p}) &= Ad_{g^{-1}} \left((R^*_{g^{-1}} \tilde{A}^\alpha)(X_{\xi_p}) \right) = Ad_{g^{-1}} \left(\tilde{A}^\alpha_{\sigma_\alpha(x)}(R_{g^{-1}*} X_\xi) \right) \\
&= Ad_{g^{-1}} \left(\tilde{A}_{\sigma_\alpha(x)}((X_{Ad_g \xi})_{\sigma_U(x)}) \right) = Ad_{g^{-1}} \left(Ad_g(\xi) \right) = \xi
\end{aligned}$$

Therefore, $\tilde{A}^\alpha$ is a connection 1-form on $\pi^{-1}(U_\alpha)$. Hence, it is enought to prove that for two trivializing neighbourhoods $U_\alpha, U_\beta \in \Lambda$ we have that $\tilde{A}^\alpha = \tilde{A}^\beta$ in $U_\alpha \cap U_\beta$. In this case, all this local forms could be pasted together to form a global conection 1-form A, as we wanted.

Let us check that $\tilde{A}^\alpha = \tilde{A}^\beta$ in $\pi^{-1}(U_\alpha \cap U_\beta)$. First of all, observe that, by the right-invariance property, it is enought to check it on $\sigma_\beta(U_\alpha \cap U_\beta)$. Moreover, cause they coincide on fundamental fields, it is enought to check that $\tilde{A}^\alpha(\sigma_{\beta_*}X_x) = \tilde{A}^\beta(\sigma_{\beta_*}X_x)$. But, in this case, by the previous computation, writting $\xi = (L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta})_*(X_x)$ we have

$$\begin{aligned}
\tilde{A}^\alpha_{\sigma_\beta(x)}(\sigma_{\beta_*}X_x) &= \tilde{A}^\alpha\left((X_\xi)_{\sigma_\beta(x)} + R_{g_{\alpha\beta}(x)_*}(\sigma_{\alpha*}(X_x))\right) = \xi + \tilde{A}^\alpha\left(R_{g_{\alpha\beta}(x)_*}(\sigma_{\alpha*}(X_x))\right) \\
&= \xi + Ad_{g_{\alpha\beta}^{-1}(x)}\left(\tilde{A}^\alpha_{\sigma_\alpha(x)}(\sigma_{\alpha*}(X_x))\right) = (L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta})_*(X_x) + Ad_{g_{\alpha\beta}^{-1}(x)}(A_\alpha(X_x)) \\
&= A^\beta_{\sigma_\beta(x)}(X_x) = \tilde{A}^\beta_{\sigma_\beta(x)}(\sigma_{\beta_*}X_x)
\end{aligned}$$

where the penultimate equality is the change of charts rule of the connection system. Therefore $\tilde{A}^\alpha_{\sigma_\beta(x)} = \tilde{A}^\beta_{\sigma_\beta(x)}$, completing the proof. ■

Finally, we can give a geometric formulation of a connection in terms of horizontal distributions.

Definition 1.4.7. Given a G-principal bundle $P \xrightarrow{\pi} M$, an **Ehresmann connection** or **horizontal distribution** over P is a subbundle H of TP such that, for every $p \in P$ and $g \in G$, we have that

$$T_pP = V_p \oplus H_p \qquad R_{g_*}(H_p) = H_{p\cdot g}$$

where $V_p := Ker\,\pi_{*p} \subseteq T_pP$ is the **vertical distribution**.

Proposition 1.4.8. *Definitions 1.4.7 and 1.4.3 are equivalent. More precisely, for $p \in P$, given a connection 1-form A, $H_p := Ker\,A_p$ is an horizontal distribution and, given an horizontal distribution $H \subseteq TP \to P$ the 1-form $A \in \Omega^1(P, \mathfrak{g})$ given by $A_p(X_{\xi_p} + X_p^h) = \xi$ where X_ξ is the fundamental field of $\xi \in \mathfrak{g}$ and $X_p^h \in H_p$.*

Proof. 1.4.3 $\Rightarrow$ 1.4.7. Taking the kernel subbundle $Ker\,A \to TP$ we observe that $H = Ker\,A$ is, in fact, a subbundle of TP.

- *Direct complement of V*: Let us take $X_p \in Ker\,A_p \cap Ker\,\pi_{*p}$. Since $X_p \in Ker\,\pi_{*p}$, we have that there exists a fundamental field X_ξ for some $\xi \in \mathfrak{g}$ such that $X_{\xi_p} = X_p$. Hence, we have that $A_p(X_p) = A_p(X_{\xi_p}) = \xi$ so $X_p \in Ker\,A_p$ if and only if $\xi = 0$ which is equivalent to $X_p = 0$. Moreover, from this computation, we have that, if $X_p^v \in V_p$ is the vertical part of some $X_p \in T_pP$, then $A_p(X_p) = A_p(X_p^v)$ so $X_p - X_p^v \in Ker\,A_p$ and, therefore $T_pP = V_p \oplus H_p$.
- *Right-invariance property*: Observe that, by dimension, it is enought to check that $R_{g_*}(Ker\,A_p) \subseteq Ker\,A_{p\cdot g}$. Let $X_p \in Ker\,A_p$. Recall that $A_{p\cdot g}(R_{g_{*p}}X_p) = Ad_{g^{-1}}(A_p(X_p))$, so, cause $A_p(X_p) = 0$ we have $A_{p\cdot g}(R_{g_{*p}}X_p) = 0$, that is, $R_{g_{*p}}X_p \in Ker\,A_{g\cdot p}$, as we wanted to prove.

1.4.3 $\Leftarrow$ 1.4.7. Given $X_p \in T_pP$, let us denote $X_p^h \in H^p$ and $X_p^v \in V_p$ its horizontal and vertical part, respectively, such that $X_p = X_p^v + X_p^h \in V_p \oplus H_p$. Observe that, since $X_p^v \in V_p = Ker\,\pi_{*p}$, there exists a unique fundamental field X_ξ such that $X_{\xi_p} = X_p^v$. Hence, the 1-form (possibly discontinous) with values in $\mathfrak{g}$ $A_p(X_p) := A_p(X_p^h + X_{\xi_p}) = \xi$ is well defined.

- *Behaviour on fundamental fields*: By construction $A(X_\xi) = \xi$ for every fundamental field X_ξ.
- *Smoothness*: Observe that for all $p \in P$, locally, there exists a neighbourhood $\tilde{U} \subseteq P$ of p, vector fields on $\tilde{U}$, $Y_1, \dots, Y_r$ and $\xi_1, \dots, \xi_s \in \mathfrak{g}$ such that $Y_{1p}, \dots, Y_{rp}$ is a basis of H_p and $(X_{\xi_1})_p, \dots, (X_{\xi_s})_p$ is a basis of V_p for all $p \in \tilde{U}$. Then, we have that $A_p(Y_{ip}) = 0$ for all $i = 1, \dots, r$ and $A_p(X_{\xi_{j\,p}}) = \xi_j$ for $j = 1, \dots, s$, which is smooth on $\tilde{U}$.
- *Right-invariance property*: Let us prove that $A_{p\cdot g}(R_{g*p}X_p) = Ad_{g^{-1}}(A_p(X_p))$ for all $g \in G, p \in P$ and $X_p \in T_pP$. To this end, observe that, if $X_p \in H_p$, the result is trivial, since $A_{p\cdot g}(R_{g*p}X_p) = 0$, (cause $R_{g*p}X_p \in H_{p\cdot g}$ by the right-invariance property of H) and $Ad_g(A_p(X_p)) = Ad_g(0) = 0$. Hence, it is enought to check it for $X_p \in V_p$, let us say $X_p = (X_\xi)_p$ for some $\xi \in \mathfrak{g}$. But, in this case, we have that

$$\begin{aligned} A_{p\cdot g}(R_{g*}X_{\xi_p}) &= A_{p\cdot g}\left(\frac{d}{dt}\bigg|_{t=0} R_g(p\cdot exp(t\xi))\right) = A_{p\cdot g}\left(\frac{d}{dt}\bigg|_{t=0} p\cdot exp(t\xi)g\right) \\ &= A_{p\cdot}\left(\frac{d}{dt}\bigg|_{t=0} (p\cdot g)\cdot g^{-1}exp(t\xi)g\right) = A_{p\cdot g}\left(\frac{d}{dt}\bigg|_{t=0} (p\cdot g)\cdot exp(tAd_{g^{-1}}(\xi))\right) \\ &= A_{p\cdot g}\left((X_{Ad_{g^{-1}}(\xi)})_{p\cdot}\right) = Ad_{g^{-1}}(\xi) = Ad_{g^{-1}}(A_p(X_{\xi_p})) \end{aligned}$$

as we wanted to prove.

■

1.4.1 Gauge Transformations on Principal Bundles

Let $P \xrightarrow{\pi} M$ and $P' \xrightarrow{\pi'} M$ be two G-principal bundles. A map $f : P \to P'$ is said to be a **principal** G**-bundle map** if f is G-equivariant (i.e. $f(p\cdot g) = f(p)\cdot g$ for all $g \in G$ and $p \in P$) and the following diagram commutes:

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

Observe that, in this case, given $x \in M$ we can find a trivializing neighbourhood of x, $U \subseteq M$, for both principal bundles, that is $\varphi_U : \pi^{-1}(U) \subseteq P \to U \times G$ and $\varphi'_U : \pi'^{-1}(U) \subseteq P' \to U \times G$. Then since $\pi' \circ f = \pi$, we have that $\varphi'_U \circ f \circ \varphi_U^{-1} : U \times G \to U \times G$ is of the form

$$\varphi'_U \circ f \circ \varphi_U^{-1}(x, g) = (x, \tilde{f}_U \cdot g)$$

for some $\tilde{f}_U : U \to G$.

Definition 1.4.9. Let $P \xrightarrow{\pi} M$ be a principal G-bundle. A **gauge transformation** is a diffeomorphism $f : P \to P$ which is a principal G-bundle map too. The group (under composition) of gauge transformations of P is call the **gauge group** of P, and denoted by $\mathcal{G}_P$.

Moreover, we can do that the gauge group acts on the set of connection 1-forms on P, $\mathcal{A}_P$.

Definition 1.4.10. Let $P \xrightarrow{\pi} M$ be a principal G-bundle with gauge group $\mathcal{G}_P$ and set of connection 1-forms $\mathcal{A}_P$. Then $\mathcal{G}_P$ acts on $\mathcal{A}_P$ by

$$f \cdot A := f^*(A)$$

for $f \in \mathcal{G}_P$ and $A \in \mathcal{A}_P$.

Remark 1.4.11. The action of the gauge group is well defined, in the sense that it sends connection 1-forms to connections 1-forms. To check it, note that, of course, $f \cdot A \in \Omega^1(P, \mathfrak{g})$ so we just need to prove that $f \cdot A$ is well behaved on fundamental fields and right-invariant.

First, let $\xi \in \mathfrak{g}$ and let X_ξ its fundamental field on P associated. Then, since f is G-equivariant and A preserves fundamental fields, we have

$$\begin{aligned}(f \cdot A)_p(X_\xi) = (f^*A)_p(X_\xi) = A_{f(p)}(f_{*p}X_x) = A_{f(p)}\left(\left.\frac{d}{dt}\right|_{t=0} f\left(p \cdot exp(t\xi)\right)\right)\\ = A_{f(p)}\left(\left.\frac{d}{dt}\right|_{t=0} f\left(p\right) \cdot exp(t\xi)\right) = A_{f(p)}\left((X_\xi)_{f(p)}\right) = \xi\end{aligned}$$

For the right-invariance, observe that, since f is G-equivariant $R_g \circ f = f \circ R_g$ and $f^*(Ad_g \circ A) = Ad_g \circ f^*A$ for all $g \in G$. Therefore, we have

$$\begin{aligned}{R_g}^*(f \cdot A) = R_g^*(f^*A) = (f \circ R_g)^*A = (R_g \circ f)^*A = f^*(R_g^*A)\\ = f^*(Ad_{g^{-1}} \circ A) = Ad_{g^{-1}} \circ f^*A = Ad_{g^{-1}} \circ (f \cdot A)\end{aligned}$$

1.4.2 Curvature on Principal Bundles

Let us suppose that we have a principal G-bundle $P \to M$ with a connection A. Let $H \subseteq TP$ be its horizontal distribution. Given a vector $X_p \in T_pP$, we will write $X_p = X_p^v + X_p^h \in V_p \oplus H_p$, where $X_p^v \in V_p$ and $X_p^h \in H_p$.

Let $\omega \in \Omega^k(P, \mathfrak{g})$, we define $\omega^h \in \Omega^k(P, \mathfrak{g})$ as

$$\omega^h(X_1, \dots, X_k) = \omega(X_1^h, \dots, X_n^h)$$

for all $X_1, \dots, X_k$ vector fields on P. With this definition, we define the **covariant exterior derivative** of P with respect to A, $d_A : \Omega^k(P, \mathfrak{g}) \to \Omega^k(P, \mathfrak{g})$ as $d_A\omega = (d\omega)^H$.

Definition 1.4.12. Let $P \to M$ be a principal G-bundle with a connection 1-form $A \in \Omega^1(P, \mathfrak{g})$. We define the **curvature** of A, $F_A \in \Omega^2(P, \mathfrak{g})$, as

$$F_A = d_A(A)$$

A very important formula that allow us to compute in a easy way is the known as structure equation for principal bundles. The proof, that is simply a computation, can be found in [7].

Theorem 1.4.13 (Structure equation in principal bundles)**.** *Let $P \to M$ be a principal G-bundle with a connection 1-form $A \in \Omega^1(P, \mathfrak{g})$. Then, its curvature can be written*

$$F_A = dA + \frac{1}{2}[A, A]$$

As we will see, this definition of curvature is analogous to the one given for vector bundles. In fact, it satisfies similar identities.

Theorem 1.4.14 (First Bianchi identity)**.** *Let $P \to M$ be a principal G-bundle with a connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ with curvature F_A. Then, we have*

$$dF_A = [F_A, A]$$

Proof. It is a computation using the structure equation. First of all, since $d[A, A] = [dA, A] - [A, dA] = 2[dA, A]$ we have

$$dF_A = d\left(dA + \frac{1}{2}[A, A]\right) = d^2A + \frac{1}{2}d[A, A] = [dA, A]$$

Moreover, since $[[A, A], A] = 0$ by the graded Jacobi identity we have

$$dF_A = [dA, A] = \left[dA + \frac{1}{2}[A, A], A\right] = [F_A, A]$$

as we wanted to prove. ■

Corollary 1.4.15 (Second Bianchi identity)**.** *Let $P \to M$ be a principal G-bundle with a connection 1-form $A \in \Omega^1(P, \mathfrak{g})$ with curvature F_A. Then, we have*

$$\boxed{d_A F_A = 0}$$

Proof. Recall that the horizontal distribution induced by A is exactly the kernel of A, so A kills every horizontal vector field. Therefore, by the first Bianchi identity, we have for all vector fields X_1, X_2, X_3

$$\begin{aligned} d_A F_A(X_1, X_2, X_3) = (dF_A)^h(X_1, X_2, X_3) = dF_A(X_1^h, X_2^h, X_3^h) = [F_A, A](X_1^h, X_2^h, X_3^h) \\ = \frac{1}{2}\sum_{\sigma \in S_3} sign(\sigma)[F_A(X_{\sigma(1)}^k, X_{\sigma(2)}^h), A(X_{\sigma(3)}^h)] = 0 \end{aligned}$$

■

Finally, let us write the connection F_A in local coordinates. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a covering of M of trivializing open sets with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Recall that, associated to each trivialization on U_α, we have a local section $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha) \subseteq P$ and such that, as in the proof of proposition 1.4.6, the local system associated to A is given by $A_\alpha = \sigma_\alpha^*(A) \in \Omega^1(U_\alpha, \mathfrak{g})$.

Analogously to this local system, we define the local curvature $F_{A\alpha} \in \Omega^2(U_\alpha, \mathfrak{g})$ by

$$F_{A\alpha} = \sigma_\alpha^*(F_A)$$

Observe that $F_{A\alpha}$ also satisfies the structure equation and first Bianchi identity since the exterior derivative and the Lie bracket commutes with pullbacks

$$F_{A\alpha} = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] \qquad\qquad dF_{A\alpha} = [F_{A\alpha}, A_\alpha]$$

For the change of chart rule, remember that, by the proof of 1.4.6, we have that, for all vector field X on $U_\alpha \cap U_\beta$

$$\sigma_{\beta_*}(X_x) = (X_\xi)_{\sigma_\beta(x)} + R_{g_{\alpha\beta}(x)_*}(\sigma_{\alpha *}(X_x))$$

for some $\xi \in \mathfrak{g}$. Observe that, since X_ξ is vertical and $F_A = (dA)^h$ we have that F_A kills every vertical field, so for every $x \in U_\alpha \cap U_\beta$

$$\begin{aligned} F_{A\beta}(X_x, Y_x) &= \sigma_\beta^*(F_A)(X_x, Y_x) = F_A(\sigma_{\beta_*}X, \sigma_{\beta_*}Y) \\ &= F_A(R_{g_{\alpha\beta}(x)_*}(\sigma_{\alpha *}(X)), R_{g_{\alpha\beta}(x)_*}(\sigma_{\alpha *}(Y))) = \sigma_\alpha^*\left(R^*_{g_{\alpha\beta}(x)} F_A(X_x, Y_x)\right) \end{aligned}$$

Let us compute the pullback under R. Recall that, since A is a connection 1-form, $R_g^* A = Ad_{g^{-1}} \circ A$ for all $g \in G$. Therefore, using that the pullback commutes with the Lie bracket, for the curvature, we have

$$\begin{aligned} R_g^* F_A = R_g^*\left(dA + \frac{1}{2}[A, A]\right) &= d(R_g^* A) + \frac{1}{2}[R_g^* A, R_g^* A] \\ &= d(Ad_{g^{-1}} \circ A) + \frac{1}{2}[Ad_{g^{-1}} \circ A, Ad_{g^{-1}} \circ A] \\ &= Ad_{g^{-1}} \circ \left(dA + \frac{1}{2}[A, A]\right) = Ad_{g^{-1}} \circ F_A \end{aligned}$$

where $Ad_{g^{-1}}$ commutes with the Lie bracket since it is a homomorphism of Lie algebras. Therefore, returning to our previous computation, we have

$$\begin{aligned} F_{A\beta}(X_x, Y_x) &= \sigma_\alpha^* \left(R^*_{g_{\alpha\beta}(x)} F_A(X_x, Y_x) \right) = \sigma_\alpha^* \left(Ad_{g^{-1}_{\alpha\beta}(x)} \circ F_A \right) \\ &= Ad_{g^{-1}_{\alpha\beta}(x)} \circ \sigma_\alpha^* \left(F_A \right) = Ad_{g^{-1}_{\alpha\beta}(x)} \circ F_{A\alpha} \end{aligned}$$

So, in particular, if G is a matrix group, we have the change of coordinates rule

$$\boxed{F_{A\beta} = g^{-1}_{\alpha\beta} F_{A\alpha} g_{\alpha\beta}}$$

Remark 1.4.16. Again, as in the case of vector bundles, if G is an abelian Lie group, then, using this local form, we can define a globally defined form $F_A \in \Omega^2(M, \mathfrak{g})$.

1.5 From Vector Bundles to Principal Bundles

Let us fix a base manifold M, a Lie group G and a finite dimensional vector space V. We define the **category of G-principal bundles** over M, $\mathbf{PB}_G^M$ to be the category whose objects are the principal G-bundles over M and whose morphisms are principal G-bundle maps.

Analogously, if there exists a representation $\rho : G \to GL(V)$, we define the **category of vector bundles** of fiber V and structure group G (seen as a subgroup of $GL(V)$ via ρ), $\mathbf{VB}_G^M(\rho, V)$. The objects of this category are the vector bundles of fiber V and structure group G and the morphisms are the G-vector bundle maps.

A very important functor between this categories is the associated bundle functor, that allow us to turn a principal bundle an viceversa. For this, we need to suppose that the representation $\rho : G \to GL(V)$ is faithful. As we will see, we are going to define two mutually inverse functors between this categories

$$\begin{tikzcd} \mathbf{PB}_G^M \arrow[r, bend left, "\mathcal{A}_\rho"] & \mathbf{VB}_G^M(\rho, V) \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

which are known as, the **associated vector bundle functor** $\mathcal{A}_\rho$ and, the **frame bundle functor** $\mathcal{F}$.

Remark 1.5.1. The requirement of ρ being faithful will appear when we want to construct the frame bundle, that is, the principal bundle associated to a vector bundle using the functor $\mathcal{F}$. However, the reverse construction, via $\mathcal{A}_\rho$, from principal bundles to vector bundles do not require to the representation to be faithful.

1.5.1 Associated Vector Bundle

For the functor $\mathcal{A}_\rho : \mathbf{PB}_G^M \to \mathbf{VB}_G^M(\rho, V)$, let $P \xrightarrow{\pi} M$ be a principal G-bundle. Then, can associate to it the bundle over M

$$\mathcal{A}_\rho(P) = P \times_\rho V := \frac{P \times V}{G}$$

where G acts on $P \times V$ on the right by $(p, v) \cdot g = (p \cdot g, \rho(g)^{-1}(v))$. In this case, we define the map $\pi' : P \times_\rho V \to M$ by $\pi'[p, v] = \pi(v)$. With this definition, $P \times_\rho V$ becomes a vector bundle.

Remark 1.5.2. Sometimes in the literature, especially when the representation ρ is clear, this associated vector bundle is simply denoted by $P \times_G V$.

Analogously, if $f : P \to P'$ is a morphism of principal G-bundles over X, let us define

$$f' : P \times V \to P' \times V$$

given by $f'(p, v) = (f(p), v)$. Observe that f' is G-equivariant, so it descends to a map

$$\mathcal{A}_\rho(f) : \mathcal{A}_\rho(P) = P \times_\rho V \to \mathcal{A}_\rho(P') = P' \times_\rho V$$

given by $\mathcal{A}_\rho(f)[p, v] = [f(p), v]$.

This construction can be stated only in terms of local trivializations of the principal bundle. Let us take a principal G-bundle $P \xrightarrow{\pi} M$, and a covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of M of trivializing neighbourhoods of the bundle, with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

With this information, we can describe completely the associated vector bundle computing its transition functions.

Proposition 1.5.3. *The associated vector bundle to P is the unique vector bundle whose transition functions on the covering $\{U_\alpha\}_{\alpha \in \Lambda}$ are $\tilde{g}_{\alpha\beta} := \rho \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$, that is*

$$P \times_\rho V \cong \frac{\bigsqcup_{\alpha \in \Lambda} U_\alpha \times V}{\sim}$$

where $(x, v) \in U_\alpha \times V \sim (x', v') \in U_\beta \times V$ if and only if $x = x'$ and $v' = \tilde{g}_{\alpha\beta}(x)(v)$.

Furthermore, the induced morphism $\mathcal{A}_\rho(f) : P \times_\rho V \to P' \times_\rho V$ is given, in local terms on U_α, by $\mathcal{A}_\rho(f)[x, v] = [x, \rho(\tilde{f}_\alpha(x))(v)]$ where f, in the trivial model, is given by $f(x, g) = (x, \tilde{f}_\alpha(x)g)$ for some $\tilde{f}_\alpha : U_\alpha \to G$.

Corollary 1.5.4. *If $P \to M$ is a principal G-bundle, then $P \times_\rho V$ has structure group G (seen as a subgroup of $GL(V)$ via ρ). Therefore, the functor $\mathcal{A}_\rho : \boldsymbol{PB}_G^M \to \boldsymbol{VB}_G^M(\rho, V)$ is well defined.*

Example 1.5.5. The most important associated bundle used in gauge theory is the well known **adjoint bundle**. Recall that, associated to any Lie group G we have the adjoint representation

$Ad : G \to GL(\mathfrak{g})$ which is faithful (the inverse of Ad_g is $Ad_{g^{-1}}$). Therefore, to every principal G-bundle $P \to M$ we can form the adjoint bundle $Ad(P) := \mathcal{A}_{Ad}(P) = P \times_{Ad} \mathfrak{g}$ which is a vector bundle with fiber $\mathfrak{g}$. Moreover, in this cases, the functor $\mathcal{A}_{Ad} : \mathbf{PB}_G^M \to \mathbf{VB}_G^M(Ad, V)$ is usually called $Ad : \mathbf{PB}_G^M \to \mathbf{VB}_G^M(Ad, V)$.

Remark 1.5.6. Let us take a G-vector bundle E. If $E = Ad(P)$ for some principal bundle P over M, then, by the previous proposition, we have that $\mathfrak{g}_E = Ad(P) = E$.

With the definition of the adjoint bundle, in [44] is proved the following proposition, that identify the algebraic structure of the space of connections as an affine space.

Proposition 1.5.7. *Given a principal G-bundle $P \xrightarrow{\pi} M$, the set of connections on P $\mathcal{A}_P$, is an affine space with underlying vector space $\Omega^1(M, Ad(P))$.*

1.5.2 Frame Bundle

The reverse functor of the associated bundle functor is the frame bundle functor $\mathcal{F} : \mathbf{VB}_G^M(\rho, V) \to \mathbf{PB}_G^M$.

To introduce this construction, let us first suppose that $G = GL(V)$. Then, the frame bundle functor asigns, to every vector bundle $E \xrightarrow{\pi} M$ with fiber V, the principal bundle of its fiberwise basis, known as the frame bundle.

In order to specify this, first of all, observe that, given a finite dimensional vector space W, we can consider $\mathcal{B}(W)$ the set of (ordered) linear basis of W. More specifically, $\mathcal{B}(W)$ is the space of all isomorfisms $\mathbb{R}^{\dim W} \to W$, topologized with any norm topology (remember that they are all equivalent) for example the norm as operator between Banach spaces or the topology as space of matrices, chosen a basis on W.

Now, returning to vector bundles, we can build the space

$$\mathcal{F}(E) = \bigsqcup_{x \in M} \mathcal{B}(\pi^{-1}(x))$$

and topologize it analogously as the tangent bundle of M is topologized. With this topology, it becomes a fiber bundle $\mathcal{F}(E) \xrightarrow{\pi'} M$. In fact, $GL(n, V) \cong GL(V)$, $n = \dim V$, acts on $\mathcal{F}(E)$ by change of basis, and this action of free an transitive fiberwise, so it becomes $\mathcal{F}(E) \xrightarrow{\pi'} M$ in a principal $GL(V)$-bundle.

In other cases, if $G = GL^+(V)$, we should take only the bundle of positive oriented basis of V, in the case of $G = SL(V)$ only basis of a fixed volume and, in the case $G = O(V)$, we have to consider only the bundle of orthonormal basis of V.

However, in the general case, the construction should be given in terms of a trivializing covering glued via transition functions. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a covering of M of trivializing neighbourhoods of the vector bundle E, with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \rho(G) \subseteq GL(V)$. Then, since ρ is faithful, it is injective, so we can form the transition functions $\tilde{g}_{\alpha\beta} := \rho^{-1} \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

Then, the G-frame bundle is the bundle

$$\mathcal{F}(E) := \frac{\bigsqcup_{\alpha\in\Lambda} U_\alpha \times G}{\sim}$$

where $(x, g) \in U_\alpha \times g \sim (x', g') \in U_\beta \times V$ if and only if $x = x'$ and $g' = \tilde{g}_{\alpha\beta}(x)g$.

With respect to the morphisms, given a map of G-vector bundles $f : E \to E'$, we can define its image under $\mathcal{F}$ as a map of G-principal bundles $\mathcal{F}(f) : \mathcal{F}(E) \to \mathcal{F}(E')$. To this end, let us define, for each $\alpha \in \Lambda$, the maps $f'_\alpha : U_\alpha \times G \to U_\alpha \times G$ given by $f'_\alpha(x, g) = (x, (\rho^{-1} \circ \tilde{f}_\alpha)(x)g)$ where f, in the trivial model U_α, is written as $f(x, v) = (x, \tilde{f}_\alpha(x)(v))$ for some $\tilde{f}_\alpha : U_\alpha \to \rho(G) \subseteq GL(V)$. Then, putting all this maps together, we can form the map

$$f' = \bigcup_{\alpha\in\Lambda} f_\alpha : \bigsqcup_{\alpha\in\Lambda} U_\alpha \times G \to \bigsqcup_{\alpha\in\Lambda} U_\alpha \times G$$

It can be checked that this map respect the equivalent relations induced by the pasting of E and E', so it descend to a map $\mathcal{F}(f) : \mathcal{F}(E) \to \mathcal{F}(E')$.

By proposition 1.5.3, this functor $\mathcal{F} : \mathbf{VB}_G^M(\rho, V) \to \mathbf{PB}_G^M$ is the inverse of the associated vector bundle functor $\mathcal{A}_\rho : \mathbf{PB}_G^M \to \mathbf{VB}_G^M(\rho, V)$, completing the relation between this two types of fiber bundles.

Remark 1.5.8. Extending remark 1.5.6, let us take a matrix Lie group $G \subseteq GL(V)$ for some vector space V and let us take E a G-vector bundle with fiber V. Then, $\mathfrak{g}_E = Ad(\mathcal{F}(E))$. Observe that, in general $Ad(\mathcal{F}(E))$ is not isomorphic to E, since $\mathcal{F}(E)$ is computed using the inclusion as representation and Ad using the adjoint representation of G.

1.5.3 Connections in Vector and Principal Bundles

Once described the relation between vector and principal bundles, we can go one step futher and study the interelation of its connections. Let us fix a base manifold M and a Lie group G together with a faithful representation $\rho : G \to GL(V)$ for some finite dimensional vector space V. Via this representation, we can identify G and $\rho(G)$ and consider G as a matrix group.

Let us take a principal G-bundle $P \to M$ and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a covering of M of trivializing open sets for P. Recall that, by proposition 1.4.6, a connection on a principal bundle $P \to M$ can be given by a connection system, that is, for every $\alpha \in \Lambda$, a $\mathfrak{g}$-valued 1-form on U_α, $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$.

Now, let us take the associated vector bundle $P \times_\rho V$ and let us take the set of 1-forms $\tilde{A}_\alpha := \rho_* \circ A_\alpha \in \Omega^1(U_\alpha, \rho_*(\mathfrak{g})) \subseteq \Omega^1(U_\alpha, \mathfrak{gl}(V))$. Recall that the change of coordinates rule for the A_α is

$$A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}$$

so, for the $\tilde{A}_\alpha$ we have that[11]

$$\tilde{A}_\beta = \tilde{g}_{\alpha\beta}^{-1} d\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}^{-1} \tilde{A}_\alpha \tilde{g}_{\alpha\beta}$$

Therefore, the $\tilde{A}_\alpha$ satisfy the change of coordinates rule for the local version of an affine connection so they define a connection ∇ on $P \times_\rho V$.

Analogously, let us take a vector bundle $E \to M$ with structure group G (strictely $\rho(G)$) and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a covering of M of trivializing open sets for E. Then, choosen a basis for V, for every affine connection on E we have an uniquely determined $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(V))$ for every $\alpha \in \Lambda$. Therefore, considering $\tilde{A}_\alpha := \rho_*^{-1} \circ A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ as before we have that the rule of transtion of the $\tilde{A}_\alpha$ is the same that the one for the A_α, that is exactly the one required for defining a connection system on the G-frame bundle $\mathcal{F}(E)$, that uniquely determines a connection on $\mathcal{F}(E)$. Thus, we have just prove

Proposition 1.5.9. *Let us fix a base manifold M and a Lie group G together with a faithful representation $\rho : G \to GL(V)$ for some finite dimensional vector space V.*

- *Given a principal G-bundle P, there is an isomorphism of affine spaces between the space of connections on P, $\mathcal{A}_P$, and the set of affine connections on the associated vector bundle $P \times_\rho V$, $\mathcal{A}^G_{P\times_\rho V}$.*
- *Given a vector bundle E with fiber V and structure group G, there is an isomorphism of affine spaces between the space of affine connections on E, $\mathcal{A}^G_E$, and the set of connections on its G-frame bundle $\mathcal{F}(E)$, $\mathcal{A}_{\mathcal{F}(E)}$.*

Furthermore, in both cases, the local form of the connection, seen as and element of $\Omega^2(U_\alpha, \mathfrak{g})$ for $\{U_\alpha\}_{\alpha\in\Lambda}$ a covering of M of trivializing open sets, agree in the original connection and the induced one. In particular, the original is flat if and only if the induced one is flat.

Example 1.5.10. If our G-vector bundle E is the adjoint bundle of some principal G-bundle P, $E = Ad(P)$, then we have that $\mathcal{A}^G_E \cong \mathcal{A}_P$. Recall that the later is an affine space modelled on $\Omega^1(M, Ad(P)) = \Omega^1(M, E)$ while the first one is modelled on $\Omega^1(M, \mathfrak{g}_E) = \Omega^1(M, E)$, as expected.

[11] This can be checked composing with ρ_* in the more invariant form

$$A_\beta(X_x) = \left(L_{g_{\alpha\beta}^{-1}(x)} \circ g_{\alpha\beta}\right)_* (X_x) + Ad_{g_{\alpha\beta}^{-1}(x)}(A_\alpha(X_x))$$

1.6 Yang-Mills Equation

Now, with our understanding of connection in both frameworks, principal and vector bundles, we can define the most important concept of this work, the Yang-Mills equation.

First of all, we need to define a very important operator known as the Hodge star operator. Here we will give a brief introduction to this map, whose complete definition will be explained in 3.1.1.

Let us take an oriented differentiable manifold M with a riemannian metric[12] g and volume form Ω. Using this metric on vector fields, we can define bundle metric on $\Omega^k(M) \to M$, g^k, for all $k \geq 0$. Then, we define the Hodge star operator, $\star : \Omega_p^k(M) \to \Omega_p^{n-k}(M)$, where $n = \dim_{\mathbb{R}} M$ as the unique map that, given $\eta_p \in \Omega_p^k(M)$, $\star\eta_p$ is the unique $(n-k)$-form such that

$$\omega_p \wedge (\star\eta)_p = g_p(\omega_p, \eta_p)\,\Omega_p$$

for all $p \in M$. The most important shorthand for computing it is the following proposition, which will be proven in 3.1.1.

Proposition 1.6.1 (Computation of the Hodge Star Operator)**.** *Let (M, g) be a compact oriented riemannian manifold of dimension n and let $p \in M$. Let $\omega_1, \ldots, \omega_n$ be a positively oriented orthonormal base of T_p^*M with respect to the induced inner product on 1-forms. Then, over k-forms, the Hodge Star operator can be computed as*

$$\star\,(\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}) = \mathit{sign}(\sigma) \cdot \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}}$$

where $\sigma = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & n \\ i_1 & i_2 & \cdots & i_k & j_1 & j_2 & \cdots & j_{n-k} \end{pmatrix}$ *is a permutation of* $\{1, \ldots, n\}$.

Remark 1.6.2. From this characterization for the Hodge Star, is very simply to observe that $\star^{-1} = (-1)^{k(n-k)}\star$, so $\star\star = (-1)^{k(n-k)}$.

With this operator $\star : \Omega^*(M) \to \Omega^*(M)$ we can extend it to the case of forms valued on any vector bundle $F \to M$, $\Omega^*(F) = \Gamma(F) \otimes \Omega^*(M)$, by

$$\star(\sigma \otimes \omega) := \sigma \otimes (\star\omega)$$

for $\sigma \in \Gamma(F)$ and $\omega \in \Omega^*(M)$.

Definition 1.6.3. Let $E \to M$ be a vector G-bundle with an affine connection ∇ on it with covariant exterior derivative $d_\nabla : \Omega^*(E) \to \Omega^{*+1}(E)$. ∇ is said to be a **Yang-Mills** connection on E, or a solution of the Yang-Mills equations on vector bundles if

$$\boxed{d_\nabla \star F_\nabla = 0}$$

[12]We can also consider the case of a semi-riemannian metric with exactly the same definition.

In order to state the Yang-Mills equations for a principal bundle $P \to M$, we need to define some kind of Hodge star on $\Omega^*(P, \mathfrak{g})$, the home of the curvature; as in the case of vector bundles, where we have defined it on $\Omega^*(M, End(E))$, the home of the curvature of vector bundles. However, as in the case of vector bundles, given a Hodge star operator on $\Omega^*(P)$ we have an unique extension to $\Omega^*(M, \mathfrak{g})$. Therefore, our problem reduces to induce a Hodge star operator in $\Omega^*(P)$ given the one on $\Omega^*(M)$.

Suppose that we have fixed a connection A on a principal bundle $P \xrightarrow{\pi} M$, from which we have defined a horizontal distribution $H \subseteq TP$ and let $n = \dim M$. Recall that $\pi_* : TP \to TM$ restricts to an isomorphism $\pi_*|_H : H \to TM$ so we can pullback the metric g on M to a bundle metric $\tilde{g} = \pi|_H^* g$ on H,[13] seen as a vector bundle $H \to P$. Likewise, given the volume form $\Omega \in \Omega^n(M)$ we can define the volume form $\tilde{\Omega} = \pi|_H^* \Omega \in \Omega^n(H)$.

From this metric and volume form, analogously to the case of a metric in the tangent bundle, for every $p \in P$ we can define a Hodge star operator $\star_p : \bigwedge^k H_p^* \to \bigwedge^{n-k} H_p^*$[14] by requiring that

$$\omega_p \wedge (\star\eta)_p = \tilde{g}_p(\omega_p, \eta_p)\, \tilde{\Omega}_p$$

for every $\omega_p, \eta_p \in \bigwedge H_p^*$.

With this operator in hand, we can extend it to a global operator $\star_p : \Omega_p^k(P) = \bigwedge^k T_p^* P \to \Omega_p^{n-k}(P) = \bigwedge^{n-k} T_p^* P$. Given $\omega_p \in \Omega_p^k(P)$ let us denote $\omega_p|_{H_p} \in \bigwedge H_p^k$ its restriction. Then, for $\omega_p \in \Omega_p^k(P)$ we define

$$\star_p \, \omega_p := (\star_p \, (\omega_p|_{H_p}))^h$$

where, given $\eta \in \bigwedge H_p^m$, $\eta^h \in \Omega_p^m(P)$ is the form $\eta^h(X_1, \ldots, X_m) = \eta(X_1^h, \ldots, X_m^h)$ for $X_1, \ldots, X_n \in T_p P$. Therefore, since it varies differentiably, we have extend the Hodge star operator to a map $\star : \Omega^k(P) \to \Omega^{n-k}(P)$ called the induced Hodge star operator on P.

Remark 1.6.4. In contrast with the Hodge star on M, this map is no longer an isomorphism, since in general

$$\dim \Omega_p^k(P) = \binom{\dim P}{k} \neq \binom{\dim P}{n-k} = \dim \Omega_p^{n-k}(P)$$

However, if we call $\Omega_h^*(P)$ the space of differential forms on P that vanish on the vertical distribution $V \subseteq TP$, $\dim\,(\Omega_h^k)_p(P) = \binom{n}{k}$, so the Hodge star is again an isomorphism $\star : \Omega_h^*(P) \to \Omega_h^*(P)$.

Finally, extending the induced Hodge star to a map $\star : \Omega^*(P, \mathfrak{g}) \to \Omega^*(P, \mathfrak{g})$ (or an isomorphism $\star : \Omega_h^*(P, \mathfrak{g}) \to \Omega_h^*(P, \mathfrak{g})$) we can define

[13] Observe that, if G would be a discrete group, then $P \to M$ would be a covering space and, only in this case, the global pullback $\pi^* g$ would be a non-degenerated symmetric 2-tensor, since $\dim P = \dim M$. If G is not discrete, we have to eliminate the degeneration induced by the vectical distribution, so we can only pull it back to the horizontal distribution.

[14] That is, $\bigwedge H^*$ is the space of differential forms on P that only can eat vectors of $H \subseteq TP$.

Definition 1.6.5. If $P \to M$ is a principal G-bundle with a connection A on it, A is said to be a **Yang-Mills** connection on P, or a solution of the Yang-Mills equations on principal bundles if

$$\boxed{d_A \star F_A = 0}$$

Remark 1.6.6. Using the translation between principal and vector G-bundles, we have that a connection on a vector bundle (resp. principal bundle) is a Yang-Mills connection if and only if its correspondend is a Yang-Mills connection in the frame bundle (resp. associated bundle).

Moreover, the Yang-Mills connections are gauge invariant, as shown in [42].

Proposition 1.6.7. *The Yang-Mills equations are gauge invariant, that is, every connection in the orbit of a Yang-Mills connection under the action of the gauge group is a Yang-Mills connection.*

Example 1.6.8. From a mathematical point of view, a G-gauge theory for G a Lie group is the study of Yang-Mills connections in a principal G-bundle (or, equivalently, a vector G-bundle). The first example of a gauge theory is electromagnetism, which can be stated as a $U(1)$-gauge theory.[15]

Indeed, let us take $M = \mathbb{R}^4$ with the Minkowski metric of signature $(1, 3)$, called it $\mathbb{R}^{1,3}$, and let us consider the trivial principal bundle $P = U(1) \times \mathbb{R}^{1,3}$. Given a connection A on P, since $U(1) \cong \mathbb{C}^*$ is an abelian Lie group, by remark 1.4.16, using the curvature of A we can define a global form $F_A \in \Omega^2(\mathbb{R}^{1,3}, \mathfrak{u}(1)) \cong \Omega^2(\mathbb{R}^{1,3})$, since $\mathfrak{u}(1) \cong \mathbb{R}$. Hence, together with the second Bianchi, A is a Yang-Mills connection if and only if

$$\boxed{d_A F_A = 0 \qquad\qquad d_A \star F_A = 0}$$

which are, exactly, the Maxwell equations in the differential form framework. Furthermore, it can be shown that, since $\mathbb{R}^{1,3}$ is contractible, $H^k(\mathbb{R}^{1,3}) = 0$ for $k > 0$, every 2-form satisfying the previous equations is the curvature of a connection A on P. Therefore, looking for solutions of the Maxwell equations is equivalent to looking for solutions of the Yang-Mills equations on $P = U(1) \times \mathbb{R}^{1,3}$, so electromagnetism is a $U(1)$-gauge theory.

1.6.1 Self-dual Connections and Instantons

A very special type of Yang-Mills connections are those that have any kind of self-similarity. These self-similar solutions are of enormous importance in mathematical-physics in general, due to its special properties, and the main concern of this work.

Let us restrict to the case of a 4-dimensional riemannian manifold (M, g) (that, in mathematical physics, play the role of spacetime). In this manifold, the Hodge star operator is an endomorphism of

[15] Those gauge theory based on an abelian Lie group are called abelian gauge theories, while, in the non abelian case, are known as non-abelian gauge theories. Therefore, electromagnetism is an abelian gauge theory, but the standard model of particle physics is a non-abelian gauge theory over $U(1) \times SU(2) \times SU(3)$.

the 2-forms, because $\star : \Omega^2(M) \to \Omega^{4-2}(M) = \Omega^2(M)$. Furthermore, by remark 1.6.2, we have that, on $\Omega^2(M)$ it satisfies

$$\star\star = (-1)^{2(4-2)} Id_{\Omega^2(M)} = Id_{\Omega^2(M)}$$

so $\star^2 = Id_{\Omega^2(M)}$, that is, $\star$ is an involution of $\Omega^2(M)$. Therefore, $\star$ has eigenvalues ± 1, with corresponding eigenspaces $\Omega^2_+(M)$ and $\Omega^2_-(M)$ for $+1$ and -1, respectively. Furthermore, given any $\omega \in \Omega^2(M)$ defining

$$\omega^+ := \frac{1}{2}(\omega + \star\omega) \qquad \omega^- := \frac{1}{2}(\omega - \star\omega)$$

we have the decomposition $\omega = \omega^+ + \omega^-$ for $\omega^+ \in \Omega^2_+(M)$ and $\omega^- \in \Omega^2_-(M)$.

Let us take, now, a G-principal bundle $P \to M$ and a connection A on it. For the induced Hodge star on P, we have $\star : \Omega^2_h(P, \mathfrak{g}) \to \Omega^{4-2}_h(P, \mathfrak{g}) = \Omega^2_h(P, \mathfrak{g})$ so $\star$ is again an involution of $\Omega^2_h(P, \mathfrak{g})$ so analogously, we have eigenspaces $(\Omega^2_h)_+(P, \mathfrak{g})$ and $(\Omega^2_h)_-(P, \mathfrak{g})$ for eigenvalues $+1$ and -1 and we can decompose $\omega = \omega^+ + \omega^-$ for every $\omega \in \Omega^2_h(P, \mathfrak{g})$ with $\omega^+ \in (\Omega^2_h)_+(P, \mathfrak{g})$ and $\omega^- \in (\Omega^2_h)_-(P, \mathfrak{g})$.

In particular, since the curvature of A $F_A \in \Omega^2_h(P, \mathfrak{g})$, we can decompose it

$$F_A = F_A^+ + F_A^-$$

for $F_A^+ \in (\Omega^2_h)_+(P, \mathfrak{g})$ and $F_A^- \in (\Omega^2_h)_-(P, \mathfrak{g})$. Using this dual and anti-dual parts of a connection we define the concept of instanton.

Definition 1.6.9. Let (M, g) be a riemannian 4-dimensional manifold and let $P \to M$ be a principal G-bundle. A connection A on P is called an **instanton** or a **self-dual connection** if $F_A^- = 0$ that is, if

$$\boxed{\star F_A = F_A}$$

Analogously, it is called an **anti-instanton** or an **anti self-dual connection** if $F_A^+ = 0$, or equivalently

$$\boxed{\star F_A = -F_A}$$

Remark 1.6.10. Analogous considerations can be done for a vector G-bundle over a 4-dimensional manifold and a connection on it. A connection on a principal bundle is self-dual (resp. anti self-dual) if and only if its induced connection on its associated adjoint vector bundle is self-dual (resp. anti self-dual), and viceversa.

Corollary 1.6.11. *Every instanton or anti-instanton is a Yang-Mills connection.*

Proof. It is enough to prove it in the case of a principal bundle $P \to M$. Let A be any self-dual (resp. anti-self-dual) connection on P. Then, by the second Bianchi identity (see corolary 1.4.15) we have

$$d_A \star F_A = \pm d_A F_A = 0$$

so A is a Yang-Mills connection. ∎

1.6.2 Dimensional Reduction and Higgs Fields

Given a principal bundle P, the study of the Yang-Mills connection on P is a extremelly difficult task, as shown, for example, in [2]. Therefore, a common strategy to deal with this situations is to simplify the problem considering some special cases, like self-dual connections or instantons. However, in this simplification procedure, we can arise to some special solutions of the Yang-Mills equations whose study leads to extremelly powerful considerations. One of the most important articles that explores this approach is [35], in which Higgs bundles are introduced and deeply studied.

Specifically, we are going to consider connections over $\mathbb{R}^4$ that are invariant in two directions. In this special-kind connections, we will discover that the equations for their self-duality can be rewriten over $\mathbb{R}^2$ in a special way that is conformally invariant. This clever trick, very common in theoretical physics, is usually called **dimensional reduction**.

Let us consider our base manifold as $M = \mathbb{R}^4$ and let G be the compact real form of a complex Lie group. . Let P be a principal G-bundle over $\mathbb{R}^4$ and $Ad(P)$ be the adjoint bundle of P, that is, the G-vector bundle associated to the adjoint representation. Finally, let us take a connection A on P with curvature F_A.

From now on, we will work on a trivializing chart $U \subseteq \mathbb{R}^4$ of P, in which $\pi_P^{-1}(U) \subseteq P \cong U \times P$ and $\pi_{Ad(P)^{-1}}(U) \subseteq Ad(P) \cong U \times \mathfrak{g}$. Hence, considering the connection system associated to this neighbourhood, we can write $A \in \Omega^1(U, \mathfrak{g})$ and $F_A \in \Omega^2(U, \mathfrak{g})$. Let us write explicitly

$$A = \sum_{i=1}^{4} A_i \, dx^i \qquad\qquad F_A = \sum_{1 \leq i < j \leq 4} F_{ij} \, dx^i \wedge dx^j$$

for some $A_i, F_{ij} : U \subseteq \mathbb{R}^4 \to \mathfrak{g}$. Hence, since

$$\star F_A = F_{12} \, dx^3 \wedge dx^4 - F_{13} \, dx^2 \wedge dx^4 + F_{14} \, dx^2 \wedge dx^3 + F_{34} \, dx^1 \wedge dx^2 - F_{24} \, dx^1 \wedge dx^3 + F_{23} \, dx^1 \wedge dx^4$$

we have that A is self-dual (i.e. $\star F_A = F_A$) if and only if

$$\begin{cases} F_{12} &= & F_{34} \\ F_{13} &= & -F_{24} \\ F_{14} &= & F_{23} \end{cases} \tag{1.1}$$

Recall that, by remark 1.3.21, if ∇ is the covariant derivative on $Ad(P)$ associated to A, then we have that

$$F_A(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

so, in particular, writing $\nabla_i = \nabla_{\frac{\partial}{\partial x^i}}$, we have that $F_{ij} = \nabla_i\nabla_j - \nabla_j\nabla_i = [\nabla_i, \nabla_j]$, so equation (1.1) can be rewriten as

$$\begin{cases} [\nabla_1, \nabla_2] & = & [\nabla_3, \nabla_4] \\ [\nabla_1, \nabla_3] & = & -[\nabla_2, \nabla_4] \\ [\nabla_1, \nabla_4] & = & [\nabla_2, \nabla_3] \end{cases} \tag{1.2}$$

Now, let us suppose that A only depends on the coordinates (x_1, x_2) of $\mathbb{R}^4$ and is independent of (x_3, x_4). This is the key simplification from which we can apply the dimensional reduction procedure. In that case, let us redefine $\phi_1 := A_3$ and $\phi_2 := A_4$, so the equations (1.2) become

$$\begin{cases} [\nabla_1, \nabla_2] & = & [\phi_1, \phi_2] \\ [\nabla_1, \phi_1] & = & -[\nabla_2, \phi_2] \\ [\nabla_1, \phi_2] & = & [\nabla_2, \phi_1] \end{cases} \tag{1.3}$$

where we have used that, since A_1 does not depend on x_3 we have

$$[\nabla_1, \nabla_3] = F_{13} = \frac{\partial}{\partial x^1}A_3 - \frac{\partial}{\partial x^3}A_1 + [A_1, A_3] = \frac{\partial}{\partial x^1}A_3 + [A_1, A_3] = \left[\frac{\partial}{\partial x^1 + A_1, A_3}\right] = [\nabla_1, \phi_1]$$

and similarly for the other terms.

There is a crafty way to rewrite these equations in a more compact form. Let us consider $G_{\mathbb{C}}$ the complexification of G with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. In this Lie algebra, we can see as complex-Lie algebra valued functions $\phi_1, \phi_2 : U \to \mathfrak{g}_{\mathbb{C}}$, so we can define the **complex Higgs field** $\phi := \phi_1 - i\phi_2 : U \to \mathfrak{g}_{\mathbb{C}}$. In this terms, if $\cdot^* : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is the anti-involution on the complex Lie algebra, using equations (1.3) we have that

$$[\phi, \phi^*] = [\phi_1 - i\phi_2, \phi_1 + i\phi_2] = [\phi_1, \phi_1] + i[\phi_1, \phi_2] - i[\phi_2, \phi_1] + [\phi_2, \phi_2] = 2i[\phi_1, \phi_2] = 2i[\nabla_1, \nabla_2] = 2iF_{12}$$

$$[\nabla_1 + i\nabla_2, \phi] = [\nabla_1, \phi_1] + [\nabla_2, \phi_2] + i\left([\nabla_2, \phi_1] - [\nabla_1, \phi_2]\right) = 0$$

Therefore, equations (1.3) are equivalent to

$$\begin{cases} F_{12} = \frac{1}{2}i[\phi, \phi^*] \\ [\nabla_1 + i\nabla_2, \phi] = 0 \end{cases} \tag{1.4}$$

Finally, we can rewrite this equations in an even more invariant form. Observe that, since A only depends on the coordinates (x_1, x_2), we can define a connection on $\mathbb{R}^2$ by

$$\hat{A} := A_1\, dx^1 + A_2\, dx^2$$

Now, let us introduce the usual complex estructure on $\mathbb{R}^2$ in the way that the holomorphic coordinate is $z = x_1 + ix_2$. We define the **Higgs field** $\Phi \in \Omega^{1,0}(\mathbb{R}^2, \mathfrak{g}_{\mathbb{C}})$ by

$$\Phi = \frac{1}{2}\phi\, dz$$

so, using the natural extension of the anti-involution of the complex Lie algebra to 2-forms, we have $\Phi^* = \frac{1}{2}\phi^*\, d\overline{z}$. Thus, equations (1.4) can be rewriten in a form known as the **self-duality equations**

$$\begin{cases} F_{\hat{A}} + [\Phi, \Phi^*] = 0 \\ \quad\overline{\partial}_{\hat{A}}\Phi = 0 \end{cases} \tag{1.5}$$

where $\overline{\partial}_{\tilde{A}} : \Omega^{p,q}(\mathbb{R}^2, \mathfrak{g}_{\mathbb{C}}) \to \Omega^{p,q+1}(\mathbb{R}^2, \mathfrak{g}_{\mathbb{C}})$ is the Dolbeault covariant exterior derivative of A on $Ad(P)$, seen as a trivial complex vector bundle on $\mathbb{R}^2$ (see section 1.3.1.1).

Since this self-duality equations are conformally invariant, they can be generalized to a general Riemann surface,

Definition 1.6.12. Let M be a compact Riemann surface, let G be the compact real form of a complex Lie group and let $E \to M$ be a C^∞-complex vector bundle with structure group G. Given a connection A on E and $\Phi \in \Omega^{1,0}(M, \mathfrak{g}_E)$, we will say that (E, A, Φ) is a **Higgs bundle** if and only if the self-duality equations

$$\begin{cases} F_A + [\Phi, \Phi^*] = 0 \\ \quad\overline{\partial}_A\Phi = 0 \end{cases} \tag{1.6}$$

hold. In that case, Φ is called the **Higgs field**.

Example 1.6.13. Given a complex vector bundle E on a compact Riemann surface M, taking $\Phi = 0$, we have that $(E, A, 0)$ is a Higgs bundle if and only if A is a flat connection. Therefore, flat bundles are particular cases of Higgs bundles.

Remark 1.6.14. Let $E \to M$ be a C^∞-complex vector bundle on a compact Riemann surface M, with canonical bundle $K_M = \Omega^{1,0}(M)$, and let A be a connection on E. A automatically induces a connection on $\mathfrak{g}_E$, also denoted by A, and we consider $\overline{\partial}_A$, the Dolbeault covariant exterior derivative associated to A on $\mathfrak{g}_E$.

Observe that, since M is a surface, $\Omega^{0,2}(E) = \Gamma(E) \otimes \Omega^{0,2}(M) = 0$, so, automatically, $\overline{\partial}_A^2 = 0$. Therefore, A satifies the integrability condition required by theorem 1.3.10, so there exists an unique complex structure on E such that $E \to M$ is an holomorphic vector bundle and $\overline{\partial}_E = \overline{\partial}_A$. Analogously, there exists an unique complex structure on $\mathfrak{g}_E$ such that $\mathfrak{g}_E \to M$ is an holomorphic vector bundle and $\overline{\partial}_{\mathfrak{g}_E} = \overline{\partial}_A$ on $\mathfrak{g}_E$.

In that case, if (E, A, Φ) is a Higgs bundle, then $\overline{\partial}_{\mathfrak{g}_E}\Phi = \overline{\partial}_A\Phi = 0$, so, by the digresion of remark 1.3.9, the last equation of (1.6) is equivalent to $\Phi \in \Omega^{1,0}(M, \mathfrak{g}_E) = \Gamma(\mathfrak{g}_E \otimes K_M)$ been holomorphic with respect to the natural complex structure on $\mathfrak{g}_E \otimes K_M$.

Therefore, an equivalent definition of a Higgs bundle on a compact Riemann surface M is a triple (E, A, Φ) with E an holomorphic vector bundle, A a connection on E compatible with the holomorphic structure and $\Phi \in H^0(\mathfrak{g}_E \otimes K_M)$ (i.e. Φ is holomorphic) such that

$$F_A + [\Phi, \Phi^*] = 0$$

Chapter 2

Non-abelian Hodge Theory

2.1 Moduli Spaces

The concept of moduli space dates back to the XVIII century, when Riemann tried to clasify all posible complex structures on a given surface of a given genus. In general, moduli spaces arise in the context of clasification problems, in whith there are a large space of non-isomorphic possibilities without any simple structure.

Moduli spaces try to solve this problem, introducing a geometric space $\mathcal{M}$ which parametrices all the posibilities of the clasification problem, and whose topology is strongly linked with some notion of closeness in the clasification problem. It should be noted that the construction of this spaces is a very dificult task, that usually requires a very deep insight in the problem itself.

Along this section, we will work over a algebraically closed field k (for our purposes, it will be $k = \mathbb{C}$), in the way that all the considered varieties will be varieties over k. Let us suppose that we are studying a colection of objects, A. Moreover, in A we have defined an equivalent relation $\sim$ so we want to understand the clasification problem of A under $\sim$, that is, we want to understand the quotient set $A/\sim$.

Remark 2.1.1. In general, we should take A to be a proper class, because A will be large enough to not be a set in a strict sense. However, this will no cause any logical problem, because, usually, our quotient $A/\sim$ (the main focus of attention) will be a set.

In order to have a better insight on the clasification problem, we can introduce some sort of closeness notion in $A/\sim$. Hence, given a variety S, suppose that we have defined a notion of a **family parametrized by** S. Setwise, a family parametrized by S is a subset of A, but the main point is that we want that the geometric properties of S translates to the families parametrized by S. Given a variety S, let us denote the set (or class) of families parametrized by S by F_S, so $F_S \subseteq \mathcal{P}(A)$. In order to have a well-possed moduli problem, we want that our families satisfy the following properties.

- For each variety S, there is an equivalent relation $\simeq_S$ on F_S.
- For any single point variety $\star$, a family parametrized by $\star$ is single-element set with an element of A. In this sense, we can consider $F_\star \subseteq A$ and, with this identification, $\simeq_\star$ must be equal to $\sim$ in $F_\star$.
- For every morphism of varieties $\phi : S \to S'$ we have a map $\phi^* : F_{S'} \to F_S$ such that
  - If $id_S : S \to S$ is the identity, then $id_S^* : F_S \to F_S$ is the identity.
  - Given $\phi_1 : S \to S'$ and $\phi_2 : S' \to S''$, then $(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^*$.
  - ϕ^* preserves the equivalent relation $\simeq$ in the sense that, given $X, X' \in F_{S'}$, if $X \simeq_{S'} X'$ then $\phi^* X \simeq_S \phi^* X'$.

Remark 2.1.2. Given a variety S and a family $X \in F_S$, let us denote, for every $s \in S$, $X_s := i_s^* X \in F_{\{s\}} = A$, where $i_s : \{s\} \to S$ is the inclusion map. Using this construction, for any $X \in F_S$ we can build a map $\phi_S : S \to X \subseteq A$ given by $\phi_S(s) = X_s$, which justifies the name *family parametrized by* S.

Remark 2.1.3. If X and X' are two families parametrized by a variety S, since $i_s^* : F_S \to F_{\{s\}} = A$ preserves the equivalent relation, we have that if $X, X' \in F_S$ are $X \simeq_S X'$, then $X_s \sim X'_s$ for all $s \in S$. However, the reciprocal is not true and we can have that, for all $s \in S$, $X_s \sim X'_s$, but $X \not\simeq_S X'$.

Remark 2.1.4. Without further modifications, the same ideas can be applied to the more general case of schemes instead of varieties. However, in the present work, we will not need this generalization, so we will focus on this more restrictive case.

Example 2.1.5 (Hypersurfaces in $\mathbb{P}^n$)**.** Suppose that we want to understand the clasification problem of hypersurfaces on a given $\mathbb{P}^n$. Given a variety S a family $X \in F_S$ is a algebraic set $X \subseteq S \times \mathbb{P}^n$ such that, for every $s \in S$, $X_s := X \cap \{s\} \times \mathbb{P}^n$ is a hypersurface of $\mathbb{P}^n$. The equivalent restriction on a family could be, for example, up to isomorphism as varieties or up to action of $PGL(n+1)$. Given $\phi : S' \to S$ the restriction $\phi^* : F_S \to F_{S'}$ is given by $\phi^*(X) = \tilde{\phi}^{-1}(X)$ where $\tilde{\phi} : S' \times \mathbb{P}^n \to S \times \mathbb{P}^n$ is the natural extension of ϕ which is the identity on $\mathbb{P}^n$.

The concept of family is necessarily very vague, cause it should be applicated to a large range of problems. Of course, it strongly depends on the clasification problem studied and determines all the constructions. Let us see some examples.

In this context, if we have defined, for every variety S, a notion of family parametrized by S, F_S, we can define a contravariant functor $\mathcal{F} : \mathbf{Var} \to \mathbf{Set}$ given, on objects, by $\mathcal{F}(S) = F_S/\simeq_S$ and, on morphisms $\phi : S' \to S$ by $\mathcal{F}(\phi) = \phi^* : \mathcal{F}(S) \to \mathcal{F}(S')$. This functor is call the family functor and captures all the information about the moduli problem, in the way that all the problem can be stated in terms of $\mathcal{F}$.

Definition 2.1.6. A **family functor** is a contravariant functor $\mathcal{F} : \mathbf{Var} \to \mathbf{Set}$ from the category of varieties to the category of sets.

Remark 2.1.7. Once given a family function, the notion of family is only philosophycal and in not need for the mathematical formulation of the problem. In fact, the moduli problem can be completely stated without any reference to families, selecting any functor $\mathcal{F} : \mathbf{Var} \to \mathbf{Set}$. However, only when the family function $\mathcal{F}$ arises via a choosing of families, the problem is well-possed, in the sense that the moduli problem solves a real clasification problem.

Hence, the moduli problem is the problem of better understanding $\mathcal{F}$. Maybe the most easy way to understand it is representing it in terms of the homomorphism to a single variety $\mathcal{M}$.

Definition 2.1.8. Let $\mathcal{F}$ be a family functor. A **fine moduli space** of $\mathcal{F}$ is a pair $(\mathcal{M}, \Phi)$ where $\mathcal{M}$ is a variety and $\Phi : \mathcal{F} \to Hom(\cdot, \mathcal{M})$ is and isomorphism of functors, that is, Φ co-represent $\mathcal{F}$ via $\mathcal{M}$.

Remark 2.1.9. If $(\mathcal{M}, \Phi)$ is a fine moduli space for $\mathcal{F}$, given a single point variety $\star$, we have that $\Phi(\star) : \mathcal{F}(\star) \to Hom(\star, \mathcal{M}) \cong \mathcal{M}$ is an isomorphism. Hence, recalling that $\mathcal{F}(\star) = F_\star / \simeq_\star = A/\sim$, we have a bijection between $\mathcal{M}$ and $A/\sim$, that is, every point of $\mathcal{M}$ is a element of the clasification problem.

Example 2.1.10. Maybe the first example of moduli space is $\mathbb{P}^n$ as the moduli space of vectorial lines (i.e. lines through 0) in k^n. As family functor, we take $\mathcal{F} : \mathbf{Var} \to \mathbf{Set}$ that, for every variety S, define $\mathcal{F}(S) = F_S$ as the set of line bundles $L \to S$ (i.e. $L \in Pic(S)$, the Picard group of S) that are contained in the trivial bundle $S \times k^n$. Of course, given a regular map of varieties $f : S \to S'$ we define $\mathcal{F}(f) : \mathcal{F}(S') \subseteq Pic(S') \to \mathcal{F}(S) \subseteq Pic(S)$ as the restriction of the pullback-of-vector-bundles mapping $f^* : Pic(S') \to Pic(S)$.

Now, observe that we can define a natural transformation $\Phi : \mathcal{F} \to Hom(\cdot, \mathbb{P}^n)$ given, for any $S \in \mathbf{Var}$, by $\Phi(S) : \mathcal{F}(S) \subseteq Pic(S) \to Hom(S, \mathbb{P}^n)$, that, for $L \in \mathcal{F}(S)$ and $s \in S$ sends

$$\Phi(S)(L)(s) := L_s$$

where $L_s \in \mathbb{P}^n$ is the fiber of L over s, seen in the trivial bundle $S \times k^n$. Since taking fibers commutes with pullbacks, Φ is a natural transformation, so Φ co-represents $\mathcal{F}$ via $\mathbb{P}^n$. In this sense, $(\mathbb{P}^n, \Phi)$ is a fine moduli space for the family functor $\mathcal{F}$, that is, it is a fine moduli space for the space of vectorial lines in k^n.

Example 2.1.11. Every variety X is a fine moduli space for some moduli problem. Indeed, let us fix a variety X and let us define the family functor $\mathcal{F} : \mathbf{Var} \to \mathbf{Set}$ by $\mathcal{F} = Hom(\cdot, X)$. Trivially, there exists a natural transformation $id : \mathcal{F} \to Hom(\cdot, X)$ so (X, id) is a fine moduli space for the moduli problem described by $\mathcal{F}$.

From this construction, we observe that, in general, not every fine moduli space is smooth, partially solving a question addressed to us by Prof. M. Logares. Thus, we should restrict our attention to geometric hypothesis, for example, restricting to family functors that arise as classes of isomorphism of families parametrized by a variety, as described above.

An equivalent way of determining a fine moduli space is via a special family.

Definition 2.1.12. Let $\mathcal{F}$ be a family functor. A **fine moduli space** of $\mathcal{F}$ is a pair $(\mathcal{M}, U)$ where $\mathcal{M}$ is a variety and $U \in \mathcal{F}(\mathcal{M})$, known as the **universal family**, such that, for every variety S and every equivalent class of family X parametrized by S (i.e. $X \in \mathcal{F}(S)$) there exists an unique regular morphism $\phi_X : S \to \mathcal{M}$ such that $X = \mathcal{F}(\phi_X)(U)$.

Proposition 2.1.13. *Definitions 2.1.8 and 2.1.12 are equivalent.*

Proof. If U is an universal family over $\mathcal{M}$ for $\mathcal{F}$, we define the natural transformation $\Phi : \mathcal{F} \to Hom(\cdot, \mathcal{M})$ by $\Phi(S)(X) = \phi_X$ for any variety S and $X \in \mathcal{F}(S)$. By uniqueness of ϕ_X, it is well defined and, by existence, it is an isomorphism.

Reciprocally, let us suppose that $\Phi : \mathcal{F} \to Hom(\cdot, \mathcal{M})$ co-represents $\mathcal{F}$. Then let us take $U := \Phi(\mathcal{M})^{-1}(id_{\mathcal{M}}) \in \mathcal{F}(\mathcal{M})$ and, for every variety S and every $X \in \mathcal{F}(S)$, let us define $\phi_X : S \to \mathcal{M}$ by $\phi_X := \Phi(S)(X)$. It is enough to prove that $\mathcal{F}(\phi_X)(U) = X$. To this end, recall that, since Φ is a natural transformation, the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(\mathcal{M}) \arrow[r, "\Phi(\mathcal{M})"] \arrow[d, "\mathcal{F}(\phi_X)"'] & Hom(\mathcal{M}, \mathcal{M}) \arrow[d, "Hom(\cdot,\mathcal{M})(\phi_X)"] \\
\mathcal{F}(\mathcal{S}) \arrow[r, "\Phi(S)"'] & Hom(S, \mathcal{M})
\end{tikzcd}$$

so we have that

$$\begin{aligned}
\mathcal{F}(\phi_X)(U) = \mathcal{F}(\phi_X)\left(\Phi(\mathcal{M})^{-1}(id_{\mathcal{M}})\right) = \mathcal{F}(\phi_X) \circ \Phi(\mathcal{M})^{-1}(id_{\mathcal{M}}) = \Phi(S)^{-1}\left(Hom(\cdot, \mathcal{M})(\phi_X)(id_{\mathcal{M}})\right) \\
= \Phi(S)^{-1}\left(id_{\mathcal{M}} \circ \phi_X\right) = \Phi(S)^{-1}(\phi_X) = \Phi(S)^{-1}\left(\Phi(S)(X)\right) = X
\end{aligned}$$

as we wanted to prove. ■

In most of cases, a fine moduli space cannot be achived cause the topology of $\mathcal{M}$ does not capture completely the complexity of $\mathcal{F}$. However, we can use a weaker version of moduli space that is enought in most of the cases.

Definition 2.1.14. Let $\mathcal{F}$ be a family functor. A **coarse moduli space** of $\mathcal{F}$ is a pair $(\mathcal{M}, \Phi)$ where $\mathcal{M}$ is a variety and $\Phi : \mathcal{F} \to Hom(\cdot, \mathcal{M})$ is a natural transformation such that:

- For every single point variety $\star \in Obj(\mathbf{Var})$, $\Phi(\star) : \mathcal{F}(\star) \to Hom(\star, \mathcal{M}) \cong \mathcal{M}$ is a bijection.
- For every manifold N and every natural transformation $\Psi : \mathcal{F} \to Hom(\cdot, N)$ there exists a unique natural transformation $\phi : Hom(\cdot, \mathcal{M}) \to Hom(\cdot, N)$ such that $\Psi = \phi \circ \Phi$, that is, the

following diagram commutes

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ Hom(\cdot, \mathcal{M}) \arrow[rr, dashed, "\phi"'] & & Hom(\cdot, N) \end{tikzcd}$$

Proposition 2.1.15. *Evary fine moduli space is a coarse moduli space.*

Proof. It is a simple check. Let $(\mathcal{M}, \Phi)$ be a fine moduli space for the family functor $\mathcal{F}$. Since Φ is a natural isomorphism, $\Phi(\star)$ is a bijection and, given $\Psi : \mathcal{F} \to Hom(\cdot, N)$ for some variety N, we can define $\phi : Hom(\cdot, \mathcal{M}) \to Hom(\cdot, N)$ by $\phi = \Psi \circ \Phi^{-1}$. ■

Moreover, playing with the definitions, we obtain that coarse moduli spaces are unique up to isomorphism, as shown in [59].

Proposition 2.1.16. *Fine and coarse moduli space, if exist, are unique up to isomorphism of varieties.*

2.1.1 Moduli Space of Stable Vector Bundles

Let us fix an algebraic variety X and let us study the space of algebraic vector bundles on X, that is, we want to study the set

$$A_{VB}(X) = \{E \to X \text{ algebraic vector bundle}\}$$

Remark 2.1.17. By GAGA theory (see appendix A.4), if X is a smooth projective variety (for example, if X is a Riemann surface) then there exists a correspondece between algebraic vector bundles on X and holomorphic vector bundles on X, with respect to the inherit complex structure. Hence, for X smooth projective, we have a natural bijection

$$A_{VB}(X) \cong \{E \to X \text{ holomorphic vector bundle}\}$$

In that way, we will simply say vector bundle when refering to an algebraic vector bundle when we see X as an algebraic variety, and to holomorphic vector bundles when X is a smooth projective variety, seen as compact Kähler manifold.

On this set A_{VB}, we define the equivalence relation $\sim$ by declaring that two algebraic vector bundles E, F are equivalent, written $E \sim F$ is and only if E and F are isomorphic as algebraic vector bundles (or, equivalently, as holomorphic vector bundles in the case X smooth projective).

In this set A_{VB}, we define the equivalence relation $\sim$ be declaring that, for any two algebraic vector bundles E, F, $E \sim F$ is and only if E and F are isomorphic as algebraic vector bundles (or, equivalently, as holomorphic vector bundles in the case X smooth projective).

In order to consider the corresponding moduli problem, let us define families of vector bundles on X. Let S be an algebraic variety (resp. smooth projective variety), a family of vector bundles on X parametrized by S is a vector bundle $E \to X \times S$. Therefore, the space of families parametrized by S, $F_S(X)$ is

$$F_S(X) = \{E \to X \times S \,\text{vector bundle}\}$$

Moreover, given a morphism $\phi : S \to S'$, we define $\phi^* : F_{S'}(X) \to F_S(X)$ by $\phi^*(E \to X \times S) = \phi^* E$, the pullback of the vector bundle E by the morphism ϕ. Observe that, in particular, for a single point variety $\star$ we have

$$F_{\{\star\}} = \{E \to X \times \star \text{ vector bundle}\} \cong \{E \to X \text{ vector bundle}\} = A_{VB}(X)$$

and, the induced map $i_s^* : F_S(X) \to F_{\{s\}}(X)$, for $s \in S$ and $i_s : \{s\} \hookrightarrow S$ the inclusion map, is just the restriction $i_s^*(E \to X \times S) = E|_{X\times\{s\}} \to X \times \{s\}$.

Therefore, in order to completely define a moduli problem, it is enough to define a equivalence relation $\cong_S$ on $F_S(X)$ for any variety S. Of course, the first idea is to use the obvious extension of $\sim$ and declare that $E \to X \times S$ and $F \to X \times S$ are equivalent via $\cong_S$ if and only if they are isomorphic as vector bundles on $X \times S$. In that case, the family functor is $\mathcal{F}_X : \mathbf{Var} \to \mathbf{Set}$, $\mathcal{F}_X(S) = F_S(X)/\cong_S$

However, with this definition, there not exists a fine moduli space for the moduli problem. Indeed, if $\mathcal{M}_{VB}$ exists, it must be $\mathcal{M}_{VB} = A_{VB}(X)/\sim$. In that case, the only posibility for $\Phi : \mathcal{F}_X \to Hom(\cdot, \mathcal{M}_{VB})$ is to define, for any variety S, $\Phi(S) : F_S(X)/\cong_S \to Hom(S, \mathcal{M}_{VB})$ given by

$$\Phi(S)([E]_{\cong_S})(s) = [E|_{X\times\{s\}}]_\sim$$

where $E \to X \times S$ is a vector bundle, $E|_{X\times\{s\}}$ is the restriction $E|_{X\times\{s\}} \to X \times \{s\} \cong X$ and $[E]_{\cong_S}$ and $[F]_\sim$ are the equivalence classes of E and F under $\cong_S$ and $\sim$, respectively.

However, this map Φ is not an isomorphism. Indeed, let S be a variety with a non-trivial line bundle $L \to S$. Let $\pi : X \times S \to S$ be the projection and let us consider the pullback line bundle on $X \times S$, $\pi^* L \to X \times S$. For a general family $E \to X \times S$ parametriced by S, we have that E and $E \otimes \pi^* L$ are not isomorphic. However, for any $s \in S$, $(\pi^* L)_s$ is a trivial line bundle, so we have

$$E|_{X\times\{s\}} \cong (E \otimes \pi^* L)\,|_{X\times\{s\}}$$

and, thus

$$\Phi(S)([E]_{\cong_S})(s) = [E|_{X\times\{s\}}]_\sim = [E \otimes \pi^* L|_{X\times\{s\}}]_\sim = \Phi(S)([E \otimes \pi^* L]_{\cong_S})(s)$$

for $[E]_{\cong_S} \neq [E \otimes \pi^* L]_{\cong_S}$. Therefore, $\Phi(S)$ is not injective so, in particular, Φ it is not an isomorphism of functors.

The solution to this pathological problem, as always in the theory of moduli spaces, is to restrict our attention to a more specific class of vector bundles. First of all, recall that, given an algebraic vector bundle on X, $E \to X$, there exists two important invariants

- The rank or E, $rk(E)$, that is the dimension of the fiber E_x for any $x \in X$.
- The degree of E, $deg(E)$. In the case of line bundles L, the divisors theory give us a well defined integer $deg(L) \in \mathbb{Z}$. Indeed, if $L \to X$ is a line bundle, it is associated to a Weil divisor $D_L \in Div(X)$, so we define $deg(L) := deg(D_L)$. For the case of vector bundles of higher rank E, we define $deg(E) := deg(det(E))$, the degree of the determinant bundle, $det(E) = \bigwedge^n E$, for $n = \dim X$, which is a line bundle.

  Recall that, for a Riemann surface X, there exists a more sofisticated definition of $deg(L)$ for a line bundle L. Recall that the first Chern class, c_1, is the map in cohomology $c_1 : Pic(X) = H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, with $Pic(X)$ the Picard group of X, that is, the group of line bundles on X. Thus, it can be proved that (see [29]), seen $c_1(L) \in \mathbb{Z}$, we have $c_1(L) = deg(L)$.

With this notions, we can define a fundamental property of vector bundles.

Definition 2.1.18. Let $E \to X$ be an algebraic vector bundle over an algebraic variety X. We say that E is **stable** if, for any subbundle $F \subseteq E \to X$ we have

$$\frac{deg(F)}{rk(F)} < \frac{deg(E)}{rk(E)}$$

and it is called **semi-stable** if, for any subbundle $F \subseteq E \to X$ we have

$$\frac{deg(F)}{rk(F)} \leq \frac{deg(E)}{rk(E)}$$

Remark 2.1.19. Due to this definition, usually, given an algebraic vector bundle $E \to X$, the **slope** of E, $\mu(E)$ is defined as

$$\mu(E) = \frac{deg(E)}{rk(E)}$$

With this definitions, we can restrict our moduli problem. Instead of considering the space of vector bundles over a fixed Riemann surface X, we will focus on the space of stable vector bundles with fixed rank n and degree d, $A^s_{VB}(X, n, d)$, that is

$$A^s_{VB}(X, n, d) = \{E \to X \text{ stable vector bundle with } rk(E) = n \text{ and } deg(E) = d\}$$

Of course, since the rank and degree is preserved under pullbacks of bundles, we can analogously define the familis as restriction of the previous, that is $F^s_S(X, n, d)$ is the set of stable bundles $E \to X \times S$ such that $rk(E) = n$ and $deg(E) = d$ and the pullback of families is simply the vector bundle pullback.

However, the equivalence relation on $F_S^s(X,n,d)$, $\cong_S$, should be slightly changed. As we shown above, using isomorphism of vector bundles as the relation $\cong_S$ is not useful, since we cannot expect to have a fine moduli space. However, that counterexample give us the correct form of $\cong_S$ that should be used. Indeed, given $E, F \in F_S^s(X,n,d)$, we define $E \cong_S F$ if there exists a line bundle $L \to X$ of degree 0 such that E and $F \otimes \pi^* L$ are isomorphic as vector bundles, i.e. $E \sim F \otimes \pi^* L$. This idea is based on the fact (see [59]) that, in the context of stable bundles, such a line bundle exists if and only if $E|_{X \times \{s\}}$ is isomorphic to $F|_{X \times \{s\}}$ for any $s \in S$.

In that case, we have a satisfactory solution of the problem, as shown in [58] and [59], or in [19] in the context of gauge theories.

Theorem 2.1.20 (Narashimhan-Seshadri)**.** *Let us fix a Riemann surface X, $n \geq 1$ and $d \in \mathbb{Z}$. The moduli space of stable vector bundles of rank n and degree d, $\mathcal{M}^s(X,n,d)$, exists and is a coarse moduli space. Furthermore, in the case of n and d co-primes, $\mathcal{M}^s(X,n,d)$ is smooth and it is a fine moduli space.*

Remark 2.1.21. For n and d not co-primes, the moduli space $\mathcal{M}^s(X,n,d)$ is not fine, as shown in [62].

2.1.2 Moduli Space of Higgs Bundles

Once studied the space of algebraic vector bundles on a fixed algebraic variety, we can enrich the moduli problem using Higgs fields.

Let us suppose that X is smooth complex variety, let G be a complex Lie group and let $H \subseteq G$ a maximal compact subgroup of G. From an algebraic point of view, given a G-holomorphic vector bundle $E \to X$, a G-**Higgs field** is a $\Phi \in H^0(X, \mathfrak{g}_E \otimes K_X)$ such that

$$\Phi \wedge \Phi = 0$$

In that case, (E, Φ) is called a **Higgs bundle**.

Remark 2.1.22. Over a compact Riemann surface, the condition $\Phi \wedge \Phi = 0$ always holds, so a G-Higgs field is just an element $\Phi \in H^0(X, \mathfrak{g}_E \otimes K_X)$.

Indeed, we can translate the stability condition from the case of holomorphic vector bundles to the more general setting of Higgs bundles.

Definition 2.1.23. Let X be a smooth algebraic variety, G be a complex Lie group and let (E, Φ) be a G-Higgs bundle on X. We say that E is **stable** if, for any subbundle $F \subseteq E \to X$ that is Φ-invariant (i.e. $\Phi(F) \subseteq F \otimes K_X$) we have

$$\mu(F) = \frac{deg(F)}{rk(F)} < \frac{deg(E)}{rk(E)} = \mu(E)$$

and it is called **semi-stable** if, for any subbundle $F \subseteq E \to X$ Φ-invariant we have

$$\mu(F) = \frac{deg(F)}{rk(F)} \leq \frac{deg(E)}{rk(E)} = \mu(E)$$

Finally, a G-Higgs bundles is called **polystable** if there exists a direct sum decomposition

$$(E, \Phi) = \bigoplus (E_i, \Phi_i)$$

with each $\mu(E_i) = \mu(E)$ and (E_i, Φ_i) are stables.

Remark 2.1.24. Let us fix a G-holomorphic vector bundle $E \to M$ with M a compact Riemann surface and $\Phi \in H^0(\mathfrak{g}_E \otimes K_M)$. In order to recover a Higgs bundle, in the gauge-theoretical sense of section 1.6.2, we have to find a connection A on E, compatible with the holomorphic structure, such that

$$F_A + [\Phi, \Phi^*] = 0$$

However, it is a general fact that we can always find such a A. The following theorem was first proven by Hitchin in [35] for the case $G = SO(3)$, and later by [67].

Theorem 2.1.25. *Let X be a Riemann surface of genus $g \geq 2$ and let (E, Φ) be a polystable G-Higgs bundle. Let us take any G-connection A_0 compatible with the holomorphic structure. Then, there exists an automorphism f of E, unique modulo H-gauge transformation, such that $(A, \Phi') = f \cdot (A, \Phi)$ satisfies the Hitchin self-duality equations*

$$F_A + [\Phi', \Phi'^*] = 0$$

Therefore, using this correspondece, we have that polystable G-Higgs bundles corresponds with our gauge-theoretical notion of Higgs bundles, as explained in section 1.6.2.

In this setting, we can form the moduli problem of G-Higgs bundles over X. In this case, given a smooth complex variety S, a family of G-Higgs bundles is a G-Higgs bundle $(E, \Phi) \to X \times S$ such that $(E|_{X\times\{s\}}, \Phi|_{X\times\{s\}}) \to X \times \{s\} \cong X$ is a G-Higgs bundle on X for all $s \in S$.

Analogously to the case of vector bundles, we define that two families of G-Higgs bundles parametrized by an smooth algebraic variety S, $(E, \Phi) \to X \times S$ and $(F, \Psi) \to X \times S$, are equivalent under $\cong_S$ if there exists a line bundle $L \to S$ and an isomorphism $f : E \overset{\cong}{\to} F \otimes \pi^* L$ such that the following diagram commutes

$$\begin{tikzcd}
E \arrow[r, "\Phi"] \arrow[d, "f"'] & E \otimes K_X \arrow[d, "f \otimes id_{K_X}"] \\
F \otimes \pi^* L \arrow[r, "\Psi"] & F \otimes \pi^* L \otimes K_X
\end{tikzcd}$$

In this setting of moduli problem, in [35] is proven the following.

Theorem 2.1.26. *Let X be a compact Riemann surface of genus $g \geq 2$ and let $L \to X$ be a fixed line bundle of degree d. The moduli space of polystable $SL(2,\mathbb{C})$-Higgs bundles $(E,\Phi) \to X$ of rank 2 such that $\det E = L$ exists and is a smooth variety of dimension $6(g-1)$. Moreover, the moduli space does not depend on L with fixed degree, so it can be denoted by $\mathcal{M}^d_{Dol}(X, SL(2,\mathbb{C}))$.*

And, in the general case, we have the theorem proven in [70].

Theorem 2.1.27. *Let $G \subseteq GL(n,\mathbb{C})$ be a complex Lie group, let X be a compact Riemann surface and let us fix $d \in \mathbb{Z}$ coprimer with n. The moduli space of polystable G-Higgs bundles $(E,\Phi) \to X$ of degree d exists and it is a smooth manifold. Moreover, the moduli space does not depend on L with fixed degree, so it can be denoted by $\mathcal{M}^d_{Dol}(X, G)$.*

Furthermore, in the case $G = GL(n,\mathbb{C})$, we have that $\mathcal{M}^d_{Dol}(X, GL(n,\mathbb{C}))$ has dimension $n^2(2g-2)+2$. Moreover, defining the map

$$\begin{array}{rccc} \lambda_{Dol}: & \mathcal{M}^d_{Dol}(X, GL(n,\mathbb{C})) & \longrightarrow & \mathcal{M}^d_{Dol}(X, GL(1,\mathbb{C})) \\ & (E,\Phi) & \longmapsto & (det\, E, tr\, \Phi) \end{array}$$

we have that $\mathcal{M}^d_{Dol}(X, SL(n,\mathbb{C})) \cong \lambda^{-1}_{Dol}(L, 0)$ for any line bundle $L \to X$. The dimension of $\mathcal{M}^d_{Dol}(X, SL(n,\mathbb{C}))$ is $2(n^2-1)(g-1)$.

2.1.2.1 Parabolic Higgs bundles

Finally, we need to understand a more general setting, in which we endow the holomorphic vector bundle with an extra structure, known as the parabollic structure. Good references for this setting are [61] and [26].

Definition 2.1.28. Let V be a finite dimensional complex vector bundle. A **parabolic structure** on V is a finite decreasing flag

$$V = V_1 \supseteq V_2 \supseteq \ldots \supseteq V_l = \{0\}$$

together with a set of real numbers, called the **parabolic weights**

$$0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_l < 1$$

We define the **multiplicity** of the parabolic structure on the k-th step by

$$m_k = dim\, V_k - dim\, V_{k+1}$$

Definition 2.1.29. Let X be a compact Riemann surface and let us take s distinct points $p_1, \ldots, p_s \in X$, called the **marked points**, the **parabolic points** or the **punctures**. This points are grouped in a effective Weil divisor $D = p_1 + p_2 + \ldots + p_s$. A **parabolic vector bundle** respect to D is an holomorphic vector bundle $E \to X$ with, for any $p \in D$, a parabolic structure on E_p. If $(\{E_{p,k}\}, \{\alpha_{p,k}\})$

is the parabolic structure on E_p, we define the **parabolic degree** of the bundle $E \to X$, $pardeg(E)$ by

$$pardeg(E) := deg(E) + \sum_{p \in D} \sum_{k=1}^{l_p} m_{p,k} \alpha_{p,k}$$

Definition 2.1.30. Let X be a compact Riemann surface and let us take s distinct points $p_1, \dots, p_s \in X$, called the **marked points**, the **parabolic points** or the **punctures**. This points are grouped in a effective Weil divisor $D = p_1 + p_2 + \ldots + p_s$. A **parabolic vector bundle** respect to D is an holomorphic vector bundle $E \to X$ with, for any $p \in D$, a parabolic structure on E_p. If $(\{E_{p,k}\}, \{\alpha_{p,k}\})$ is the parabolic structure on E_p, we define the **parabolic degree** of the bundle $E \to X$, $pardeg(E)$ by

$$pardeg(E) := deg(E) + \sum_{p \in D} \sum_{k=1}^{l_p} m_{p,k} \alpha_{p,k}$$

In this setting, the condition of Higgs bundles translates as follows.

Definition 2.1.31. Let G be a complex Lie group, let X be a compact Riemann surface and let us an effective Weil divisor $D = p_1 + p_2 + \ldots + p_s$. Let us denoted $K_X(D)$ the twisted line bundle via the divisor D. Given a parabolic G-vector bundle $E \to X$, a morphism $\Phi \in H^0(X, \mathfrak{g}_E \otimes K_X(D))$ is called a **Higgs field** if Φ is **parabolic**, that is, if it preserves the parabolic structure on the parabolic points, that is

$$\Phi_p(E_{p,k}) \subseteq E_{p,k} \otimes K_X(D)_p$$

for all $p \in D$. Analogously, Φ is called **strongly parabolic** if

$$\Phi_p(E_{p,k}) \subseteq E_{p,k+1} \otimes K_X(D)_p$$

In that case, (E, Φ) is called a **parabolic G-Higgs bundle**.

Finally, the generalized notion of stability is the following.

Definition 2.1.32. Let G be a complex Lie group, let X be a compact Riemann surface and let us take an effective Weil divisor $D = p_1+p_2+\ldots+p_s$. Given a parabolic G-Higgs bundle $(E, \Phi) \to X$, is is called **stable** if, for any parabolic subbundle $F \subseteq E \to X$ that is Φ-invariant (i.e. $\Phi(F) \subseteq F \otimes K_X(D)$) we have

$$\mu_p(F) = \frac{pardeg(F)}{rk(F)} < \frac{pardeg(E)}{rk(E)} = \mu_p(E)$$

and it is called **semi-stable** if, for any parabolic subbundle $F \subseteq E \to X$ Φ-invariant we have

$$\mu_p(F) = \frac{pardeg(F)}{rk(F)} \leq \frac{pardeg(E)}{rk(E)} = \mu_p(E)$$

Finally, a parabolic G-Higgs bundle is called **polystable** if there exists a direct sum decomposition

$$(E, \Phi) = \bigoplus (E_i, \Phi_i)$$

on parabolic G-Higgs bundles, such that $\mu_p(E_i) = \mu_p(E)$ and (E_i, Φ_i) are stables.

In this case, we can also state a moduli problem for parabolic Higgs bundles, obtaining the following result, proven in [77].

Theorem 2.1.33. *Let $G \subseteq GL(n, \mathbb{C})$ be a complex Lie group, let X be a compact Riemann surface, let us take an effective Weil divisor $D = p_1 + p_2 + \ldots + p_s$, a parabolic systiem of weights α on the parabolic points and let us fix $d \in \mathbb{Z}$. The moduli space of polystable parabolic G-Higgs bundles $(E, \Phi) \to X$ of degree d and parabolic weights α, $\mathcal{M}_{Dol}^{d,\alpha}(X, G)$ exists and it is normal, quasi-projective variety. For $G = GL(n, \mathbb{C})$, the dimension of $\mathcal{M}_{Dol}^{d,\alpha}(X, GL(n, \mathbb{C}))$ is $(2g - 2 + s)n^2 + 1$*

Remark 2.1.34. For the case of degree $d = 0$ (i.e. topologically trivial bundles) we will simply use $\mathcal{M}_{Dol}(X, G)$ to denote the moduli space of polystable G-Higgs bundles $(E, \Phi) \to X$ of degree 0 and, in the parabolic case $\mathcal{M}_{Dol}^{\alpha}(X, G)$ denote the moduli space of polystable parabolic G-Higgs bundles $(E, \Phi) \to X$ of degree 0 and parabolic weights α.

2.2 Character Varieties

2.2.1 Representations of Algebraic Groups

Let G be a complex algebraic group and let Γ be a finitely generated group. A group homomorphism $\rho : \Gamma \to G$ is called a **representation** of Γ into G. In that case, ρ is called **irreducible** if $\rho(\Gamma)$ is not contained in any proper parabolic subgroup of G and ρ is called **completely reducible** or **semi-simple** if for every parabolic subgroup $P \leq G$ such that $\rho(\Gamma) \subseteq P$ then $\rho(\Gamma) \subseteq L \subseteq P$ for L the Levi subgroup of P.

Let us explain briefly the notions appearing in the reducibility conditions. For general references, see [9] and [36]. First of all, recall that a group G is called **solvable** if there exists a decomposition series of subgroups, each one normal in the next

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_n = G$$

such that every factor group G_k/G_{k+1} is abelian. Remember that, in Galois theory, this concept captures the idea of *solvability by radicals* by means of the Galois group.

Then, given an algebraic group G, a subgroup $B \leq G$ is called a **Borel subgroup** if B is a maximal Zariski closed solvable connected algebraic subgroup of G. For example, in the case $G = GL(n, \mathbb{C})$, a Borel subgroup is the subgroup of invertible upper triangular matrices (and it can be proved that every other Borel subgroup is conjugated to it).

With this notion, a **parabolic subgroup** of G is a algebraic subgroup $P \leq G$ such that there exists a Borel subgroup $B \leq G$ with $B \leq P \leq G$. It can be proved that, for G affine, a closed algebraic subgroup $P \leq G$ is parabolic if and only if the quotient G/P is a projective variety. In this sense, Borel subgroups are minimal parabolic subgroups. Finally, it can be proved that, again for G affine, every parabolic subgroup $P \leq G$ admits a semi-direct decomposition, called a Levi decomposition of P, as $P = R \rtimes L$, where R is the unipotent radical of P and L is a closed reductive[1] group, which is called the **Levi subgroup** of P.

In the case of $G = GL(V)$ for some finite dimensional complex vector space V (the important one for our purposes) it can be proved that, since the Borel subgroup of upper triangular invertible matrices is the unique Borel subgroup up to conjugation, then the parabolic subgroups of $GL(V)$ are the subgroups that preserve flags. More precisely, given a closed algebraic subgroup $P \leq GL(V)$, P is parabolic if there exists a flag

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_r = V$$

such that all the elements of P are exactly the automorphism of V that preserve this fixed flag. Observe that, in particular, the Borel subgroup of $GL(V)$ of upper triangular invertible matrices corresponds to the parabolic subgroup of automorphism that preserves a full flag of 1-dimensional steps, which is minimal among them.

Analogously, from this description we have that the maximal proper parabolic subgroups $M \leq GL(V)$ are exactly the subgroups of automorphism for which there exists a proper subspace $0 \subset W \subset V$ preserved by M (i.e. $f(W) \subseteq W$ for all $f \in M$). Hence, a subgroup $H \leq GL(V)$ is not contained in any parabolic subgroup of $GL(V)$ is H has no proper invariant subspaces.

In particular, a representation $\rho : \Gamma \to GL(V)$, or in general $\rho : \Gamma \to G$ with G a linear group, is irreducible if $\rho(\Gamma)$ has no proper invariant subspaces. Therefore, for linear representations $\rho : \Gamma \to G \subseteq GL(V)$, the concept of irreducibility correspond to the usual one used in representation theory, that is V is a simple Γ-module via ρ. In the same spirit, it can be proved that $\rho : \Gamma \to G \subseteq GL(V)$ is semi-simple if and only if there exists a decomposition $V = \bigoplus_i V_i$ with V_i invariant under $\rho(\Gamma)$ such that $\rho|_{V_i} : \Gamma \to GL(V_i)$ is irreducible. In this case, V itself is called a semi-simple Γ-module.

2.2.2 Representation Varieties

Let Γ be a discrete finitely generated group and let G be a complex algebraic group. Let $Hom(\Gamma, G)$ be the group of representations of Γ into G. Let us choose a finite set of generators of Γ, $S = \{\gamma_1, \ldots, \gamma_N\}$ so $\Gamma = \langle S \rangle$. Recall that any representation is uniquely determined by the image of the set of generators S. Hence, we can give to $Hom(\Gamma, G)$ the structure of an algebraic variety, known as the **representation variety**, via the injection

$$\begin{array}{cccc} \varphi_S & Hom(\Gamma, G) & \longrightarrow & G^N \\ & \rho & \longmapsto & (\rho(\gamma_1), \ldots, \rho(\gamma_N)) \end{array}$$

[1]This notion will be explained later.

Indeed, if S has relations $\{R_\alpha\}_{\alpha\in\Lambda} \subseteq \mathbb{N}[x_1, x_1^{-1}, \dots, x_N, x_N^{-1}]$, (i.e. if $\Gamma = Free(S)/\langle R_\alpha(\gamma_1, \dots, \gamma_N)\rangle_{\alpha\in\Lambda}$) then, we have that

$$\varphi_S(Hom(\Gamma, G)) = \left\{(g_1, \dots, g_N) \in G^N \mid \ R_\alpha(g_1, \dots, g_N) = 1, \forall \alpha \in \Lambda\right\}$$

which is an algebraic subvariety of G^N and, thus, an algebraic variety itself. Moreover, observe that, by the Hilbert's basis theorem, G^N is a nöetherian space. Hence, the, possible infinity, set of equations $R_\alpha(g_1, \dots, g_N) = 1$ for $\alpha \in \Lambda$ reduces to

$$\varphi_S(Hom(\Gamma, G)) = \left\{ (g_1, \dots, g_N) \in G^N \ \middle| \ \begin{array}{rcl} R_{\alpha_1}(g_1, \dots, g_N) & = & 1 \\ & \vdots & \\ R_{\alpha_r}(g_1, \dots, g_N) & = & 1 \end{array} \right\}$$

for some $\alpha_1, \dots, \alpha_r \in \Lambda$.

Example 2.2.1. If $\Gamma = Free(\gamma_1, \dots, \gamma_N)$ is the free group in N generators, then every N-tuple $(g_1, \dots, g_N)$ determines a representation via $\rho(\gamma_k) = g_k$ for $k = 1, \dots, N$, since there is no relations between the γ_k. In that case, we have that $Hom(\Gamma, G) \cong G^N$, inheriting its algebraic structure.

Example 2.2.2. Let us take the group $\Gamma = \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\rangle$ for some $g \geq 1$ with an unique relation

$$R(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g) = \prod_{k=1}^{g} [\alpha_k, \beta_k] = 1$$

then, by the previous digression, we have that

$$Hom(\Gamma, G) \cong \left\{(A_1, \dots, A_g, B_1, \dots, B_g) \in G^{2g} \mid \prod_{k=1}^{g} [A_k, B_k] = 1\right\}$$

This example will be of crucial importance in the following sections, since we are going to study character varieties with Γ of this form.

Proposition 2.2.3. *The algebraic structure given to $Hom(\Gamma, G)$ does not depend on the choosen set of generators.*

Proof. Let $S' = \{\gamma_1', \dots, \gamma_M'\}$ be another sets of generators of Γ, we will prove that the map $\varphi_{S'} \circ \varphi_S^{-1}$: $\varphi_S(Hom(\Gamma, G)) \subseteq G^N \to \varphi_S'(Hom(\Gamma, G)) \subseteq G^M$ is a biregular mapping, so both varieties will be isomorphic as algebraic varieties. By symmetry, it is enough to chech that $\varphi_{S'} \circ \varphi_S^{-1}$ is regular.

Since S is a set of generators, for all $k = 1, \dots, M$, there exists monic monomials $p_k \in \mathbb{Z}[x_1, \dots, x_N]$ such that

$$\gamma_k' = p_k(\gamma_1, \dots, \gamma_N)$$

Let $f : U \to \mathbb{C}$ be a regular function on an open set (in the Zariski topology) $U \subseteq G^M$. Then, we have that, for all representation $\rho : \Gamma \to G$

$$\begin{aligned} f \circ \left(\varphi_{S'} \circ \varphi_S^{-1}\right)(\varphi_S(\rho)) &= f(\varphi_{S'}(\rho)) = f\left(\rho(\gamma'_k)\right)_k = f\left(\rho(p_k(\gamma_i)_i)\right)_k \\ &= f\left(p_k(\rho(\gamma_i)_i)\right)_k = f(p_1(\varphi_S(\rho)), \dots, p_M(\varphi_S(\rho))) \end{aligned}$$

which is a regular function, since the product in G is a regular function. Therefore, $f \circ \left(\varphi_{S'} \circ \varphi_S^{-1}\right)$ is a regular function for every regular function f, so $\varphi_{S'} \circ \varphi_S^{-1}$ is regular, as we wanted to prove. ■

2.2.3 The Conjugation Action

A very important action that will have to consider in our space of representations is the following.

Definition 2.2.4. Let Γ be a discrete finitely generated group, let G be a complex algebraic group and let $Hom(\Gamma, G)$ be space of representations of Γ into G. We have that G acts on the right by conjugation on $Hom(\Gamma, G)$ by

$$g \cdot \rho(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G, \rho \in Hom(\Gamma, G)$ and $\gamma \in \Gamma$.

Remark 2.2.5. Writing down coordinates, it can be seen that the conjugation action of G on $Hom(\Gamma, G)$ is an algebraic action, that is, the induced map $G \times Hom(\Gamma, G) \to Hom(\Gamma, G)$ is a regular map.

Remark 2.2.6. For linear groups, the conjugation has a very important geometric meaning. Suppose that G is a linear group, so the representations are group homomorphisms $\rho : \Gamma \to GL(V)$ for some complex finite dimensional vector space V. Then V becomes a Γ-module via ρ by $\gamma \cdot v := \rho(\gamma)(v)$. Hence, given two representations of Γ, $\rho_1 : \Gamma \to GL(V)$ and $\rho_2 : \Gamma \to GL(W)$, a map $f : V \to W$ is called **Γ-equivariant** or **intertwining** if $f(\gamma \cdot_1 v) = \gamma \cdot_2 f(v)$ for $v \in V$, $\gamma \in \Gamma$ and the action given by the respective representations.

In this context, two representations $\rho_1 : \Gamma \to GL(V)$ and $\rho_2 : \Gamma \to GL(V)$ are called **isomorphic** if there exists a Γ-equivariant linear isomorphism $f : V \to V$. In this case, this is equivalent to have, for all $\gamma \in \Gamma$ and $v \in V$

$$f \circ \rho_1(\gamma)(v) = f(\rho_1(\gamma)(v)) = f(\gamma \cdot_1 v) = \gamma \cdot_2 f(v) = \rho_2(\gamma)(f(v)) = \rho_2(\gamma) \circ f(v)$$

so, for all $\gamma \in \Gamma$

$$\rho_2(\gamma) = f \circ \rho_1(\gamma) \circ f^{-1} = f \cdot \rho_1(\gamma)$$

seen $f \in GL(V)$. Hence, for linear groups, two representations are isomorphic if and only if they are conjugated.

Therefore, by the previous remark, if we want to study the space of representations of some finitely generated group Γ into an algebraic complex group G (that we can think as linear), we have to kill the redundacy induced by isomorphic representations or, equivalently, by conjugations. Hence, our first candidate would be the quotient

$$Hom(\Gamma, G)/G$$

However, in general this quotient will not be an algebraic variety nor a complex manifold. Briefly, this happens because some of the orbits of the action of G are too much closed such that, in the quotient topology, they are topologically pasted together despite they are different points. More precisely, there are two orbits such that the open neighbourhoods of one always contains the other, violating the T_1 separation axiom, required in any complex manifold or algebraic variety.

2.2.4 A Brief about Geometric Invariant Theory

The solution to this problem of bad-behaved quotients is studied by a powerful algebraic technique known as **Geometric Invariant Theory** (or GIT abbreviated). The idea is to detect this kind of phenomena and make them collapse. For this, GIT uses invariant functions under the action of G, since, if two orbits are too close, then the G-invariant functions will not see any difference between them and, automatically, they identify them.

Example 2.2.7. Let k be any field (for our purposes, the important case is $k = \mathbb{C}$, but this is irrelevant for this example). Let us take the (affine) variety $X = k^2$ and let us define the action of $G = k^*$ on X by $\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y)$ for $\lambda \in k^*$ and $(x, y) \in X$.

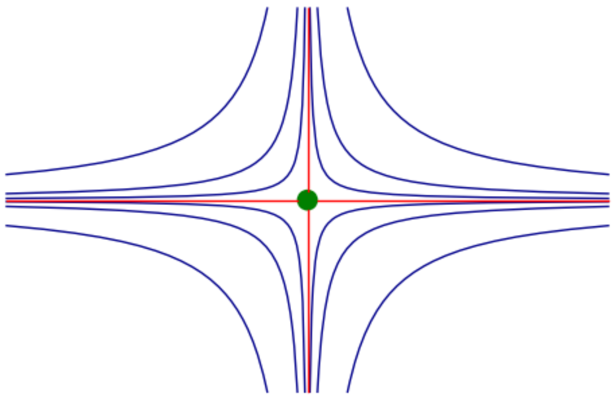

FIGURE 2.1: GIT problem for the action of k^* on k^2.

Observe that, in this case, the orbits are the hyperbolas $H_c := \{xy = c\}$ for all $c \in k^*$, shown in blue in figure 2.1, the orbits $A_x := \{(x, 0) \,|\, x \in k^*\}$ and $A_y := \{(0, y) \,|\, y \in k^*\}$, shown in red, and the point $\{(0, 0)\}$, shown in green. The orbits H_c are Zariski closed sets. However, the axis A_x and A_y are not Zariski closed, and, in its Zariski closure is the point $(0, 0)$. Therefore, in order to have a quotient with good properties, we have to identify this three orbits in just one, in a procedure called the S-equivalence. Therefore, under this identification, the GIT quotient is the orbits H_c plus one more corresponding to the S-equivalence. Hence, the GIT quotient is just a k-line.

We will focus in the case of affine varieties, the one needed for our purposes. Let X be an affine algebraic variety with an algebraic action of an algebraic group G. Using the completation functor $\tau : \mathbf{Var}_{\mathbb{C}} \to \mathbf{Sch}_{\mathbb{C}}$ from the category of complex varieties to the category of schemes of finite type over $\mathbb{C}$ (see [30]), we can complete X to an scheme $\tilde{X} := \tau(X)$. In the particular case of affine varieties, this functor can be easily decribed. Indeed, since X is affine, it is $X = V(I) \subseteq \mathbb{A}^n_{\mathbb{C}}$ for some $n \geq 0$ and some ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$. Then, taking the coordinate ring $A := \mathbb{C}[x_1, \dots, x_n]/I$ (a finitely generated $\mathbb{C}$-algebra) of regular functions on X, we can identify the scheme $\tilde{X} = Spec(A)$.

Now, let us consider the action of G on A by

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

for $f \in A$ (seen as a regular function on X), $g \in G$ and $x \in X$. Let us take the ring of invariants $A^G := \{f \in A \,|\, g \cdot f = f, \forall g \in G\}$.

Definition 2.2.8. Let X be an complex affine algebraic variety with coordinate ring A and let G be a complex algebraic group acting algebraically on X. We define the **GIT quotient** of X by G, denoted by $X /\!\!/ G$ as the affine scheme

$$X /\!\!/ G := Spec(A^G)$$

The digression of when $X /\!\!/ G$ is, in fact, an affine variety is a very deep question. Of course, it can be reduced to the question of when A^G is a finitely generated $\mathbb{C}$-algebra. Indeed, if A^G would be a finitelly generated $\mathbb{C}$-algebra, it will be $A^G \cong \mathbb{C}[x_1, \dots, x_m]/J$ for some $m \geq 0$ and ideal $J \subseteq \mathbb{C}[x_1, \dots, x_m]$. Then, the associated affine variety to the scheme $Spec(A^G)$ (via τ) would be $X /\!\!/ G := V(J) \subseteq \mathbb{A}^m_{\mathbb{C}}$.

The problem of when A^G is a finitely generated $\mathbb{C}$-algebra is, in fact, a version of the well known Hilbert's 14th problem. In general, the answer is no, as Nagata shows in [55] and [56].

However, it can be proved (see [59]) that, if G is a complex reductive group, then A^G is, in fact, a finitely generated $\mathbb{C}$-algebra. So, in this case, we can improve our previous definition.

Definition 2.2.9. Let X be a complex affine algebraic variety with coordinate ring A and let G be a complex reductive algebraic group acting algebraically on X. We define the **GIT quotient** of X by G, as the affine variety $X /\!\!/ G$ whose coordinate ring is A^G.

Remark 2.2.10. Recall that a algebraic group G is reductive if it is a linear algebraic group whose unipotent radical is trivial. Equivalently, if G is complex as in our case, seen as a complex Lie group, G is reductive if and only if its Lie algebra $\mathfrak{g}$ can be decomposed $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$ with $\mathfrak{a}$ abelian (i.e. the Lie bracket is trivial there) and $\mathfrak{h}$ semi-simple (i.e. direct sum of simple Lie algebras, Lie algebras with no proper subalgebras). For example, the classical complex linear groups $GL(n, \mathbb{C}), SL(n, \mathbb{C}), PGL(n, \mathbb{C})$ and $PSL(n, \mathbb{C})$ are reductive.

2.2.4.1 Properties of the GIT quotient

In some sense, the GIT quotient is universal in a categorical framework. In order to make this idea precise, we have to introduce some categorical machinery.

Recall that a category $\mathcal{C}$ is call **locally small** if for every $X, Y \in Obj(\mathcal{C})$, $Hom(X, Y)$ is actually a set (not a proper class). Given two objects $X, Y \in Obj(\mathcal{C})$, a **product** of X, Y, denoted by $X \times Y$ is an object of $\mathcal{C}$ with two morphism $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ such that, given morphisms $Z \overset{f_1}{\to} X$ and $Z \overset{f_2}{\to} Y$, there exists an unique morphism $f = (f_1, f_2) : Z \to X \times Y$ such that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f_1"] \arrow[dr, dashed, "f"] \arrow[ddr, bend right, "f_2"'] & & \\
& X \times Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \\
& Y &
\end{tikzcd}$$

Using an universal-type argument, it can be proven that, if exists, the product is unique up to isomorphism. A category is said to **admit products**, or simply **with products**, if any two objects have a product.

Now, let us take a locally small category $\mathcal{C}$ with products and a terminal object $\star$ (i.e. $\star \in Obj(\mathcal{C})$ and, for every $X \in Obj(\mathcal{C})$ there exists a map $X \to \star$). An object of $\mathcal{C}$, G, is called a **group object** if there exists maps $1 : \star \to G$ (from which we can form the map $1 : G \to \star \to G$), $m : G \times G \to G$ and $\cdot^{-1} : G \to G$ such that the following diagrams commute

$$\begin{tikzcd}
G \times G \times G \arrow[r, "id \times m"] \arrow[d, "m \times id"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "(id,1)"] \arrow[d, "(1,id)"'] \arrow[dr, "id"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "(\cdot^{-1},id)\circ\Delta"] \arrow[d, "(id,\cdot^{-1})\circ\Delta"'] \arrow[dr, "1"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

where $\Delta : G \to G \times G$ is the diagonal morphism.

Remark 2.2.11. Recall that the diagonal morphism is the unique morphism defined by the limiting property of the following diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "id"] \arrow[dr, dashed, "\Delta"] \arrow[ddr, bend right, "id"'] & & \\
& X \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \\
& X &
\end{tikzcd}$$

Analogously, given maps $f : X \to Y$ and $g : X' \to Y'$, we can define a map $f \times g : X \times X' \to Y \times Y'$ as the limiting map

$$\begin{tikzcd}
X \times X' \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, dashed, "f \times g"] & X \arrow[dr, bend left, "f"] & \\
X' \arrow[dr, bend right, "g"'] & Y \times Y' \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \\
& Y' &
\end{tikzcd}$$

Finally, let us take $\mathcal{C}$ a locally small category with products and a terminal object. Let us take $G \in Obj(\mathcal{C})$ a group object of $\mathcal{C}$ and $X \in Obj(\mathcal{C})$. An **action** (in the categorical sense) of G in X is a morphism $\rho : G \times X \to X$.

Definition 2.2.12. Let $\mathcal{C}$ be a locally small category with products and a terminal object. Let us take $G \in Obj(\mathcal{C})$ a group object of $\mathcal{C}$, an object $X \in Obj(\mathcal{C})$ and an action $\rho : G \times X \to X$. A **categorical quotient** of X by G is a object $Y \in Obj(\mathcal{C})$ together with a morphism $\pi : X \to Y$ such that:

- π is G-invariant: That is, $\pi \circ \rho = \pi \circ \pi_2$, where $\pi_2 : G \times X \to X$ is the second projection.
- Y is universal: In the sense that, given any G-invariant morphism $f : X \to Z$ for some $Z \in Obj(\mathcal{C})$, there exists an unique morphism $\tilde{f} : Y \to Z$ such that $\tilde{f} \circ \pi = f$

$$\begin{tikzcd}
& X \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\
Y \arrow[rr, dashed, "\tilde{f}"'] & & Z
\end{tikzcd}$$

Remark 2.2.13. Using the usual universal-type argument, it can be proven that the categorical quotient, if exists, is unique up to isomorphism.

Definition 2.2.14. In the category of algebraic varieties with regular morphisms, a categorical quotient $\pi : X \to Y$ under the algebraic action of an algebraic group G is called a **orbit space** if for every $y \in Y$, $\pi^{-1}(y)$ is a single orbit of G.

Definition 2.2.15. Let X be a variety and let G be a algebraic group acting algebraically on X. A **good quotient** of X by G is a variety Y with a regular morphism $\pi : X \to Y$ such that

- π is a quotient map, that is, π is surjective and every $U \subseteq Y$ is open if and only if $\pi^{-1}(U) \subseteq X$ is open.
- π is affine, that is, if $U \subseteq Y$ is an affine open set, then $\pi^{-1}(U) \subseteq X$ is affine.
- π is G-invariant.

- For every open set $U \subseteq Y$ the induced map

$$\pi^* : A(U) \to A(\pi^{-1}U)^G$$

is an isomorphism.

- If $C \subseteq X$ is closed and G-invariant, then $\pi(C) \subseteq Y$ is closed.

- If $C_1, C_2 \subseteq X$ are closed and G-invariant with $C_1 \cap C_2 = \emptyset$, then $\pi(C_1) \cap \pi(C_2) = \emptyset$.

Maybe the most important consequence of this definition is that, in the affine case, a good quotient is also a categorical quotient. The proof of this statement can be found in [59].

Theorem 2.2.16. *Let X be an affine variety and let $\pi : X \to Y$ be a good quotient of X by a algebraic group G. Then, Y is a categorical quotient by G in the category of algebraic varieties and regular maps. Moreover, for every open set $U \subseteq Y$, U is a categorical quotient of $\pi^{-1}(U) \subseteq X$ by G.*

In fact, in a good quotient, some other important consequences follows easily from its properties. First of all, automatically, we have some improvements of the last property of good quotients.

Proposition 2.2.17. *Let $\pi : X \to Y$ be a good quotient by a algebraic group G. If $x, y \in X$ satisfy $\pi(x) = \pi(y)$ then $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$.*

Proof. Let us take $C_1 := \overline{G \cdot x}$ and $C_2 = \overline{G \cdot y}$ the closed G-invariant subsets of X. Then, if $C_1 \cap C_2 = \emptyset$, then, by the last property of good quotients, $\pi(C_1) \cap \pi(C_2) = \emptyset$. But $\pi(x) \in \pi(C_1)$ and $\pi(y) \in \pi(C_2)$ so it must be $\pi(x) \neq \pi(y)$. ∎

Proposition 2.2.18. *Let $\pi : X \to Y$ be a good quotient for an action of an algebraic group G on X, and let us fix an open set $U \subseteq Y$. If $C_1, C_2 \subseteq \pi^{-1}(U)$ are closed in $\pi^{-1}(U)$ and G-invariant with $C_1 \cap C_2 = \emptyset$, then $\pi(C_1) \cap \pi(C_2) = \emptyset$.*

Proof. Let us take $\overline{C}_1, \overline{C}_2 \subseteq X$ be the closures of C_1 and C_2 in X. Suppose that there exists $x \in \pi(C_1) \cap \pi(C_2)$. Then, $Z := \pi^{-1}(x) \cap \overline{C}_1 \subseteq X$ is a closed G-invariant set of X. Hence, together with $\overline{C}_2$, by the last property of a good quotient, since $\pi(Z) \cap \pi(\overline{C}_2) \neq \emptyset$, then $Z \cap \overline{C}_2 = \pi^{-1}(x) \cap \overline{C}_1 \cap \overline{C}_2 \neq \emptyset$. But this is impossible, because the superset

$$\pi^{-1}(U) \cap \overline{C}_1 \cap \overline{C}_2 = C_1 \cap C_2 = \emptyset$$

since C_1 and C_2 are closed in $\pi^{-1}(U)$. ∎

Corollary 2.2.19. *Let X be an affine variety and let $\pi : X \to Y$ be a good quotient for some algebric action of some algebraic group G. If $U \subseteq Y$ is an open set satisfying that $\pi^{-1}(U) \subseteq X$ is G-invariant with closed orbits on $\pi^{-1}(U)$, then U is an orbit space and it is homeomorphic to $\pi^{-1}(U)/G$.*

Proof. First of all, since we known (prop 2.2.16) that in the affine case, any good quotient is categorical, we have to prove that, if $y \in U$ then $\pi^{-1}(y) \subseteq \pi^{-1}(U)$ is exactly a G-orbit. Of course, since π is G-invariant, $\pi^{-1}(y)$ contains complete orbits of G so it is enought to prove that if $x, y \in \pi^{-1}(U)$ are not in the same orbit, then $\pi(x) \neq \pi(y)$. To this end, let us take $C_1 = G \cdot x$ and $C_2 = G \cdot y$. Then, if $\pi(x) = \pi(y)$, we have that

$$\pi(C_1) = \pi(G \cdot x) = \pi(x) = \pi(y) = \pi(G \cdot y) = \pi(C_2)$$

Hence, in particular, $\pi(C_1) \cap \pi(C_2) \neq \emptyset$ so, by the previous proposition 2.2.18, $C_1 = C_2$. Therefore, $G \cdot x = G \cdot y$ so x and y are in the same orbit, as we wanted to show.

Finally, for the homeomorphism $U \cong \pi^{-1}(U)/G$, observe that the surjective map $\pi : \pi^{-1}(U) \to U$ is G-invariant, so it descends to a surjective map $\tilde{\pi} : \pi^{-1}(U)/G \to U$. Moreover, by the properties of the quotient topology, $\tilde{\pi}$ is open and continuous, so it is enough to prove that $\tilde{\pi} : \pi^{-1}(U)/G \to U$ is injective. But this is exactly the previous checking, so the proof is finished. ■

Remark 2.2.20. Suppose that X is an affine algebraic variety with an algebraic action of an algebraic group G on it. Suppose that there exists a good quotient for the action of G on X. Then, if all the orbits of the action of G are closed, then, by the previous corolary, X/G (with the quotient topology) is homeomorphic to an algebraic variety, unique up to isomorphism. Identifying this spaces, we will say that we have endow X/G with the structure of an algebraic variety. This type of identifications will be intensively used in the computation of chapter 4.

The most important result of GIT that we will use is the existence of good quotients for affine varieties. Of course, this good quotient is, in fact, the GIT quotient. See [59] for further references.

Theorem 2.2.21. *Let X be a complex affine reductive variety and let G be a complex reductive algebraic group acting on X algebraically. The GIT quotient $X \xrightarrow{\pi} X /\!\!/ G$ is a good quotient and, in particular, a categorical quotient.*

Remark 2.2.22. Continuing with this type of descriptions of quotients, if X is a variety with an algebraic action of an algebraic group G whose orbit space X/G is a variety, X/G is called a **geometric quotient** if X/G is also a good quotient. We will not need this fact anywhere in this work.

The case of general projective varieties is so rather more dificult and requires the analysis of some special points in the variety that *behaves* well under the action of G, known as stable and semi-stable points. In this particular sets, the GIT behaves well and can be applied as in the affine case. For a complete introduction to this fascinating area, see, for example [59].

2.2.5 Character Varieties via GIT Quotients

With this notion of GIT quotient, we can finally define what is a character variety. However, to this end, we need to restrict our attention to algebraic groups G which are reductive, in order to obtain a well behaved GIT quotient, as explained before. Observe that, in this case, taking G as a complex reductive group, G is affine, so for any finitely generated group Γ, $Hom(\Gamma, G)$ is an affine variety and the previous Geometric Invariant Theory can be applied.

Definition 2.2.23. Let Γ be a finitely generated group and let G be a complex reductive algebraic group. Let $Hom(\Gamma, G)$ be space of representations of Γ into G with its structure of affine algebraic variety. Then, the **character variety** of Γ into G, $R_G(\Gamma)$ is the GIT quotient

$$R_G(\Gamma) := Hom(\Gamma, G) /\!\!/ G$$

where G acts on $Hom(\Gamma, G)$ by conjugation.

Remark 2.2.24. Let us take $G = GL(V)$ for some complex vector space V and let us consider a representation $\rho : \Gamma \to GL(V)$. The **character** associated to ρ is the homomorphism $\chi_\rho : \Gamma \to \mathbb{C}$ given by $\chi_\rho(g) = tr(\rho(g))$, the trace of the induced map.

Of course, since the trace is invariant under change of basis, we have that, if $\rho_1, \rho_2 : \Gamma \to GL(V)$ are isomorphic representations (or equivalently, conjugated) then $\chi_{\rho_1} = \chi_{\rho_2}$. However, if we restric our attention to irreducible representations, then the reciprocal is also true, that is, two representations are isomorphic if and only if they have the same character. Therefore, if $Hom_0(\Gamma, GL(V))$ denotes the set of irreducible representations of Γ, then we have that the space of characters can be identified with the quotient

$$Hom_0(\Gamma, G)/G$$

Hence, in this sense, the character variety $R_G(\Gamma) = Hom(\Gamma, G) /\!\!/ G$ can be seen as a extension of the space of characters, becoming a algebraic variety, which justifies its name.

Definition 2.2.25. Let X be manifold with finitely generated fundamental group $\pi_1(X)$ and let G be a complex reductive algebraic group. The G**-character variety** of X, $R_G(X)$ is the algebraic variety

$$R_G(X) := R_G(\pi_1(X)) = Hom(\pi_1(X), G) /\!\!/ G$$

Remark 2.2.26. Every compact manifold has a finitely generated fundamental group and, moreover, every compact manifold with a finite number of removed points has a finitely generated fundamental group. In particular, we can take X to be a compact Riemann surface, or a compact Riemann surface with a finite number of removed points (called the **punctures**, the **parabolic points** or the **marked points**).

Remark 2.2.27. Since the compact orientable surfaces are topologically clasified in terms of its genus, the G-character variety only depends on the genus of X. Hence, if X is a compact Riemann surface of genus $g \geq 0$, then X is homeomorphic to Σ_g, the orientable surface of genus g, so its fundamental group is

$$\pi_1(\Sigma_g) = \left\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \mid \prod_{k=1}^{g} [\alpha_k, \beta_k] = 1 \right\rangle$$

Therefore, using this presentation, we have that the G-character variety is

$$R_G(\Sigma_g) = \left\{ (A_1, \dots, A_g, B_1, \dots, B_g) \in G^{2g} \mid \prod_{k=1}^{g} [A_k, B_k] = 1 \right\} /\!\!/ \, G$$

Usually, due to its relation with other moduli spaces, it is denoted $\mathcal{M}_B^g(G) = \mathcal{M}_B(\Sigma_g, G) := R_G(\Sigma_g)$. When $G = SL(2,\mathbb{C})$ we will simply write $\mathcal{M}^g$.

Example 2.2.28. Concretely, we will discuss the case of $SL(2,\mathbb{C})$-character variety of an elliptic curve (i.e. a compact Riemann surface of genus $g = 1$). In this case, we will denote $\mathcal{M} = \mathcal{M}^1$, having

$$\mathcal{M} = \left\{ (A, B) \in SL(2,\mathbb{C})^2 \mid [A, B] = Id \right\} /\!\!/ \, SL(2,\mathbb{C})$$

2.2.5.1 Parabolic character varieties

In this work we will study deeply the case of a $SL(2,\mathbb{C})$-character variety of an elliptic curve, that is a smooth complex projective curve of genus 1 or, equivalently, a compact Riemann surface of genus 1, with a finite number of punctures, sometimes called $SL(2,\mathbb{C})$**-parabolic character varieties**. Let us suppose that our elliptic curve X has s punctures, that is, s removed points, so the fundamental group of this surface is

$$\pi_1(X) = \pi_1(\Sigma_1 - \{p_1, \dots, p_s\}) = \left\langle \alpha, \beta, \gamma_1, \dots, \gamma_s \mid [\alpha, \beta] \prod_{i=1}^{s} \gamma_i = 1 \right\rangle$$

so, analogously to the previous example, the $SL(2,\mathbb{C})$-character variety, that we will called $\mathcal{M}_s$ is

$$\mathcal{M}_s = R_{SL(2,\mathbb{C})}(\Sigma_1 - \{p_1, \dots, p_s\}) = \left\{ (A, B, C_1, \dots, C_s) \in SL(2,\mathbb{C})^{2+s} \mid [A, B] \prod_{i=1}^{s} C_i = Id \right\} /\!\!/ SL(2,\mathbb{C})$$

where the action of $SL(2,\mathbb{C})$ is by simultaneous conjugation. In particular, we will focus on the case of only one puncture, that is, the variety

$$\mathcal{M}_1 = \left\{ (A, B, C) \in SL(2,\mathbb{C})^3 \mid [A, B]C = Id \right\} /\!\!/ \, SL(2,\mathbb{C})$$

Remark 2.2.29. Of course, the genus 1 case is only important in order to simplify the computations, so, analogously, we can define the $SL(2,\mathbb{C})$-parabolic character variety of a curve of genus $g \geq 1$ with s punctures

$$\mathcal{M}_s^g := \left\{ (A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_s) \in SL(2,\mathbb{C})^{2g+s} \mid \prod_{k=1}^{g} [A_k, B_k] \prod_{i=1}^{s} C_i = Id \right\} /\!\!/ \, SL(2,\mathbb{C})$$

with $SL(2,\mathbb{C})$ acting by simultaneous conjugation.

Remark 2.2.30. Even more general, for an affine reductive complex group G, the parabolic character variety with s punctures on a compact Riemann surface X is

$$\mathcal{M}_{B,s}(X,G) := \left\{(A_1,\dots,A_g,B_1,\dots,B_g,C_1,\dots,C_s) \in G^{2g+s} \mid \prod_{k=1}^{g}[A_k,B_k]\prod_{i=1}^{s} C_i = Id\right\} /\!\!/ G$$

with G acting by simultaneous conjugation.

However, due to its relations with other moduli spaces, we will have to restrict the possible representations of the loop arround the puncture, γ. In particular, we will fix a conjugacy class on $SL(2,\mathbb{C})$, called it $\mathcal{C} \subseteq SL(2,\mathbb{C})$, and we will only focus on representations $\rho : \pi_1(\Sigma_1 - \{\star\}) \to SL(2,\mathbb{C})$ with $\rho(\gamma) \in \mathcal{C}$. In this setting, it is usually said that the loop arround the puncture has prescribed monodromy. In this case, the $SL(2,\mathbb{C})$-character variety of this special representations will be called $\mathcal{M}_{\mathcal{C}}$, being the space

$$\boxed{\mathcal{M}_{\mathcal{C}} \;\; = \left\{(A,B,C) \in SL(2,\mathbb{C})^3 \;\middle|\; \begin{array}{c}[A,B]C = Id \\ C \in \mathcal{C}\end{array}\right\} /\!\!/ SL(2,\mathbb{C})}$$

This space admits two possible isomorphic (as complex varieties) presentations. First of all, observe that the map

$$\begin{array}{ccc} \{(A,B) \in SL(2,\mathbb{C}) \,|\, [A,B] \in \mathcal{C}\} & \longleftrightarrow & \big\{(A,B,C) \in SL(2,\mathbb{C})^2 \times \mathcal{C}\,[A,B]C = id\big\} \\ (A,B) & \longmapsto & (A,B,[A,B]^{-1}) \end{array}$$

is an algebraic isomorphism. Moreover, since $[PAP^{-1}, PBP^{-1}] = P[A,B]P^{-1}$ for $A,B,P \in SL(2,\mathbb{C})$, this map respects the conjugation, so it descends to the quotient under the action of $SL(2,\mathbb{C})$ by conjugation, inducing an isomorphism of algebraic varieties

$$\boxed{\mathcal{M}_{\mathcal{C}} \cong \big\{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] \in \mathcal{C}\big\} /\!\!/ SL(2,\mathbb{C})}$$

For the other presentation, let us take some element $\xi \in \mathcal{C}$ and define

$$\boxed{\mathcal{M}_{\xi} = \big\{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = \xi\big\} /\!\!/ Stab(\xi)}$$

where $Stab(\xi)$ is the stabilizer of ξ under conjugation on $SL(2,\mathbb{C})$.

Remark 2.2.31. In this case, we should restrict the action of $SL(2,\mathbb{C})$ to the action of $Stab(\xi)$. Indeed, if $P \in SL(2,\mathbb{C})$ and $A,B \in SL(2,\mathbb{C})$ satisfies $[A,B] = \xi$, then $[PAP^{-1}, PBP^{-1}] = \xi$ if and only if $P\xi P^{-1} = \xi$, that is $P \in Stab(\xi)$. Therefore, the action is well-defined only restricting to $Stab(\xi)$.

To see that this space $\mathcal{M}_\xi$ if algebraic isomorphic to $\mathcal{M}_\mathcal{C}$, observe that, the map

$$\begin{array}{cccc} \phi : & \mathcal{M}_\xi & \longleftrightarrow & \mathcal{M}_\mathcal{C} \\ & (A,B)\cdot Stab(\xi) & \longmapsto & (A,B)\cdot SL(2,\mathbb{C}) \end{array}$$

is clearly well defined since $Stab(\xi) \leq SL(2,\mathbb{C})$ and, if $(A,B)\cdot Stab(\xi) \in \mathcal{M}_\xi$ then $[A,B] = \xi$ so $[A,B] \in \mathcal{C}$. Moreover, it is surjective since, if $(A,B)\cdot SL(2,\mathbb{C}) \in \mathcal{M}_\mathcal{C}$, then, there exists $P \in SL(2,\mathbb{C})$ such that $[PAP^{-1}, PBP^{-1}] = P[A,B]P^{-1} = \xi$ so $(A,B)\cdot SL(2,\mathbb{C}) = (PAP^{-1}, PBP^{-1})\cdot SL(2,\mathbb{C})$ and $(PAP^{-1}, PBP^{-1}) \cdot Stab(\xi) \in \mathcal{M}_\xi$ is a contraimage via ϕ.

Finally, for the injectivity, suppose that $(A,B)\cdot Stab(\xi)$ and $(A',B')\cdot Stab(\xi)$ satisfy $\phi((A,B)\cdot Stab(\xi)) = \phi((A',B')\cdot Stab(\xi))$. Then, we have that $(A,B)\cdot SL(2,\mathbb{C}) = (A',B')\cdot SL(2,\mathbb{C})$ so there exists $P \in SL(2,\mathbb{C})$ such that $A' = PAP^{-1}$ and $B' = PBP^{-1}$. In this case, P should satisfies

$$\xi = [A',B'] = [PAP^{-1}, PBP^{-1}] = P[A,B]P^{-1} = P\xi P^{-1}$$

so $\xi \in Stab(\xi)$ and, therefore $(A,B)\cdot Stab(\xi) = (A',B')\cdot Stab(\xi)$.

Remark 2.2.32. This kind of arguments will be extensively used in the computations of chapter 4 without further details. The complete proof of those statements is a straightforward application of these ideas.

Finally, we will also study the case of two puntures on a elliptic curve. Then, the desired $SL(2,\mathbb{C})$-parabolic character variety with prescribed monodromy on conjugacy classes $\mathcal{C}_1, \mathcal{C}_2 \subseteq SL(2,\mathbb{C})$ around the punctures is

$$\boxed{\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2} = \left\{ (A,B,C_1,C_2) \in SL(2,\mathbb{C})^4 \,\middle|\, \begin{array}{c} [A,B]C_1C_2 = Id \\ C_1 \in \mathcal{C}_1, \quad C_2 \in \mathcal{C}_2 \end{array} \right\} /\!\!/ \, SL(2,\mathbb{C})}$$

with $SL(2,\mathbb{C})$ acting by simultaneous conjugation.

2.3 Relations between Moduli Spaces

The relations between the moduli spaces of Higgs bundles, moduli spaces of flat connections and character varieties is a very deep and active area of reseach, known as **non-abelian Hodge theory**. In this section, we will sketch the fundamental points of the theory, maybe in a little non-rigurous way. For a detailed account on his wide subject, please check [69] or [31].

As the name sugests, the starting point of this theory is the following interplay between algebro-geometric objects. Let us fix a compact Riemann surface X and denote $H^*_{DR}(X,\mathbb{C})$ the de Rham cohomology of X with complex coefficients. Since every compact Riemann surface is a compact

Kähler manifold, classical Hodge theory (see theorem 3.1.31) give us a decomposition

$$H^1_{DR}(X,\mathbb{C}) \cong H^{1,0}_{Dol}(X) \oplus H^{0,1}_{Dol}(X) \tag{2.1}$$

where $H^{p,q}_{Dol}(M)$ is the Dolbeault cohomology of X.

Now, observe that, considering Dolbeault cohomology as a sheaf cohomology (see remark A.1.3), if Ω^p denote the sheaf of holomorphic p-forms, we have isomorphisms

$$H^{0,1}_{Dol}(X) \cong H^0(X,\Omega^1) \qquad\qquad H^{1,0}(X) \cong H^1(X,\Omega^0) = H^1(X,\mathcal{O}_X)$$

where we have used that $\Omega^0 = \mathcal{O}_X$, the sheaf of holomorphic funcions on X (also known as the structure sheaf). Finally, observe that

$$H^1(X,\mathcal{O}_X) = Pic(X)$$

the Picard group of X, that is, the group of holomorphic line bundles on X.

Therefore, via this isomorphisms, 2.1 can be reinterpreted as

$$H^1_{DR}(X,\mathbb{C}) \cong Pic(X) \oplus H^0(X,\Omega^1)$$

that is, is the same to have a 1-cohomology class as to have a pair of an algebraic line bundle and a holomorphic 1-form.

On the other hand, if $H_B(X,\mathbb{C})$ is the singular cohomology of X with coefficients in X (also called Betti cohomology), then de Rham theorem give us an isomorphism $H_B(X,\mathbb{C}) \cong H_{DR}(X,\mathbb{C})$. However, by Hurewicz theorem (see [18]) we have that $H_B(X,\mathbb{C}) \cong \frac{\pi_1(X)}{[\pi_1(X),\pi_1(X)]} \otimes_{\mathbb{Z}} \mathbb{C} \cong Hom(\pi_1(X),\mathbb{C})$.

Philosofically, non-abelian Hodge theory translates this abelian framework to the more general setting of moduli spaces. Along all this section, let $G \subseteq GL(n,\mathbb{C})$ be a complex reductive linear group. In the sense of non-abelian Hodge theory, the non-abelian analogues of the previous one are the following:

- De Rham cohomology: Its non-abelian analogous is the moduli space of flat G-connections, $\mathcal{M}_{DR}(X,G)$.
- Dolbeault cohomology: Its non-abelian analogous is the moduli space of G-Higgs bundles, $\mathcal{M}_{Dol}(X,G)$.
- Betti cohomology: Its non-abelian analogous is character variety, $\mathcal{M}_B(X,G)$.

The first step in this non-abelian Hodge theory is to understand the relation between flat G connections and representations $\pi_1(X) \to G$. The key point here is a geometric tool called the **monodromy representation**.

Definition 2.3.1. Let $\Pi_1(X)$ be the fundamental groupoid of X, that is, the category whose objects are the points of X and whose morphism between $x, y \in X$ are the homotopy classes of paths between x and y. For example, if $x = y$, then $Hom_{\Pi_1(X)}(x,x) = \pi_1(X,x)$. A **local system** of R-modules is a functor

$$\mathcal{V} : \Pi_1(X) \to R - \mathbf{Mod}$$

from the fundamental groupoid of X to the category of R-modules.

Given a local system $\mathcal{V}$ of complex vector spaces on X (usually just called a local system), and fixed any $x_0 \in X$, denote $V_{x_0} := \mathcal{V}(x_0)$. Then, we define a group homomorphism $\rho_{\mathcal{V}} : \pi_1(X) \to GL(V_{x_0})$ given, for $[\gamma] \in \pi_1(X) = Hom_{\Pi_1(X)}(x_0, x_0)$

$$\rho([\gamma]) = \mathcal{V}([\gamma]) \in GL(V_{x_0}) = Hom(V_{x_0}, V_{x_0})$$

A local system $\mathcal{V}$ of complex vector spaces is called a G-local system if $\rho_{\mathcal{V}} : \pi_1(X) \to G \subseteq GL(V_{x_0})$.

Proposition 2.3.2. *There is an injective mapping from G-local systems on X, modulo natural equivalence, and representations $\pi_1(X) \to G$ modulo conjugation.*

$$\frac{\{G - \textit{local systems on } X\}}{\sim} \hookrightarrow \frac{\{\pi_1(X) \to G\}}{G}$$

Moreover, given a G-local system of complex vector spaces $\mathcal{V}$, using it, we can define a sheaf on X, $\mathcal{F}_{\mathcal{V}}$, whose stalks are $(\mathcal{F}_{\mathcal{V}})_x = \mathcal{V}(x)$. This sheaf has a very special property

Definition 2.3.3. Let X be a complex manifold and let $\mathcal{F}$ be a sheaf on X. $\mathcal{F}$ is called a **locally constant sheaf** if there exists a covering $\{U_i\} \subseteq X$ such that, for all U_i and $x \in U_i$, the passing-to-stalk morphism $\rho_i : \mathcal{F}(U_i) \to \mathcal{F}_x$ is an isomorphism.

In this case, $\mathcal{F}_{\mathcal{V}}$ is a locally constant sheaf. This is because, since X has a basis of simply-connected open sets, then for every $x, y \in X$ close enough, the unique class of path between x and y give us an isomorphism between $(\mathcal{F}_{\mathcal{V}})_x$ and $(\mathcal{F}_{\mathcal{V}})_y$. Moreover, quotienting by isomorphisms, we obtain the following result.

Proposition 2.3.4. *There is a* 1-1 *correspondece between G-local systems on X, modulo isomorphism, and locally G-constant sheaves on X, modulo sheaf isomorphism.*

Finally, since a locally constant sheaf of modules is automatically locally free (because $\mathcal{F}(U_i)$ is isomorphic to $\mathcal{F}_x$ for any $x \in U_i$) using the relation between locally free sheaves and vector bundles with a flat connection (see [60]), we have the following result.

Proposition 2.3.5. *There is a* 1-1 *correspondece between locally constant sheaves on X, modulo sheaf isomorphism, and flat G-vector bundles, modulo gauge equivalence.*

Therefore, putting together these correspondences, we have a mapping from flat G-vector bundles, modulo gauge equivalence, to representations $\pi_1(X) \to G$ modulo conjugation. This identification which respects the topology, is the so called **Riemann-Hilbert correspondence**, whose proof can be found in [71].

Theorem 2.3.6 (Riemann-Hilbert correspondece)**.** *The moduli spaces of G-flat connections $\mathcal{M}_{DR}(X, G)$ and the character variety $R_G(X) = \mathcal{M}_B(X, G)$ are analytically isomorphic.*

Even more, we can twist this spaces in order to obtain an even more general Riemann-Hilbert correspondence. For this, let us define the twisted character variety

$$\mathcal{M}_B^d(X, G) := \left\{ (A_1, \ldots, A_g, B_1, \ldots, B_g) \in G^{2g} \mid \prod_{k=1}^{g} [A_k, B_k] = e^{\frac{2\pi i d}{n}} Id \right\} /\!\!/ G$$

Theorem 2.3.7 (Riemann-Hilbert correspondece, twisted case)**.** *The moduli spaces of G-flat connections on $X - \{p_0\}$ holonomy $e^{\frac{2\pi i d}{n}} Id$ around p_0, $\mathcal{M}_{DR}^d(X, G)$ and the character variety $\mathcal{M}_B^d(X, G)$ are analytically isomorphic.*

Now, we can also focus our attention to the moduli space of G-Higgs bundles or, more restrictive, of holomorphic vector bundles. In this context, the starting point of the theory was a result of Narasimahan and Seshadri that relates polystable holomorphic vector bundles of rank n and unitary character varieties. In the original proof in [58], they used only algebraic methods to stated the theorem. However, Donaldson, in a later paper [19] gave a new proof of this theorem using gauge-theoretical methods, that iniciate the study of this theorem from the point of view of Higgs bundles.

Theorem 2.3.8 (Narasimhan-Seshadri)**.** *The moduli space of polystable holomorphic vector bundles of rank n degree 0, $\mathcal{M}_{VB}(M, n, 0)$, is homeomorphic to the character variety $R_{U(n)}(X) = \mathcal{M}_B(X, U(n))$. Analogously, for general degree $d \in \mathbb{Z}$ we have that $\mathcal{M}^s(M, n, d)$ is homeomorphic to $\mathcal{M}_B^d(X, U(n))$.*

Hitchin in [35] proved a generalization of this theorem for the case of $SU(2)$-Higgs bundles using a totally different proof based on gauge theory. Later, the combined work of Donaldson, Corlette and Simpson in [20], [14] and [70], among others, proved the following version.

Theorem 2.3.9. *Let $G \subseteq GL(n, \mathbb{C})$ be a reductive Lie group. The moduli space of polystable G-vector bundles n degree 0, $\mathcal{M}_{Dol}(M, G)$ is homeomorphic to the character variety $R_G(X) = \mathcal{M}_B(X, G)$. Analogously, for general degree $d \in \mathbb{Z}$ we have that $\mathcal{M}_{Dol}^d(M, G)$ is homeomorphic to $\mathcal{M}_B^d(X, G)$.*

Therefore, with this result, the relation between moduli spaces is the following, where C^0 indicates continuos isomorphism (that is, homeomorphism) and C^ω analytical isomorphism.

$$\begin{tikzcd}
 & \mathcal{M}_B^d(X, G) \arrow[dl, leftrightarrow, "C^\omega"'] \arrow[dr, leftrightarrow, "C^0"] & \\
\mathcal{M}_{DR}^d(X, G) \arrow[rr, leftrightarrow, "C^0"'] & & \mathcal{M}_{Dol}^d(X, G)
\end{tikzcd}$$

Remark 2.3.10. This three different points of view allow us to introduce a very special structure on this spaces, known as a hyperkähler structure, which consists of three compatible Kähler structures.

Finally, in the context of parabolic G-Higgs bundles, Metha and Seshadri in [53], first, and later Simpson in [68], proved the following non-abelian Hodge correspondence for parabolic Higgs bundles.

Theorem 2.3.11. *Let $G \subseteq GL(n, \mathbb{C})$. Let us choose parabolic points $p_1, \ldots, p_s \in X$, and let us define the effective Weil divisor $D = p_1 + \ldots + p_s$. Let us fix conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_s \subseteq G$ given by semisimple elements. Then, we have*

- *The moduli space of parabolic G-bundles of parabolic degree 0 with parabolic structures α on D $\mathcal{M}^{\alpha}_{Dol}(X, G)$ is homeomorphic to the parabolic character variety of X with s marked points and holonomies in $\mathcal{C}_1, \ldots, \mathcal{C}_s$ around $p_1, \ldots, p_s$, respectively, $\mathcal{M}_{\mathcal{C}_1,\ldots,\mathcal{C}_s}(X, G)$.*
- *The moduli space of flat logarithmic G-bundles with poles in D $\mathcal{M}_{DR,s}(X, G)$ is analytically isomorphic to parabolic character variety of X with s marked points and holonomies in $\mathcal{C}_1, \ldots, \mathcal{C}_s$ around $p_1, \ldots, p_s$, respectively, $\mathcal{M}_{\mathcal{C}_1,\ldots,\mathcal{C}_s}(X, G)$.*

$$\begin{tikzcd} & \mathcal{M}_{\mathcal{C}_1,\ldots,\mathcal{C}_s}(X,G) \arrow[dl, leftrightarrow, "C^{\omega}"'] \arrow[dr, leftrightarrow, "C^0"] & \\ \mathcal{M}_{DR,s}(X,G) \arrow[rr, leftrightarrow, "C^0"'] & & \mathcal{M}^{\alpha}_{Dol}(X,G) \end{tikzcd}$$

Corollary 2.3.12. *Let us take $SL(2, \mathbb{C}) \subseteq GL(2, \mathbb{C})$. Let us choose parabolic points $p_1, p_2 \in X$, and let us fix different conjugacy classes $\mathcal{C}_1, \mathcal{C}_2 \subseteq SL(2, \mathbb{C})$ of semisimple elements. Then, $\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2}(X) := \mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2}(X, SL(2, \mathbb{C}))$ is homeomorphic to the moduli space of traceless parabolic Higgs bundles of rank 2 and parabolic degree 0 with a fixed parabolic structures α, $\mathcal{M}^{\alpha}_{Dol}(X, SL(2, \mathbb{C})$.*

2.3.1 Nahm Transform

In this interplay between moduli spaces, there is a fundamental tool, known as the **Nahm transform** that allow us to relate different types of solutions to the Yang-Mills equations (of which Higgs bundles are a special case). Good references for this topic are [41], [40] and [38].

The first example of the Nahm transform appears in the Atiyah-Drinfeld-Hitchin-Mani construction (usually shortened to ADHM construction) of instantons in $\mathbb{R}^4$, see [3] or [21]. Later, Nahm adapted this method for constructing time-invariant anti self-dual solutions of the Yang-Mills equations, which he called monopoles [57]. This paper, based on physical arguments, was later formalised in a paper of Hitchin [34].

Corrigan, Goddard, Braam and van Baal realized that these constructions are special cases of a more general construction, which they called the Nahm transform. The Nahm trasform, at least in its primary state, is a mechanism that transforms anti self-dual connections on $\mathbb{R}^4$ that are invariant under some subgroup of translations $\Lambda \subseteq \mathbb{R}^4$ into dual instantons on $(\mathbb{R}^4)^*$ which are invariant under

$$\Lambda^* = \left\{\alpha \in (\mathbb{R}^4)^* \,|\, \alpha(\lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda\right\}$$

In this context, some constructions of instantons arise in this way.

- For $\Lambda = \{0\}$, the Nahm transforms is related to the ADHM construction.
- For $\Lambda = \mathbb{R}$, it reduces to the study of monopoles, as studied by Hitchin in [34].
- For $\Lambda = \mathbb{Z}^4$, it defines an hyperkähler isometry on the moduli space of instantons over two dual 4-tori, as explained in [10] and [21].
- For $\Lambda = \mathbb{Z}^2$, the Λ-invariant anti self-dual connections on $\mathbb{R}^4$ are called **doubly periodic instantons**. Therefore, the Nahm transform gives a correspondence between doubly periodic instantons and certain tame solutions of Hitchin's equations on a punctured two-torus. See [40], [39] and [6].
- For $\Lambda = \mathbb{R} \times \mathbb{Z}$, we obtain periodic monopoles, as studied in [13].

Indeed, using the Nahm transform, in [40] is proven the following theorem, that justifies the name of this work.

Theorem 2.3.13. *The moduli space of doubly periodic instantons over an elliptic curve X is diffeomorphic to the moduli space of traceless parabolic Higgs bundles of rank 2 and parabolic degree 0 with a fixed parabolic structures α, $\mathcal{M}_{Dol}^{\alpha}(X, SL(2,\mathbb{C})$.*

Therefore, using corolary 2.3.12

Corollary 2.3.14. *Let X be an elliptic curve with two marked points $p_1, p_2 \in X$. Let us fix distinct conjugacy classes $\mathcal{C}_1, \mathcal{C}_2 \subseteq SL(2,\mathbb{C})$ of semisimple elements. Then, $\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2}(X)$ is homeomorphic to the moduli space of doubly periodic instantons over X.*

Remark 2.3.15. In the notation of chapter 4, the moduli space of doubly periodic instantons over an elliptic curve is homeomorphic to the parabolic character variety $\mathcal{M}_{[D_{\lambda_1}],[D_{\lambda_2}]}$ with $\lambda_1 \neq \lambda_2, \lambda_2^{-1}$, studied in sections 4.3.3.1, 4.3.3.2 and 4.3.4.6.

Chapter 3

Hodge Structures

3.1 Classical Hodge Theory

3.1.1 L^2 Product on Manifolds and the Hodge Star Operator

Let (M, g) be a differentiable compact manifold without boundary, oriented with volume form Ω. Using the inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ for each $p \in M$, we can define a product in $\Omega^k_p(M)$, $g^k_p : \Omega^k_p(M) \times \Omega^k_p(M) \to \mathbb{R}$ for each $p \in M$ given, on elemental forms, by

$$g^k_p(\omega^1_p \wedge \ldots \wedge \omega^k_p, \eta^1_p \wedge \ldots \wedge \eta^k_p) := det\left(g_p({\omega^i_p}^\sharp, {\eta^j_p}^\sharp)\right)$$

where $\cdot^\sharp : T^*_pM \to T_pM$ is the isomorphism with the dual space induced by g_p, i.e. $\omega^\sharp_p$ is the unique vector such that $\omega_p(X) = g_p(\omega^\sharp_p, X)$ for all $X \in T_pM$. Furthermore, we can extend this product to a product $g_p : \Omega^*_p(M) \to \Omega^*_p(M)$, decreeing that $\Omega^{k_1}_p(M)$ is orthogonal to $\Omega^{k_2}_p(M)$ for $k_1 \neq k_2$.

With this pointwise product, we can define a global product in $\Omega^*(M)$, known as the L^2 **product** by

$$\langle \omega, \eta \rangle_{L^2} := \int_M g_p(\omega_p, \eta_p)\, \Omega$$

We can introduce a shorthand for this product defining the **Hodge star operator** $\star : \Omega^k_p(M) \to \Omega^{n-k}_p(M)$, where $n = \dim_{\mathbb{R}} M$. Indeed, given $\eta_p \in \Omega^k_p(M)$, $\star\eta_p$ is the unique $(n-k)$-form such that

$$\omega_p \wedge (\star\eta)_p = g_p(\omega_p, \eta_p)\, \Omega_p$$

Moreover, we can extend this operator to a global operator $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ and, with this operator, we have that

$$\langle \omega, \eta \rangle_{L^2} = \int_M \omega \wedge \star\eta$$

Remark 3.1.1. $\star$ preserves the orthogonality law between form of different degree, that is, if $\omega \in \Omega^{k_1}(M)$ and $\eta \in \Omega^{k_2}(M)$ with $k_1 \neq k_2$, we have that $\omega \wedge \star\eta = 0$.

In fact, there is a very simple way of compute the Hodge star operator in local coordinates, using the following proposition.

Proposition 3.1.2 (Computation of the Hodge Star Operator)**.** *Let (M, g) be a compact oriented riemannian manifold of dimension n and let $p \in M$. Let $\omega_1, \dots, \omega_n$ be a positively oriented orthonormal base of T_p^*M with respect to the induced inner product on 1-forms. Then, over k-forms, the Hodge Star operator can be computed as*

$$\star\left(\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}\right) = sign(\sigma) \cdot \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}}$$

where $\sigma = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & n \\ i_1 & i_2 & \cdots & i_k & j_1 & j_2 & \cdots & j_{n-k} \end{pmatrix}$ *is a permutation of* $\{1, \dots, n\}$*.*

Proof. Observe that, since $\omega_1, \dots, \omega_n$ is a positively oriented orthonormal base, we obtain that it holds $g_p^k\left(\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}, \operatorname{sign}(\sigma) \cdot \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}}\right) = 1$ and $\omega_1 \wedge \cdots \wedge \omega_n = \Omega$. Hence, in this way

$$(\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}) \wedge \left(\operatorname{sign}(\sigma) \cdot \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}}\right) = \operatorname{sign}(\sigma)^2 \omega_1 \wedge \cdots \wedge \omega_n = \Omega$$

Therefore, $\operatorname{sign}(\sigma) \cdot \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}}$ satisfies the property required to be de Hodge Star of $\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$. ∎

Remark 3.1.3. From this characterization for the Hodge Star, is very simply to observe that $\star^{-1} = (-1)^{k(n-k)}\star$, so $\star\star = (-1)^{k(n-k)}$.

Remark 3.1.4. Using the same definition, we can extend the definition of the Hodge Star operator to semi-riemannian manifolds. In this case, the previous proposition is analogous up to a sign that apears when we act on time-like covectors. Therefore, it can be shown that, in this case, we have $\star\star = s(-1)^{k(n-k)}$, where $s = \pm 1$ is the signature of the semi-riemannian metric.

3.1.2 Laplace-Beltrami Operator

Definition 3.1.5. Let (M, g) be a differentiable oriented compact manifold. Let $T : \Omega^*(M) \to \Omega^*(M)$ be a linear operator (not necessarely bounded), we say that $T^* : \Omega^*(M) \to \Omega^*(M)$ is a **formal adjoint** of T over $\Omega^*(M)$ if, for all $\omega, \eta \in \Omega^*(M)$ we have

$$\langle \omega, T\eta \rangle_{L^2} = \langle T^*\omega, \eta \rangle_{L^2}$$

Moreover, if $T^* = T$ we say that T is a **symmetric operator**.

Proposition 3.1.6. *Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior differential over a compact oriented riemannian manifold (M, g). Then, the linear operator $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$ given by*

$$d^* = (-1)^{n(k+1)+1} \star d \star$$

is the formal adjoint of d over $\Omega^*(M)$ with respect to the L^2 inner product.

Proof. Let $\omega, \eta \in \Omega^k(M)$, then, using the distributivity of d and the remark 3.1.3 we have

$$d\,(\eta \wedge \star\omega) = d\eta \wedge \star\omega + (-1)^k \eta \wedge d(\star\omega) = d\eta \wedge \star\omega - \eta \wedge \star(d^*\omega)$$

So, integrating over M and using the Stokes theorem in its boundaryless version

$$0 = \int\limits_{\partial M} \eta \wedge \star\omega = \int\limits_M d\,(\eta \wedge \star\omega) = \int\limits_M d\eta \wedge \star\omega - \int\limits_M \eta \wedge \star(d^*\omega) = \langle d\eta, \omega \rangle_{L^2} - \langle \eta, d^*\omega \rangle_{L^2}$$

as we wanted to show. ■

Definition 3.1.7. Let (M, g) be a compact oriented riemannian manifold with exterior differential $d : \Omega^*(M) \to \Omega^{*+1}(M)$, whose formal adjoint operator, with respect to the L^2 norm, is $d^* : \Omega^*(M) \to \Omega^{*-1}(M)$. The **Laplace-Beltrami operator**, $\Delta : \Omega^k(M) \to \Omega^k(M)$, is given by

$$\Delta = d\,d^* + d^*\,d$$

Moreover, a differential form $\omega \in \Omega^*(M)$ is said **harmonic** if $\Delta\omega = 0$.

Remark 3.1.8. Using the explicit formula for d^* in terms of the Hodge Star, we can rewrite the Laplace-Beltrami operator $\Delta : \Omega^k(M) \to \Omega^k(M)$ in its classic form

$$\Delta = (-1)^{n(k+1)+1} d \star d \star + (-1)^{nk+1} \star d \star d$$

Moreover, with this explicit expresion, its easy to recover the classic laplacian operator for C^∞ functions (i.e. elements of $\Omega^0(M)$) in a flat manifold. Let us suppose that, there exists a local isometry $\varphi : U \to \mathbb{R}^n$ for some open set $U \subset M$[1]. Therefore, using this map, we can take coordinates $(x_1, \ldots, x_n)$ in U, such that $\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p$ is a orthogonal basis of T_pM, for all p in U.

[1]This can be achieved if, for example, the sectional curvature is zero in an open set of M.

Let $f \in C^\infty(U)$ and note that $d^*(f) = 0$, so we have

$$\begin{aligned}
\Delta f = d^* d(f) = - \star d \star d(f) &= - \sum_{i=0}^{n} \star d \star \frac{\partial f}{\partial x_i}\, dx_i \\
&= - \sum_{i=0}^{n} (-1)^i \star d \left(\frac{\partial f}{\partial x_i}\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \\
&= - \sum_{i=0}^{n} (-1)^i \star \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= - \sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i^2} \star (dx_1 \wedge \cdots \wedge dx_n) = - \sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i^2}
\end{aligned}$$

where, in the third line, the sum is cut because the i, j term is not null if and only if $i = j$ (otherwise, it contains two dx_j). Hence, in summary, over $\Omega^0(U)$ we have

$$\Delta = - \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)$$

as usual in analysis (up to sign).

Furthermore, if we use the adjointness of the operators used in the definition of the laplacian, we obtain that, for all $\omega, \eta \in \Omega^*(M)$

$$\langle \Delta\omega, \eta \rangle_{L^2} = \langle (d\, d^* + d^*\, d)\, \omega, \eta \rangle_{L^2} = \langle \omega, (d^*\, d + d\, d^*)\, \eta \rangle_{L^2} = \langle \omega, \Delta\eta \rangle_{L^2}$$

Hence, with this simple computation, we have just prove

Corollary 3.1.9. *The Laplace-Beltrami operator is symmetric with respect to the L^2 product, that is*

$$\langle \Delta\omega, \eta \rangle_{L^2} = \langle \omega, \Delta\eta \rangle_{L^2}$$

for all $\omega, \eta \in \Omega^*(M)$.

Indeed, repeating the computation with the same form, we obtain a characterization of the harmonic forms, which by definition are solutions of a second-order PDE, in terms of a system of first-order PDE.

Corollary 3.1.10. *A differential form $\omega \in \Omega^*(M)$ is harmonic if and only if $d\omega = 0$ and $d^*\omega = 0$.*

Proof.

$$\langle \Delta\omega, \omega \rangle_{L^2} = \langle (d\, d^* + d^*\, d)\, \omega, \eta \rangle_{L^2} = \langle d\omega, d\omega \rangle_{L^2} + \langle d^*\omega, d^*\omega \rangle_{L^2}$$

Hence, cause the inner product is positive defined, $\Delta\omega = 0$ if and only if $d\omega = 0$ and $d^*\omega = 0$. ■

3.1.3 Hodge Decomposition Theorem

The most important result in classical Hodge theory is the theorem known as the Hodge Decomposition, that allow us to have a better understanding of the space of differentiable forms. As we shall see, this insight becomes very useful for topological and geometric considerations.

Recall that a differentiable k-form $\omega \in \Omega^k(M)$ is call harmonic if $\Delta\,\omega = 0$, and let us denote the space of harmonic differentiable k-forms as $\mathcal{H}^k(M)$.

Theorem 3.1.11 (Hodge Decomposition)**.** *Let (M, g) be a compact oriented riemannian manifold of dimension n, with Laplace-Beltrami operator $\Delta : \Omega^*(M) \to \Omega^*(M)$. Then, for each $0 \leq k \leq n$, $\mathcal{H}^k(M)$ is finite dimensional and we have the split*

$$\Omega^k(M) = \Delta\,\Omega^k(M) \oplus \mathcal{H}^k(M)$$

Furthermore, this decomposition is orthogonal with respect to the L^2 norm.

Corollary 3.1.12. *For each $0 \leq k \leq n$ we have the orthogonal decomposition*

$$\Omega^k(M) = d\,\Omega^{k-1}(M) \oplus d^*\,\Omega^{k+1}(M) \oplus \mathcal{H}^k(M)$$

Before its proof, let us discuss some of its consecuences. Maybe, the most evident one is that it solves the Poisson problem in compact manifolds.

Corollary 3.1.13. *Let (M, g) be a compact oriented riemannian manifold and let us consider $\pi_{\mathcal{H}}$: $\Omega^*(M) \to \mathcal{H}^*(M)$ the orthogonal projection of the space of forms onto the space of harmonic forms, given by the Hodge Decomposition Theorem.*

Given $\eta \in \Omega^*(M)$ the Poisson problem $\Delta\omega = \eta$ has solution if and only if $\pi_{\mathcal{H}}(\eta) = 0$. Furthermore, if it has solution, one and only one solution lives in $\mathcal{H}^*(M)^{\perp}$.

Proof. The first part is evident from the Hodge decompostion of $\Omega^*(M)$, because, by the directness of the sum, $\eta \in \mathrm{Im}\ \Delta$ if and only if $\pi_{\mathcal{H}}(\eta) = 0$.

For the uniqueness, let us suppose, that $\omega_1, \omega_2 \in \mathcal{H}^*(M)^{\perp}$ are two solutions of $\Delta\omega = \eta$, then $\Delta\,(\omega_1 - \omega_2) = \eta - \eta = 0$ so $\omega_1 - \omega_2 \in \mathcal{H}^*(M)$. Moreover, bt hypothesis, $\omega_1 - \omega_2 \in \mathcal{H}^*(M)^{\perp}$ so, by orthogonality, it must be $\omega_1 = \omega_2$. ∎

Remark 3.1.14. Note that, without boundary conditions, the uniqueness of the Poisson problem is an utopy. Indeed, if ω is a solution of $\Delta\omega = \eta$ and $\alpha \in \mathcal{H}^*(M)$, then $\omega + \alpha$ is also a solution. However, as we will se below, depending of the topology of M, we can reach uniqueness up to constant.

Thanks to this proposition, we can define the operator that asigns, to every differential forms, its *non-harmonic part*.

Definition 3.1.15. Let $\eta \in \Omega^*(M)$, we define the **Green operator** of η, $\mathcal{G}(\eta)$ as the unique $\omega \in \mathcal{H}^*(M)^\perp$ such that $\Delta\omega = \eta - \pi_{\mathcal{H}}(\eta)$.

The Green operator satisfies a crucial property.

Proposition 3.1.16. *$\mathcal{G}$ commutes with every linear operator T that commutes with Δ and such that its formal adjoint $T^* : \Omega^*(M) \to \Omega^*(M)$ is defined.*

Proof. Let $T : \Omega^*(M) \to \Omega^*(M)$ be a linear operator such that $\Delta \circ T = T \circ \Delta$, then

$$\Delta T\mathcal{G}\left(\eta\right) = T\Delta\mathcal{G}\left(\eta\right) = T\left(\eta - \pi_{\mathcal{H}}(\eta)\right) = T\eta - \pi_{\mathcal{H}}(T\eta) = \Delta\mathcal{G}\left(T\eta\right)$$

where T commutes with $\pi_{\mathcal{H}}$ because, if $\eta = \omega_{\mathcal{H}^\perp} \oplus \omega_{\mathcal{H}}$, then $T\eta = T\omega_{\mathcal{H}^\perp} \oplus T\omega_{\mathcal{H}}$. Therefore, both $\mathcal{G}\left(T\eta\right)$ and $T\mathcal{G}\left(\eta\right)$ are solutions of $\Delta\omega = \eta - \pi_{\mathcal{H}}(\eta)$.

By uniqueness of the Poisson equation, it is enought to show that $T\mathcal{G}\left(\eta\right) \in \mathcal{H}^*(M)$. To this end, let $\alpha \in \mathcal{H}(M)$ be any harmonic form, then

$$\langle T\mathcal{G}\left(\eta\right), \alpha\rangle_{L^2} = \langle \mathcal{G}\left(\eta\right), T^*\alpha\rangle_{L^2} = 0$$

because, by the commutativity of T with Δ, $T^*\alpha$ is harmonic.

■

Corollary 3.1.17. *$\mathcal{G}$ commutes with d.*

After this technical lemma, we can proof our desired result.

Theorem 3.1.18. *Let M be a compact oriented riemannian manifold. Then, every cohomology class of the de Rham cohomology contains one and only one harmonic representator.*

Proof. Let $\omega \in H^k_{dR}(M)$. Then, by the Hodge theorem, there exists $\eta \in \mathcal{H}^k(M)$ such that $\omega = \Delta\mathcal{G}(\omega) + \eta$. But, then

$$\omega = \left(d\, d^* + d^*\, d\right)\mathcal{G}(\omega) + \eta = d\, d^*\mathcal{G}(\omega) + d^*\mathcal{G}(d\omega) + \eta = d\, d^*\mathcal{G}(\omega) + \eta$$

Therefore, η is an harmonic form in the cohomology class ω.

For uniqueness, let us suppose that η_1, η_2 are two harmonic forms in the same chomology class, that is, $\eta_1 = \eta_2 + d\alpha$ for some differential form α. Observe that, $d\alpha \in \mathcal{H}^k(M)^\perp$, cause, for every harmonic β we have

$$\langle \beta, d\alpha\rangle_{L^2} = \langle d^*\beta, \alpha\rangle_{L^2} = 0$$

Therefore, $d\alpha$ is an harmonic form orthogonal to $\mathcal{H}^k(M)$, so it must be $d\alpha = 0$ and, hence, $\eta_1 = \eta_2$. ■

From this result, we can deduce some very important conclusions.

Corollary 3.1.19. *$H^k_{dR}(M)$ is isomorphic (as a $\mathbb{R}$-vector space) to $\mathcal{H}^k(M)$. In particular, we have $\dim_{\mathbb{R}} H^k_{dR}(M) < \infty$.*

Corollary 3.1.20 (Poincaré duality)**.** *If M is a compact orientable differentiable manifold, then*

$$H^k_{dR}(M) \cong \left(H^{n-k}_{dR}(M)\right)^*$$

Proof. Let us define the pairing $\varphi : H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}$ by

$$\varphi([\omega], [\eta]) = \int_M \omega \wedge \eta$$

Note that, by the Stokes' theorem, φ is well defined. Moreover, if φ were non-degenerated, then $\omega \mapsto \varphi(\omega, \cdot)$ will define the desired isomorphisim between $H^k_{dR}(M)$ and $\left(H^{n-k}_{dR}(M)\right)^*$.

To check it, observe that if $[\omega] \in H^k_{dR}(M)$, with ω harmonic, then $\star\omega \in H^{n-k}_{dR}(M)$ is also closed. Indeed, cause $\Delta\star = \star\Delta$, we have that $\star\omega$ is also harmonic, which, in particular, means that $d(\star\omega) = 0$. Taking this into acount, we have

$$\varphi([\omega], [\star\omega]) = \int_M \omega \wedge \star\omega = \|\omega\|_{L^2} \neq 0$$

except for $\omega = 0$. ■

Example 3.1.21 (Cohomology of S^1)**.** Observe the general fact that, for every differentiable manifold, cause $\mathrm{Im}\,(d : \Omega^{-1}(M) \to \Omega^0(M)) = 0$, then

$$H^0(M) = Ker\, d : \Omega^0(M) \to \Omega^1(M)$$

But, for $f \in \Omega^0(M) = C^\infty(M)$, $df = 0$ if and only if f is locally constant, and all the constant are linearly dependent over $\mathbb{R}$. Therefore, $H^0(M) \cong \mathbb{R}^N$, where N is the number of conected components of M.

In particular, we have that S^1 is conected, so $H^0(S^1) \cong \mathbb{R}$. Therefore, using the Poincaré duality, we have that $H^1(S^1) \cong \left(H^{1-1}(M)\right)^* \cong \left(H^0(M)\right)^* \cong \mathbb{R}^* \cong \mathbb{R}$. Hence, only using analytical methods, we just have computed the cohomology of S^1.

3.1.4 Hodge Decomposition on Kähler Manifolds

3.1.4.1 The adjoint operatos of ∂ y $\overline{\partial}$ and their laplacians

In the same way than for the exterior derivative, d, the existence of an hermitian metric on a complex manifold allows us to define formal adjoints for ∂ y $\overline{\partial}$.

First, observe that the Hodge star operator can be extended to $\Omega^*_{\mathbb{C}}(M)$ by $\mathbb{C}$-linearity and preserves the bigrading, i.e. $\star : \Omega^{p,q}(M) \to \Omega^{n-p,n-q}(M)$, with $n = \dim_{\mathbb{C}} M$. Hence, using this star, we can define an hermitian product on $\Omega^*_{\mathbb{C}}(M)$, also called the L^2 product by

$$\langle \omega, \eta \rangle_{L^2} := \int_M \omega \wedge \overline{\star\eta}$$

for any $\omega, \eta \in \Omega^*_{\mathbb{C}}(M)$. Then, we obtain the following characterization of the adjoints operators of the Dolbeault operators.

Proposition 3.1.22. *Let (M, g) be a compact hermitian complex manifold and let ∂ y $\overline{\partial}$ be its anti-Dolbeault and Dolbeault operators, respectively. Then, on $\Omega^{p,q}(M)$, the formal adjoints $\partial^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M)$ and $\overline{\partial}^* : \Omega^{p,q}(M) \to \Omega^{p-1,q}(M)$, respect to the L^2 product, are*

$$\partial^* = -\star \overline{\partial} \star \qquad\qquad \overline{\partial}^* = -\star \partial \star$$

Proof. We will prove it for $\overline{\partial}^*$, and the other case is analoguous. Let $\omega \in \Omega^{p,q}(M)$ and $\eta \in \Omega^{p-1,q}(M)$, then, using the Leibniz rule for $\overline{\partial}$, the fact $\overline{\partial \star \omega} = \overline{\partial}(\overline{\star\omega})$ and remark B.1.12 we have

$$\overline{\partial}\,(\eta \wedge \overline{\star\omega}) = \overline{\partial}\eta \wedge \overline{\star\omega} + (-1)^k \eta \wedge \overline{\partial}(\overline{\star\omega}) = \overline{\partial}\eta \wedge \overline{\star\omega} - \eta \wedge \overline{\star(\overline{\partial}^* \omega)}$$

Now, if $\alpha \in \Omega^{n-1,n}(M)$, then $d\alpha = \partial\alpha + \overline{\partial}\alpha = \overline{\partial}\alpha$, cause $\partial\alpha \in \Omega^{n-1,n+1}(M) = 0$. Thus, integrating on M and using Stokes' theorem in the boundaryless version

$$0 = \int_M d\,(\eta \wedge \overline{\star\omega}) = \int_M \overline{\partial}\,(\eta \wedge \overline{\star\omega}) = \int_M \overline{\partial}\eta \wedge \overline{\star\omega} - \int_M \eta \wedge \overline{\star(\overline{\partial}^* \omega)} = \langle \overline{\partial}\eta, \omega \rangle_{L^2} - \langle \eta, \overline{\partial}^* \omega \rangle_{L^2}$$

as we wanted to prove. ∎

Using this operators, and their adjoints, we can generalize even more the notion of laplace operator and consider the operators $\Delta_\partial, \Delta_{\overline{\partial}} : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$, given by

$$\Delta_\partial = \partial\partial^* + \partial^*\partial \qquad\qquad \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

In analogy with the Laplace-Beltrami operator, we denote $\mathcal{H}^{p,q}_\partial(M)$ and $\mathcal{H}^{p,q}_{\overline{\partial}}(M)$ the set of harmonic (p,q)-form, with respect to Δ_∂ and $\Delta_{\overline{\partial}}$, respectively. In that case, using similar techniques to the proof of the Hodge decomposition theorem, we have

Theorem 3.1.23. *Let M be a compact hermitian complex manifold with Dolbeault-typle Laplace-Beltrami operator $\Delta_{\overline{\partial}} : \Omega^{*,*}(M) \to \Omega^{*,*}(M)$. Then, for any $0 \leq p, q \leq n$, $\mathcal{H}^{p,q}_{\overline{\partial}}(M)$ is finite dimensional, and the following decomposition holds*

$$\Omega^{p,q}_{\mathbb{C}}(M) = \Delta\,\Omega^{p,q}_{\mathbb{C}}(M) \oplus \mathcal{H}^{p,q}_{\overline{\partial}}(M)$$

Corollary 3.1.24. *Let M be a compact hermitian complex manifold. In every Dolbeault-cohomology class there exists one and only one $\Delta_{\overline{\partial}}$-harmonic form. Furthermore, we have the $\mathbb{C}$-vector space isomorphism*

$$H^{p,q}(M) \cong \mathcal{H}^{p,q}_{\overline{\partial}}(M)$$

Corollary 3.1.25 (Serre duality)**.** *If M is a compact orientable complex manifold, then*

$$H^{p,q}(M) \cong \left(H^{n-p,n-q}(M)\right)^*$$

Proof. Let us pick any hermitian metric for M and let us define the pairing $\varphi : H^{p,q}(M)\times H^{n-p,n-q}(M) \to \mathbb{C}$ by

$$\varphi([\omega]\,,[\eta]) = \int_M \omega \wedge \overline{\eta}$$

By a similar argument to the one of 3.1.22 by Stokes' theorem, φ is well-defined. Moreover, if we prove that φ is non-degenerated, then $\omega \mapsto \varphi(\omega, \cdot)$ will be the desired isomorphism between $H^{p,q}(M)$ and $(H^{n-p,n-q}(M))^*$.

In order to check it, observe that, if $[\omega] \in H^{p,q}(M)$, with ω $\overline{\partial}$-harmonic, then $\star\omega \in H^{n-p,n-q}(M)$ is $\overline{\partial}$-closed too. Indeed, since $\Delta_{\overline{\partial}}\star = \star\Delta_{\partial}$, we have that $\star\omega$ is also $\overline{\partial}$-harmonic, which, in particular, means $\overline{\partial}\,(\star\omega) = 0$. Hence,

$$\varphi([\omega]\,,[\star\omega]) = \int_M \omega \wedge \overline{\star\omega} = \|\omega\|_{L^2} \neq 0$$

except for $\omega = 0$. ∎

3.1.4.2 Kähler identities

One of the most important properties of a Kähler manifold is that, only using the osculation of its Kähler metric, we can relate the Dolbeault operators with the adjoint operators of the anti-Dolbeault (and viceversa), what is known as the Kähler identities[2].

Definition 3.1.26. Let M be a Kähler manifold with $\omega \in \Omega^2(M)$. We define the **Lefschetz operator** $L : \Omega^*_{\mathbb{C}}(M) \to \Omega^{*+2}_{\mathbb{C}}(X)$ given by $L(\eta) = \omega \wedge \eta$.

Proposition 3.1.27. *The operator $\Lambda := \star^{-1}L\star : \Omega^*_{\mathbb{C}}(M) \to \Omega^{*-2}_{\mathbb{C}}(M)$ is the formal adjoint of L with respect to the L^2 metric.*

[2]Indeed, if we extend our analysis, it can be shown that the Kähler property implies that we can perfectly identify the super Lie algebra generated by Δ_d, ∂, $\overline{\partial}, \partial^*$ and $\overline{\partial}$ as $\mathbb{C}$-vector space, observing that it is finite dimensional and we can identify its generators.

Proof. It is a simple computation, observing that, for all $\alpha, \beta \in \Omega^*(M)$

$$\begin{aligned}\langle L\beta, \alpha\rangle_{L^2} &= \int_M L\beta \wedge \star\alpha = \int_M \omega \wedge \beta \wedge \star\alpha = \int_M \beta \wedge \omega \wedge \star\alpha \\ &= \int_M \beta \wedge L \star \alpha = \int_M \beta \wedge \star(\star^{-1} L \star \alpha) = \beta, \langle \star^{-1} L \star \alpha\rangle_{L^2}\end{aligned}$$

∎

Proposition 3.1.28 (Kähler Identities)**.** *If M is a Kähler manifold, we have*

$$[\Lambda, \overline{\partial}] = -i\partial^* \qquad\qquad [\Lambda, \partial] = i\overline{\partial}^*$$

Proof. For a detailed proof, see [37] or [73]. It is enough to prove the first identity, since the second one follows from the first by conjugation and recalling that Λ is real. Moreover, taking adjoints, it is enough to prove that$[L, \overline{\partial}^*] = -i\partial$.

By proposition A.3.5, given $p \in M$, there exists local coordinates that maps p to $0 \in \mathbb{C}^n$ and the hermitina metric is, locally

$$g = \sum_k dz_k \otimes dz_k + d\overline{z}_k \otimes d\overline{z}_k + O(|z|^2)$$

Now, observe that the operators $\Lambda, \overline{\partial}$ y ∂^* only requiere the Taylor series expansion of the metric up to order one. Thus, after applying the operators and evaluate in $0 \in \mathbb{C}^n$, the result only depends on the metric up to order one.

Therefore, it is enough to prove the result for the eucliden metric $g = \sum_k dz_k \otimes dz_k + d\overline{z}_k$. For this metric, the Kähler form is

$$\omega = \frac{i}{2} \sum dz_l \wedge d\overline{z}_l$$

Let us consider $\alpha = \sum_{I,J} a_{I,J}\, dz_I \wedge d\overline{z}_J \in \Omega^{p,q}$ with $p = |I|$ and $q = |J|$. Then, considering that $\overline{\partial}^* = - \star \partial \star$, a computation shows

$$[L, \overline{\partial}^*]\alpha = L\overline{\partial}^*\alpha - \overline{\partial}^* L\alpha = -L \star \partial \star (\alpha) + \star\partial \star L(\alpha)$$

In order to compute the Hodge star operator without going crazy, let us introduce the following notation. Let us suppose that the complex dimension of M is n. Given a multiindex without repetitions $I = \{i_1 < i_2 < \ldots < i_k\}$, we will denote $I^c := \{1, \ldots, n\} - I$. Moreover, if $1 \leq r \leq n$, we will denote by $I + k := I \cup \{k\}$ if $k \notin I$ and $\emptyset$ if $k \in I$. Analogously, we will denote $I - k := I - \{k\}$.

With this notation, we have that $\star dz_I \wedge d\overline{z}_J = \pm dz_{I^c} \wedge d\overline{z}_{J^c}$, where the signs only depend on $|I|$ y $|J|$. Thus, computing

$$\begin{aligned} L \star \partial \star (\alpha) &= L \star \partial \star \left(\sum_{I,J} a_{I,J}\, dz_I \wedge d\overline{z}_J \right) = \sum_{I,J} \pm L \star \partial \left(a_{I,J}\, dz_{I^c} \wedge d\overline{z}_{J^c} \right) \\ &= \sum_{I,J,l} \pm L \star \frac{\partial a_{I,J}}{\partial z_k}\, dz_k \wedge dz_{I^c} \wedge d\overline{z}_{J^c} = \sum_{I,J,l} \pm L \left(\frac{\partial a_{I,J}}{\partial z_k}\, dz_{I-k} \wedge d\overline{z}_J \right) \\ &= \frac{i}{2} \sum_{I,J,k,l} \pm \frac{\partial a_{I,J}}{\partial z_k}\, dz_l \wedge d\overline{z}_l \wedge dz_{I-k} \wedge d\overline{z}_J = \frac{i}{2} \sum_{I,J,k,l} \pm \frac{\partial a_{I,J}}{\partial z_k}\, dz_{I+l-k} \wedge d\overline{z}_{J+l} \end{aligned}$$

Analogously, for the other term we have

$$\begin{aligned} \star \partial \star L(\alpha) &= \star \partial \star L \left(\sum_{I,J} a_{I,J}\, dz_I \wedge d\overline{z}_J \right) = \frac{i}{2} \sum_{I,J,l} \star \partial \star \left(a_{I,J}\, dz_l \wedge d\overline{z}_l \wedge dz_I \wedge d\overline{z}_J \right) \\ &= \frac{i}{2} \sum_{I,J,l} \pm \star \partial \left(a_{I,J}\, dz_{(I+l)^c} \wedge d\overline{z}_{(J+l)^c} \right) = \frac{i}{2} \sum_{I,J,k,l} \pm \star \frac{\partial a_{I,J}}{\partial z_k}\, dz_k \wedge dz_{(I+l)^c} \wedge d\overline{z}_{(J+l)^c} \\ &= \frac{i}{2} \sum_{I,J,k,l} \pm \frac{\partial a_{I,J}}{\partial z_k}\, dz_{I+l-k} \wedge d\overline{z}_{J+l} \end{aligned}$$

Therefore, putting all together and taking care of signs and index, we have

$$[L, \overline{\partial}^*]\alpha = -L \star \partial \star (\alpha) + \star \partial \star L(\alpha) = -i \sum_{I,J} \frac{\partial a_{I,J}}{\partial z_k}\, dz_k \wedge dz_I \wedge d\overline{z}_J = i\partial(\alpha)$$

as we wanted to prove. ■

3.1.4.3 Hodge decomposition in cohomology

Finally, all the previous work allow us to prove the desired Hodge decomposition in cohomology. The main point of this decomposition is that the de Rham cohomology of a complex manifold can be computed using the Dolbeault cohomology, and do not depend on the complex structure choosen. For a more extensive introduction to the topic, see [37], [73] and [74].

Proposition 3.1.29. *Let M be a compact Kähler manifold and let $\Delta_d, \Delta_\partial$ and $\Delta_{\overline{\partial}}$ be its Laplace-Beltrami operators, defined in terms of d, ∂ y $\overline{\partial}$. Then, we have*

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_\partial \qquad\qquad \Delta_\partial = \Delta_{\overline{\partial}}$$

Proof. It is enough to show $\Delta_\partial = \Delta_{\overline{\partial}}$ and $\Delta_d = \Delta_\partial + \Delta_{\overline{\partial}}$. For the first one, recall that, since the almost complex structure is integrable, we have $\partial\overline{\partial} + \overline{\partial}\partial = 0$ so, using this and the Kähler identities, we obtain

$$\begin{aligned}\Delta_\partial &= \partial^*\partial + \partial\partial^* = i[\Lambda, \overline{\partial}]\partial + i\partial[\Lambda, \overline{\partial}] = i(\Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial + \partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda) \\ &= i(\Lambda\overline{\partial}\partial - (\overline{\partial}[\Lambda, \partial] + \overline{\partial}\partial\Lambda) + (-[\Lambda, \partial]\overline{\partial} + \Lambda\partial\overline{\partial}) - \partial\overline{\partial}\Lambda) \\ &= i(\Lambda\overline{\partial}\partial - i\overline{\partial}\overline{\partial}^* - \overline{\partial}\partial\Lambda - i\overline{\partial}^*\overline{\partial} + \Lambda\partial\overline{\partial} - \partial\overline{\partial}\Lambda) = \Delta_{\overline{\partial}} + [\Lambda, \partial\overline{\partial} + \overline{\partial}\partial] = \Delta_{\overline{\partial}}\end{aligned}$$

For the second equality, recall that, again by the Kähler identities, we have

$$\partial\overline{\partial}^* + \overline{\partial}^*\partial = -i(\partial[\Lambda, \partial] + [\Lambda, \partial]\partial) = -i(\partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial) = 0$$

and, therefore

$$\begin{aligned}\Delta_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= \Delta_\partial + \Delta_{\overline{\partial}} + (\partial\overline{\partial}^* + \overline{\partial}^*\partial) + \overline{\partial\overline{\partial}^* + \overline{\partial}^*\partial} = \Delta_\partial + \Delta_{\overline{\partial}}\end{aligned}$$

as we wanted to prove. ■

Corollary 3.1.30. *Let M be a compact Kähler manifold. Then, for all $0 \leq k \leq \dim_{\mathbb{R}} M$, the following decomposition of harmonic forms holds*

$$\mathcal{H}^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\overline{\partial}}(M)$$

Proof. Let $\alpha \in \mathcal{H}^k_{\mathbb{C}}(M)$ and let us decompose it in its (p,q)-components, let us say $\alpha = \sum_{p+q=k} \alpha_{p,q} \in \bigoplus_{p+q=k} \Omega^{p,q}(M)$. It is enough to show that the $\alpha_{p,q}$ are harmonic. For this purpose, observe that

$$0 = \Delta_d\,\alpha = \sum_{p+q=k} \Delta_d\,\alpha_{p,q}$$

Since $\Delta_d = 2\Delta_\partial$ is bihomogeneous with bidegree $(0,0)$ we have that $\Delta_d\,\alpha_{p,q} \in \Omega^{p,q}(M)$, so it should vanish component by component. In that way, $\Delta_d\,\alpha_{p,q} = 0$ for all $p + q = k$, or, equivalently $\alpha_{p,q} \in \mathcal{H}^{p,q}_{\overline{\partial}}(M)$. ■

Corollary 3.1.31 (Hodge decomposition in cohomology)**.** *Let M be a compact Kähler manifold. Then, we have the following decomposition in cohomology*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M)$$

Moreover, this decomposition does not depend on the choosen Kähler structure.

Proof. The existence part of the decomposition is clear remembering that $H^{p,q}(M) \cong \mathcal{H}^{p,q}_{\overline{\partial}}(M)$ and $H^k(M, \mathbb{C}) \cong \mathcal{H}^k_{\mathbb{C}}(M)$ (see theorems 3.1.19 and 3.1.24).

For uniqueness, let $K^{p,q} \subseteq H^k(M,\mathbb{C})$ be the set of de Rham cohomology classes that contains any closed (p,q)-form. We will show that $H^{p,q}(M) = K^{p,q}$. For the inclusion $H^{p,q}(M) \subseteq K^{p,q}$, recall that, if $[\alpha] \in H^{p,q}(M)$ then there exists a (p,q)-form $\Delta_{\overline{\partial}}$-harmonic, let us say $\alpha' \in [\alpha]$. However, since $\Delta_d \alpha' = 2\Delta_{\overline{\partial}}\alpha' = 0$, in particular $d\alpha' = 0$, so $\alpha' \in [\alpha]$ is the desired (p,q)-form.

For the contrary inclusion, let $[\alpha] \in K^{p,q}$ with $\alpha \in \Omega^{p,q}(M)$ closed. By the Hodge decomposition, we have $\alpha = \Delta_d \beta + \alpha'$, for $\alpha' \in \Omega^{p,q}(M)$ Δ_d-harmnic (recall that Δ_d is bihomogeneous with bidegree $(0,0)$). Now, in that case, $\Delta_d \beta = dd^*\beta + d^*d\beta$ is closed, so $dd^*d\beta = 0$. However, Im $d^* \perp Ker\, d$, so we must have $d^*d\beta = 0$. In that case, we have $\alpha = d(d^*\beta) + \alpha'$, so $[\alpha] = [\alpha'] \in H^{p,q}(M)$.

Therefore, we have just prove that $K^{p,q} = H^{p,q}$ and, thus, we have the metric-independent decomposition

$$H^k(M,\mathbb{C}) \cong \bigoplus_{p+q=k} K^{p,q}$$

■

Corollary 3.1.32. *Let M^{2n} be a compact Kähler manifold. We define the **Poincaré** and **Hodge polynomials**, respectively*

$$P_M(t) = \sum_{k=1}^{2n} b_k(M)t^k \qquad\qquad h_M(u,v) = \sum_{0\leq p,q\leq n} h^{p,q}(M)u^p v^q$$

Then, we have $P_M(t) = h_M(t,t)$.

Corollary 3.1.33. *If M is a compact Kähler manifold, conjugation on forms induces an isomorphism $\overline{H^{p,q}}(M) = H^{q,p}(M)$.*

Proof. It is obvious for $K^{p,q} = H^{p,q}(M)$. ■

In particular, if k is odd, then $H^k_{\mathbb{C}}(M)$ decomposes in a sum of $k+1$ (which is an even number) pairwise isomorphic terms. Therefore, its dimension should be an even number. Of course, this introduces a strong restriction on the topology of compact Kähler manifolds.

Corollary 3.1.34. *In a compact Kähler manifold, all the odd Betti numbers, b_{2k+1}, are even.*

Corollary 3.1.35. *If M^{2n} es a compact Kähler manifold with Kähler form ω, then, for all $0 \leq p \leq n$, $0 \neq [\omega^p] \in H^{p,p}(M)$ for every $0 < p \leq n$. In particular, $h^{p,p} \neq 0$ for all $0 \leq p \leq n$.*

3.2 Pure Hodge Structures

Let M_R be a R-module for some ring R with $\mathbb{Z} \subseteq R$ (in our cases, it will be $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) and let $k \supseteq R$ be a field (usually $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$). We define the k-fication, M_k as the k-vector space

$M_k := M_R \otimes_{\mathbb{Z}} k$. Moreover, given a homomorphism of R-modules $f : M_R \to N_R$ we can also define its k-fication $f_k : M_k \to N_k$ given by $f_k\left(\sum m_i \otimes k_i\right) := \sum f(m_i) \otimes k_i$.

If $k = \mathbb{C}$ then $M_{\mathbb{C}} = M_R \otimes \mathbb{C}$ has a natural complex conjugation $\bar{\cdot} : M_{\mathbb{C}} \to M_{\mathbb{C}}$ given forms by $\overline{\sum m_i \otimes z_i} := \sum m_i \otimes \overline{z_i}$.

Definition 3.2.1. Let R be a ring containing $\mathbb{Z}$, usually $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and H_R finitely generated R-module. An **pure R-Hodge structure** of weight k on H_R (usually simply called a Hodge structure of weight k) is a direct sum decomposition of $H_{\mathbb{C}} = H_R \otimes_{\mathbb{Z}} \mathbb{C}$

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

with $\overline{H^{p,q}} = H^{q,p}$. A **morphism** of Hodge structures of weight k is a R-module homomorphism $f : H_R \to H'_R$ such that, for all $p, q \in \mathbb{Z}$ with $p + q = k$ we have that $f_{\mathbb{C}} : H^{p,q} \to H'^{p,q}$, that is, $f_{\mathbb{C}}$ respects the bigrading induced by the Hodge structures.

Remark 3.2.2. If a finitely generated R-module H_R has a grading

$$H_R = \bigoplus H_R^k$$

such that each H_R^k has a pure Hodge structure of weight k, H_R is said to have a pure Hodge structure, without any reference to the weight. With this definitions, the pure R-Hodge structures form a category denoted by $\mathbf{HS}_R$. It is a subcategory of $R - \mathbf{Mod}$, the category of R-modules (or $\mathbf{Ab}$ the category of abelian groups in the case $R = \mathbb{Z}$).

Remark 3.2.3. A $\mathbb{Z}$-Hodge structure is usually called a **integral** Hodge structure and $\mathbf{HS}_{\mathbb{Z}}$ is denoted $\mathbf{HS}$, while a $\mathbb{R}$-Hodge structure is called a **real** Hodge structure, a $\mathbb{Q}$-Hodge structure is called a **rational** Hodge structure and a $\mathbb{C}$-Hodge structure is called a **complex** Hodge structure.

Definition 3.2.4. Given a pure R-Hodge structure on H_R, we define the **Hodge numbers** associated to this Hodge structure as

$$h^{p,q}(H_R) = \dim_{\mathbb{C}} H^{p,q}$$

With this numbers, we can form the **Hodge polynomial** of H_R, $h_H \in \mathbb{Z}[u, v, u^{-1}, v^{-1}]$ by

$$h_H(u, v) = \sum_{p+q=k} h^{p,q}(H_R) u^p v^q$$

Remark 3.2.5. The sum in the definition of the Hodge polynomial is finite because H_R is finitely generated. If the pure Hodge structure of H_R lives in the first quadrant (i.e. if $h^{p,q}(H_R) = 0$ for $p < 0$ or $q < 0$) then $h_H \in \mathbb{Z}[u, v]$.

Example 3.2.6. Every R-module M_R has a pure Hodge estructure of weight 0 by decreeing $M^{0,0} := M_{\mathbb{C}} = M_R \otimes \mathbb{C}$ and zero otherwise.

Example 3.2.7. One of the simplest integral pure Hodge structures that can be defined is the **Tate Hodge structure** $\mathbb{Z}(1) := 2\pi i\mathbb{Z} \subseteq \mathbb{C}$ with a pure Hodge structure of weight -2 by defining $\mathbb{Z}(1)^{-1,-1} := \mathbb{Z}(1)_{\mathbb{C}}$ and $\mathbb{Z}(1)^{p,q} = 0$ for $p, q \neq -1$.

The main reason for living of the pure Hodge structures is to restate the Hodge decomposition theorem in a more general framework.

Corollary 3.2.8. *On every compact Kähler manifold M, its k-th de Rham cohomology ring $H^k_{dR}(M)$ has a real pure Hodge structure of weight k living on the first quadrant, given by*

$$H^k_{dR}(M)_{\mathbb{C}} = H^k_{dR}(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$$

Moreover, since $H^k(M, \mathbb{Z}) \otimes \mathbb{C} \cong H^k_{dR}(M, \mathbb{C})$ by the de Rham theorem, its integral cohomology ring $H^k(M, \mathbb{Z})$ also has an integral pure Hodge structure of weight k. Furthermore, the Hodge polynomial of M is the sum of the Hodge polynomials of the Hodge structures on $H^k(M)$ for $k = 0, \ldots, \dim_{\mathbb{R}} M$.

Furthermore, since the induced maps in cohomology factorices through the Dolbeaut cohomology, we have a stronger result.

Corollary 3.2.9. *Let $\boldsymbol{K\ddot{a}h}_c$ be the category of compact Kähler manifolds and C^∞ maps and let us define the contravariant functor $H_{\mathbb{Z}} : \boldsymbol{K\ddot{a}h}_c \to \boldsymbol{Ab}$ given by $H_{\mathbb{Z}}(M) = H^*(M, \mathbb{Z})$ and, for $f : M \to N$ a C^∞-map, $H_{\mathbb{Z}}(f) = f^* : H^*(N, \mathbb{Z}) \to H^*(M, \mathbb{Z})$. Then $H_{\mathbb{Z}}$ factorices through the inclusion $\boldsymbol{HS} \hookrightarrow \boldsymbol{Ab}$, that is, there exists a contravariant functor $\tilde{H}_{\mathbb{Z}} : \boldsymbol{K\ddot{a}h}_c \to \boldsymbol{HS}$ such that the following diagram commutes*

$$\begin{tikzcd}
\boldsymbol{K\ddot{a}h}_c \arrow[rr, "H_{\mathbb{Z}}"] \arrow[dr, "\tilde{H}_{\mathbb{Z}}"'] & & \boldsymbol{Ab} \\
& \boldsymbol{HS} \arrow[ur, hook] &
\end{tikzcd}$$

With a pure Hodge structure we can use the general contructions of linear algebra to build some associated Hodge structures.

- If H_R and H'_R have pure R-Hodge structures of weights k and k', respectively, then we can define a pure R-Hodge structure of weight $k + k'$ on $H_R \otimes H'_R$ by defining

$$(H \otimes H')^{p,q}_{\mathbb{C}} := \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H^{p_1,q_1} \otimes H^{p_2,q_2}$$

- If H_R has a pure R-Hodge structure of weight k and $H^*_R := Hom(H_R, R)$ is its dual, then H^*_R has a pure R-Hodge structure of weight $-k$ by decreeting the decomposition

$$H^{*\,p,q}_{\mathbb{C}} := \bigoplus_{p+q=-k} H^{-p,-q}$$

- More generally, if H_R and H'_R have pure R-Hodge structures of weights k and k', respectively, then we can define a pure R-Hodge structure of weight $k'-k$ on $Hom_R(H,H') = H^* \otimes H$ by defining
$$Hom(H,H')^{p,q}_{\mathbb{C}} := \bigoplus_{\substack{p_2-p_1=p\\ q_2-q_1=q}} Hom_{\mathbb{C}}(H^{p_1,q_1}, H^{p_2,q_2})$$

Example 3.2.10. Given the Tate Hodge structure $\mathbb{Z}(1)$, the m-th Tate Hodge structure is the Hodge structure on $\mathbb{Z}(m) := \underbrace{\mathbb{Z}(1)\otimes \ldots \otimes \mathbb{Z}(1)}_{m\ \textbf{times}}$. It is a Hodge structure of weight $-2m$ on $\mathbb{Z}(m) = (2\pi i)^m\,\mathbb{Z}$ with decomposition $\mathbb{Z}(m)^{-m,-m} = \mathbb{Z}(m)_{\mathbb{C}}$ and zero otherwise. If $m<0$, we define $\mathbb{Z}(m) := \mathbb{Z}(-m)^*$. Tensoring by a ring $R \subseteq \mathbb{Z}$ we can define the R-Tate m-th Hodge structure $R(m) := \mathbb{Z}(m)\otimes R$ for $m \in \mathbb{Z}$.

Definition 3.2.11. Given a pure Hodge structure H_R, we define its m**-th Tate twist** as the induced pure Hodge structure on $H_R(m) := H_R \otimes R(m)$. A morphism of R-modules $f : H_R \to H'_R$ is said to be a **morphism of pure Hodge structures of type** m if $f(-m) : H_R \to H'_R(-m)$ is a morphism of Hodge structures.

Remark 3.2.12. Observe that, using the definitions of the Hodge structure induced in the tensor product, we have that, if H_R has a pure Hodge estructure of weight k, then the Hodge structure on $H_R(m)$ has weight $k-2m$ and satisfies

$$H(m)^{p,q} = H^{p+m,q+m}$$

Note that this is coherent with the usual definition of the shift of grading in a graded ring.

Example 3.2.13. Let M be a compact Kähler manifold of real dimension $2n$. Observe that the integration map in top cohomology

$$\begin{array}{rccc} Tr: & H^{2n}(M,\mathbb{C}) & \to & \mathbb{C} \\ & [\omega] & \mapsto & \dfrac{1}{(2\pi i)^n}\displaystyle\int_M \omega \end{array}$$

do not respect the Hodge structures on $H^{2n}(M,\mathbb{C}) = H^{n,n}(M)$ and $\mathbb{C} = \mathbb{C}^{0,0}$, since $Tr(H^{n,n}(M)) \not\subseteq \mathbb{C}^{n,n} = 0$. However, if we twist the grading in $\mathbb{C}$ we have for $Tr(-n) : H^{2n}(M,\mathbb{C}) \to \mathbb{C}(-n)$ that $Tr(-n)(H^{n,n}(M)) = \mathbb{C}(-n)^{n,n} = \mathbb{C}^{0,0}$. Therefore, $H^{2n}(M,\mathbb{C})$ and $\mathbb{C}(-n)$ are isomorphic as Hodge structures and Tr is an isomorphism of Hodge structures of type n. Moreover, using this shifted integration map, we have that the bilinear form used in the Poincaré duality

$$H^k(M,\mathbb{C}) \otimes H^{2n-k}(M,\mathbb{C}) \overset{\frown}{\to} H^{2n}(M,\mathbb{C}) \overset{Tr(-n)}{\to} \mathbb{C}(-n)$$

respects de Hodge structures so the Poicaré isomorphism induced by it

$$H^k(M,\mathbb{C}) \overset{\cong}{\to} Hom(H^{2n-k}(M,\mathbb{C}),\mathbb{C}(-n))$$

is an isomorphism of Hodge structures.

Finally, we can also generalize the concept of polarization to this general framework.

Definition 3.2.14. Let H_R be a pure R-Hodge structure of weight k. A **polarization** on H_R is a R-valued bilinear form

$$Q : H_R \otimes_R H_R \to R$$

such that

- Q is symmetric for k even and antisymmetric for k odd.
- With respect to its complexification $Q_{\mathbb{C}} : H_{\mathbb{C}} \otimes_{\mathbb{C}} H_{\mathbb{C}} \to \mathbb{C}$ the spaces $H^{p,q}$ and $H^{p',q'}$ are orthogonal for $p \neq p'$ or $q \neq q'$.
- $Q^{p,q} : H^{p,q} \otimes H^{p,q} \to \mathbb{C}$ given by $Q^{p,q}(x, y) = i^{p-q} Q_{\mathbb{C}}(x, \overline{y})$ is positive-defined.

A pure Hodge structure that admits a polarization is said to be **polarizable**. A pure Hodge structure with a polarization is called a **polarized pure Hodge structure**.

3.2.1 Pure Hodge Structures via Filtrations

We can use an alternative way for defining pure Hodge structures which, in some contexts, is useful and will allow us to generalize it to the mixed Hodge structures framework. Let us fix a R-module H_R with a pure Hodge structure of weight k, $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$. Let us define the submodules

$$F_p H_{\mathbb{C}} := \bigoplus_{r \geq p} H^{r,k-r}$$

Observe that the F_p form a decreasing filtration

$$H_{\mathbb{C}} \supseteq \ldots \supseteq F_{p-1} H_{\mathbb{C}} \supseteq F_p H_{\mathbb{C}} \supseteq F_{p+1} H_{\mathbb{C}} \supseteq \ldots \supseteq \{0\}$$

called the associated **Hodge filtration**. Observe that, since $H_{\mathbb{C}}$ is a finite dimensional $\mathbb{C}$-vector space, the Hodge filtration must have finite length. Moreover, we have

$$\overline{F_p H_{\mathbb{C}}} = \bigoplus_{r \geq p} \overline{H^{r,k-r}} = \bigoplus_{r \geq p} H^{k-r,r} = \bigoplus_{s \leq n-p} H^{s,k-s}$$

so we can recover the Hodge structure by

$$H^{p,q} = F_p H_{\mathbb{C}} \cap \overline{F_q H_{\mathbb{C}}}$$

Furthermore, the condition that $\bigcup_{p=-\infty}^{\infty} F_pH_{\mathbb{C}} = H_{\mathbb{C}}$ can be checked to be equivalent to $F_pH_{\mathbb{C}} \oplus \overline{F_{k-p+1}H_{\mathbb{C}}} = H_{\mathbb{C}}$ for all $p \in \mathbb{Z}$. Therefore, we can reformulate the property of having a Hodge structure of weigh k.

Definition 3.2.15 (Equivalent to 3.2.1)**.** Given a finitely generated R-module H_R, for some ring $R \subseteq \mathbb{C}$, a **pure Hodge structure** on H_R of weight k is a decreasing filtration of $H_{\mathbb{C}} = H_R \otimes_{\mathbb{Z}} \mathbb{C}$, $\{F_pH_{\mathbb{C}}\}_{p\in\mathbb{Z}}$

$$H_{\mathbb{C}} \supseteq \ldots \supseteq F_{p-1}H_{\mathbb{C}} \supseteq F_pH_{\mathbb{C}} \supseteq F_{p+1}H_{\mathbb{C}} \supseteq \ldots \supseteq \{0\}$$

such that, for all $p \in \mathbb{Z}$

$$F_pH_{\mathbb{C}} \oplus \overline{F_{k-p+1}H_{\mathbb{C}}} = H_{\mathbb{C}}$$

3.2.2 Pure Hodge Structures via Representations

Another equivalent way of specifying a pure Hodge structure is via a real representation. Let us define the real algebraic group $\mathbb{S}(\mathbb{R})$ by

$$\mathbb{S}(\mathbb{R}) = \left\{ M(u,v) := \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in GL(2,\mathbb{R}) \right\}$$

Observe that, the homomorphism $f : \mathbb{S}(\mathbb{R}) \to \mathbb{C}^*$ by $f(M(u,v)) = u+iv$ is an isomorphism, so $\mathbb{S}(\mathbb{R})$ is nothing more that $\mathbb{C}^*$ seen as real algebraic group. In particular, we can see $\mathbb{R} \hookrightarrow \mathbb{S}(\mathbb{R})$ as the points of the form $M(t,0)$ for $t \in \mathbb{R}$.

Analogously, we define the complex algebraic group $\mathbb{S}(\mathbb{C})$ by

$$\mathbb{S}(\mathbb{C}) = \left\{ \tilde{M}(z,w) := \begin{pmatrix} z & -w \\ w & z \end{pmatrix} \in GL(2,\mathbb{C}) \right\}$$

In this case, we have that $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$ by $\tilde{M}(z,w) \mapsto (z+iw, z-iw)$.

Proposition 3.2.16. *Let $H_{\mathbb{Q}}$ be finite dimensional $\mathbb{Q}$-vector space. There is a natural biyective correspondence between rational Hodge structures of weight k on $H_{\mathbb{Q}}$ and real algebraic representations $\rho : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$ such that, $\rho|_{\mathbb{R}^*} : \mathbb{R}^* \to GL(H_{\mathbb{R}})$ is of the form $\rho(t)(v) = t^k v$.*

Proof. $\Rightarrow$) If $H_{\mathbb{Q}}$ has a pure Hodge structure of weight k

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

we can decompose every $v \in H_{\mathbb{C}}$ in $v = \sum\limits_{p+q=k} v_{p,q}$ with $v_{p,q} \in H^{p,q}$ in an unique way. Therefore, we can define the complex representation $\rho : \mathbb{C}^* \to GL(H_{\mathbb{C}})$ by

$$\rho(z)(v) = \sum_{p+q=k} z^p \overline{z}^q v_{p,q}$$

This is, in fact, a real representation because

$$\overline{\rho(z)}(v) = \overline{\rho(z)(\overline{v})} = \overline{\sum_{p+q=k} z^p \overline{z}^q \overline{v}_{p,q}} = \overline{\sum_{p+q=k} z^p \overline{z}^q \overline{v_{q,p}}} = \sum_{p+q=k} z^q \overline{z}^p v_{q,p} = \overline{\rho(z)}(v)$$

so $\overline{\rho(z)} = \rho(z)$. Hence, it can be seen as a real algebraic representation $\rho : \mathbb{S}(\mathbb{R}) \cong \mathbb{C}^* \to GL(H_{\mathbb{R}})$ that, for $t \in \mathbb{R}^* \hookrightarrow \mathbb{S}(\mathbb{R})$ is given by

$$\rho(t)(v) = \sum_{p+q=k} t^p \overline{t}^q v_{p,q} = \sum_{p+q=k} t^{p+q} v_{p,q} = t^k v$$

$\Leftarrow$) Let $\rho : \mathbb{S}(\mathbb{R}) \to GL(H_{\mathbb{R}})$ be a real algebraic representation such that, on $\mathbb{R}^*$ is of the form $\rho(t)(v) = t^k v$ and let take its complexification $\rho_{\mathbb{C}} : \mathbb{C}^* \to GL(H_{\mathbb{C}})$.

Since $\mathbb{C}^*$ is an abelian Lie group, by the Schur lemma, every irreducible representation is 1-dimensional. Hence, taking its irreducible representations, we have a splitting

$$H_{\mathbb{C}} = \sum_{s=1}^{\dim_{\mathbb{C}} H_{\mathbb{C}}} H^s \qquad \rho = \sum_{s=1}^{\dim_{\mathbb{C}} H_{\mathbb{C}}} \rho_s$$

where the H^s are 1-dimensional and $\rho_s : \mathbb{C}^* \to GL(H^s)$ is of the form $\rho_s(z)(v) = f_s(z)v$ for some $f_s : \mathbb{C}^* \to \mathbb{C}^*$. However, since ρ is real algebraic, $f_s : \mathbb{C}^* \subseteq \mathbb{R}^2 \to \mathbb{C}^* \subseteq \mathbb{R}^2$ must be a real algebraic function, ergo of the form $f_s(z) = z^{p_s} \overline{z}^{q_s}$ for some $p_s, q_s \in \mathbb{Z}$. Hence, defining

$$H^{p,q} := \bigoplus_{\substack{p_s = p \\ q_s = q}} H^s$$

we have that $\rho_{\mathbb{C}} : \mathbb{C}^* \to GL(H_{\mathbb{C}})$ splits $\rho = \sum\limits_{p,q} \rho_{p,q}$ with $\rho_{p,q} : \mathbb{C}^* \to GL(H^{p,q})$ of the form $\rho_{p,q}(v) = z^p \overline{z}^q v$. Moreover, since ρ on $\mathbb{R}^*$ is

$$t^k v = \rho(t)(v) = \left(\sum_{p,q} \rho_{p,q}(t) \right)(v) = \sum_{p,q} \rho_{p,q}(t)(v_{p,q}) = \sum_{p,q} t^p \overline{t}^q v_{p,q} = \sum_{p,q} t^{p+q} v_{p,q}$$

we must have $p+q = k$, so $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$, that is, the Hodge structure has weight k. Furthermore, since ρ is real, $\overline{\rho_{\mathbb{C}}(z)} = \rho_{\mathbb{C}}(z)$ so $\overline{H^{p,q}} = H^{q,p}$, completing the check that this is a rational pure Hodge structure of weight k. ∎

3.3 Mixed Hodge Structures

In order to study the Weil conjectures, in a serie of articles published between 1971 and 1974 ([15], [17] and [16]), Deligne extended the notion of pure Hodge structures into a larger category, rather abstract and artificial, known as mixed Hodge structures. With this general definition, he could proof that the cohomology ring of a large range of geometric spaces is naturally endowed with this mixed Hodge structure.

As we will see, this mixed Hodge structures, and especially an integral polynomial that traces them, will be the main tool of study of $SL(2,\mathbb{C})$ character varieties allowing us to compute some numerical invariants.

3.3.1 Review of category theory

A category is an abstraction of the fundamental structures that lies in the fundamentals of mathematics. In some sense, they try to capture the properties that can be defined only refering to mathematical objects an morphisms between them. For a more general introduction to this topic, see, for example [75], [48] or [65].

Definition 3.3.1. A **category**, $\mathcal{C}$, is made of the following elements:

- A class [3], $Obj(\mathcal{C})$, whose elements are called the **objets** of the category.
- For each $A, B \in Obj(\mathcal{C})$, a class, $Hom(A,B)$, whose elments are called the **morphisms** between A and B. An element $f \in Hom(A,B)$ is denoted by $f : A \to B$. Moreover, the classes $\{Hom(A,B)\}_{A,B}$ must be pairwise disjoints.
- A binary associated operation with unit, $\circ$, called the **composition** of morphisms, such that, for all $A, B, C \in Obj(\mathcal{C})$, we have a map $\circ : Hom(A,B) \times Hom(B,C) \to Hom(A,C)$ denoted $(f,g) \mapsto g \circ f$.

Example 3.3.2. Some examples of categories are

- **Set**, the category whose objects is the class of all sets and its morphisms are all the maps between sets.
- **Gr**, the category of all groups with group homomorphisms and, inside it, the category **Ab** of all the abelian groups.

[3] A class is a generalization of the notion of set, in the sense of the Zermelo-Fraenkel axioms. As we will see, we will need to form the class of all the sets and, by Russell's paradox, it cannot be a set. Essentially, a class behaves as a set, except for the fact that they cannot be elements of another class, eliminating the Russell's paradox. For a complete formalization, see Von Neumann-Bernays-Gödel's axioms.

- **Top**, the category of topological spaces and continous maps between them. More generally, we have **Diff** the category of diferentiable manifolds and differentiable maps between them.
- **Var**$_k$, the category of algebraic varieties over the field k. We $k = \mathbb{C}$, we usually elipse the subscript.

Definition 3.3.3. A category $\mathcal{C}$ is called **small** if $Obj(\mathcal{C})$ is a set. Analogously, $\mathcal{C}$ is called **locally small** if, for each $A, B \in Obj(\mathcal{C})$, $Hom(A, B)$ is a set.

Example 3.3.4. With this definition, we can consider **Cat**, the category whose objects are small categories and functors between them. In that case, **Cat** is not a small nor a locally small category. ni localmente pequeña.

Definition 3.3.5. In a category $\mathcal{C}$, a morphism $f : A \to B$ is called a **monomorphism** if cancels by left, i.e., if for all $g_1, g_2 : C \to A$ it holds that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. Analogously, $f : A \to B$ is called an **epimorphism** if cancels by right, i.e., if forl all $g_1, g_2 : B \to C$, it holds that $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

3.3.1.1 Abelian categories

In the theory of mixed Hodge structures, one of the most important facts is that the mixed Hodge structures behaves well under kernels and cokernels. This well-behaviour can be captured under the notion of an abelian category. Roughly speaking, an abelian category tries to mimic the most important properties of the category of abelian groups of modules.

In order to extend this notions to general categoris, first of all, let us study how to define Ker y $Coker$ in a categorical setting.

Definition 3.3.6. In a category $\mathcal{C}$, an object $0 \in Obj(\mathcal{C})$ is called the **zero object** if, for all object $A \in Obj(\mathcal{C})$, we have two unique morphisms $0 \to A$ and $A \to 0$. In this sense, a morphism $A \to B$ between $A, B \in Obj(\mathcal{C})$ is called the **zero morphism** if it factorices through $A \to 0 \to B$.

Definition 3.3.7. Let $\mathcal{C}$ be a category and let $f : A \to B \in Hom(A, B)$ for $A, B \in Obj(\mathcal{C})$. We say that $k : K \to A$ is the **kernel** of f if $f \circ k = 0$ and, for any other morphism $\tilde{k} : \tilde{K} \to A$ with $f \circ \tilde{k} = 0$ there exists a morphism $h : \tilde{K} \to K$ such that $\tilde{k} = k \circ h$.

$$\begin{tikzcd}
K \arrow[dr, "k"] \arrow[ddr, bend right, "0"'] & & \tilde{K} \arrow[ll, dashed, "h"'] \arrow[dl, "\tilde{k}"'] \arrow[ddl, bend left, "0"] \\
& A \arrow[d, "f"] & \\
& B &
\end{tikzcd}$$

Analogously, the **cokernel** of $f : A \to B \in Hom(A, B)$ is a dual element to kernel, that is, is a morphism $k^* : B \to K^*$ such that $k^* \circ f = 0$ and, for any other morphism $\tilde{k}^* : B \to \tilde{K}^*$ with $\tilde{k}^* \circ f = 0$ there exists a morphism $h : K^* \to \tilde{K}^*$ such that $k^{*\prime} = h \circ k^*$.

$$
\begin{tikzcd}
K^* \arrow[rr, dashed, "h"] & & \tilde{K}^* \\
& B \arrow[ul, "k^*"] \arrow[ur, "\tilde{k}^*"'] & \\
& A \arrow[u, "f"] \arrow[uul, bend left, "0"] \arrow[uur, bend right, "0"'] &
\end{tikzcd}
$$

Let us suppose that, in our category $\mathcal{C}$, every morphism has kernel and cokernel. Then, given a morphism $f : A \to B$, we define the **image** of f, Im f as Im $f = Ker\, k^*$, where $B \xrightarrow{k^*} Coker\, f$ is the cokernel of f.

Definition 3.3.8. A pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is called **exact** in B if Im $f = Ker\, g$.

Definition 3.3.9. A category $\mathcal{C}$ is called **abelian** if it satifies

- For each $A, B \in Obj(\mathcal{C})$, $Hom(A, B)$ is an abelian group and the composition of morphisms is bilateral linear respecto to the group operation, that is

$$f \circ (g + h) = f \circ g + f \circ h \qquad (g + h) \circ f = g \circ f + h \circ f$$

- $\mathcal{C}$ has zero object.
- For every objects A_1 y A_2 there exists an object B and morphisms

$$
\begin{tikzcd}
A_1 \arrow[r, bend right, "i_1"'] & B \arrow[l, bend right, "p_1"'] \arrow[r, bend left, "p_2"] & A_2 \arrow[l, bend left, "i_2"]
\end{tikzcd}
$$

  such that

$$
\begin{array}{cc}
p_1 \circ i_1 = id_{A_1} & p_2 \circ i_2 = id_{A_2} \\
i_1 \circ p_1 + i_2 \circ p_2 = id_B & p_2 \circ i_1 = p_1 \circ i_2 = 0
\end{array}
$$

- Every morphism $f : A \to B$ has kernel and cokernel.
- For every morphism $f : A \to B$, we have the following decomposition in $f = j \circ i$

$$0 \to Ker\, f \xrightarrow{k} A \xrightarrow{i} \text{Im } f \xrightarrow{j} B \xrightarrow{k^*} Coker\, f \to 0$$

  where i is and epimorphism, j is a monomorphism and Im $f \cong Coker\, k \cong Ker\, k^*$. This decomposition is, usually, called the **canonical decomposition** of f.

In an abelian category, monomorphisms and epimorphisms can be characterized as in the case of groups.

Proposition 3.3.10. *In an abelian category, a morphism $f : A \to B$ is a monomorfism if and only if $Ker\, f = 0$. Analogously, it is an epimorphism if and only if $Coker\, f = 0$. Moreover, f is an isomorfism if and only if $Ker\, f = 0$ y $Coker\, f = 0$.*

Proof. Let us suppose that f is a monomorphism and $K \xrightarrow{k} A$ is its kernel, so $f \circ k = 0 = f \circ 0$. Hence, since f cancels by the left, we have $k = 0$, and, thus, $0 \to A$ satisfies the universal property of been the kernel of f. The checking for the cokernel is analogous.

Reciprocally, let us suppose that $Ker\, f = 0$. In that case, we have that $id_A : A \to A$ is the cokernel of k and, thus, there exists an isomorphism $h : A \to \text{Im } f$. Therefore, using the canonical decomposition of f, we have

$$\begin{tikzcd} 0 \arrow[r, "0"] & A \arrow[r, "i"] \arrow[d, "id_A"'] & \text{Im } f \arrow[r, "j"] & B \arrow[r, "k^*"] & 0 \\ & A \arrow[ur, "h"'] & & & \end{tikzcd}$$

so i is an isomorfism. Therefore, since j is a monomorfism, we have that $f = j \circ i$ is a monomorphis. Analogously, if $Coker\, f = 0$, then $id_B : B \to B$ is the kernel k^* and, thus, there exists an isomorfismo $h' : B \to \text{Im } f$ that resticts j to be and isomorfism. Together with i been an epimorfismo implies that f is an epimorfism. ■

Every axiom of abelian category is natural, except the thirth one. This axiom is related with the existence of direct sums and direct products, that can be formalized by means of the notion of pullback and pushout of a diagram.

Definition 3.3.11. Given morphisms $f : A \to Z$ and $g : B \to Z$, a **pullback** is a pair of morphisms $p_1 : P \to A$ and $p_2 : P \to B$ such that $f \circ p_1 = g \circ p_2$

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & Z \end{tikzcd}$$

and they are universals for this diagram, i.e., given morphisms $q_1 : P' \to A$ y $q_2 : P' \to B$, there exists a morphisms $h : P' \to P$ such that $p_1 \circ h = q_1$ and $p_2 \circ h = q_2$.

$$\begin{tikzcd}
P' \arrow[drr, bend left, "q_1"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "q_2"'] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & Z
\end{tikzcd}$$

Analogously, given morphisms $f : Z \to A$ y $g : Z \to B$ a **pushout** is a pair of morphisms $i_1 : A \to P$ and $i_2 : B \to P$ such that $i_1 \circ f = i_2 \circ g$

$$\begin{tikzcd}
P & A \arrow[l, "i_1"'] \\
B \arrow[u, "i_2"] & Z \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

and they are universals for this diagram, i.e., given morphisms $j_1 : A \to P'$ y $j_2 : B \to P'$, there exists a morphism $h : P \to P'$ such that $p_1 \circ h = q_1$ and $p_2 \circ h = q_2$.

$$\begin{tikzcd}
P' & & \\
& P \arrow[ul, dashed, "h"'] & A \arrow[l, "i_1"'] \arrow[ull, bend right, "j_1"'] \\
& B \arrow[u, "i_2"] \arrow[uul, bend left, "j_2"] & Z \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

In this way, in an abelian category, given two objects, we can form its *direct sum* and its *direct product*, and both agree. For the proof, see [24].

Proposition 3.3.12. *In an abelian category $\mathcal{C}$, for every $A, B \in Obj(\mathcal{C})$ the pullbak of the morphisms $A_1 \to 0$ y $A_2 \to 0$ and the pushout of the morphisms $0 \to A_1$ and $0 \to A_2$ agree.*

3.3.1.2 Filtrations

Definition 3.3.13. Let $\mathcal{C}$ be an abelian category[4] and let $A \in Obj(\mathcal{C})$ be and object of $\mathcal{C}$. A **decreasing filtration** of A, F_*A, is a sequence of subobjects of A, $\{F_pA\}_{p\in\mathbb{Z}}$ such that

$$A \supseteq \ldots \supseteq F_{p-1}A \supseteq F_pA \supseteq F_{p+1}A \supseteq \ldots \supseteq 0$$

[4] Usually, we will take $\mathcal{C}$ to be the category of R-modules, the category of abelian groups or the category of k-vector spaces.

In this case, (A, F_*) is called a **filtered object**. The associated **graded complex** of F_*A, $Gr_*^F A$, is the graded object $Gr_*^F A = \bigoplus_p Gr_p^F A$ where the pieces are given by

$$Gr_p^F A := \frac{F_p A}{F_{p+1} A}$$

Analogously, an **increasing filtration** of A, W^*A is a sequence of subobjects of A, $\left\{W^k A\right\}_{k\in\mathbb{Z}}$ such that

$$0 \subseteq \ldots \subseteq W^{k-1}A \subseteq W^k A \subseteq W^{k+1}A \subseteq \ldots \subseteq A$$

and de associated **graded complex** of W^*A, $Gr_W^* A$ is the graded object $Gr_W^* A = \bigoplus_p Gr_W^k A$ where the pieces are given by

$$Gr_W^k A := \frac{W^k A}{W^{p-1} A}$$

Given two objects $A, B \in Obj(\mathcal{C})$ with decreasing (resp. increasing) filtrations F_*A and F_*B (resp. W^*A and W^*B), a morphism $f : A \to B$ is called a **filtered morphism** or **morphism of filtrations** if

$$f(F_p A) \subseteq F_p B \qquad \Big(\text{resp. } f(W^k A) \subseteq W^k B\Big)$$

for all $p \in \mathbb{Z}$. In that case, we will write $f : (A, F_*) \to (B, F_*)$ (resp. $f : (A, W^*) \to (B, W^*)$). A filtration F_*A (resp. W^*A) is called **finite** if there exist $\alpha, \omega \in \mathbb{Z}$ such that

$$F_\alpha A = A \quad F_\omega A = 0 \qquad (\text{resp. } W^\alpha A = 0 \quad W^\omega A = A)$$

Remark 3.3.14. Given an increasing filtration W^*A of an object A, we can define a decreasing filtration F_*A by

$$F_p A := W^{-p} A$$

so, in practice, without lost of generality, we can suppose that our filtration is a decreasing filtration.

Example 3.3.15 ($\mathbb{R}^n$)**.** Taking $\mathcal{C}$ to be the category of $\mathbb{R}$-vector spaces, and $A = \mathbb{R}^n$, we have the decreasing filtration

$$\mathbb{R}^n \supseteq \mathbb{R}^{n-1} \times \{0\} \supseteq \ldots \supseteq \mathbb{R}^{n-k} \times \{0\}^k \supseteq \ldots \supseteq \mathbb{R}^1 \times \{0\}^{n-1} \supseteq \{0\}^n$$

In general, in a vector space V, a decreasing filtration is a sequence of decreasing vector spaces $\{V_p\}_{p\in\mathbb{Z}}$ such that, taking $F_p = V_p$

$$V \supseteq \ldots \supseteq V_{p-1} \supseteq V_p \supseteq V_{p+1} \supseteq \ldots \supseteq 0$$

Definition 3.3.16. Let $A \in Obj(\mathcal{C})$ with a decreasing filtration F_*A and let $B \subseteq A$ be a subobject of A. Then, we can define a filtration on B, called the **induced filtration in subobjects**, F_*B by taking

$$F_p B := F_p A \cap B$$

Analogously, in a quotient A/B for $B \subseteq A$, we have the **induced quotient filtration** (or simply the **quotient filtration**), F_*A/B, given by

$$F_p\left(\frac{A}{B}\right) := \pi(F_pA) = \frac{F_pA + B}{B} \cong \frac{F_pA}{F_pA \cap B} = \frac{F_pA}{F_pB}$$

where $\pi : A \to A/B$ is the passing to the quotient map.

3.3.2 The Category of Mixed Hodge Structures

Definition 3.3.17. Let $R = \mathbb{Z}$ or $\mathbb{Q}$ and let H_R be a finitely generated R-module[5]. A **mixed Hodge structure** on H_R is pair of

- A finite increasing filtration of the $\mathbb{Q}$-vector space $H_\mathbb{Q}$, $W^*H_\mathbb{Q}$, called the **weight filtration**

$$0 \subseteq \ldots \subseteq W^{k-1}H_\mathbb{Q} \subseteq W^kH_\mathbb{Q} \subseteq W^{k+1}H_\mathbb{Q} \subseteq \ldots \subseteq H_\mathbb{Q}$$

- A finite decreasing filtration of the $\mathbb{C}$-vector space $H_\mathbb{C}$, $F_*H_\mathbb{C}$, called the **Hodge filtration**

$$H_\mathbb{C} \supseteq \ldots \supseteq F_{p-1}H_\mathbb{C} \supseteq F_pH_\mathbb{C} \supseteq F_{p+1}H_\mathbb{C} \supseteq \ldots \supseteq 0$$

Such that, for each $k \in \mathbb{Z}$, the induced filtration of F_* on $Gr^k_WH_\mathbb{Q} \otimes \mathbb{C}$ gives a pure rational Hodge structure of weight k on $Gr^k_WH_\mathbb{Q}$. If $R = \mathbb{Z}$ the mixed Hodge structure is called **integral** and, if $R = \mathbb{Q}$, the Hodge structure is called **rational**.

Remark 3.3.18. Recall that the induced quotient filtration by F_* on the graded complex $Gr^k_WH_\mathbb{Q} \otimes \mathbb{C}$ is given by

$$F_p(G^k_WH_\mathbb{Q} \otimes \mathbb{C}) = \pi\left(F_pH_\mathbb{C} \cap \left(W^kH_\mathbb{Q} \otimes \mathbb{C}\right)\right) = \frac{\left(F_pH_\mathbb{C} \cap \left(W^kH_\mathbb{Q} \otimes \mathbb{C}\right)\right) + W^{k-1}H_\mathbb{Q} \otimes \mathbb{C}}{W^{k-1}H_\mathbb{Q} \otimes \mathbb{C}}$$

Remark 3.3.19. Analogously to the previous definition, we can define, when $R = \mathbb{R}$ (resp. $R = \mathbb{C}$), a **real mixed Hodge structure** (resp. **complex Hodge structure**) over a finite dimensional $\mathbb{R}$-vector space, $H_\mathbb{R}$ (resp. $\mathbb{C}$-vector space $H_\mathbb{C}$). In that case, we should take the weight filtration to be a filtration of the $\mathbb{R}$-vector space $H_\mathbb{R}$ (resp. $\mathbb{C}$-vector space $H_\mathbb{C}$) and the Hodge filtration should induce a pure real Hodge structure on $Gr^k_WH_\mathbb{R}$ (resp. a pure complex Hodge structure on $Gr^k_WH_\mathbb{C}$).

Definition 3.3.20. Given two R-modules H_R and H'_R with respective R-mixed Hodge structures, and homomorphism $f : H_R \to H'_R$ is called a **morphism of mixed Hodge structures** if $f_\mathbb{Q} : H_\mathbb{Q} \to H'_\mathbb{Q}$ (resp. $f_\mathbb{R} : H_\mathbb{R} \to H'_\mathbb{R}$ in the real case) is a filtered morphism with respect to $W^*H_\mathbb{Q}$ (resp. $W^*H_\mathbb{R}$) and $f_\mathbb{C} : H_\mathbb{C} \to H'_\mathbb{C}$ is a filtered morphism with respect to $F_*H_\mathbb{C}$.

[5] A finitely generated abelian group if $R = \mathbb{Z}$ and a finite dimensional $\mathbb{Q}$-vector space if $R = \mathbb{Q}$

With this definitions, we can form the category of R-mixed Hodge structures, denoted by $\mathbf{MHS}_R$, whose objects are finitely generated R-modules with a R-mixed Hodge structues and its morphism are the morphisms of mixed Hodge structures. It is a subcategory of $R-\mathbf{Mod}$.

One of the most importants results about the category of mixed Hodge structures is that it is well behaved with respect to kernels and cokernels. The proof of the following theorem can be found in [12] or [15].

Theorem 3.3.21 (Deligne)**.** *The category of R-mixed Hodge structures, $\boldsymbol{MHS}_R$ is an abelian category.*

Example 3.3.22. Let H_R be a pure Hodge structure of weight k, induced by a decreasing filtration $F_*H_{\mathbb{C}}$ of $H_{\mathbb{C}}$. Then H_R also has a mixed Hodge structure by taking the Hodge filtration as $F_*H_{\mathbb{C}}$ and the weight filtration as $W^sH_{\mathbb{Q}} := H_{\mathbb{Q}}$ for $s \geq k$ and $W^rH_{\mathbb{Q}} := 0$ for $r < k$. In this case, the associated graded complex of W is $Gr_W^k H_{\mathbb{Q}} = H_{\mathbb{Q}}$ and vanish otherwise, so $Gr_W^k H_{\mathbb{Q}} \otimes \mathbb{C} = H_{\mathbb{C}}$ and, indeed, the decreasing filtration $F_*H_{\mathbb{C}}$ induces a pure Hodge structure on $Gr_W^k H_{\mathbb{Q}} \otimes \mathbb{C}$. Thus, this filtrations form a mixed Hodge structure on H_R, as expected.

Example 3.3.23. More general, suppose that, a R-module H_R has a pure Hodge structure, that is, we have a grading $H_R = \oplus_k H_R^k$ and every H_R^k has a pure Hodge structure of weight k. Then, H_R also has a mixed Hodge structure. For building it, first define the weight filtration of $H_{\mathbb{Q}}$ as

$$W^k H_{\mathbb{Q}} = \bigoplus_{s \leq k} H_{\mathbb{Q}}^s$$

where $H_{\mathbb{Q}}^s = H_R^s \otimes \mathbb{Q}$. Observe that, with this definition, $Gr_W^k H_{\mathbb{Q}} = H_{\mathbb{Q}}^r$.

For the Hodge filtration, let $F_*^k H_{\mathbb{C}}^k$ be the decreasing filtration of H_R^k that induces, in H_R^k its pure Hodge structure of weight k. Let us define the double grading of subspaces

$$F_{k,q}\, H_{\mathbb{C}} = \bigoplus_{s > k} H_{\mathbb{C}}^s \oplus F_q^k H_{\mathbb{C}}^k$$

Let $A \subseteq \mathbb{Z} \times \mathbb{Z}$ be the pairs of possibles pairs (k, q) such that $F_q^k H_{\mathbb{C}}^k$ is not trivial (that is, the filtration of $H_{\mathbb{C}}^k$ has not stabilized yet in the q-th step). Since every filtration $F_*^k H_{\mathbb{C}}^k$ is finite, we can find a biyection $\sigma : \mathbb{Z} \to A$ that preserves the order with respect to the direct lexicographic order induced in $A \subseteq \mathbb{Z} \times \mathbb{Z}$[6]. Then, using this σ, we define the Hodge filtration as the decreasing filtration

$$F_p H_{\mathbb{C}} := F_{\sigma(p)} H_{\mathbb{C}}$$

[6]That, is, we want that the subspaces $F_{k,q}\, H_{\mathbb{C}}$ comes in order and before $F_{k',q}\, H_{\mathbb{C}}$ for $k < k'$, i.e.

$$\ldots \supseteq F_{k,q-1}\, H_{\mathbb{C}} \supseteq F_{k,q}\, H_{\mathbb{C}} \supseteq F_{k,q+1}\, H_{\mathbb{C}} \supseteq \ldots \supseteq F_{k',q'-1}\, H_{\mathbb{C}} \supseteq F_{k',q'}\, H_{\mathbb{C}} \supseteq F_{k',q'+1}\, H_{\mathbb{C}} \supseteq \ldots$$

for $k < k'$ and every q, q'.

In this case, observe that the induced quotient filtration of $F_*H_{\mathbb{C}}$ on $Gr^k_W H_{\mathbb{Q}} \otimes \mathbb{C} = H^k_{\mathbb{C}}$ is

$$F_p(G^k_W H_{\mathbb{Q}} \otimes \mathbb{C}) = \begin{cases} H^k_{\mathbb{C}} & \text{for } p \text{ small enough} \\ 0 & \text{for } p \text{ big enough} \\ F^k_p H^k_{\mathbb{Q}} & \text{for } p \text{ intermediate} \end{cases}$$

Therefore, the induced filtration of $F_*H_{\mathbb{C}}$ on $G^k_W H_{\mathbb{Q}} \otimes \mathbb{C} = H^k_{\mathbb{C}}$ is exactly the filtration $F^k_* H^k_{\mathbb{C}}$ (maybe shifted) so it induces a pure Hodge structure of weight k on $G^k_W H_{\mathbb{Q}} \otimes \mathbb{C}$, as expected. As a result, via this construction, we can see mixed Hodge structures as a generalization of pure Hodge structures.

Example 3.3.24. Let M be a compact Kähler manifold and $H^*(M, \mathbb{C})$ its cohomology ring, $H^*(M, \mathbb{C}) = \bigoplus_{k \geq 0} H^k(M, \mathbb{C})$. By corolary 3.2.8, every $H^k(M, \mathbb{C})$ has a pure Hodge structure of weight k so, by the previous example, $H^*(M, \mathbb{C})$ has a mixed Hodge structure. Therefore, the cohomology ring of a compact Kähler manifold has a mixed Hodge structure.

Example 3.3.25. Moreover, using the same ideas than example 3.3.22, we can build a mixed Hodge structures on direct sums of mixed Hodge structures. Suppose that we have two finitely generated R-modules H^1_R and H^2_R, with respective mixed Hodge structures. Then, we can put a canonical mixed Hodge structure on $H_R := H^1_R \oplus H^2_R$ in the following way. Let $W^*H^1_{\mathbb{Q}}$ and $W^*H^2_{\mathbb{Q}}$ be the weight filtrations and $F_*H^1_{\mathbb{C}}$, $F_*H^2_{\mathbb{C}}$ be the Hodge filtrations for H^1_R and H^2_R, respectively. Then, we define the filtrations W^*H_R and $F_*H_{\mathbb{C}}$ by

$$W^k H_R := W^k H^1_R \oplus W^k H^2_R \qquad F_p M_R := F_p H^1_R \oplus F_p H^2_R$$

Using the same technique that in example 3.3.22 we see that this filtration can be used as weight and Hodge filtrations for a mixed Hodge structure on H_R. Furthermore, in the case of a pure Hodge structure $H_R = \bigoplus_k H^k_R$, this mixed Hodge structure on H_R coincides with the one described on example 3.3.22.

Analogous considerations can be done to equip tensor products, dual spaces and homomorphisms of mixed Hodge structures with a mixed Hodge structure.

The main important theorem in this area, and the reason of living of mixed Hodge structures is the following result, whose proof can be found in [12] or [17]-[16].

Theorem 3.3.26 (Deligne)**.** *The cohomology ring of any complex algebraic variety admits a mixed Hodge structure.*

In fact, what Deligne proved is even stronger.

Theorem 3.3.27 (Deligne)**.** *Let $\textbf{Var}_{\mathbb{C}}$ be the category of complex algebraic varieties with regular morphisms and let us define the contravariant functor $H_{\mathbb{Q}} : \textbf{Var}_{\mathbb{C}} \to \textbf{Ab}$ given by $H_{\mathbb{Q}}(X) = H^k(X, \mathbb{Q})$. Moreover, for regular maps $f : X \to Y$ we define $H_{\mathbb{Q}}(f) = f^* : H^k(Y, \mathbb{Q}) \to H(X, \mathbb{Q})$. Then H*

factorices through the inclusion $\boldsymbol{MHS}_{\mathbb{Q}} \hookrightarrow \mathbb{Q} - \boldsymbol{Vect}$*, that is, there exists a contravariant functor* $\tilde{H}_{\mathbb{Q}} : \boldsymbol{Var}_{\mathbb{C}} \to \boldsymbol{MHS}_{\mathbb{Q}}$ *such that the following diagram commutes*

$$\begin{tikzcd} \boldsymbol{Var}_{\mathbb{C}} \arrow[rr, "H_{\mathbb{Q}}"] \arrow[dr, "\tilde{H}_{\mathbb{Q}}"'] & & \mathbb{Q} - \boldsymbol{Vect} \\ & \boldsymbol{MHS}_{\mathbb{Q}} \arrow[ur, hook] & \end{tikzcd}$$

Analogous considerations can be done for the compactly-supported cohomology $X \mapsto H_c^k(X, \mathbb{Q})$.

Remark 3.3.28. Noting that $H(X, \mathbb{C}) = H(X, \mathbb{Q}) \otimes \mathbb{C}$ (resp. $H_c(X, \mathbb{C}) = H_c(X, \mathbb{Q}) \otimes \mathbb{C}$), we have that the previous theorem is also valid for complex coefficients and complex Hodge structures.

$$\begin{tikzcd} \mathbf{Var}_{\mathbb{C}} \arrow[rr, "H_{\mathbb{C}}"] \arrow[dr, "\tilde{H}_{\mathbb{C}}"'] & & \mathbb{C} - \mathbf{Vect} \\ & \mathbf{MHS}_{\mathbb{C}} \arrow[ur, hook] & \end{tikzcd}$$

As we will see, this complex version of the Deligne's theorem will be the most important for our purposes. Therefore, except explicit denotation, hereon $H^*(X)$ will means complex cohomology of X, $H^*(X) := H^*(X, \mathbb{C})$.

Remark 3.3.29. Using the mapping cone of a map, the previous theorem can be extended for the category of pairs of complex algebraic varieties $\mathbf{PVar}_{\mathbb{C}}$ whose objects are elements of the form (X, U) with X a complex algebraic variety and $U \subseteq X$ a subvariety; and whose maps are regular maps of pairs $f : (X, U) \to (Y, V)$. Then, the pair cohomology functor $(X, U) \mapsto H^*(X, U; \mathbb{Q})$ also factorices through mixed Hodge structures

$$\begin{tikzcd} \mathbf{PVar}_{\mathbb{C}} \arrow[rr, "H_{\mathbb{Q}}"] \arrow[dr, "\tilde{H}_{\mathbb{Q}}"'] & & \mathbb{Q} - \mathbf{Vect} \\ & \mathbf{MHS}_{\mathbb{Q}} \arrow[ur, hook] & \end{tikzcd}$$

and analogously for complex cohomology.

Remark 3.3.30. By GAGA theory, every compact Kähler manifold whose Kähler form is integral (which are called Hodge manifolds) is a projective algebraic variety so, by theorem 3.3.27, its cohomology ring has a mixed Hodge structure. This observation agrees with the fact that, as a compact Kähler manifold, its cohomology ring is has a pure Hodge structure (see corolary 3.2.9) which induces a mixed Hodge structure.

A very useful tool that we will use in order to compute the mixed Hodge structures of the cohomology of some complex varieties is a long sequence satisfied by the mixed Hodge structures. The proof can be found in [60] and the cohomological background can be readed in [18].

Theorem 3.3.31. *Let X be a complex algebraic variety and $U \subseteq X$ be a subvariety. Then, the induced map in cohomology by the inclusion map $i : U \subseteq X$ gives us a long exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,U) \arrow[r] & H^0(X) \arrow[r] & H^0(U) \arrow[dll, "\delta"] \\
& H^1(X,U) \arrow[r] & H^1(X) \arrow[r] & H^1(U) \arrow[dll, dashed] \\
& H^n(X,U) \arrow[r] & H^n(X) \arrow[r] & H^n(U) \arrow[r] & \cdots
\end{tikzcd}$$

that is also a long exact sequence of mixed Hodge structures.

Moreover, given a tiple (X, U, V) with $V \subseteq U \subseteq X$, the inclusion maps of pairs

$$(U, \emptyset) \hookrightarrow (U, V) \overset{j}{\hookrightarrow} (X, V) \overset{i}{\hookrightarrow} (X, U)$$

induced a long exact sequence in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,U) \arrow[r, "i^*"] & H^0(X,V) \arrow[r, "j^*"] & H^0(U,V) \arrow[dll, "\delta"] \\
& H^1(X,U) \arrow[r, "i^*"] & H^1(X,V) \arrow[r, "j^*"] & H^1(U,V) \arrow[dll, dashed] \\
& H^n(X,U) \arrow[r, "i^*"] & H^n(X,V) \arrow[r, "j^*"] & H^n(U,V) \arrow[r] & \cdots
\end{tikzcd}$$

that is also a long exact sequence of mixed Hodge structures.

Now, recall that, by the compactly-supported excision property, we have that, if X is an algebraic variety (or manifold, or even more general spaces) and $Y \subseteq X$ is closed, then the inclusion map of pairs $(X - Y, \emptyset) \hookrightarrow (X, Y)$ induces in cohomology an isomorphism

$$H^*(X, Y) \overset{\cong}{\to} H_c^*(X - Y)$$

Moreover, since the mixed Hodge structure on the compactly supported cohomology is induced from the one on pairs (see [60]) this isomorphism is also an isomorphism of mixed Hodge structures. Therefore, we have the following corolary.

Corollary 3.3.32. *Let X be a complex algebraic variety and $Y \subseteq X$ a closed subvariety. Then, we have a long exact sequence of mixed Hodge structures in compactly-supported cohomology*

$$\begin{tikzcd}
0 \arrow[r] & H^0_c(X-Y) \arrow[r] & H^0_c(X) \arrow[r] & H^0_c(Y) \arrow[dll, rounded corners, "\delta"] & \\
 & H^1_c(X-Y) \arrow[r] & H^1_c(X) \arrow[r] & H^1_c(Y) \arrow[ddll, dashed, rounded corners] & \\
 & & & & \\
 & H^n_c(X-Y) \arrow[r] & H^n_c(X) \arrow[r] & H^n_c(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Let $\tilde{X} \supseteq X$ be a compactification of X (for example, by projectivization) and let $\tilde{Y} \subseteq Y$ the closure of Y in $\tilde{X}$. Then, via the previously described isomorphism, the desired long exact sequence is the long exact sequence for the triple $(\tilde{X}, \tilde{Y} \cup (\tilde{X} - X), \tilde{X} - X)$. ■

3.3.3 Deligne-Hodge Polynomials

Let X be a complex variety and let $H^*(X, \mathbb{C})$ be its complex cohomology ring. A very important invariant that can be computed using the mixed Hodge structure of $H^*(X, \mathbb{C})$ is the well known as Deligne-Hodge polynomial, or E-polynomial. In order to define it, we have to define some numerical invariants associated to the mixed Hodge structure.

Definition 3.3.33. Let H_R be a finitely generated R-module with a R-mixed Hodge structure on it, given by weight filtration W^*H and Hodge filtration F_*H. We define the **Hodge pieces** associated to this mixed Hodge structures as the $\mathbb{C}$-vector spaces

$$H^{p,q}(H_R) := Gr_p^F \left(Gr_W^{p+q} H \otimes \mathbb{C} \right)$$

and we define the **mixed Hodge numbers**, or simply **Hodge numbers**, as

$$h^{p,q}(H_R) := \dim_{\mathbb{C}} H^{p,q}(H_R)$$

Remark 3.3.34. Recall that, in general, given a finite filtration (increasing or decreasing) F_*V for some finite dimensional vector space V, we have that

$$Gr_*^F V = \bigoplus_p Gr_p^F V = \bigoplus_p \frac{F_p V}{F_{p+1} V} \cong V$$

so, in particular, $\dim Gr^* V = \dim V$. Now, if we take a finitely generated R-module H_R with a mixed Hodge structure on it, we have

$$\bigoplus_{p,q} H^{p,q}(H_R) = \bigoplus_p \bigoplus_q Gr_p^F \left(Gr_W^{p+q} H \otimes \mathbb{C} \right) \cong \bigoplus_p Gr_p^F \left(\bigoplus_q Gr_W^{p+q} H \otimes \mathbb{C} \right) \cong \bigoplus_p Gr_p^F (H_{\mathbb{C}}) \cong H_{\mathbb{C}}$$

so, in particular, taking dimensions

$$\sum_{p,q} h^{p,q}(H_R) = \dim_{\mathbb{C}} H_{\mathbb{C}}$$

Definition 3.3.35. Let X be a complex variety and let $H^k(X, \mathbb{C})$ be its k-th complex cohomology group and $H_c^k(X, \mathbb{C})$ its compactly-supported k-th complex cohomology group, both endowed with the mixed Hodge structures given by theorem 3.3.26. We define the **Hodge pieces** of X as the $\mathbb{C}$-vector spaces

$$H^{k;p,q}(X) := H^{p,q}(H^k(X, \mathbb{C})) \qquad H_c^{k;p,q}(X) := H^{p,q}(H_c^k(X, \mathbb{C}))$$

and we define the **mixed Hodge numbers**, or simply **Hodge numbers**, of X as

$$h^{k;p,q}(X) := h^{p,q}(H^k(X, \mathbb{C})) \qquad h_c^{k;p,q}(X) := h^{p,q}(H_c^k(X, \mathbb{C}))$$

Moreover, we define the **Euler-Hodge characteristic** as

$$\chi_c^{p,q}(X) := \sum_k (-1)^k h_c^{k;p,q}(X)$$

Definition 3.3.36. Let X be a complex variety and let $H^k(X, \mathbb{C})$ be its k-th complex cohomology group and $H_c^k(X, \mathbb{C})$ its compactly-supported k-th complex cohomology group, both endowed with their respective mixed Hodge structures. We define the **mixed Hodge polynomial** of X (resp. with compact support) as $H(X) \in \mathbb{Z}[t, u, v]$ (resp. $H_c(X)$) given by

$$H(X)(t, u, v) = \sum_{k,p,q} h^{k;p,q}(X)\, t^k u^p v^q \qquad \left(\text{resp. } H_c(X)(u, v, t) = \sum_{k,p,q} h_c^{k;p,q}(X)\, t^k u^p v^q \right)$$

From this polynomial, we define the **Deligne-Hodge polynomial** or the E-**polynomial** of X as the polynomial $e(X) \in \mathbb{Z}[u, v]$ given by

$$e(X)(u, v) = H_c(X)(-1, u, v) = \sum_{p,q} \chi_c^{p,q}(X)\, u^p v^q$$

Remark 3.3.37. Sometimes, in the literature the Deligne-Hodge polynomial is refered as $e(X)(-u, -v)$. We will not use this criterion anytime.

Remark 3.3.38. By remark 3.3.34, we have that, for all $k \geq 0$

$$\sum_{p,q} h^{k;p,q}(X) = b^k(X) \qquad \sum_{p,q} h_c^{k;p,q}(X) = b_c^k(X) \qquad \sum_{p,q} \chi_c^{p,q}(X) = \chi_c(X)$$

where $b^k(X)$ is the k-th Betti number, $b_c^k(X)$ is the k-th Betti number with compact support and $\chi_c(X)$ is the compactly supported Euler characteristic of X. In particular, if $P(X) \in \mathbb{Z}[t]$ is the Poincaré polynomial of X and $P_c(X)$ the compactly supported one, we have

$$P(X)(t) = H(t, 1, 1) \qquad P_c(X)(t) = H_c(t, 1, 1)$$

3.3.3.1 Properties of Deligne-Hodge polynomials

One of the main properties of the Euler characteristic of a topological space X is that it is additive, that is, if $X = X_1 \sqcup X_2$ then $\chi(X) = \chi(X_1) + \chi(X_2)$. This property can be extended to the case of mixed Hodge structures obtaining the following theorem, whose proof can be found in [16].

Theorem 3.3.39 (Deligne)**.** *The Deligne-Hodge polynomial is additive. That is, if X is a complex variety that can be writen as $X = X_1 \sqcup X_2$, where X_1 and X_2 are locally closed in X, then*

$$e(X) = e(X_1) + e(X_2)$$

Remark 3.3.40. Recall that a subspace $Y \subseteq X$ is locally closed in X if Y is a closed set of an open set of X. In the case of algebraic varieties, $Y \subseteq X$ is locally closed means that Y is an open set of a subvariety of X (a quasi-subvariety). In the practice, this means that Y is the space determined by a set of polynomials equaties and negation of equalities that contains the ones of X.

Corollary 3.3.41. *The following polynomials hold:*

- $e(\mathbb{C}^n)(u, v) = (uv)^n$.
- *If $A \subseteq \mathbb{C}^n$ is a finite set of points, then $e(\mathbb{C}^n - A)(u, v) = (uv)^n - |A|$.*
- $e(\mathbb{P}^n)(u, v) = 1 + uv + (uv)^2 + \ldots + (uv)^n = \frac{1-(uv)^n}{1-uv}$

Proof. First of all, let us compute the Hodge polynomial of $\mathbb{P}^1$. Recall that, using basic techniques of algebraic topology (for example, an argument using a Mayer-Vietoris sequence) we have that

$$H^0_{\mathbb{C}}(\mathbb{P}^1) \cong \mathbb{C} \qquad H^1_{\mathbb{C}}(\mathbb{P}^1) \cong 0 \qquad H^2_{\mathbb{C}}(\mathbb{P}^1) \cong \mathbb{C}$$

Now, let us endow $\mathbb{P}^1$ with the Fubini-Study metric, becoming a Kähler manifold. By the Hodge decomposition theorem for pure Hodge structures on compact Kähler manifolds 3.2.8 we have that $H^0_{\mathbb{C}}(\mathbb{P}^1) \cong H^{0,0}(\mathbb{P}^1)$ and $H^2_{\mathbb{C}}(\mathbb{P}^1) \cong H^{2,0}(\mathbb{P}^1) \oplus H^{1,1}(\mathbb{P}^1) \oplus H^{0,2}(\mathbb{P}^1)$. However, since $\mathbb{P}^1$ has complex dimension 1, $\Omega^{2,0}(\mathbb{P}^1) = \Omega^{0,2}(\mathbb{P}^1) = 0$, so $H^{2,0}(\mathbb{P}^1) = H^{0,2}(\mathbb{P}^1) = 0$. Hence, the only non-trivial Dolbeaut-cohomology groups of $\mathbb{P}^1$ are

$$H^{0,0}(\mathbb{P}^1) \cong \mathbb{C} \qquad\qquad H^{1,1}(\mathbb{P}^1) \cong \mathbb{C}$$

or, in terms of the induced mixed Hodge structure, we have the Hodge numbers

$$h^{0;0,0}(\mathbb{P}^1) = h_c^{0;0,0}(\mathbb{P}^1) = 1 \qquad\qquad h^{2;1,1}(\mathbb{P}^1) = h_c^{2;1,1}(\mathbb{P}^1) = 1$$

so the only non-trivial Euler characteristics are $\chi_c^{0,0} = 1$ and $\chi_c^{1,1} = 1$. Thus, we have obtain the Deligne-Hodge polynomial

$$e(\mathbb{P}^1)(u,v) = 1 + uv$$

Furthermore, since $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$ we have that, by additivity of the Deligne-Hodge polynomial

$$e(\mathbb{C})(u,v) = e(\mathbb{P}^1) - e(\star) = 1 + uv - 1 = uv$$

as expected.

For the general case of the n-dimensional projective space $\mathbb{P}^n$, we can use a similar argument. By the same reason that the previous case, it can be shown that the cohomology of $\mathbb{P}^n$ is $H^{2k}(\mathbb{P}^n) \cong \mathbb{R}$ if $k = 0, \ldots, n$ and vanish otherwise. Now, if we endow $\mathbb{P}^n$ with the Fubini-Study metric, it becomes a Kähler manifold with Kähler form $\omega \in \Omega^2(\mathbb{P}^n)$. Observe that, for $k = 0, \ldots, n$, $[\omega^k] \in H^{2k}(\mathbb{P}^n)$ is the generator of the corresponding cohomology group. Indeed, since ω is a symplectic form, it is closed, so $d\omega^k = 0$. Moreover, ω^k is not exact. To see this, observe that, if ω^k would be exact for some k, let us say $\omega^k = d\eta$ for $\eta \in \Omega^{2k-1}(\mathbb{P}^1)$, then we will have

$$\omega^n = \omega^k \wedge \omega^{n-k} = d\eta \wedge \omega^{n-k} = d\left(\eta \wedge \omega^{n-k}\right)$$

But this is impossible, because, since ω is not degenerated, ω^n is a volume form so, in particular $\int_{\mathbb{P}^1} \omega^n \neq 0$, contradicting the Stokes' theorem. Hence, ω^k is a closed non-exact $2k$-form and, since $H^{2k}(\mathbb{P}^n) \cong \mathbb{R}$, we have that $H^{2k}(\mathbb{P}^n) = [\omega^k]$.

For the induced pure Hodge structure, observe that every Kähler form lives in the $(1,1)$-part of the forms, (see remark A.3.4), so $\omega \in \Omega^{1,1}(\mathbb{P}^n)$ and $\omega^k \in H^{k,k}(\mathbb{P}^n)$ for $k = 0, \ldots, n$. Hence, in terms of the Dolbeaut cohomology we have the non-trivial Dolbeaut class $0 \neq [\omega^k] \in H^{k,k}(\mathbb{P}^n)$. Therefore, using the pure decomposition given by 3.2.8, we have $H^{2k}_{\mathbb{C}}(\mathbb{P}^n) \cong \bigoplus_{p+q=2k} H^{p,q}(\mathbb{P}^n)$, so we obtain that the only non-trivial Dolbeaut cohomology groups are, for $k = 0, \ldots, n$

$$H^{k,k}(\mathbb{P}^n) \cong \mathbb{C}$$

or, in terms of Hodge numbers $h^{2k;k,k}(\mathbb{P}^n) = h_c^{2k;k,k}(\mathbb{P}^n) = 1$ and vanishing otherwise. Thus, $\chi_c^{k,k}(\mathbb{P}^n) = 1$ for $k = 0, \ldots, n$ and vanish otherwise, giving the Deligne-Hodge polynomial

$$e(\mathbb{P}^n)(u,v) = 1 + uv + (uv)^2 + \ldots + (uv)^n$$

as expected.

For the case of $\mathbb{C}^n$, observe that, removing an affine hypersurface of $\mathbb{P}^n$, we have the decomposition

$$\mathbb{P}^n = \mathbb{P}^{n-1} \sqcup \mathbb{C}^n$$

so, in particular, by theorem 3.3.39, $e(\mathbb{P}^n) = e(\mathbb{P}^{n-1}) + e(\mathbb{C}^n)$ obtaining

$$e(\mathbb{C}^n)(u,v) = e(\mathbb{P}^n)(u,v) - e(\mathbb{P}^{n-1})(u,v) = (uv)^n$$

Finally, if $A \subseteq \mathbb{C}^n$ is a finite set with cardinal $|A|$ then, since the Deligne-Hodge polynomial is additive and $e(A)(u,v) = |A|$ we have

$$(uv)^n = e(\mathbb{C}^n)(u,v) = e(\mathbb{C}^n - A)(u,v) + e(A)(u,v) = e(\mathbb{C}^n - A)(u,v) + |A|$$

■

Another important property of the Euler characteristic is that it is multiplicative. That is, if $X = Y \times Z$ then $\chi(X) = \chi(Y)\chi(Z)$. This property translates into mixed Hodge structures via some kind of Künneth formula. The proof of this result can be found in [60].

Theorem 3.3.42 (Künneth formula)**.** *Let X, Y be complex algebraic varieties endowed with their respective mixed Hodge structures. Then, for any $k \in \mathbb{N}$ and $p, q \in \mathbb{Z}$ we have*

$$H_c^{K;p,q}(X \times Y) \cong \bigoplus_{\substack{k=k_1+k_2 \\ p=p_1+p_2 \\ q=q_1+q_2}} H_c^{k_1;p_1,q_1}(X) \otimes H_c^{k_2;p_2,q_2}(Y)$$

In particular, we have

$$h_c^{k;p,q}(X \times Y) = \sum_{k,p,q} h_c^{k_1;p_1,q_1}(X) \cdot h_c^{k_2;p_2,q_2}(Y)$$

Corollary 3.3.43. *Let X, Y be complex algebraic varieties endowed with Deligne-Hodge polynomials $e(X), e(Y) \in \mathbb{Z}[u,v]$. Then, we have*

$$e(X \times Y) = e(X)\, e(Y)$$

Furthermore, this multiplicative property of the Euler characteristic (and, thus, of the Deligne-Hodge polynomial) can be extended using spectral sequences, it can be proved (see [72]) that, if we have an orientable fibration $X \to B$ with fiber F and B path connected, then again $\chi(X) = \chi(B)\chi(F)$. This property of the Euler characteristic again can be extended to the case of mixed Hodge structures under rather general hypothesis. The proof can be found in [47].

Definition 3.3.44. Let X, B and F be smooth algebraic varieties and let $\pi : X \to B$. π is called a **semi-algebraic fibration** if π is an algebraic map and $\pi : X \to B$ is an holomorphic bundle with fiber F. This means that, seen X, B and F as complex manifolds with the analytical topology, for every $b \in B$, there exists an neighbourhood (in the analytical topology) $U \subseteq B$ of b such that $\pi^{-1}(U)$ is biholomorphic to $U \times F$.

Theorem 3.3.45 (Logares-Muñoz-Newstead)**.** *Let X, B and F be smooth algebraic varieties and let $X \to B$ be a semi-algebraic fibration with fiber F. If the action of $\pi_1(B)$ on $H^*_c(F)$ is trivial, then*

$$e(X) = e(B)e(F)$$

Remark 3.3.46. Since the previously theorem will be heavily used, we are going to call ***E*-fibration** to any fibration $\pi : X \to B$ with fiber F that satisfies the hypotesis of theorem 3.3.45. Supposing that π is an algebraic morphism, some cases where the hypothesis are satisfied are:

- B is simply-connected.
- B is irreducible and π is an algebraic bundle, that is, locally trivial in the Zariski topology.
- $F = \mathbb{P}^n$ for some $n > 0$. This was proven in [54] by Muñoz, Ortega and Vázquez-Gallo.
- π is a principal G-bundle with G a connected algebraic group. Indeed, let us fix $b \in B$ and observe that any loop on B around b is associated, up to homotopy, to an automorphism of F_b, which is the action of some element $g \in G$. Since G is connected, if $\gamma : [0,1] \to G$ is a path between g and the identity element $e \in G$, then $H : F_b \times [0,1] \to F_b$ given by $H(y,t) = \gamma(t) \cdot y$ is an homotopy between $g\cdot : F_b \to F_b$ and $id_{F_b} : F_b \to F_b$. Therefore, the action of $\pi_1(B)$ on $H^*(F)$ is trivial.

Remark 3.3.47. The case of principal bundles, apart from the obvious use, will be also used in the following way. Let us suppose that we have algebraic varieties $Y \subseteq X$, an algebraic group G and a connected subgroup $H \subseteq G$. Suppose that G acts algebraically and freely on X such that $G \cdot Y = X$ and, for all $y \in Y$

$$H \cdot y = G \cdot y \cap Y$$

Observe that, in particular, $H \cdot Y = Y$ and $X/G \cong Y/H$.

In this situation, we define the right action of H on $G \times Y$ by $(g, y) \cdot h = (gh, h^{-1} \cdot y)$. This is a free action and the *action of G on Y* map $\rho : G \times Y \to X$ is an H-invariant map. Moreover, if we take $x_0 \in X$, let us say $x_0 = g_0 \cdot y_0$ for some $y_0 \in Y$ and $g_0 \in G$. Then, we have

$$\rho^{-1}(x_0) = \{(g, y) \in G \times Y \,|\, g \cdot y = x_0\} = \{(g, y) \in G \times Y \,|\, g \cdot y = y_0\} = \rho^{-1}(y_0) \cong H \cdot y_0$$

so H acts transitively on the fibers and, locally, ρ is a H-equivariant fiber bundle.

Therefore, we have an H-principal bundle

$$H \to G \times Y \xrightarrow{\rho} X$$

with ρ algebraic. Hence, by remark 3.3.46 this is an E-fibration, so

$$e(G)e(Y) = e(G \times Y) = e(X)e(H)$$

or, equivalently

$$e(X) = e(Y)e\left(G/H\right)$$

Remark 3.3.48. The hypothesis of theorem 3.3.45 of $\pi_1(B)$ acting trivially on $H_c^*(F)$ is absolutely necessary. For example, let us take the affine hyperbola

$$X = \{4xy = 1\} = \left\{(x+y)^2 - (x-y)^2 = 1\right\} \subseteq \mathbb{C}^2$$

Of course, the map $t \mapsto (t, 4t^{-1})$ for $t \in \mathbb{C} - \{0\}$ is a biregular isomorphism $X \cong X$ so $e(X) = e(\mathbb{C}^*) = uv - 1$.

However, let us consider the fibration $\pi : X \to \mathbb{C}$ given by $\pi(x, y) = x - y$. If $t \in \mathbb{C} - \{\pm i\}$ we have that $\pi^{-1}(t) = \left\{(x, y) \in \mathbb{C}^2 \mid (x+y)^2 = 1 + t^2, x - y = t\right\} \cong \mathbb{C} \times \mathbb{Z}_2$, Therefore, taking $X_1 = \pi^{-1}(\mathbb{C} - \{\pm i\})$ we have the semi-algebraic fibration

$$\mathbb{C} \times \mathbb{Z}_2 \to X_1 \xrightarrow{\pi} \mathbb{C} - \{\pm 1\}$$

Moreover, for $\pm i$, we have $\pi^{-1}(\pm i) = \left\{(x, y) \in \mathbb{C}^2 \mid (x+y)^2 = 0, x - y = \pm i\right\} \cong \mathbb{C}$. Hence, if we define $X_2 = X - X_1$, then $X_2 = \pi^{-1}(i) \sqcup \pi^{-1}(-i) \cong \mathbb{C} \sqcup \mathbb{C}$ so, by additivity, $e(X_2) = e(\mathbb{C}) + e(\mathbb{C}) = 2uv$.

Now, if π was an E-fibration on $\mathbb{C} - \{\pm 1\}$, then we will have $e(X_1) = e(\mathbb{C} - \{\pm 1\})e(\mathbb{C} \times \mathbb{Z}_2) = 2uv(uv - 2) = 2u^2v^2 - 4$. This, together with the true computation $e(X_2) = 2uv$ will give us $e(X) = e(X_1) + e(X_2) = 2u^2v^2 - 4 + 2uv \neq uv - 1$. Of course the application of theorem 3.3.45 fails because $\pi_1(\mathbb{C} - \{\pm i\})$ does not act trivially on $H_c^*(\mathbb{C} \times \mathbb{Z}_2) = H_c^*(\mathbb{C}) \otimes H_c^*(\mathbb{C})$ since the non-trivial elemental loops interchange the two copies.

Remark 3.3.49. Using this multiplicative property of the Deligne-Hodge polynomials, the computation of this polynomials for $\mathbb{C}^n$ and $\mathbb{P}^n$ can be substantially simplified. Indeed, in the first part of the proof of corolary 3.3.41, we obtained

$$e(\mathbb{C})(u, v) = uv \qquad\qquad e(\mathbb{P}^1)(u, v) = 1 + uv$$

Now, by the multiplicative property

$$e(\mathbb{C}^n)(u,v) = e(\overbrace{\mathbb{C}\times\ldots\times\mathbb{C}}^{n \text{ times}})(u,v) = (e(\mathbb{C})(u,v))^n = (uv)^n$$

and, using the decomposition

$$\mathbb{P}^n = \{\star\} \sqcup \mathbb{C} \sqcup \mathbb{C}^2 \sqcup \ldots \sqcup \mathbb{C}^n$$

the Deligne-Hodge polynomial of $\mathbb{P}^n$ easily follows.

This two properties of the Deligne-Hodge polynomial, additivity and multiplicativity, are the main ingredients of a powerful technique that allow us to study the cohomological property of algebraic varieties by stratifying the space in simpler pieces to which we can easily compute their Hodge-Deligne polynomial. This is known as the **stratification technique**, first developed by P. Newstead, M. Logares and V. Muñoz in [47] and extended in [46] or [51], [49], [50] and [52], and the main tool used for studing the character varieties of this Master's thesis.

Finally, a very important simplification that we will encounter in our computations is that all the Deligne-Hodge polynomials will only depends on uv. In general, if the Deligne-Hodge polynomial of X only depends on uv, then X is said to be of **balanced type** or of **Hodge-Tate type**. In that case, the Deligne-Hodge polynomial of X is writen using the change of variables $q := uv$, considering $e(X)(q)$. For example, by the previous example 3.3.41, we have that $\mathbb{C}^n$ and $\mathbb{P}^n$ are of balanced type and

$$e(\mathbb{C}^n)(q) = q^n \qquad e(\mathbb{P}^n)(q) = 1 + q + q^2 + \ldots + q^n = \frac{1-q^n}{1-q}$$

Moreover, if we only use spaces of balanced type, then all the spaces that can be constructed form them are going to be of balanced type. In practice, this allow us to assure that all the spaces that will appear in the computations of chapter 4 will be of balanced type. More precisely, the assertion is the following, and its proof can be found in [47].

Proposition 3.3.50. *Let X, Y, U be algebraic varieties with $Y, U \subseteq X$, Y closed in X and $X = Y \sqcup U$. Then, if two of the spaces are of balanced type, then is the third. Moreover, if $F \to X \to B$ is a E-fibration and B, F are of balanced type, then X is of balanced type.*

3.3.3.2 Deligne-Hodge polynomials via equivariant methods

Another important tool that we will need for the computations of chapter 4 is a method for computing the Deligne-Hodge polinomial of a variety X quotiented by an action of $\mathbb{Z}_2$. This method is an application of a more general setting known as equivariant cohomology.

Let us suppose that we have a variety X and an action of $\mathbb{Z}_2$ on X. Let us introduce the auxiliar polynomials

$$e(X)^+ := e(X/\mathbb{Z}_2) \qquad e(X)^- := e(X) - e(X)^+$$

Then, the key point is that we can extend our theorem of E-fibrations to this equivariant context. For the proof of this result, see [47].

Theorem 3.3.51. *Let X be an algebraic variety with an action of $\mathbb{Z}_2$ on it. Let B a smooth irreducible variety and let F be a variety. Suppose that there exists an E-fibration $F \to X \stackrel{\pi}{\to} B$, an algebraic fibration $F \to X/\mathbb{Z}_2 \stackrel{\tilde{\pi}}{\to} B/\mathbb{Z}_2$ and $2:1$-maps $X \stackrel{\rho}{\to} X/\mathbb{Z}_2$ and $B \stackrel{\tilde{\rho}}{\to} B/\mathbb{Z}_2$ such that the following diagram holds*

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "\rho"] \arrow[d, "\pi"'] & & X/\mathbb{Z}_2 \arrow[d, "\tilde{\pi}"] \\ B \arrow[rr, "\tilde{\rho}"] & & B/\mathbb{Z}_2 \end{tikzcd}$$

Then, we have that

$$e(X)^+ = e(X/\mathbb{Z}_2) = e(F)^+ e(B)^+ + e(F)^- e(B)^-$$

where the action of $\mathbb{Z}_2$ on F is the induced action.

Remark 3.3.52. In the hypotesis of theorem 3.3.51. Since $\pi : X \to B$ is an E-fibration, we have

$$e(X) = e(F)e(B)$$

and, since $e(Y) = e(Y)^+ + e(Y)^-$ for $Y = F, B$ we also obtain that

$$e(X)^- = e(X) - e(X)^+ = e(F)^+ e(B)^- + e(F)^- e(B)^+$$

3.3.3.3 Mayer-Vietories type arguments for mixed Hodge structures

Finally, let us observe that, in the notation of this section, the long exact sequence of corollary 3.3.32 can be restated in the following useful terms.

Proposition 3.3.53. *Let X be a complex algebraic variety and $Y \subseteq X$ a closed subvariety. Then, for every $p, q \in \mathbb{Z}$, we have a long exact sequence of Hodge pieces*

$$\begin{tikzcd} 0 \arrow[r] & H_c^{0;p,q}(X-Y) \arrow[r] & H_c^{0;p,q}(X) \arrow[r] & H_c^{0;p,q}(Y) \arrow[dll, "\delta"'] \\ & H_c^{1;p,q}(X-Y) \arrow[r] & H_c^{1;p,q}(X) \arrow[r] & H_c^{1;p,q}(Y) \arrow[dll, dashed] \\ & H_c^{n;p,q}(X-Y) \arrow[r] & H_c^{n;p,q}(X) \arrow[r] & H_c^{n;p,q}(Y) \arrow[r] & \cdots \end{tikzcd}$$

Example 3.3.54. We will finish this section with some computations of Hodge numbers for very simple and important spaces. First of all, recall that, since $\mathbb{P}^n$ is a compact Kähler manifold, it has a pure Hodge structure that is also a mixed Hodge structure. From the computations of corollary 3.3.41, we know that

$$H_c^{2k;k,k}(\mathbb{P}^n) = \mathbb{C}$$

for $k = 0, \dots, n$ and vanish otherwise. In particular, for $\mathbb{P}^1$ we have that the only non-trivial vector spaces are $H_c^{0;0,0}(\mathbb{P}^1) = H_c^{2;1,1}(\mathbb{P}^1) = \mathbb{C}$.

Using this information and the long exact sequence of proposition 3.3.53, we can compute the mixed Hodge structure of $\mathbb{C}$. To this end, observe that $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$. Thus, taking the long exact sequence of proposition 3.3.53 with $(p, q) = (0, 0)$ have that the only non-trivial part of this sequence is

$$\begin{tikzcd} 0 \arrow[r] & H_c^{0;0,0}(\mathbb{C}) \arrow[r] & H_c^{0;0,0}(\mathbb{P}^1) \arrow[r] \arrow[d, equal] & H_c^{0;0,0}(\{\infty\}) \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{C} & \mathbb{C} & \end{tikzcd}$$

so $H^{0;0,0}(\mathbb{C}) = 0$. Analogously, for $(p, q) = (1, 1)$, the only non-trivial part of the long exact sequence is

$$\begin{tikzcd} 0 \arrow[r] & H_c^{2;1,1}(\mathbb{C}) \arrow[r] & H_c^{2;1,1}(\mathbb{P}^1) \arrow[r] \arrow[d, equal] & H_c^{2;1,1}(\{\infty\}) \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{C} & 0 & \end{tikzcd}$$

so $H_c^{2;1,1}(\mathbb{C}) \cong H_c^{2;1,1}(\mathbb{P}^1) \cong \mathbb{C}$. Thus, the unique non-trivial mixed Hodge group in the induced mixed Hodge structure of $\mathbb{C}$ is

$$H_c^{2,1,1}(\mathbb{C}) \cong \mathbb{C}$$

Finally, using the same ideas, we can compute the mixed Hodge structure of $\mathbb{C}^* = \mathbb{C} - \{0\}$. Again, for $(p, q) = (1, 1)$, the only non-trivial part of the induced long exact sequence is

$$\begin{tikzcd} 0 \arrow[r] & H_c^{2;1,1}(\mathbb{C}^*) \arrow[r] & H_c^{2;1,1}(\mathbb{C}) \arrow[r] \arrow[d, equal] & H_c^{2;1,1}(\{0\}) \arrow[r] \arrow[d, equal] & 0 \\ & & \mathbb{C} & 0 & \end{tikzcd}$$

so $H_c^{2;1,1}(\mathbb{C}^*) \cong H_c^{2;1,1}(\mathbb{C}) \cong \mathbb{C}$. However, for $(p,q) = (0,0)$, the situation is slightly more dificult, because we have the long exact sequence

$$\begin{tikzcd}
 & & 0 \arrow[d, equal] & \mathbb{C} \arrow[d, equal] \\
0 \arrow[r] & H_c^{0;0,0}(\mathbb{C}^*) \arrow[r] & H_c^{0;0,0}(\mathbb{C}) \arrow[r] & H_c^{0;0,0}(\{0\}) \arrow[dll, rounded corners] \\
 & H_c^{1;0,0}(\mathbb{C}^*) \arrow[r] & H_c^{1;0,0}(\mathbb{C}) \arrow[r] \arrow[d, equal] & H_c^{1;0,0}(\{0\}) \arrow[d, equal] \\
 & & 0 & 0
\end{tikzcd}$$

so $H_c^{0;0,0}(\mathbb{C}^*) = 0$ and $H_c^{1;0,0}(\mathbb{C}^*) \cong H_c^{1;0,0}(\{0\}) \cong \mathbb{C}$. In this way, we have that the unique non-trivial mixed Hodge structure groups of $\mathbb{C}^*$ are

$$H_c^{1;0,0}(\mathbb{C}^*) \cong \mathbb{C} \qquad\qquad H_c^{2;1,1}(\mathbb{C}^*) \cong \mathbb{C}$$

Example 3.3.55. In fact, inspecting the previous computations, the situation is completelly general. Let X be any algebraic variety and let us take $\star \in X$. Then, using proposition 3.3.53, we have that, for $(p,q) \neq (0,0)$ and $k = 0, \ldots$ it holds

$$H_c^{k;p,q}(X - \{\star\}) \cong H_c^{k;p,q}(X)$$

Moreover, for $(p,q) = (0,0)$, in the case of $k \geq 2$ we again have

$$H_c^{k;0,0}(X - \{\star\}) \cong H_c^{k;0,0}(X)$$

However, for the other two groups, we can only say that the following exact sequence holds

$$\begin{tikzcd}
 & & & \mathbb{C} \arrow[d, equal] \\
0 \arrow[r] & H_c^{0;0,0}(X - \{\star\}) \arrow[r] & H_c^{0;0,0}(X) \arrow[r] & H_c^{0;0,0}(\{\star\}) \arrow[dll, rounded corners] \\
 & H_c^{1;0,0}(X - \{\star\}) \arrow[r] & H_c^{1;0,0}(X) \arrow[r] & H_c^{1;0,0}(\{\star\}) = 0
\end{tikzcd}$$

In particular, we have that

$$h_c^{0;0,0}(X - \{\star\}) + h_c^{1;0,0}(X) + 1 = h_c^{1;0,0}(X - \{\star\}) + h_c^{0;0,0}(X)$$

Chapter 4

$SL(2,\mathbb{C})$-Character Varieties

4.1 Stratification of $SL(2,\mathbb{C})$

Recall that $SL(2,\mathbb{C})$, the **special linear group** of order 2, is the group of complex-valued square matrices of order 2 with determinant 1. That is,

$$SL(2,\mathbb{C}) = \{M \in GL(2,\mathbb{C}) \mid det(M) = 1\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}) \mid ad - bc = 1 \right\}$$

where $\mathbb{M}_2(\mathbb{C})$ is the space of complex-valued square matrices of order 2.

In a differentiable setting, $SL(2,\mathbb{C})$ is a complex Lie group of complex dimension 3 (i.e. real dimension 6), seen as a closed subgroup of the Lie group $GL(2,\mathbb{C})$. In this case, its analytic topology is the subspace topology when we look $SL(2,\mathbb{C}) \subseteq \mathbb{C}^4$, $\mathbb{C}^4$ with its analytic topology. Its Lie algebra, known as $\mathfrak{sl}(2,\mathbb{C})$, is the vector space

$$\mathfrak{sl}(2,\mathbb{C}) = \{A \in \mathbb{M}_2(\mathbb{C}) \mid tr(A) = 0\}$$

with Lie bracket the ring-commutator $[A,B] := AB - BA$.

Furthermore, in algebraic terms, $SL(2,\mathbb{C})$ is also a complex algebraic affine variety. To this end, let us look $SL(2,\mathbb{C}) \subseteq \mathbb{C}^4$ and, with this identification and coordinates (a,b,c,d) in $\mathbb{C}^4$, we have that

$$SL(2,\mathbb{C}) = V(ad - bc - 1)$$

so $SL(2,\mathbb{C})$ is an affine variety of $\mathbb{C}^4$. In this sense, seen $SL(2,\mathbb{C}) \subseteq GL(2,\mathbb{C})$, $SL(2,\mathbb{C})$ is also a subvariety of the quasi-affine variety $GL(2,\mathbb{C}) \subseteq \mathbb{C}^4$.

Strongly related with $GL(2,\mathbb{C})$ and $SL(2,\mathbb{C})$ is the **general projective group** of order 2, $PGL(2,\mathbb{C})$. It is defined as the quotient of $GL(2,\mathbb{C})$ by the diagonal automorphisms

$$PGL(2,\mathbb{C}) := \frac{GL(2,\mathbb{C})}{\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C}^* \right\}} = \frac{GL(2,\mathbb{C})}{\mathbb{C}^*}$$

seen $\mathbb{C}^* \hookrightarrow GL(2,\mathbb{C})$ as the subgroup of diagonal automorphisms.

Furthermore, we can endow $PGL(2,\mathbb{C})$ with the structure of a quasi-projective variety. Observe that the equivalence relation on $GL(2,\mathbb{C})$ defining $PGL(2,\mathbb{C})$ means that $M \equiv M'$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $M' = \lambda M$. Therefore, the embedding $GL(2,\mathbb{C}) \hookrightarrow \mathbb{C}^4$ descends to an embedding $PGL(2,\mathbb{C}) \hookrightarrow \mathbb{P}^4$ as quasi-projective variety. Therefore, $PGL(2,\mathbb{C})$, with this structure, is an algebraic group.

Analogously, we can restrict our attention to $SL(2,\mathbb{C}) \subseteq GL(2,\mathbb{C})$ and quotient by the diagonal automorphisms, obtaining the **special projective group** of order 2, $PSL(2,\mathbb{C})$. However, since the only diagonal automorphisms of deteminant 1 are $Id, -Id \in SL(2,\mathbb{C})$ we have

$$PSL(2,\mathbb{C}) = \frac{SL(2,\mathbb{C})}{\{Id, -Id\}} = \frac{SL(2,\mathbb{C})}{\mathbb{Z}^2}$$

However, in the complex case, this two groups are isomorphic. Indeed, using the inclusion map $SL(2,\mathbb{C}) \hookrightarrow GL(2,\mathbb{C})$, consider the morphism

$$\begin{array}{rccc} \varphi : & SL(2,\mathbb{C}) & \longrightarrow & PGL(2,\mathbb{C}) \\ & A & \longmapsto & A \cdot \mathbb{C}^* \end{array}$$

Observe that φ is surjective, since, if $M \cdot \mathbb{C}^* \in PGL(2,\mathbb{C})$ for some $M \in GL(2,\mathbb{C})$, then, taking

$$\tilde{M} := M \begin{pmatrix} \frac{1}{\sqrt{det(M)}} & 0 \\ 0 & \frac{1}{\sqrt{det(M)}} \end{pmatrix}$$

we have $M \cdot \mathbb{C}^* = \tilde{M} \cdot \mathbb{C}^*$ and $\tilde{M} \in SL(2,\mathbb{C})$, so $M \cdot \mathbb{C}^* = \varphi(\tilde{M})$. Therefore, since the kernel of φ are the diagonal morphisms of $SL(2,\mathbb{C})$ it induces an isomorphisms

$$\tilde{\varphi} : PSL(2,\mathbb{C}) = \frac{SL(2,\mathbb{C})}{\mathbb{Z}^2} \overset{\cong}{\to} PGL(2,\mathbb{C})$$

Remark 4.1.1. Of course, the dimension and the ground field do not matter anything at all, so, analogously, we can define groups $GL(n,k), SL(n,k)$, $PGL(n,k)$ and $PSL(n,k)$ for any field k and $n > 0$. Observe that, in the previous argument, the only algebraic fact that we need was that every element of $\mathbb{C}^*$ has a square-root. Therefore, if every element of k^* has a n-th root, we have that

$$PGL(n,k) \cong PSL(n,k)$$

In particular, $PGL(n,\mathbb{C}) \cong PSL(n,\mathbb{C})$ for every $n > 0$. In the other cases, we just has an inclusion $PSL(n,k) \hookrightarrow PGL(n,k)$. This is, for example, the case of $PSL(2,\mathbb{R}) \subsetneq PGL(2,\mathbb{R})$ where the automorphism $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2,\mathbb{R})$ has no element of $SL(2,\mathbb{R})$ in its $\mathbb{R}^*$-orbit.

4.1.1 First Deligne-Hodge Polynomials

Using the properties of the Deligne-Hodge polynomial, we can easily compute these polynomials for the groups $GL(2,\mathbb{C}), SL(2,\mathbb{C})$ and $PGL(2,\mathbb{C})$.

For the first case, let us fix a basis $\{e_1, e_2\}$ of $\mathbb{C}^2$. Then, we can define the surjective map $\pi : GL(2,\mathbb{C}) \to \mathbb{C}^2 - \{0\}$ given by $\pi(M) = M(e_1)$ (i.e. the image of the first vector of the basis). The fiber of this map in a point $v \in \mathbb{C}^2$ are the possible elements $w \in \mathbb{C}^2$ such that $\{v, w\}$ is a basis of $\mathbb{C}^2$ and this is $\mathbb{C}^2 - \langle v \rangle \cong \mathbb{C}^2 - \mathbb{C}$. Therefore, we have the fibration

$$\mathbb{C}^2 - \mathbb{C} \to GL(2,\mathbb{C}) \xrightarrow{\pi} \mathbb{C}^2 - \{0\}$$

This fibration is locally trivial in the Zariski topology and $\mathbb{C}^2 - \{0\}$ is irreducible. Hence, it is an E-fibration and, thus, by theorem 3.3.45,

$$\boxed{e(GL(2,\mathbb{C})) = e(\mathbb{C}^2 - \{0\})e(\mathbb{C}^2 - \mathbb{C}) = (q^2-1)(q^2-q) = q(q-1)^2(q+1)}$$

For $SL(2,\mathbb{C})$ we can repeat the argument given for $GL(2,\mathbb{C})$ counting properly. Again, let us fix a basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ and define the surjection $\pi : SL(2,\mathbb{C}) \to \mathbb{C}^2 - \{0\}$ given by $\pi(A) = A(e_1)$. However, in this case, fixed $v \in \mathbb{C}^2$, its fiber under π is not the entire space $\mathbb{C}^2 - \mathbb{C}$.

In fact, given a vector $w \in \mathbb{C}^2 - \langle v \rangle$, there exists an automorphism $A \in SL(2,\mathbb{C})$ such that $A(e_1) = v$ and $A(e_2) = w$ if and only if the volume of the basis $\{v, w\}$ is equal to the volume of the basis $\{e_1, e_2\}$. This can be achived re-scalling w so there exists one and only one posible vector in possible direction, so the expected fiber is $\mathbb{P}(\mathbb{C}^2 - \mathbb{C}) = \mathbb{P}^1 - \{\star\} \cong \mathbb{C}$.

More precisely, let us take the Zariski open set $U_1 := \left\{(z_1, z_2) \in \mathbb{C}^2 \,|\, z_1 \neq 0\right\} \subseteq \mathbb{C}^2$. Then, we have the fiber

$$\pi^{-1}(U_1) = \left\{ \begin{pmatrix} z_1 & w_1 \\ z_2 & 1 + \frac{w_1 z_2}{z_1} \end{pmatrix} \middle| \; z_1 \neq 0 \right\} \cong U_1 \times \mathbb{C}$$

Analogous considerations can be done taking the Zariski open set $U_2 := \{z_2 \neq 0\} \subseteq \mathbb{C}^2$. Therefore, we have the locally trivial fibration in the Zariski topology

$$\mathbb{C} \to SL(2,\mathbb{C}) \xrightarrow{\pi} \mathbb{C}^2 - \{0\}$$

In addition, $\mathbb{C}^2 - \{0\}$ is an irreducible variety, so π is an E-fibration . Hence, by theorem 3.3.45,

$$\boxed{e(SL(2,\mathbb{C})) = e(\mathbb{C}^2 - \{0\})e(\mathbb{C}) = q(q^2-1)}$$

Remark 4.1.2. Another attempt to compute this Deligne-Hodge polynomial could be using the map $\psi : GL(2,\mathbb{C}) \to SL(2,\mathbb{C})$ given by $\psi(M) = \frac{1}{\sqrt{det(M)}}M$. Then, in this case, we would have a well behaved fibration

$$\mathbb{C}^* \to GL(2,\mathbb{C}) \to SL(2,\mathbb{C})$$

so the Deligne-Hodge polynomial would be

$$e(SL(2,\mathbb{C})) = \frac{e(GL(2,\mathbb{C})}{e(\mathbb{C}^*)} = q(q^2-1)$$

as expected. However, we cannot use this argument, since ψ does not satisfy the hypotesis of theorem 3.3.45, at least in its present form. The reason is that, due to the square-root, ψ is not an algebraic map.

Finally, for $PGL(2,\mathbb{C})$ observe that the quotient map

$$\mathbb{C}^* \to GL(2,\mathbb{C}) \to PGL(2,\mathbb{C})$$

is a principal $\mathbb{C}^*$-bundle map, so it is an E-fibration and, again by theorem 3.3.45,

$$\boxed{e(PGL(2,\mathbb{C})) = \frac{e(GL(2,\mathbb{C})}{e(\mathbb{C}^*)} = \frac{q(q-1)^2(q+1)}{q-1} = q(q^2-1) = q^3 - q}$$

4.1.2 The Conjugation Action and the Commutator

Maybe most important action on $SL(2,\mathbb{C})$ that we will study is the action on itself by conjugation, that is $P \cdot M = PMP^{-1}$ for $P, M \in SL(2,\mathbb{C})$. Using the Jordan canonical forms, we obtain that possible Jordan forms are

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad -Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for $\lambda \in \mathbb{C}^* - \{\pm 1\}$. Hence, $SL(2,\mathbb{C})$ has five types of conjugacy classes, let us call them $[Id], [-Id], [J_+], [J_-]$ and $[D_\lambda]$ for $\lambda \in \mathbb{C} - \{\pm 1\}$. We also define the set of orbits

$$[D] := \bigsqcup_{\lambda \in \mathbb{C}^* - \{\pm 1\}} [D_\lambda] = \{A \in SL(2,\mathbb{C}) \mid tr(A) \neq \pm 2\}$$

Remark 4.1.3. In $SL(2,\mathbb{C})$, each conjugacy class is a quasi-affine subvariety of $SL(2,\mathbb{C})$. Indeed, since $[Id] = \{Id\}$ and $[-Id] = \{-Id\}$, we have that $[Id]$ and $[-Id]$ are points, and, in particular, algebraic subvarieties.

For $[J_+]$, observe that, given $A \in SL(2,\mathbb{C})$, $A \in [J_+]$ if and only if A has a single eigenvalue 1 and is not diagonalizable. However, since the unique diagonalizable matrix with single eigenvalue 1 is Id, we have that $A \in [J_+]$ if and only if A has a single eigenvalue 1 and $A \neq Id$. Furthermore, a matrix $A \in SL(2,\mathbb{C})$ has a single eigenvalue 1 if and only if $tr(A) = 2$. To check this, observe that, the characteristic polynomial of $A \in SL(2,\mathbb{C})$ is of the form $char(A)(\lambda) = \lambda^2 - tr(A)\lambda + 1$ which is equal to $(\lambda - 1)^2$ if and only if $tr(A) = 2$. Hence, summarizing, $A \in [J_+]$ if and only if $tr(A) = 2$ and $A \neq Id$, so

$$[J_+] = \{A \in SL(2,\mathbb{C}) \,|\, tr(A) = 2,\ A \neq Id\}$$

which is a quasi-affine subvariety of $SL(2,\mathbb{C})$. Analogously, $A \in [J_-]$ if and only if A has a single eigenvalue -1 and $A \neq -Id$ if and only if $tr(A) = -2$ and $A \neq -Id$ so

$$[J_-] = \{A \in SL(2,\mathbb{C}) \,|\, tr(A) = -2,\ A \neq -Id\}$$

which, again, is a quasi-affine subvariety of $SL(2,\mathbb{C})$. Finally, for $[D_\lambda]$ observe that $A \in [D_\lambda]$ if and only if $tr(A) = \lambda + \lambda^{-1} \neq \pm 2$, so $[D_\lambda]$ is the affine subvariety

$$[D_\lambda] = \left\{A \in SL(2,\mathbb{C}) \,|\, tr(A) = \lambda + \lambda^{-1},\ A \neq -Id\right\}$$

and, for $[D]$ we can write

$$[D] = \{A \in SL(2,\mathbb{C}) \,|\, tr(A) \neq 2,\ A \neq -Id\}$$

which is a quasi-affine subvariety of $SL(2,\mathbb{C})$.

Remark 4.1.4. For subsequent computations, we will need to observe that, under the action of $SL(2,\mathbb{C})$ on itself by conjugation it can be shown that the stabilizer of the canonical matrices are the subgroups

$$U := Stab(J_+) = Stab(J_-) = \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C} \right\} \cong \mathbb{C} \sqcup \mathbb{C}$$

$$D := Stab(D_\lambda) = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \;\middle|\; \mu \in \mathbb{C}^* \right\} \cong \mathbb{C}^*$$

Remark 4.1.5. Using this stabilizers, we can even compute the Deligne-Hodge polynomial of each conjugation class:

- For $[Id]$, observe that $[Id] = \{Id\}$, that is, just a point, so $e([Id]) = 1$.
- For $[-Id]$, analogously to $[Id]$, we have that $e([-Id]) = 1$.
- For $[J_+]$, observe that $[J_+] \cong SL(2,\mathbb{C})/Stab(J_+)$, with $Stab(J_+)$ acting by conjugation. Since $\pm Id$ acts trivially, we have that $[J_+] \cong SL(2,\mathbb{C})/(Stab(J_+)/\{\pm Id\})$. But, in this case, we have that $Stab(J_+)/\{\pm Id\} \cong \mathbb{C}$ acts freely on $SL(2,\mathbb{C})$ so we have that

$$e(J_+) = \frac{e(SL(2,\mathbb{C}))}{e\left(Stab(J_+)/\{\pm Id\}\right)} = \frac{e(SL(2,\mathbb{C}))}{e\left(\mathbb{C}\right)} = q^2 - 1$$

- For $[J_-]$, analogously to $[J_+]$, we have that $e([J_-]) = q^2 - 1$.
- For $[D_\lambda]$, the reasoning is analogous to the one of $[J_+]$ but taking $Stab(D_\lambda)$. Hence, we have that $[D_\lambda] \cong SL(2,\mathbb{C})/(Stab(D_\lambda)/\{\pm Id\})$ with $Stab(D_\lambda)/\{\pm Id\} = \mathbb{C}^*$, so we have

$$e(J_+) = \frac{e(SL(2,\mathbb{C}))}{e\left(Stab(D_\lambda)/\{\pm Id\}\right)} = \frac{e(SL(2,\mathbb{C}))}{e\left(\mathbb{C}^*\right)} = q^2 + q$$

- For $[D]$, since $SL(2,\mathbb{C}) = [Id] \sqcup [-Id] \sqcup [J_+] \sqcup [J_-] \sqcup [D]$, we obtain

$$e([D]) = e(SL(2,\mathbb{C})) - e([Id]) - e([-Id]) - e([J_+]) - e([J_-]) = q^3 - 2q^2 - q$$

Now, using the group structure, let us define the group-commutator map in $SL(2,\mathbb{C})$

$$\boxed{\begin{array}{rccc} [\cdot,\cdot]: & SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) & \longrightarrow & SL(2,\mathbb{C}) \\ & (A,B) & \longmapsto & [A,B] := ABA^{-1}B^{-1} \end{array}}$$

This algebraic map will be our main concern of this section. Specifically, using the Jordan canonical forms, we will be interested in the algebraic varieties

- $X_{Id} := [\cdot,\cdot]^{-1}(Id) = \{(A,B) \in SL(2,\mathbb{C}) \mid AB = BA\}$.
- $X_{-Id} := [\cdot,\cdot]^{-1}(-Id) = \{(A,B) \in SL(2,\mathbb{C}) \mid AB = -BA\}$.
- $X_{J_+} := [\cdot,\cdot]^{-1}(J_+) = \left\{(A,B) \in SL(2,\mathbb{C}) \mid ABA^{-1}B^{-1} = J_+\right\}$.
- $X_{J_-} := [\cdot,\cdot]^{-1}(J_-) = \left\{(A,B) \in SL(2,\mathbb{C}) \mid ABA^{-1}B^{-1} = J_-\right\}$.
- $X_{D_\lambda} := [\cdot,\cdot]^{-1}(D_\lambda) = \left\{(A,B) \in SL(2,\mathbb{C}) \mid ABA^{-1}B^{-1} = D_\lambda\right\}$ for $\lambda \in \mathbb{C} - \{\pm 1\}$ and, more generally

$$X_D := \bigsqcup_{\lambda \in \mathbb{C}-\{\pm 1\}} X_{D_\lambda}$$

Remark 4.1.6. Since an algebraic map is a continous map in the Zariski topology and a single-element subset of an algebraic variety is a closed set, we have that $X_{Id}, X_{-Id}, X_{J_+}, X_{J_-}$ and X_{D_λ} for $\lambda \neq \pm 1$ are closed subvarieties of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Moreover, X_D, as the complement of an algebraic subvariety, is a quasi-affine subvariety of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$.

Furthermore, we will also need the preimage of each conjugacy class, given by

- $\overline{X}_{[J_+]} := [\cdot,\cdot]^{-1}[J_+] = \{(A,B) \in SL(2,\mathbb{C}) \mid ABA^{-1}B^{-1} \in [J_+]\}$.
- $\overline{X}_{[J_-]} := [\cdot,\cdot]^{-1}[J_-] = \{(A,B) \in SL(2,\mathbb{C}) \mid ABA^{-1}B^{-1} \in [J_-]\}$.
- $\overline{X}_{[D_\lambda]} := [\cdot,\cdot]^{-1}[D_\lambda] = \{(A,B) \in SL(2,\mathbb{C}) \mid ABA^{-1}B^{-1} \in [D_\lambda]\}$ for $\lambda \in \mathbb{C} - \{\pm 1\}$ and, more generally

$$\begin{aligned}\overline{X}_D := \bigsqcup_{\lambda \in \mathbb{C}-\{\pm 1\}} \overline{X}_{[D_\lambda]} &= \{(A,B) \in SL(2,\mathbb{C}) \mid ABA^{-1}B^{-1} \in [D]\} \\ &= \{(A,B) \in SL(2,\mathbb{C}) \mid tr(ABA^{-1}B^{-1}) \neq \pm 2\}\end{aligned}$$

Remark 4.1.7. Since $[Id] = \{Id\}$ and $[-Id] = \{-Id\}$, it is unnecessary to define $\overline{X}_{[Id]}$ and $\overline{X}_{[-Id]}$.

Remark 4.1.8. Again, since an algebraic map sends quasi-affine subvarieties onto quasi-affine subvarieties, we have that all the subsets $\overline{X}_{[J_+]}, \overline{X}_{[J_-]}, \overline{X}_{[D_\lambda]}$ and $\overline{X}_{[D]}$ are quasi-affine subvarieties of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Thus, in particular, they are algebraic varieties.

Therefore, using this varieties, we have the stratification of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ in algebraic varieties

$$SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) = X_{Id} \sqcup X_{-Id} \sqcup \overline{X}_{[J_+]} \sqcup \overline{X}_{[J_-]} \sqcup \overline{X}_D$$

Finally, observe that, fixed a conjugacy class $\mathcal{C} \subseteq SL(2,\mathbb{C})$, the space $\overline{X}_{\mathcal{C}}$ is very related to X_ξ for $\xi \in \mathcal{C}$. Suppose that there exists a subgroup $K \subseteq Stab(\xi) \subseteq SL(2,\mathbb{C})$ such that the action of K on $\overline{X}_{\mathcal{C}}$ by simultaneous conjugation is trivial and $SL(2,\mathbb{C})/K$ acts freely on $\overline{X}_{\mathcal{C}}$. Usually, it will be $K = \{Id, -Id\}$, so $SL(2,\mathbb{C})/K = PGL(2,\mathbb{C})$.

In that case, observe that, considering $X_\xi \subseteq \overline{X}_{\mathcal{C}}$, it holds:

- $SL(2,\mathbb{C})/K \cdot X_\xi = \overline{X}_{\mathcal{C}}$. Indeed, given $(A,B) \in \overline{X}_{\mathcal{C}_1,\mathcal{C}_2}$, let $P \in SL(2,\mathbb{C})$ such that $P[A,B]P^{-1} = \xi$. Then, $y := (PAP^{-1}, PBP^{-1}) \in X_\xi$ and $P^{-1}K \cdot y = (A,B)$.
- For all $y \in X_\xi$ we have that

  $$Stab(\xi)/K \cdot y = SL(2,\mathbb{C})/K \cdot y \cap X_\xi$$

  This is because, if $(A,B) \in X_\xi$ and $P \in SL(2,\mathbb{C})$ satisfies $P \cdot (A,B) \in X_\xi$ then it should satify

  $$P\xi P^{-1} = P[A,B]P^{-1} = \xi$$

so $P \in Stab(\xi)$.

Therefore, the algebraic groups $Stab(\xi)/K \subseteq SL(2,\mathbb{C})/K$ satisfies the hypotesis of proposition 3.3.47 for the varieties $X_\xi \subseteq \overline{X}_\mathcal{C}$ so we have a $Stab(\xi)/K$-principal bundle

$$Stab(\xi)/K \to SL(2,\mathbb{C})/K \times X_\xi \to \overline{X}_\mathcal{C}$$

Hence, since it is an E-fibration, it holds

$$e\left(\overline{X}_\mathcal{C}\right) = e\left(X_\xi\right) e\left(\frac{SL(2,\mathbb{C})/K}{Stab(\xi)/K}\right) = e\left(X_\xi\right) e\left(\frac{SL(2,\mathbb{C})}{Stab(\xi)}\right)$$

4.1.3 Deligne-Hodge Polynomial of X_{Id}

Let us compute the Deligne-Hodge polynomial of

$$\boxed{X_{Id} = \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid AB = BA\right\}}$$

First of all, observe that we have the degenerated cases

$$X_{Id}^A := \{\pm Id\} \times SL(2,\mathbb{C}) \subseteq X_{Id} \qquad\qquad X_{Id}^B := SL(2,\mathbb{C}) \times \{\pm Id\} \subseteq X_{Id}$$

so we have that, defining

$$\tilde{X}_{Id} = \left\{(A,B) \in (SL(2,\mathbb{C}) - \{\pm Id\})^2 \mid AB = BA\right\} = X_{Id} - X_{Id}^A - X_{Id}^b$$

we have that $X_{Id} = \tilde{X}_{Id} \sqcup \left(X_{Id}^A \cup X_{Id}^b\right)$. Hence, by the additivity of the Deligne-Hodge polynomial, we have

$$e(X_{Id}) = e(\tilde{X}_{Id}) + e\left(X_{Id}^A \cup X_{Id}^b\right)$$

For $X_{Id}^A \cup X_{Id}^b$, observe that

$$\begin{aligned} X_{Id}^A \cup X_{Id}^b &= \{\pm Id\} \times ((SL(2,\mathbb{C}) - \{\pm Id\}) \sqcup \{\pm Id\} \times ((SL(2,\mathbb{C}) - \{\pm Id\}) \sqcup \{(\pm Id, \pm Id)\} \\ &\cong \mathbb{Z}_4 \times (SL(2,\mathbb{C}) - \{2 \text{ points}\}) \sqcup \{4 \text{ points}\} \end{aligned}$$

so its Deligne-Hodge polynomial is

$$e\left(X_{Id}^A \cup X_{Id}^b\right) = 4\left(e(SL(2,\mathbb{C})) - 2\right) + 4 = 4q^3 - 4q - 4$$

In order to study $\tilde{X}_{Id}$, let us consider the trace of A map $t : \tilde{X}_{Id} \to \mathbb{C}$ given by $t(A,B) = tr\, A$. Depending of the value of t we have different strata.

- $Z_2 = t^{-1}(2)$: In this case, since $A \neq Id$, we should have $A \sim J_+$, let us say $PAP^{-1} = J_+$ for some $P \in SL(2, \mathbb{C})$. Hence, since $BAB^{-1} = A$ we have

$$J_+ = PAP^{-1} = PBAB^{-1}P^{-1} = \left(PBP^{-1}\right)\left(PAP^{-1}\right)\left(PBP^{-1}\right)^{-1} = \left(PBP^{-1}\right) J_+ \left(PBP^{-1}\right)^{-1}$$

and, thus, since $B \neq \pm Id$, we have

$$PBP^{-1} \in Stab(J_+) - \{\pm Id\} = \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C}^* \right\}$$

Let us define the subvariety of Z_2

$$\hat{Z}_2 = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C}^* \right\} \cong \mathbb{C}^* \times \mathbb{Z}_2$$

and observe that, for $PSL(2, \mathbb{C})$ acting on Z_2 by conjugation, we have just prove that $PSL(2, \mathbb{C}) \cdot \hat{Z}_2 = Z_2$ and that, for all $y \in \hat{Z}_2$

$$\frac{Stab(J_+)}{\{\pm Id\}} \cdot y = PSL(2, \mathbb{C}) \cdot y \cap \hat{Z}_2$$

Therefore, by remark 3.3.47, we obtain an E-fibration

$$\frac{Stab(J_+)}{\{\pm Id\}} \to PSL(2, \mathbb{C}) \times \hat{Z}_2 \to Z_2$$

so, using that $Stab(J_+)/\{\pm Id\} \cong \mathbb{C}$,

$$e(Z_2) = \frac{e(PSL(2, \mathbb{C}))}{e(Stab(J_+)/\{\pm Id\})} e(\hat{Z}_2) = 2q^3 - 2q^2 - 2q + 2$$

- $Z_{-2} = t^{-1}(-2)$: Observe that the map $\phi : Z_2 \to Z_{-2}$ given by $\phi(A, B) = (-A, -B)$ is an isomorphism, so

$$e(Z_{-2}) = e(Z_2) = 2q^3 - 2q^2 - 2q + 2$$

- $\tilde{Z} = t^{-1}(\mathbb{C} - \{\pm 2\})$: In this case, we have that $tr\, A \neq \pm 2$, so A diagonalizes in some basis. Let us suppose that $PAP^{-1} = D_\lambda$ for some $\lambda \in \mathbb{C}^* - \{\pm 1\}$ and $P \in SL(2, \mathbb{C})$. In this case, we have that $BAB^{-1} = A$ so

$$D_\lambda = PAP^{-1} = PBAB^{-1}P^{-1} = \left(PBP^{-1}\right)\left(PAP^{-1}\right)\left(PBP^{-1}\right)^{-1} = \left(PBP^{-1}\right) D_\lambda \left(PBP^{-1}\right)^{-1}$$

Hence, $PBP^{-1} \in Stab(D_\lambda) = U$, that is, B also diagonalizes via P. Moreover, since $B \neq \pm Id$, it should be $PBP^{-1} = D_\mu$ for some $\mu \in \mathbb{C}^* - \{\pm 1\}$.

Now, let us define the morphism $\tilde{\pi} : \tilde{Z} \to [D]$ given by $\pi(A, B) = A$. The problem is that we have not control on the action of $\pi_1([D])$ on $H^*(\tilde{Z})$, so we cannot claim that $\tilde{\pi}$ is not an

E-fibration[1]. Therefore, in order to understand $\tilde{\pi}$, we define the auxiliar varieties

$$[\hat{D}] := \mathbb{C}^* - \{\pm 1\} \times \frac{SL(2,\mathbb{C})}{D} \qquad\qquad \hat{Z} := (\mathbb{C}^* - \{\pm 1\})^2 \times \frac{SL(2,\mathbb{C})}{D}$$

with $D = Stab(D_\lambda) = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \middle| \; \mu \in \mathbb{C}^* \right\} \cong \mathbb{C}^*$. Observe that, if we define the action of $\mathbb{Z}_2$ on $[\hat{D}]$ by $-1 \cdot (\lambda, P) = (\lambda^{-1}, P_0 P P_0^{-1})$ with

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then we have that $[\hat{D}]/\mathbb{Z}_2 \cong [D]$. Analogously, if we define the action of $\mathbb{Z}_2$ on $\hat{Z}$ by $-1 \cdot (\lambda, \mu, P) = (\lambda^{-1}, \mu^{-1}, P_0 P P_0^{-1})$, then $\hat{Z}/\mathbb{Z}_2 \cong \tilde{Z}$.

Let us define the morphism $\pi : \hat{Z} \to [\hat{D}]$ by $\pi(\lambda, \mu, P) = (\lambda, P)$. In this case, π do is an E-fibration with fiber $\mathbb{C}^* - \{\pm 1\}$. Therefore, if we define the morphisms $\rho : \hat{Z} \to \tilde{Z}$ by $\rho(\lambda, \mu, P) = (P D_\lambda P^{-1}, P D_\mu P^{-1})$ and $\tilde{\rho} : [\hat{D}] \to [D]$ by $\tilde{\rho}(\lambda, P) = P D_\lambda P^{-1}$, then we have the diagram of fibrations

$$\begin{tikzcd}
 & \mathbb{C}^* - \{\pm 1\} \arrow[dl] \arrow[dr] & \\
\hat{Z} \arrow[rr, "\rho"] \arrow[d, "\pi"'] & & \tilde{Z} \arrow[d, "\tilde{\pi}"] \\
{[\hat{D}]} \arrow[rr, "\tilde{\rho}"] & & {[D]}
\end{tikzcd}$$

Hence, by theorem 3.3.51, we have that

$$\begin{aligned} e(\tilde{Z}) = e(\hat{Z})^+ &= e([\hat{D}])^+ \, e(\mathbb{C}^* - \{\pm 1\})^+ + e([\hat{D}])^- \, e(\mathbb{C}^* - \{\pm 1\})^- \\ &= e([D]) e(\mathbb{C}^* - \{\pm 1\})^+ + \left(e([\hat{D}]) - e([D]) \right) e(\mathbb{C}^* - \{\pm 1\})^- \end{aligned}$$

For compute this polynomials, observe that, since the action of $D/\{\pm Id\} \cong \mathbb{C}^*$ is free, we have

$$e([\hat{D}]) = e(\mathbb{C}^* - \{\pm 1\}) e\left(\frac{SL(2,\mathbb{C})}{D} \right) = e(\mathbb{C}^* - \{\pm 1\}) \frac{e\,(SL(2,\mathbb{C}))}{e\,(D)} = q^3 - 2q^2 - 3q$$

Moreover, the induced action of $\mathbb{Z}_2$ on $\mathbb{C}^* - \{\pm 1\}$ is $-1 \cdot \lambda = \lambda^{-1}$ so we have that $(\mathbb{C}^* - \{\pm 1\})/\mathbb{Z}_2 \cong \mathbb{C}^* - \{1\}$. Thus,

$$e(\mathbb{C}^* - \{\pm 1\})^+ = e(\mathbb{C}^* - \{1\}) = q - 2 \qquad\qquad e(\mathbb{C}^* - \{\pm 1\})^- = e(\mathbb{C}^* - \{\pm 1\}) - e(\mathbb{C}^* - \{\pm 1\})^+ = -1$$

[1] Indeed, as we will compute, it is not.

So, finally, using the computation of $e([D]) = q^3 - 2q^2 - q$ from remark 4.1.5, we have that

$$e(\tilde{Z}) = e([D])e(\mathbb{C}^* - \{\pm 1\})^+ + \left(e([\hat{D}]) - e([D])\right) e(\mathbb{C}^* - \{\pm 1\})^- = q^4 - 4q^3 + 3q^2 + 4q$$

Therefore, putting all together, we obtain the Deligne-Hodge polynomial of $\tilde{X}_{Id}$

$$e(\tilde{X}_{Id}) = e(Z_2) + e(Z_{-2}) + e(\tilde{Z}) = q^4 - q^2 + 4$$

and, with this computation, we finally obtain

$$\boxed{e(X_{Id}) = e(\tilde{X}_{Id}) + e\left(X_{Id}^A \cup X_{Id}^b\right) = q^4 + 4q^3 - q^2 - 4q}$$

4.1.4 Deligne-Hodge Polynomial of X_{-Id}

Let us compute the Deligne-Hodge polynomial of

$$\boxed{X_{-Id} = \left\{(A, B) \in SL(2, \mathbb{C})^2 \mid AB = -BA\right\}}$$

To this end, let us fix $(A, B) \in X_{-Id}$. Since $-A = B^{-1}AB$ and the trace is invariant under change of basis, we have that

$$tr(A) = tr(BAB^{-1}) = tr(-A) = -tr(A)$$

so $tr(A) = 0$ (by symmetry in the argument, $tr(B) = 0$ too). Hence, if A has eigenvalues λ and λ^{-1}, then they should satisfy $\lambda + \lambda^{-1} = 0$, ergo $\lambda = \pm i$. Therefore, there exists $P \in SL(2, \mathbb{C})$ such that

$$\tilde{A} := PAP^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

But, in this basis, $\tilde{B} = PBP^{-1} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ should satisfy $\tilde{A}\tilde{B} = -\tilde{B}\tilde{A}$ so

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = - \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \Leftrightarrow \begin{pmatrix} ix & iy \\ -iz & -it \end{pmatrix} = \begin{pmatrix} -ix & iy \\ -iz & it \end{pmatrix} \Leftrightarrow x = t = 0$$

so, in this basis, $\tilde{B}$ must be of the form

$$\tilde{B} = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

Moreover, since the conjugation by a diagonal matrix in $SL(2,\mathbb{C})$ left invariant $\tilde{A}$ but rescale $\tilde{B}$, we can find $P' \in SL(2,\mathbb{C})$ such that

$$P'AP'^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad P'BP'^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus, summarizing, we have just prove that the action of $SL(2,\mathbb{C})$ on X_{-Id} by conjugation is transitive, since all the elements can be moved to a fixed one. However, this action is not free. Since the isotropy group of the action can be computed as the stabilizer of any element of the acted set, choosing $\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \in X_{-Id}$, we have

$$Iso = Stab \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \{Id, -Id\}$$

Therefore, the action of $SL(2,\mathbb{C})/Iso = SL(2,\mathbb{C})/\mathbb{Z}_2 = PSL(2,\mathbb{C}) = PGL(2,\mathbb{C})$ on X_{-Id} is transitive and free, so, algebraically

$$X_{-Id} \cong PGL(2,\mathbb{C})$$

and, in particular

$$\boxed{e(X_{-id}) = e(PGL(2,\mathbb{C})) = q^3 - q}$$

4.1.5 Deligne-Hodge Polynomial of X_{J_+}

Recall that the variety X_{J_+} is

$$\boxed{X_{J_+} = \{(A,B) \in SL(2,\mathbb{C}) \mid AB = J_+BA\}}$$

where

$$J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

First of all let us restric the form of the elements of X_{J_+}. Let us take $(A,B) \in X_{J_+}$, which means that $ABA^{-1} = J_+B$. Then, taking traces we have

$$tr(B) = tr(ABA^{-1}) = tr(J_+B)$$

Explicity, if $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, then $J_+B = \begin{pmatrix} x+z & y+t \\ z & t \end{pmatrix}$ so B must satisfy $x + t = tr(B) = tr(J_+B) = x + z + t$, that is, $z = 0$. Same considerations can be done for A, so, if $(A,B) \in X_{J_+}$ A and B should have the form

$$A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \qquad\qquad B = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$$

for some $a, x \in \mathbb{C}^*$ and $b, y \in \mathbb{C}$.

In this restrictive form, we can write down explicit equations for the commutation relation. That is, $(A, B) \in X_{J_+}$ if and only if

$$AB = \begin{pmatrix} ax & ay + bx^{-1} \\ 0 & a^{-1}x^{-1} \end{pmatrix} = \begin{pmatrix} ax & xb + a^{-1}(x^{-1} + y) \\ 0 & a^{-1}x^{-1} \end{pmatrix} = J_+BA$$

and this happens if and only if $ay + bx^{-1} = xb + a^{-1}(x^{-1} + y)$. Therefore, simplifying the equation, we have the explicit description of X_{J_+} as the quasi-affine variety in $\mathbb{C}^4$

$$\begin{aligned} X_{J_+} &\cong \left\{(x, a, y, b) \in (\mathbb{C}^*)^2 \times \mathbb{C}^2 \,|\, y(x(a^2 - 1)) + b(a(1 - x^2)) = 1\right\} \\ &= V\left(y(x(a^2 - 1)) + b(a(1 - x^2)) - 1\right) \cap \{x \neq 0, a \neq 0\} \end{aligned}$$

From this description, it is very easy to compute the Deligne-Hodge polynomial of X_{J_+}. In fact, observe that, taking the projection $\pi : X_{J_+} \to (\mathbb{C}^*)^2 - \{(\pm 1, \pm 1)\}$, $\pi(x, a, y, b) = (x, a)$, the fiber under π of some $(x, a) \in (\mathbb{C}^*)^2 - \{(\pm 1, \pm 1)\}$ is a complex line. Therefore, we have the algebraic line bundle

$$\mathbb{C} \to X_{J_+} \xrightarrow{\pi} (\mathbb{C}^*)^2 - \{(\pm 1, \pm 1)\}$$

Thus, since an algebraic line bundle over an irreducible variety is a E-fibration, by theorem 3.3.45 we have

$$\boxed{e(X_{J_+}) = e(\mathbb{C})e\left((\mathbb{C}^*)^2 - \{(\pm 1, \pm 1)\}\right) = q((q-1)^2 - 4) = q^3 - 2q^2 - 3q}$$

Finally, by the argument in section 4.1.2, taking $K = \{\pm Id\}$, we have $SL(2, \mathbb{C})/K = PGL(2, \mathbb{C})$ and, since $PGL(2, \mathbb{C})$ acts freely on $\overline{X}_{[J_+]}$ we obtain an E-fibration

$$Stab(J_+)/K \to PGL(2, \mathbb{C}) \times X_{J_+} \to \overline{X}_{[J_+]}$$

Now, since $Stab(J_+)/K \cong \mathbb{C}$, we have $e\left(\frac{PGL(2,\mathbb{C})}{Stab(J_+)/K}\right) = \frac{e(PGL(2,\mathbb{C}))}{e(\mathbb{C})} = q^2 - 1$. Therefore, we obtain

$$\boxed{e\left(\overline{X}_{[J_+]}\right) = e\left(X_{J_+}\right) e\left(\frac{PGL(2, \mathbb{C})}{Stab(J_+)/K}\right) = q^5 - 2q^4 - 4q^3 + 2q^2 + 3q}$$

4.1.6 Deligne-Hodge Polynomial of X_{J_-}

Now, let us study the variety X_{J_-}, that is

$$\boxed{X_{J_-} = \{(A, B) \in SL(2, \mathbb{C}) \mid AB = J_-BA\}}$$

where

$$J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

As before, we can obtain some restrictions using the constrains on the trace. Let us fix $(A,B) \in X_{J_-}$. Since $ABA^{-1} = J_-B$ we must have $tr(B) = tr(ABA^{-1}) = tr(J_-B)$ and $tr(A) = tr(BAB^{-1}) = tr(J_-^{-1}A)$. Explicity, let us write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

with $ad - bc = 1$ and $xt - yz = 1$. Then, we have

$$J_-^{-1}A = \begin{pmatrix} -a-c & -b-d \\ -c & -d \end{pmatrix} \qquad\qquad J_-B = \begin{pmatrix} z-x & t-y \\ -z & -t \end{pmatrix}$$

so $(A,B) \in X_{J_-}$ should satisfy

$$c = -2(a+d) = -2tr(A) \qquad\qquad z = 2(x+t) = 2tr(B)$$

Using these relations, a straighforward computation shows that, given $A, B \in SL(2,\mathbb{C})$, $AB = J_-BA$ if and only if the previous relations hold and

$$2dt + bz + cy = 0$$

Thus, we have the explicit description

$$X_{J_-} = \left\{ \begin{matrix} 2dt + bz + cy = 0 \\ c = -2(a+d) \\ z = 2(x+t) \\ ad - bc = 1 \\ xt - yz = 1 \end{matrix} \right\} \subseteq \mathbb{C}^8$$

or, eliminating the components c and z

$$X_{J_-} = \left\{ \begin{matrix} dt + b(x+t) = y(a+d) \\ 2b(a+d) = 1 - ad \\ 2y(x+t) = xt - 1 \end{matrix} \right\} \subseteq \mathbb{C}^6$$

As we can see, the conjugate invariants $tr(A) = a + d$ and $tr(B) = x + t$ appear everywhere in this formulas, so it is a good idea to stratify this space based on this invariants. So, taking $\alpha := tr(A) = a+d$

and $\beta := tr(B) = x + t$ we have the very useful description

$$X_{J_-} = \left\{ \begin{matrix} y\alpha = b\beta + (\alpha - a)(\beta - x) \\ 2b\alpha = 1 - a(\alpha - a) \\ 2y\beta = x(\beta - x) - 1 \end{matrix} \right\} \subseteq \mathbb{C}^6 \tag{4.1}$$

Now, we can stratify $X_{J_-} = X_{J_-}^\alpha \sqcup X_{J_-}^\beta \sqcup \tilde{X}_{J_-}$, with $X_{J_-}^\alpha := X_{J_-} \cap \{\alpha = 0\}$, $X_{J_-}^\beta \cap \{\beta = 0\}$ and $\tilde{X}_{J_-} = X_{J_-} \cap \{\alpha, \beta \neq 0\}$, Observe that $X_{J_-}^\alpha \cap X_{J_-}^\beta = \emptyset$ since it cannot occur $\alpha = \beta = 0$.

Thus, for $\tilde{X}_{J_-}$ we have $b = \frac{1-a(\alpha-a)}{2\alpha}$ and $y = \frac{x(\beta-x)-1}{2\beta}$ so, replacing in (4.1) we have

$$\begin{aligned} \tilde{X}_{J_-} &\cong \left\{\alpha^2(1 - x(\beta - x)) + \beta^2(1 - a(\alpha - a)) + 2(\alpha - a)(\beta - a) = 0\right\} \\ &= \left\{\alpha^2 x^2 + \beta^2 a^2 2\alpha\beta x a - 3\alpha^2\beta x - 3\alpha\beta^2 a + \left(\alpha^2 + \beta^2 + 2\alpha^2\beta^2\right) = 0\right\} \\ &= \left\{ \begin{pmatrix} x & a & 1 \end{pmatrix} \begin{pmatrix} \alpha^2 & \alpha\beta & -\frac{3}{2}\alpha^2\beta \\ \alpha\beta & \beta^2 & -\frac{3}{2}\alpha\beta^2 \\ -\frac{3}{2}\alpha^2\beta & -\frac{3}{2}\alpha\beta^2 & \alpha^2 + \beta^2 + 2\alpha^2\beta^2 \end{pmatrix} \begin{pmatrix} x \\ a \\ 1 \end{pmatrix} = 0 \right\} \subseteq \mathbb{C}^2 \times (\mathbb{C}^*)^2 \end{aligned} \tag{4.2}$$

4.1.6.1 Traceless cases

First, let us study $X_{J_-}^\alpha$. Replacing $\alpha = 0$ we have

$$X_{J_-}^\alpha = \left\{ \begin{matrix} b\beta - a(\beta - x) = 0 \\ 2y\beta = x(\beta - x) - 1 \\ a^2 = 1 \end{matrix} \right\} \cong \underbrace{\left\{ \begin{matrix} b\beta - (\beta - x) = 0 \\ 2y\beta = x(\beta - x) - 1 \end{matrix} \right\}}_{Y_+} \bigsqcup \underbrace{\left\{ \begin{matrix} b\beta + (\beta - x) = 0 \\ 2y\beta = x(\beta - x) - 1 \end{matrix} \right\}}_{Y_-} \subseteq \mathbb{C}^4$$

Observe that the projection map $\pi : Y_\pm \to \mathbb{C}^*$, $\pi((b, x, y, \beta)) = \beta$ is surjective and, for $\beta_0 \in \mathbb{C}^*$ we have the fiber

$$\pi^{-1}(\beta_0) = \left\{ \begin{pmatrix} \pm\frac{\beta_0 - x}{\beta_0} \\ x \\ \frac{x(\beta_0 - x)}{2\beta_0} \\ \beta_0 \end{pmatrix} \;\middle|\; x \in \mathbb{C} \right\} \cong \mathbb{C}$$

so $\pi : Y_\pm \to \mathbb{C}^*$ is a algebraic fibration, locally trivial in the Zariski topology, such that

$$\mathbb{C} \to Y_\pm \xrightarrow{\pi} \mathbb{C}^*$$

Hence, π is an E-fibration and, therefore

$$e(Y_\pm) = e(\mathbb{C}^*)e(\mathbb{C}) = q(q - 1)$$

so, finally

$$\boxed{e(X_{J_-}^\alpha) = e(Y_+) + e(Y_-) = 2q(q - 1)}$$

Analogously, for $X^{\beta}_{J_-}$ we have

$$X^{\beta}_{J_-} = \left\{ \begin{matrix} y\alpha + x(\alpha - a) = 0 \\ 2b\alpha = 1 - a(\alpha - a) \\ x^2 = 1 \end{matrix} \right\} \cong \underbrace{\left\{ \begin{matrix} y\alpha + (\alpha - a) = 0 \\ 2b\alpha = 1 - a(\alpha - a) \end{matrix} \right\}}_{Z_+} \bigsqcup \underbrace{\left\{ \begin{matrix} y\alpha - (\alpha - a) = 0 \\ 2b\alpha = 1 - a(\alpha - a) \end{matrix} \right\}}_{Z_-} \subseteq \mathbb{C}^4$$

and the projection over α, $\pi : Z_\pm \to \mathbb{C}^*$ is a surjective E-fibration

$$\mathbb{C} \to Z_\pm \to \mathbb{C}^*$$

so, again, $e(Z_\pm) = q(q-1)$ and therefore

$$\boxed{e(X^{\beta}_{J_-}) = e(Z_+) + e(Z_-) = 2q(q-1)}$$

4.1.6.2 Orbit space analysis

Now, let us study $\tilde{X}_{J_-} = X_{J_-} \cap \{\alpha, \beta \neq 0\}$. Recall from remark 4.1.4 that, for the action of $SL(2,\mathbb{C})$ on itself by conjugation we have

$$Stab(J_-) = \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C} \right\}$$

Let us consider the action of $Stab(J_-)$ on $\tilde{X}_{J_-}$ by simultaneous conjugation. We will prove that the orbit space $S := \tilde{X}_{J_-}/Stab(J_-)$ has a natural structure of algebraic variety. Then, by uniqueness of good quotients, we will have that $\tilde{X}_{J_-} /\!\!/ Stab(J_-) \cong S$. Furthermore, using this description of the orbit space, we will find that $Stab(J_-)$ have isomorphic isotropy groups for any element of $\tilde{X}_{J_-}$, let us call this group $Iso(Stab(J_-))$. Thus, the action of $Stab(J_-)/Iso(Stab(J_-))$ on $\tilde{X}_{J_-}$ is free, obtaining a principal bundle

$$\frac{Stab(J_-)}{Iso(Stab(J_-))} \to \tilde{X}_{J_-} \to S$$

In particular, this is an E-fibration, so

$$e(\tilde{X}_{J_-}) = e(S)e\left(\frac{Stab(J_-)}{Iso(Stab(J_-))}\right)$$

In order to describe $S = \tilde{X}_{J_-}/Stab(J_-)$ let us take $(A,B) \in \tilde{X}_{J_-}$. Observe that, for all $\lambda \in \mathbb{C}$ we have

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x + \lambda z & y + \lambda t - \lambda x - \lambda^2 x \\ z & t - \lambda z \end{pmatrix}$$

Therefore, for any $(A,B) \in SL(2,\mathbb{C})$ taking $P_B = \begin{pmatrix} 1 & \frac{t}{z} \\ 0 & 1 \end{pmatrix} \in Stab(J_-)$ (recall that $z \neq 0$ since $z = 2\beta \neq 0$)), we have

$$P_B B P_B^{-1} = \left\{ \begin{pmatrix} x' & -\frac{1}{2x'} \\ 2x' & 0 \end{pmatrix} \right\}$$

for some $x' \in \mathbb{C}^*$. So, if $P_B A P_B^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, since $P_B \cdot (A,B) = (P_b A P_B^{-1}, P_B B P_B^{-1}) \in X_{J_-}$, equations (4.2) become equivalent to

$$a'(\alpha - a') = 1 + \frac{\alpha^2}{\beta^2}$$

Therefore, taking $\lambda = \frac{\alpha}{\beta}$, if we define the quasi-affine variety

$$S = \left\{ ad = 1 + \lambda^2, a + d \neq 0 \right\} \subseteq \mathbb{C}^2 \times \mathbb{C}^*$$

then, the map $\phi : \tilde{X}_{J_-} \to S$ given by $\phi(A,B) = P_B \cdot (A,B)$ is a good quotient for the action of $Stab(J_-)$ on $\tilde{X}_{J_-}$ by simultaneous conjugation, so $\tilde{X}_{J_-} /\!\!/ Stab(J_-) \cong S$.

For analysing S, let us consider its Zariski closure

$$\overline{S} = \left\{ ad = 1 + \lambda^2 \right\} \subseteq \mathbb{C}^2 \times \mathbb{C}^*$$

so $\overline{S} = S \sqcup S_0$ with $S_0 := \left\{ ad = 1 + \lambda^2, a + d = 0 \right\}$. Since S_0 is the hyperbola with removed points

$$S_0 = \left\{ -a^2 = 1 + \lambda \right\} - \{(\pm i, 0)\}$$

and any hyperbola is isomorphic to $\mathbb{C}^*$ we have $e(S_0) = e(\mathbb{C}^*) - e(\{(\pm i, 0)\}) = q - 3$ and, therefore

$$e(S) = e(\overline{S}) - e(S_0) = e(\overline{S}) - q + 3$$

Thus, the problem reduces to compute the Deligne-Hodge polynomial of $\overline{S}$. For this, let us consider its projective completion

$$\hat{S} = \left\{ ((x_0 : x_1 : x_2), \lambda) \in \mathbb{P}^2 \times \mathbb{C}^* \mid x_1 x_2 = (1 + \lambda^2) x_0^2 \right\}$$

in such a way that $\overline{S} = \hat{S} - \hat{S}_\infty$, where $\hat{S}_\infty$ are the points at infinity of $\hat{S}$, i.e. $\hat{S}_\infty := \hat{S} \cap \{x_0 = 0\}$. The Deligne-Hodge polynomial of this points at infinity is easy to compute, since we have

$$\begin{aligned} \hat{S}_\infty = \hat{S} \cap \{x_0 = 0\} &= \left\{ ((0 : x_1 : x_2), \lambda) \in \mathbb{P}^2 \times \mathbb{C}^* \mid x_1 x_2 = 0 \right\} \\ &= \left\{ ((0:0:1), \lambda) \in \mathbb{P}^2 \times \mathbb{C}^* \right\} \sqcup \left\{ ((0:1:0), \lambda) \in \mathbb{P}^2 \times \mathbb{C}^* \right\} \cong \mathbb{C}^* \sqcup \mathbb{C}^* \end{aligned}$$

so $e(\hat{S}_\infty) = 2(q-1)$.

Finally, for $\hat{S}$, let us see it as a conic fibration over the projection map $\pi : \hat{S} \to \mathbb{C}^*$, $\pi(a, d, \lambda) = \lambda$. This fibration has two different types of fibers:

- The degenerated fibers: They correspond to the fibers over $\lambda = \pm i$. In these cases, we have

$$\begin{aligned}\pi^{-1}(\pm i) &= \left\{((x_0 : x_1 : x_2), \pm i) \in \mathbb{P}^2 \times \mathbb{C}^* \mid x_1 x_2 = 0\right\} \\ &\cong \left\{(x_0 : 0 : x_2) \in \mathbb{P}^2\right\} \cup \left\{(x_0 : x_1 : 0) \in \mathbb{P}^2\right\} \cong \mathbb{P}^1 \sqcup \left(\mathbb{P}^1 - \{(1:0:0)\}\right)\end{aligned}$$

  so if $\hat{S}_D = \pi^{-1}(i) \sqcup \pi^{-1}(-i)$ are the degenerated fibers, we have $e(\hat{S}_D) = 2((q+1)+((q+1)-1)) = 4q + 2$.

- The non-degenerated fibers: In this case, for $\lambda_0 \in \mathbb{C}^* - \{\pm i\}$ we have that $\pi^{-1}(\lambda_0) = \{x_1 x_2 = (1 + \lambda_0)x_0\}$ is a projective non-degenerated conic, so $\pi^{-1}(\lambda_0) \cong \mathbb{P}^1$. Therefore, π is and algebraic bundle on the non-degenerated set $\hat{S}_{ND} = \hat{S} - \hat{S}_D$

$$\mathbb{P}^1 \to \hat{S}_{ND} \xrightarrow{\pi} \mathbb{C}^* - \{\pm i\}$$

  so, in particular, it is an E-fibration, ergo, by theorem 3.3.45

$$e(\hat{S}_{ND}) = e(\mathbb{C}^* - \{\pm i\})e(\mathbb{P}^1) = (q - 3)(q + 1)$$

Thus, summarizing, we have

$$\boxed{e(\hat{S}) = e(\hat{S}_D) + e(\hat{S}_{ND}) = 4q + 2 + (q - 3)(q + 1) = q^2 + 2q - 1}$$

From this, we have

$$e(\overline{S}) = e(\hat{S}) - e(\hat{S}_\infty) = q^2 + 2q - 1 - 2\,(q - 1) = q^2 + 1$$

And, therefore, the Deligne-Hodge polynomial of the orbit space is

$$\boxed{e(\tilde{X}_{J_-} \mathbin{/\!\!/} Stab(J_-))e(S) = e(\overline{S}) - e(S_0) = q^2 + 1 - (q - 3) = q^2 - q + 4}$$

Remark 4.1.9. In fact, we have just prove that S stratifies as

$$S = \hat{S} - \hat{S}_\infty - S_0 = \hat{S}_{ND} \sqcup \hat{S}_D - \hat{S}_\infty - S_0$$

Description that will be very valuable in the following.

With this computation in hand, in order to complete our computation of the Deligne-Hodge polynomial of $\tilde{X}_{J_-}$ we need to compute the isotropy groups of the action of $Stab(J_-)$ by simultaneous conjugations. Since the isotropy groups of elements in the same orbit are isomorphic it is enough to compute the isotopy group of a complete set of elements of each orbit. For this, recall that, above, we proved that every orbit contains an element $(A,B) \in \tilde{X}_{J_-}$ with B of the form

$$B = \begin{pmatrix} x & -\frac{1}{2x} \\ 2x & 0 \end{pmatrix}$$

for some $x \in \mathbb{C}^*$. However, given $P = \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \in Stab(J_-)$ with $\lambda \in \mathbb{C}$ we have

$$PBP^{-1} = \begin{pmatrix} x \pm 2x\lambda & \mp x\lambda - 2x\lambda^2 - \frac{1}{2x} \\ 2x & \mp 2x\lambda \end{pmatrix}$$

so $PBP^{-1} = B$ if and only if $\lambda = 0$. Moreover, since these elements belong to the center of $SL(2,\mathbb{C})$ we have that $Stab(A,B) = \{Id, -Id\}$. Thus, all the isotropy groups are isomorphic and, therefore, the isotropy group of $Stab(J_-)$ is

$$Iso(Stab(J_-)) = \{Id, -Id\}$$

and, therefore

$$\frac{Stab(J_-)}{\{Id, -Id\}} \cong \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C} \right\} \cong \mathbb{C}$$

Hence, summarizing, we have the $\mathbb{C}$-principal bundle

$$\mathbb{C} \to \tilde{X}_{J_-} \to S$$

which, automatically, is an E-fibration, so

$$\boxed{e(\tilde{X}_{J_-}) = e(\mathbb{C})e(S) = q(q^2 - q + 4) = q^3 - q^2 + 4q}$$

Furthermore, we have just compute the Deligne-Hodge polynomials of every stratum of X_{J_-} so we have

$$\boxed{e(X_{J_-}) = e(X^{\alpha}_{J_-}) + e(X^{\beta}_{J_-}) + e(\tilde{X}_{J_-}) = 2q(q-1) + 2q(q-1) + q^3 - q^2 + 4q = q^3 + 3q^2}$$

Moreover, by the argument in section 4.1.2, taking $K = \{\pm Id\}$, we have $SL(2,\mathbb{C})/K = PGL(2,\mathbb{C})$ and, since $PGL(2,\mathbb{C})$ acts freely on $\overline{X}_{[J_-]}$ we obtain an E-fibration

$$Stab(J_-)/K \to PGL(2,\mathbb{C}) \times X_{J_-} \to \overline{X}_{[J_-]}$$

Now, since $Stab(J_-)/K \cong \mathbb{C}$, we have $e\left(\frac{PGL(2,\mathbb{C})}{Stab(J_-)/K}\right) = \frac{e(PGL(2,\mathbb{C}))}{e(\mathbb{C})} = q^2 - 1$. Therefore, we obtain

$$\boxed{e\left(\overline{X}_{[J_-]}\right) = e\left(X_{J_-}\right) e\left(\frac{PGL(2,\mathbb{C})}{Stab(J_-)/K}\right) = q^5 + 3q^4 - q^3 - 3q^2}$$

4.1.6.3 A geometric viewpoint

We can understand geometrically $\tilde{X}_{J_-}$ by means of a fibration with some singular fibers. Recall that, from the description (4.2) we have that

$$\tilde{X}_{J_-} \cong \left\{ (x, a, \alpha, \beta) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 \;\middle|\; \begin{pmatrix} x & a & 1 \end{pmatrix} \begin{pmatrix} \alpha^2 & \alpha\beta & -\frac{3}{2}\alpha^2\beta \\ \alpha\beta & \beta^2 & -\frac{3}{2}\alpha\beta^2 \\ -\frac{3}{2}\alpha^2\beta & -\frac{3}{2}\alpha\beta^2 & \alpha^2+\beta^2+2\alpha^2\beta^2 \end{pmatrix} \begin{pmatrix} x \\ a \\ 1 \end{pmatrix} = 0 \right\}$$

For simplicity, we will consider its projective completion

$$\hat{X}_{J_-} := \left\{ \begin{pmatrix} x_1 & x_2 & x_0 \end{pmatrix} \begin{pmatrix} \alpha^2 & \alpha\beta & -\frac{3}{2}\alpha^2\beta \\ \alpha\beta & \beta^2 & -\frac{3}{2}\alpha\beta^2 \\ -\frac{3}{2}\alpha^2\beta & -\frac{3}{2}\alpha\beta^2 & \alpha^2+\beta^2+2\alpha^2\beta^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_0 \end{pmatrix} = 0 \right\} \subseteq \mathbb{P}^2 \times (\mathbb{C}^*)^2$$

Then, using the projection $\pi : \hat{X}_{J_-} \to \mathbb{C}^* \times \mathbb{C}^*$ given by $\pi((x_0 : x_1 : x_2), (\alpha, \beta)) = (\alpha, \beta)$ we have that for all $(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^*$, the fibers $C_{\alpha,\beta} := \pi^{-1}(\alpha, \beta)$, seen as affine varieties of $\mathbb{P}^2$ are projective conics. Thus,

$$\hat{X}_{J_-} \xrightarrow{\pi} \mathbb{C}^* \times \mathbb{C}^*$$

is a projective conic fibration.

For the structure of the fibers, let us call

$$A_{\alpha,\beta} := \begin{pmatrix} \alpha^2 & \alpha\beta & -\frac{3}{2}\alpha^2\beta \\ \alpha\beta & \beta^2 & -\frac{3}{2}\alpha\beta^2 \\ -\frac{3}{2}\alpha^2\beta & -\frac{3}{2}\alpha\beta^2 & \alpha^2+\beta^2+2\alpha^2\beta^2 \end{pmatrix}$$

and observe that $det(A_{\alpha,\beta}) = 0$ for all $\alpha, \beta \in \mathbb{C}^*$. Thus, $Rg(A_{\alpha,\beta}) \leq 2$ for all $\alpha, \beta \in \mathbb{C}^*$, so all the projective conics $C_{\alpha,\beta}$ are degenerated. Thus, since it cannot happend $Rg(A_{\alpha,\beta}) = 0$, we have the following casuistic:

- $Rg(A_{\alpha,\beta}) = 1$: In this case, the degenerated conic $C_{\alpha,\beta}$ is a double projective line, so $C_{\alpha,\beta} \cong \mathbb{P}^1$. Let us call
$$B_D := \{(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^* \,|\, Rg(A_{\alpha,\beta}) = 1\}$$

and $\hat{X}^D_{J_-} := \pi^{-1}(B_D)$, in the way that we have an analytical fiber bundle

$$\mathbb{P}^1 \to \hat{X}^D_{J_-} \to B_D$$

Since the fiber is the projective space $\mathbb{P}^1$ and π is algebraic, this fiber bundle is an E-fibration, so

$$e(\hat{X}^D_{J_-}) = e(\mathbb{P}^1)e(B_D)$$

For computing B_D, observe that, for all $(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}^*$, the first and the second columns of $A_{\alpha,\beta}$ are linearly dependent. Therefore, $Rg(A_{\alpha,\beta}) = 1$ if and only if

$$det \begin{pmatrix} \alpha\beta & -\frac{3}{2}\alpha^2\beta \\ -\frac{3}{2}\alpha\beta^2 & \alpha^2+\beta^2+2\alpha^2\beta^2 \end{pmatrix} = \alpha\beta\left(\alpha^2+\beta^2-\frac{1}{4}\alpha^2\beta^2\right) = 0$$

$$det \begin{pmatrix} \beta^2 & -\frac{3}{2}\alpha\beta^2 \\ -\frac{3}{2}\alpha\beta^2 & \alpha^2+\beta^2+2\alpha^2\beta^2 \end{pmatrix} = \beta^2\left(\alpha^2+\beta^2-\frac{1}{4}\alpha^2\beta^2\right) = 0$$

so, since $\alpha, \beta \neq 0$ we have that

$$B_D := \left\{(\alpha,\beta) \in \mathbb{C}^* \times \mathbb{C}^* \mid \alpha^2+\beta^2-\frac{1}{4}\alpha^2\beta^2 = 0\right\}$$

For identifiying B_D observe that, defining the map $\varphi(\alpha,\beta) = \left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$ is an isomorphisms between

$$B_D \overset{\varphi}{\longleftrightarrow} \left\{x^2+y^2-\frac{1}{4}=0\right\} - \left\{\left(0,\pm\frac{1}{2}\right),\left(\pm\frac{1}{2},0\right)\right\}$$

which is an affine hyperbola with four removed points. Therefore, $B_D \cong \mathbb{C}^* - \{p,q,r,s\}$ and, consequently

$$e(B_D) = e(\mathbb{C}^* - \{p,q,r,s\}) = q-5$$

Therefore, since $e(\mathbb{P}^1) = q+1$ we have

$$\boxed{e(\hat{X}^D_{J_-}) = (q+1)(q-5) = q^2-4q-5}$$

- $Rg(A_{\alpha,\beta}) = 2$: In this case, the degenerated conic $C_{\alpha,\beta}$ is a pair of projective lines (which necessarily intecepts in a point). Thus, if $B_{ND} := \mathbb{C}^* \times \mathbb{C}^* - B_D$ and $\hat{X}^{ND}_{J_-} := \pi^{B_{ND}}$ we have the analytical fiber bundle

$$\mathbb{P}^1 \cup \mathbb{P}^1 = \mathbb{P}^1 \sqcup \left(\mathbb{P}^1 - \{\star\}\right) \to \hat{X}^{ND}_{J_-} \to B_{ND}$$

Finally, to recover $\tilde{X}_{J_-}$ from $\hat{X}_{J_-}$ we have to remove the points at infinity of $\hat{X}_{J_-}$, $\hat{X}^{\infty}_{J_-} := \hat{X}_{J_-} \cap \{x_0 = 0\}$ so $\tilde{X}_{J_-} = \hat{X}_{J_-} - \hat{X}^{\infty}_{J_-}$. Computing

$$\begin{aligned}
\hat{X}^{\infty}_{J_-} &= \left\{((0 : x_1 : x_2), (\alpha, \beta)) \in \mathbb{P}^2 \times (\mathbb{C}^*)^2 \mid \alpha^2 x_1^2 + \beta^2 x_2^2 + 2\alpha\beta x_1 x_2 = 0\right\} \\
&= \left\{((0 : x_1 : x_2), (\alpha, \beta)) \in \mathbb{P}^2 \times (\mathbb{C}^*)^2 \mid (\alpha x_1 + \beta x_2)^2 = 0\right\} \\
&= \left\{((0 : \beta : -\alpha), (\alpha, \beta)) \in \mathbb{P}^2 \times (\mathbb{C}^*)^2\right\} \cong \mathbb{C}^* \times \mathbb{C}^*
\end{aligned}$$

In fact, from the previous computation, we observe that for all $\alpha, \beta \in \mathbb{C}^*$, $C_{\alpha,\beta}$ contains one and only one point at infinity. In particular, for $(\alpha, \beta) \in B_{ND}$, since each of the two lines in $C_{\alpha,\beta}$ intercepts the line of infinity in at least one point, the point of interception between these two lines should be the unique point at infinity of $C_{\alpha,\beta}$, so, in the affine plane, they are a pair of parallel lines.

Thus, restricting our fibration to the affine case $\pi : \tilde{X}_{J_-} \to \mathbb{C}^* \times \mathbb{C}^*$ if $\tilde{X}^{D}_{J_-} := \pi^{-1}(B_D)$ and $\tilde{X}^{ND}_{J_-} := \pi^{-1}(B_{ND})$ we have stratification $\tilde{X}_{J_-} = \tilde{X}^{D}_{J_-} \sqcup \tilde{X}^{ND}_{J_-}$ and the fibrations

$$\mathbb{C} \to \tilde{X}^{D}_{J_-} \xrightarrow{\pi} B_D \qquad\qquad \mathbb{C} \sqcup \mathbb{C} \to \tilde{X}^{ND}_{J_-} \xrightarrow{\pi} B_{ND}$$

Remark 4.1.10. The analytical fiber bundle in the case of double lines

$$\mathbb{P}^1 \cup \mathbb{P}^1 = \mathbb{P}^1 \sqcup \left(\mathbb{P}^1 - \{\star\}\right) \to \hat{X}^{ND}_{J_-} \to B_{ND}$$

cannot be an E-fibration. Indeed, if it would be, we will have

$$\begin{aligned}
e(\hat{X}^{ND}_{J_-}) &= e(\mathbb{P}^1 \cup \mathbb{P}^1) e(B_{ND}) = e(\mathbb{P}^1 \sqcup \left(\mathbb{P}^1 - \{\star\}\right)) e(\mathbb{C}^* \times \mathbb{C}^* - B_D) \\
&= ((q+1) + q)\left((q-1)^2 - (q-5)\right) = 2q^3 - 5q^2 + 9q + 6
\end{aligned}$$

So, together with the previous computation $e(\hat{X}^{D}_{J_-}) = q^2 - 4q - 5$ we will have

$$e(\hat{X}_{J_-}) = e(\hat{X}^{D}_{J_-}) + e(\hat{X}^{ND}_{J_-}) = 2q^3 - 4q^2 + 5q + 1$$

and

$$e(\tilde{X}_{J_-}) = e(\hat{X}_{J_-}) - e(\hat{X}^{\infty}_{J_-}) = 2q^3 - 5q^3 + 7q$$

which is impossible, since, by the previous section, we know that $e(\hat{X}_{J_-}) = q^3 - q^2 + 4q$. In fact, inverting the reasoning, we can compute the correct Deligne-Hodge polynomial of $\hat{X}^{ND}_{J_-}$, being

$$\boxed{e(\hat{X}^{ND}_{J_-}) = e(\tilde{X}_{J_-}) + e(\hat{X}^{\infty}_{J_-}) - e(\hat{X}^{D}_{J_-}) = q^3 - q^2 + 6q + 6}$$

Observe that this polynomial is irreducible over $\mathbb{Z}[q]$, so $\hat{X}^{ND}_{J_-}$ cannot be written as any non-trivial E-fibration.

4.1.7 Deligne-Hodge Polynomial of X_{D_λ}

Finally, let us fix $\lambda \in \mathbb{C}^* - \{\pm 1\}$ and let us consider the variety

$$\boxed{X_{D_\lambda} = \{(A,B) \in SL(2,\mathbb{C}) \mid AB = D_\lambda BA\}}$$

where

$$D_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix}$$

As usual, we can restrict our attention to some spacial type of matrices using the constrains imposed by the trace. Let us fix $(A,B) \in X_{D_\lambda}$. Since $ABA^{-1} = D_\lambda B$ we must have $tr(B) = tr(ABA^{-1}) = tr(D_\lambda B)$ and $tr(A) = tr(BAB^{-1}) = tr(D_\lambda^{-1}A)$. In orther to make it explicit, let us consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

with $ad - bc = 1$ and $xt - yz = 1$. Then, we have

$$D_\lambda^{-1}A = \begin{pmatrix} \lambda^{-1}a & \lambda^{-1}b \\ \lambda c & \lambda d \end{pmatrix} \qquad D_\lambda B = \begin{pmatrix} \lambda x & \lambda y \\ \lambda^{-1}z & \lambda^{-1}t \end{pmatrix}$$

Thus, we have that $(A,B) \in D_\lambda$ should satisfy

$$a + d = tr(A) = tr(D_\lambda^{-1}A) = \lambda^{-1}a + \lambda d \qquad x + t = tr(B) = tr(D_\lambda B) = \lambda x + \lambda^{-1}t$$

which, for $\lambda \neq 1$ satisfy if and only if $d = \lambda^{-1}a$ and $t = \lambda x$, so every $(A,B) \in D_\lambda$ must be of the form

$$A = \begin{pmatrix} a & b \\ c & \lambda^{-1}a \end{pmatrix} \qquad B = \begin{pmatrix} x & y \\ z & \lambda x \end{pmatrix}$$

with $\lambda^{-1}a^2 - bc = 1$ and $\lambda x^2 - yz = 1$.

With this special form, the equations of D_λ can be drastically simplified. Indeed, observe that, $(A,B) \in D_\lambda$ if and only if

$$\begin{pmatrix} ax + bz & ay + \lambda bx \\ cx + \lambda^{-1}az & cy + ax \end{pmatrix} = AB = D_\lambda BA = \begin{pmatrix} \lambda ax + \lambda cy & ay + \lambda bx \\ cx + \lambda^{-1}az & \lambda^{-1}ax + \lambda^{-1}bz \end{pmatrix}$$

which holds if and only if $ax + bz = \lambda(ax + cy)$. Thus, together with the restriction on the determinant, we have that

$$X_{D_\lambda} = \left\{ \begin{matrix} ax + bz = \lambda(ax + cy) \\ \lambda^{-1}a^2 - bc = 1 \\ \lambda x^2 - yz = 1 \end{matrix} \right\} \subseteq \mathbb{C}^6$$

Now, we are going to stratify X_{D_λ} in four strata $X^{\alpha}_{D_\lambda}, X^{\beta}_{D_\lambda}, X^{R}_{D_\lambda}$ and $\tilde{X}_{D_\lambda}$ in the way that

$$X_{D_\lambda} = X^{\alpha}_{D_\lambda} \sqcup X^{\beta}_{D_\lambda} \sqcup X^{R}_{D_\lambda} \sqcup \tilde{X}_{D_\lambda}$$

and, therefore

$$e(X_{D_\lambda}) = e(X^{\alpha}_{D_\lambda}) + e(X^{\beta}_{D_\lambda}) + e(X^{R}_{D_\lambda}) + e(\tilde{X}_{D_\lambda})$$

4.1.7.1 The variety $X^{\alpha}_{D_\lambda}$

Let us define $X^{\alpha}_{D_\lambda} := X_{D_\lambda} \cap \{b = 0, c = 0\}$. In this case, we have that $X^{\alpha}_{D_\lambda}$ has the simplified form

$$X^{\alpha}_{D_\lambda} = \left\{ \begin{array}{c} ax = \lambda a x \\ \lambda^{-1} a^2 = 1 \\ \lambda x^2 - yz = 1 \end{array} \right\} = \left\{ \begin{array}{c} x = 0 \\ a^2 = \lambda \\ yz = -1 \end{array} \right\} \subseteq \mathbb{C}^4$$

and this space can be decomposed as the disjoint union of the algebraic varieties

$$X^{\alpha}_{D_\lambda} = \left\{ \begin{array}{c} x = 0 \\ a^2 = \lambda \\ yz = -1 \end{array} \right\} \cong \big\{(\lambda_1, y, z) \in \mathbb{C}^3 \mid yz = -1\big\} \sqcup \big\{(\lambda_2, y, z) \in \mathbb{C}^3 \mid yz = -1\big\} \cong \mathbb{C}^* \sqcup \mathbb{C}^*$$

where λ_1, λ_2 are the two (different) square roots of $\lambda \neq 0$. Thus, we have

$$\boxed{e(X^{\alpha}_{D_\lambda}) = e(\mathbb{C}^*) + e(\mathbb{C}^*) = 2q - 2}$$

4.1.7.2 The variety $X^{\beta}_{D_\lambda}$

Now, we take $X^{\beta}_{D_\lambda} := X_{D_\lambda} \cap \{y = 0, z = 0\}$ so the equations are

$$X^{\beta}_{D_\lambda} = \left\{ \begin{array}{c} ax = \lambda a x \\ \lambda^{-1} a^2 - bc = 1 \\ \lambda x^2 = 1 \end{array} \right\} = \left\{ \begin{array}{c} a = 0 \\ bc = 1 \\ x^2 = \lambda^{-1} \end{array} \right\} \subseteq \mathbb{C}^4$$

In this case, by symmetry in the equations, the map $(a, b, c, x, y, z) \mapsto (x, -y, z, a, b, c)$ is an isomorphism between $X^{\alpha}_{D_\lambda}$ and $X^{\beta}_{D_\lambda}$ so

$$\boxed{e(X^{\beta}_{D_\lambda}) = e(\mathbb{C}^*) + e(\mathbb{C}^*) = 2q - 2}$$

4.1.7.3 The variety $X_{D_\lambda}^R$

For the case of $X_{D_\lambda}^R$ (R stands for *residual*) we define

$$X_{D_\lambda}^R := X_{D_\lambda} \cap (\{bcyz = 0\} - \{b = 0, c = 0\} - \{y = 0, z = 0\}) = X_{D_\lambda} \cap \{bcyz = 0\} - X_{D_\lambda}^\alpha - X_{D_\lambda}^\beta$$

For understanding it, we define the auxiliar varieties

$$\begin{array}{ll} Y_1 = X_{D_\lambda} \cap \{b = 0, c \neq 0\} & Y_2 = X_{D_\lambda} \cap \{c = 0, b \neq 0\} \\ Y_3 = X_{D_\lambda} \cap \{y = 0, z \neq 0\} & Y_4 = X_{D_\lambda} \cap \{z = 0, y \neq 0\} \end{array}$$

so we have the decomposition

$$X_{D_\lambda}^R = Y_1 \cup Y_2 \cup Y_2 \cup Y_4$$

However, this varieties are not disjoint so, for the Deligne-Hodge polynomial we have

$$e(X_{D_\lambda}^R) = e(Y_1) + e(Y_2) + e(Y_3) + e(Y_4) - \sum_{i \neq j} e(Y_i \cap Y_j)$$

For Y_1, observe that the equations of X_{D_λ} restricts to

$$Y_1 = \left\{ \begin{matrix} (1-\lambda)ax = \lambda cy \\ a^2 = \lambda \\ \lambda x^2 - yz = 1 \end{matrix} \right\} = \left\{ \begin{matrix} a^2 = \lambda \\ c = \frac{(1-\lambda)ax}{\lambda y} \\ z = \frac{\lambda x^2 - 1}{y} \end{matrix} \right\} \cong \bigsqcup_{i=1,2} \left\{ \begin{pmatrix} \lambda_i \\ x \\ y \end{pmatrix} \in \mathbb{C}^3 \,|\, x, y \in \mathbb{C}^* \right\} \cong (\mathbb{C}^* \times \mathbb{C}^*) \sqcup (\mathbb{C}^* \times \mathbb{C}^*)$$

where λ_1, λ_2 are the two different square roots of $\lambda \neq 0$ and we have used that it is impossible $y = 0$ and, since $c \neq 0$, it should be $x \neq 0$.

Analogously, we can obtain that $Y_2 \cong Y_3 \cong Y_4 \cong (\mathbb{C}^* \times \mathbb{C}^*) \sqcup (\mathbb{C}^* \times \mathbb{C}^*)$, so, for $k = 1, 2, 3, 4$ we have

$$e(Y_k) = e\left(\mathbb{C}^* \times \mathbb{C}^*\right) + e\left(\mathbb{C}^* \times \mathbb{C}^*\right) = 2(q-1)^2$$

For computing the intersections $Y_i \cap Y_j$, recall that $X_{D_\lambda} \cap \{b = 0, y = 0\} = X_{D_\lambda} \cap \{c = 0, z = 0\} = \emptyset$ so we have

$$\begin{array}{ll} Y_1 \cap Y_2 = \emptyset & Y_1 \cap Y_3 = \emptyset \\ Y_2 \cap Y_4 = \emptyset & Y_3 \cap Y_4 = \emptyset \end{array}$$

and, therefore, the only non-trivial intersections are

$$Y_1 \cap Y_4 = X_{D_\lambda} \cap \{b = 0, z = 0\} \qquad Y_2 \cap Y_3 = X_{D_\lambda} \cap \{c = 0, y = 0\}$$

For $Y_1 \cap Y_4$ we have the equations

$$Y_1 \cap Y_4 = \left\{ \begin{array}{c} \frac{1-\lambda}{\lambda} ax = cy \\ a^2 = \lambda \\ x^2 = \lambda^{-1} \end{array} \right\} \cong \bigsqcup_{i,j=1}^{2} \left\{ \begin{array}{c} \frac{1-\lambda}{\lambda} \lambda_i \lambda_j = cy \\ a = \lambda_i \\ x = \lambda_j \end{array} \right\} \cong \bigsqcup_{i,j=1}^{2} \left\{ \tfrac{1-\lambda}{\lambda} \lambda_i \lambda_j = cy \right\}$$

where, again, λ_1, λ_2 are the two different square roots of $\lambda \neq 0$. This variety is the disjoint union of four hyperbolas, and thus isomorphic to $\mathbb{C}^*$, so we have

$$Y_1 \cap Y_4 \cong \bigsqcup_{i=1}^{4} \mathbb{C}^*$$

Analogously, $Y_2 \cap Y_3 \cong \bigsqcup_{i=1}^{4} \mathbb{C}^*$, so

$$e(Y_1 \cap Y_4) = e(Y_2 \cap Y_3) = 4(q-1)$$

Therefore, summarizing, we have

$$\begin{aligned} e(X^R_{D_\lambda}) &= e(Y_1) + e(Y_2) + e(Y_3) + e(Y_4) - e(Y_1 \cap Y_4) - e(Y_2 \cap Y_3) \\ &= 4e(Y_1) - 2e(Y_1 \cap Y_4) = 8(q-1)^2 - 8(q-1) \end{aligned}$$

that is

$$\boxed{e(X^R_{D_\lambda}) = 8(q-1)(q-2) = 8q^2 - 24q + 16}$$

4.1.7.4 The variety $\tilde{X}_{D_\lambda}$

In this case, we have to consider the variety

$$\tilde{X}_{D_\lambda} = X_{D_\lambda} \cap \{b, c, y, z \neq 0\}$$

In order to study it, let us multiply the first equation of X_{D_λ} by $bz \neq 0$ obtaining the equivalent equations

$$X_{D_\lambda} = \left\{ \begin{array}{c} (1-\lambda)abxy + b^2yz = \lambda bcy^2 \\ bc = \lambda^{-1}a^2 - 1 \\ yz = \lambda x^2 - 1 \\ bc \neq 0, yz \neq 0 \end{array} \right\} = \left\{ \begin{array}{c} \lambda b^2x^2 + (\lambda - a^2)y^2 + (1-\lambda)abxy - b^2 = 0 \\ a^2 \neq \lambda, \ \ \lambda x^2 \neq 1 \end{array} \right\}$$

where we have used that $bc \neq 0$ if and only if $a^2 \neq \lambda$ and $yz \neq 0$ if and only if $x^2 \neq 0$. Equivalently, considering X_{D_λ} as a bundle of affine conics in the (x, y)-plane parametriced by (a, b), we have the matricial form

$$\tilde{X}_{D_\lambda} \cong \left\{ (x,y,a,b) \in \mathbb{C}^3 \times \mathbb{C}^* \;\middle|\; \begin{array}{c} \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \lambda b^2 & \frac{(1-\lambda)}{2}ab & 0 \\ \frac{(1-\lambda)}{2}ab & \lambda - a^2 & 0 \\ 0 & 0 & -b^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0 \\ a^2 \neq \lambda, \qquad \lambda x^2 \neq 1 \end{array} \right\}$$

As usual, in order to study this variety, we consider the projection map $\pi : \tilde{X}_{D_\lambda} \to (\mathbb{C}\times\mathbb{C}^*) - \left\{a^2 = \lambda\right\}$ given by $\pi(x,y,a,b) = (a,b)$. Given $(a,b) \in (\mathbb{C} \times \mathbb{C}^*) - \left\{a^2 = \lambda\right\}$, we denoted $C_{a,b} = \pi^{-1}(a,b)$ the parametrized conic in (a,b).

First of all, let us remove the degenerated fibers. Observe that fixed (a,b), the discriminant of the resulting conic is

$$D_{(a,b)} = -b^2 \left(\lambda b^2(\lambda - a^2) - \frac{(\lambda-1)^2}{4} a^2 b^2 \right) = -\frac{b^4}{4} \left[((\lambda+1)a + 2\lambda)\left((\lambda+1)a - 2\lambda\right) \right]$$

so

$$D_{(a,b)} = 0 \Leftrightarrow a = \pm \frac{2\lambda}{\lambda+1}$$

and this are two diferent points because $a^2 = \left(\frac{2\lambda}{\lambda+1}\right)^2 \neq \lambda$ for $\lambda \in \mathbb{C}^* - \{\pm 1\}$. In that case, since the rank of the matrix defining $C_{\pm\frac{2\lambda}{\lambda+1},b}$, for $b \in \mathbb{C}^*$, is always positive, we have that $C_{\pm\frac{2\lambda}{\lambda+1},b}$ is always a pair of parallel lines, except the two points in each line corresponding to the excluded values $x = \pm\frac{1}{\sqrt{\lambda}}$. Thus, the fiber is $\mathbb{C} - \{p,q\} \sqcup \mathbb{C} - \{p,q\}$ with $p \neq q$.

Hence, if we define $\tilde{X}^D_{D_\lambda} := \tilde{X}_{D_\lambda} \cap \left\{ D_{(a,b)} = 0 \right\}$ we have an E-fibration

$$(\mathbb{C} - \{p,q\}) \sqcup (\mathbb{C} - \{p,q\}) \to \tilde{X}^D_{D_\lambda} \to \left\{ \left(\pm\frac{2\lambda}{\lambda+1}, b \right) \mid b \in \mathbb{C}^* \right\} \cong \mathbb{C}^* \sqcup \mathbb{C}^*$$

so

$$e\left(\tilde{X}^D_{D_\lambda}\right) = 4e\left(\mathbb{C}^*\right) e\left(\mathbb{C} - \{p,q\}\right) = 4q^2 - 12q + 8$$

Now, once we have removed the degenerated fibers, let us denote $\tilde{X}^{ND}_{D_\lambda} := \tilde{X}_{D_\lambda} - \tilde{X}^D_{D_\lambda}$ the conic bundle with non-degenerated fibers. For it study, we complete it to its projective completion

$$\hat{X}^{ND}_{D_\lambda} \cong \left\{ ([x_0 : x_1 : x_2], a, b)) \in \mathbb{P}^2 \times S \;\middle|\; \begin{pmatrix} x_1 & x_2 & x_0 \end{pmatrix} \begin{pmatrix} \lambda b^2 & \frac{(1-\lambda)}{2}ab & 0 \\ \frac{(1-\lambda)}{2}ab & \lambda - a^2 & 0 \\ 0 & 0 & -b^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_0 \end{pmatrix} = 0 \right\}$$

with S the space of parameters for (a,b), that is

$$S = \left(\mathbb{C} - \left\{ \pm\frac{2\lambda}{\lambda+1}, \pm\sqrt{\lambda} \right\} \right) \times \mathbb{C}^*$$

In that case, since $C_{a,b}$ is a non-degenerated conic for any $(a,b) \in S$, we have an E-fibration

$$\mathbb{P}^1 \to \hat{X}_{D_\lambda}^{ND} \to S$$

and, since $e(S) = (q-4)(q-1)$, we have

$$e\left(\hat{X}_{D_\lambda}^{ND}\right) = e(S)e(\mathbb{P}^1) = q^3 - 4q^2 - q + 4$$

However, $\hat{X}_{D_\lambda}^{ND}$ contains more points than $\tilde{X}_{D_\lambda}^{ND}$, from two different sources:

- The points at infinity, $\hat{X}_{D_\lambda}^{\infty}$: They correspond to impose $x_0 = 0$ in $\hat{X}_{D_\lambda}^{ND}$, so it is

  $$\hat{X}_{D_\lambda}^{\infty} \cong \left\{([x_1 : x_2], a, b) \in \mathbb{P}^1 \times S \;\middle|\; \lambda b^2 x_1^2 + (\lambda - a^2)x_2^2 + (1-\lambda)ab\, x_1 x_2 = 0\right\}$$

  Observe that, since $a^2 \neq \lambda$, if $x_1 = 0$, then $x_2 = 0$, so in fact, $\hat{X}_{D_\lambda}^{\infty}$ lives in the affine open set $\{x_2 = 0\}$. Therefore, seen $\hat{X}_{D_\lambda}^{\infty}$ as an affine variety via de change of variables $x = \frac{x_2}{x_1}$, we have that

  $$\begin{aligned}\hat{X}_{D_\lambda}^{\infty} &\cong \left\{(x, a, b) \in \mathbb{C} \times S \;\middle|\; (\lambda - a^2)x^2 + (1-\lambda)ab\, x + \lambda b^2 = 0\right\} \\ &\cong \left\{(x, a, b) \in \mathbb{C} \times S \;\middle|\; \left(x + \frac{1}{2}\frac{ab(1-\lambda)}{\lambda - a^2}\right)^2 = \frac{D_{a,b}}{(\lambda - a^2)^2}\right\}\end{aligned}$$

  Let us consider the fibration $\pi : \hat{X}_{D_\lambda}^{\infty} \to \mathbb{C}^*$ given by $\pi(x, a, b) = b$, then, we have

  $$\pi^{-1}(b) \cong \left\{(x, a) \in \mathbb{C} \times \mathbb{C} - \left\{\pm\frac{2\lambda}{\lambda+1}, \pm\sqrt{\lambda}\right\} \;\middle|\; \left(x + \frac{1}{2}\frac{ab(1-\lambda)}{\lambda - a^2}\right)^2 = \frac{D_{a,b}}{(\lambda - a^2)^2}\right\}$$

  Now, using the change of variables $(x, a) \mapsto (\alpha, \beta)$ with

  $$\alpha = \frac{x + \frac{1}{2}\frac{ab(1-\lambda)}{\lambda - a^2}}{b^2}\sqrt{\lambda(\lambda - a^2)} \qquad\qquad \beta = \frac{(1-\lambda)a}{2}\sqrt{\lambda(\lambda - a^2)}$$

  we have that $\pi^{-1}(b)$ can be written as

  $$\pi^{-1}(b) = \left\{(\alpha, \beta) \in \mathbb{C} \times \mathbb{C} - \{6 \text{ points}\} \mid \alpha^2 = \beta^2 + 1\right\}$$

  Hence, defining $F = \left\{(\alpha, \beta) \in \mathbb{C} \times \mathbb{C} - \{6 \text{ points}\} \mid \alpha^2 = \beta^2 + 1\right\}$ we have an E-fibration

  $$F \to \hat{X}_{D_\lambda}^{\infty} \xrightarrow{\pi} \mathbb{C}^*$$

  Since F is a conic with 6 removed points, we have $e(F) = (q-1) - 6 = q - 7$ and, therefore

  $$e\left(\hat{X}_{D_\lambda}^{\infty}\right) = e(F)e(\mathbb{C}^*) = q^2 - 8q + 7$$

- The points with $\lambda x^2 = 1$ in $\tilde{X}_{D_\lambda}$, $\tilde{X}^0_{D_\lambda}$: In this case, we have that $x = \pm\frac{1}{\sqrt{\lambda}}$, so

$$\tilde{X}^0_{D_\lambda} \cong \left\{ (y,a,b) \in \mathbb{C} \times S \;\middle|\; y\left((\lambda - a^2)y \pm (1-\lambda)ab\frac{1}{\sqrt{\lambda}}\right) = 0 \right\}$$

Observe that $\tilde{X}^0_{D_\lambda} \cap \{y=0\} = S \times \mathbb{Z}_2$ and, for its complement, we have

$$\begin{aligned}\tilde{X}^0_{D_\lambda} - \tilde{X}^0_{D_\lambda} \cap \{y=0\} &\cong \left\{ (y,a,b) \in \mathbb{C} \times S \;\middle|\; y = \pm\frac{(1-\lambda)ab}{\sqrt{\lambda}(\lambda - a^2)} \ ,\ y \neq 0 \right\} \\ &\cong (S - \{y=0\}) \times \mathbb{Z}_2 = \left(\mathbb{C}^* - \left\{\pm\frac{2\lambda}{\lambda+1}, \pm\sqrt{\lambda}\right\}\right) \times \mathbb{C}^* \times \mathbb{Z}_2\end{aligned}$$

so, adding

$$e\left(\tilde{X}^0_{D_\lambda}\right) = 2e(S) + 2e\left(\left(\mathbb{C}^* - \left\{\pm\frac{2\lambda}{\lambda+1}, \pm\sqrt{\lambda}\right\}\right) \times \mathbb{C}^*\right) = 4q^2 - 22q + 18$$

Therefore, considering all the contributions, we have

$$e\left(\tilde{X}^{ND}_{D_\lambda}\right) = e\left(\hat{X}^{ND}_{D_\lambda}\right) - e\left(\tilde{X}^0_{D_\lambda}\right) - e\left(\hat{X}^\infty_{D_\lambda}\right) = q^3 - 9q2 + 29q - 21$$

and, thus

$$e\left(\tilde{X}_{D_\lambda}\right) = e\left(\tilde{X}^D_{D_\lambda}\right) + e\left(\tilde{X}^{ND}_{D_\lambda}\right) = q^3 - 5q2 + 17q - 13$$

Therefore, putting all together, we obtain the Deligne-Hodge polynomial

$$\boxed{e\left(X_{D_\lambda}\right) = e\left(X^\alpha_{D_\lambda}\right) + e\left(X^\beta_{D_\lambda}\right) + e\left(X^R_{D_\lambda}\right) + e\left(\tilde{X}_{D_\lambda}\right) = q^3 + 3q^2 - 3q - 1}$$

Finally, by the argument in section 4.1.2, taking $K = \{\pm Id\}$, we have $SL(2,\mathbb{C})/K = PGL(2,\mathbb{C})$ and, since $PGL(2,\mathbb{C})$ acts freely on $\overline{X}_{[D_\lambda]}$ we obtain an E-fibration

$$Stab(D_\lambda)/K \to PGL(2,\mathbb{C}) \times X_{D_\lambda} \to \overline{X}_{[D_\lambda]}$$

Now, since $Stab(D_\lambda)/K \cong \mathbb{C}^*$, we have $e\left(\frac{PGL(2,\mathbb{C})}{Stab(D_\lambda)/K}\right) = \frac{e(PGL(2,\mathbb{C}))}{e(\mathbb{C}^*)} = q^2 + q$. Therefore, we obtain

$$\boxed{e\left(\overline{X}_{[D_\lambda]}\right) = e\left(X_{D_\lambda}\right) e\left(\frac{PGL(2,\mathbb{C})}{Stab(D_\lambda)/K}\right) = q^5 + 4q^4 - 4q^2 - q}$$

4.1.8 The Varieties X_D and $X_D/\mathbb{Z}_2$

Now, let us focus on the variety

$$X_D = \bigsqcup_{\lambda \in \mathbb{C}^* - \{\pm 1\}} X_{D_\lambda} = \left\{(A,B,\lambda) \in SL(2,\mathbb{C})^2 \times (\mathbb{C}^* - \pm 1) \mid [A,B] = D_\lambda\right\}$$

Repeating the statification analysis of section 4.1.7 but considering λ variable living in $\mathbb{C}^* - \{\pm 1\}$, in [47] is proven that

$$\boxed{e(X_D) = q^4 - 3q^3 - 6q^2 + 5q + 3}$$

The variety X_D can be shown in a rather more general way. Indeed, let us take any regular morphism $P : \mathbb{C}^* - \{\pm 1\} \to SL(2, \mathbb{C})$, so, via the map $(A, B, \lambda) \mapsto (P^{-1}(\lambda)AP(\lambda), P^{-1}(\lambda)BP(\lambda), \lambda)$ we have the isomorphism

$$X_D \cong \left\{(A, B, \lambda) \in SL(2, \mathbb{C})^2 \times (\mathbb{C}^* - \pm 1) \mid P(\lambda)[A, B]P^{-1}(\lambda) = D_\lambda\right\}$$

Remark 4.1.11. In order to give some interpretation to this morphism, observe that $P(\lambda)$ can be interpreted as a choose of an ordered basis for each first eigenvalue λ. Thus, X_D is exactly the set of triples of matrices $A, B \in SL(2, \mathbb{C})$ and eigenvalues $\lambda \in \mathbb{C}^* - \{\pm 1\}$ such that $[A, B]$ diagonalizes, in the ordered basis selected by $P(\lambda)$ with first eigenvalue λ.

However, in this interpretation, the ordering of the selected basis is crucial. This arbitrary selection can be removed by considering the action of $\mathbb{Z}_2$ on X_D. Concretely, let us take

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and let us define the action of $\mathbb{Z}_2$ on X_D by $-1 \cdot (A, B, \lambda) = (P_0AP_0^{-1}, P_0BP_0^{-1}, \lambda^{-1})$. Hence we can consider the quotient

$$\boxed{X_D/\mathbb{Z}_2 = \left\{(A, B, \lambda) \in SL(2, \mathbb{C})^2 \times (\mathbb{C}^* - \pm 1) \mid [A, B] = D_\lambda\right\}/\mathbb{Z}_2}$$

Observe that this can be described as

$$X_D/\mathbb{Z}_2 \cong \left\{ P_0 \cdot (A, B) \in \frac{SL(2, \mathbb{C})^2}{\langle P_0 \rangle} \,\middle|\, \begin{array}{c} \lambda + \lambda^{-1} = tr\,[A, B] \\ [A, B] = \{D_\lambda, D_{\lambda^{-1}}\} \end{array} \right\}$$

via the isomorphism $\varphi : [A, B, \lambda] \mapsto P_0 \cdot (A, B)$. Indeed, this is well defined since $\varphi([A, B, \lambda]) = P_0 \cdot (A, B) = \varphi[P_0AP_0^{-1}, P_0BP_0^{-1}, \lambda^{-1}]$ and the inverse map is given by $\varphi^{-1}(P_0 \cdot (A, B)) = [A, B, \lambda]$ where λ is the $(1, 1)$-entry of $[A, B]$.

One of the most important properties of this quotient is that its elements $P_0 \cdot (A, B) \in X_D/\mathbb{Z}_2$ do not really depend on the arbitrary choose of the first eigenvalue λ of $[A, B]$, but on the trace of $[A, B]$, that is $\lambda + \lambda^{-1}$, which is an invariant of $[A, B]$. Hence, defining $\tilde{D}_t := \{D_\lambda, D_{\lambda^{-1}}\}$ with $\lambda + \lambda^{-1} = t$ we have the isomorphism

$$X_D/\mathbb{Z}_2 \cong \left\{ P_0 \cdot (A, B) \in \frac{SL(2, \mathbb{C})^2}{\langle P_0 \rangle} \,\middle|\, [A, B] = \tilde{D}_{tr\,[A,B]} \right\}$$

Remark 4.1.12. In this form, we can give a more geometric interpretation to $X_D/\mathbb{Z}_2$. To this purpose, let us choose a regular mapping $P : \mathbb{C} - \{\pm 2\} \to SL(2,\mathbb{C})/\langle P_0 \rangle$. Then, the map $P_0 \cdot (A,B) \mapsto P(tr\,[A,B])[A,B]P^{-1}(tr\,[A,B])$ give us an isomorphism

$$X_D/\mathbb{Z}_2 \cong \left\{ (A,B) \in \frac{SL(2,\mathbb{C})^2}{\langle P_0 \rangle} \;\middle|\; P(tr\,[A,B])[A,B]P^{-1}(tr\,[A,B]) = \tilde{D}_{tr\,[A,B]} \right\}$$

Analogously to the case of X_D, this mapping $P(t)$ should be interpreted as picking a basis for each possible trace $t \in \mathbb{C} - \{\pm 2\}$ but, now, unordered. Hence, $(A,B) \in X_D/\mathbb{Z}_2$ if and only if $[A,B]$ diagonalizes in the corresponding basis $P(tr\,[A,B])$ with diferent eigenvalues. Equivalently, in terms of endomorphisms, given two volume preserving endomorphisms $f,g : \mathbb{C}^2 \to \mathbb{C}^2$, $(f,g) \in X_D/\mathbb{Z}_2$ if and only if $[f,g]$ diagonalizes in the basis $P(tr\,[f,g])$

4.1.8.1 Deligne-Hodge polynomial of $X_D/\mathbb{Z}_2$

In order to compute de Deligne-Hodge polynomial of $X_D/\mathbb{Z}_2$, let us define the auxiliar variety

$$\hat{X}_D = \left\{ (A,B,l) \in SL(2,\mathbb{C})^2 \times \mathbb{P}^1 \mid tr\,[A,B] \neq \pm 2, \;\; l \text{ eigenspace of } [A,B] \right\}$$

Observe that, if we define the action of $\mathbb{Z}_2$ on $\hat{X}_D$ by $-1 \cdot (A,B,l) = (P_0AP_0^{-1}, P_0BP_0^{-1}, l')$ where

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and l' is the eigenspace of $[A,B]$ that is not l, then, we have that $\overline{X}_D = \hat{X}_D/\mathbb{Z}_2$. Now, let us define $\pi : PGL(2,\mathbb{C}) \times X_D \to \hat{X}_D$ by $\pi(P,A,B,\lambda) = (PAP^{-1}, PBP^{-1}, \langle P(0,1) \rangle)$. Since the fiber is

$$\pi^{-1}(A,B,l) \cong \left\{ \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; \mu \in \mathbb{C}^* \right\} \cong \mathbb{C}^*$$

we have that $\pi : PGL(2,\mathbb{C}) \times X_D \to \hat{X}_D$ is a $\mathbb{C}^*$-principal bundle, so we have an E-fibration

$$\mathbb{C}^* \to PGL(2,\mathbb{C}) \times X_D \xrightarrow{\pi} \hat{X}_D$$

and, in particular

$$e\left(\hat{X}_D\right) = \frac{e(PGL(2,\mathbb{C}))e(X_D)}{e(\mathbb{C}^*)} = q^6 - 2q^5 - 9q^4 - q^3 + 8q^2 + 3q$$

Now, we can define an action of $\mathbb{Z}_2$ on $PGL(2,\mathbb{C}) \times X_D$ by imposing

$$-1 \cdot (P,A,B,\lambda) = (PP_0^{-1}, P_0AP_0^{-1}, P_0BP_0^{-1}, \lambda^{-1})$$

With respect to this action, the map $\pi : PGL(2,\mathbb{C}) \times X_D \to \hat{X}_D$ descends to the quotient, giving us a morphism $\tilde{\pi} : (PGL(2,\mathbb{C}) \times X_D)/\mathbb{Z}_2 \to \hat{X}_D/\mathbb{Z}_2 \cong \overline{X}_D$. Hence, we have the diagram of fibrations

$$\begin{tikzcd}
& \mathbb{C}^* \arrow[dl] \arrow[dr] & \\
PGL(2,\mathbb{C}) \times X_D \arrow[rr] \arrow[d, "\pi"] & & (PGL(2,\mathbb{C}) \times X_D)/\mathbb{Z}_2 \arrow[d, "\tilde{\pi}"] \\
\hat{X}_D \arrow[rr] & & \overline{X}_D
\end{tikzcd}$$

where the horizontal maps are the passing-to-quotient morphisms. Therefore, by theorem 3.3.51, remembering that $\hat{X}_D/\mathbb{Z}_2 = \overline{X}_d$, we have

$$\begin{aligned}
e((PGL(2,\mathbb{C}) \times X_D)/\mathbb{Z}_2) &= e(\hat{X}_D)^+ e(\mathbb{C}^*)^+ + e(\hat{X}_D)^- e(\mathbb{C}^*)^- \\
&= e(\overline{X}_D) e(\mathbb{C}^*)^+ + (e(\hat{X}_D) - e(\overline{X}_D)) e(\mathbb{C}^*)^-
\end{aligned}$$

For computing the Deligne-Hodge polynomial of $\overline{X}_D$, recall that we have an stratification of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ by

$$SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) = X_{Id} \sqcup X_{-Id} \sqcup \overline{X}_{[J_+]} \sqcup \overline{X}_{[J_-]} \sqcup \overline{X}_D$$

so, using the computations of the previous sections, we have

$$e(\overline{X}_D) = e(SL(2,\mathbb{C}))^2 - e(X_{Id}) - e(X_{-Id}) - e(\overline{X}_{[J_+]}) - e(\overline{X}_{[J_-]}) = q^6 - 2q^5 - 4q^4 + 3q^2 + 2q$$

Therefore, returning to $(PGL(2,\mathbb{C}) \times X_D)/\mathbb{Z}_2$, and remembering that $e(\mathbb{C}^*)^+ = q$ and $e(\mathbb{C}^*)^- = -1$, we obtain

$$\begin{aligned}
e((PGL(2,\mathbb{C}) \times X_D)/\mathbb{Z}_2) &= e(\overline{X}_D) e(\mathbb{C}^*)^+ + (e(\hat{X}_D) - e(\overline{X}_D)) e(\mathbb{C}^*)^- \\
&= q^7 - 2q^6 - 4q^5 + 5q^4 + 4q^3 - 3q^2 - q
\end{aligned}$$

And, now, the final trick. Observe that the action of $\mathbb{Z}_2$ on $PGL(2,\mathbb{C})$ by left multiplication extends to an action of $\mathbb{Z}_2$ on $GL(2,\mathbb{C})$ by left multiplication. However, since $GL(2,\mathbb{C})$ is connected, the induced map is homotopy to the identity and, thus, the action of $\mathbb{Z}_2$ on cohomology is trivial. Hence, we have that $e(PGL(2,\mathbb{C}))^+ = e(PGL(2,\mathbb{C})/\mathbb{Z}_2) = e(PGL(2,\mathbb{C}))$ and, thus $e(PGL(2,\mathbb{C}))^- = 0$. Therefore, we can recompute $e((PGL(2,\mathbb{C}) \times X_D)/\mathbb{Z}_2)$ as

$$\begin{aligned}
e((PGL(2,\mathbb{C}) \times X_D)/\mathbb{Z}_2) &= e(PGL(2,\mathbb{C}) \times X_D)^+ = e(PGL(2,\mathbb{C}))^+ e(X_D)^+ + e(PGL(2,\mathbb{C}))^- e(X_D)^- \\
&= e(PGL(2,\mathbb{C})) e(X_D)^+ = e(PGL(2,\mathbb{C})) e(X_D/\mathbb{Z}_2)
\end{aligned}$$

Therefore, we have obtained

$$\boxed{e(X_D/\mathbb{Z}_2) = \frac{e((PGL(2,\mathbb{C}) \times X_D)/\mathbb{Z}_2)}{e(PGL(2,\mathbb{C}))} = q^4 - 2q^3 - 3q^2 + 3q + 1}$$

4.2 Moduli of $SL(2,\mathbb{C})$-Representations of Elliptic Curves with 1 marked point

With these computations in hand, we can study the main concern of this work, the parabolic $SL(2,\mathbb{C})$-character varieties. Recall from section 2.2 that this varieties appear as representation varieties into $SL(2,\mathbb{C})$, modulo conjugation, of a elliptic curve (i.e. a compact Riemann surface of genus 1) with some removed points (called the punctures, the parabolic points or the marked points) with prescribed monodromy. The constrains in these spaces is due to their relation with other important moduli spaces that appear in mathematical-physics, like the moduli space of parabolic Higgs-bundles.

In particular, in the case of a single marked point, we must consider, for each conjugacy class $\mathcal{C} \subseteq SL(2,\mathbb{C})$ the parabolic character variety

$$\boxed{\mathcal{M}_{\mathcal{C}} = \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] \in \mathcal{C}\right\} /\!\!/ SL(2,\mathbb{C})}$$

or, equivalently (see section 2.2.5.1), for any $\xi \in \mathcal{C}$ we have $\mathcal{M}_{\mathcal{C}} \cong \mathcal{M}_{\xi}$, where

$$\boxed{\mathcal{M}_{\xi} = \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = \xi\right\} /\!\!/ Stab(\xi)}$$

being $Stab(\xi)$ the stabilizer of $\xi \in SL(2,\mathbb{C})$ under the action of $SL(2,\mathbb{C})$ on itself by conjugation. Again, recall that the conjugacy classes of $SL(2,\mathbb{C})$ are uniquely determined by one and only one of the following elements

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad -Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for $\lambda \in \mathbb{C}* - \{\pm 1\}$.

4.2.1 The parabolic character variety $\mathcal{M}_{Id}$

Let us study the $SL(2,\mathbb{C})$-character variety

$$\mathcal{M}_{Id} = \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid AB = BA\right\} /\!\!/ Stab(Id) = X_{Id} /\!\!/ SL(2,\mathbb{C})$$

where $SL(2,\mathbb{C})$ acts by simultaneous conjugation and we have used that, since Id lives in the center of $SL(2,\mathbb{C})$, $Stab(Id) = SL(2,\mathbb{C})$. Moreover, since $\pm Id$ acts trivially on X_{Id}, we have

$$\mathcal{M}_{Id} = X_{Id} /\!\!/ PGL(2,\mathbb{C})$$

However, observe that every element of X_{Id} is reducible. Indeed, from the analysis of X_{Id} in section 4.1.3, we obtained that all the orbits for the action of $PGL(2,\mathbb{C})$ on X_{Id} are of the form

$$PGL(2,\mathbb{C})\cdot(\pm Id, P) \qquad PGL(2,\mathbb{C})\cdot(P,\pm Id) \qquad PGL(2,\mathbb{C})\cdot(J_\pm, Q) \qquad PGL(2,\mathbb{C})\cdot(D_\lambda, D_\mu)$$

for some $P \in SL(2,\mathbb{C})$, $Q \in Stab(J_\pm)$ and $\lambda,\mu \in \mathbb{C}^* - \{\pm 1\}$. In any of the orbits, choosing an appropiate sequence of elements in $SL(2,\mathbb{C})$, we have that its Zariski closure contains a diagonal element. Therefore, identifying each orbit with its Zariski closure (as the S-equivalence procedure says), we have that

$$\mathcal{M}_{Id} = X_{Id} /\!\!/ PGL(2,\mathbb{C}) = \{(D_\lambda, D_\mu)\,|\,\lambda,\mu\in\mathbb{C}^*\}/\mathbb{Z}_2 \cong (\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts by $-1\cdot(D_\lambda, D_\mu) = (D_{\lambda^{-1}}, D_{\mu^{-1}})$, or, equivalently, in $\mathbb{C}^*\times\mathbb{C}^*$ by $-1\cdot(\lambda,\mu) = (\lambda^{-1},\mu^{-1})$.

In order to compute its Deligne-Hodge polynomial, observe that, taking $\pi : \mathbb{C}^*\times\mathbb{C}^* \to \mathbb{C}^*$ to be the first projection $\pi(\lambda,\mu)=\mu$, and $\tilde{\pi} : (\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{Z}_2 \to \mathbb{C}^*/\mathbb{Z}_2$ by $\tilde{\pi}([\lambda,\mu]) = [\lambda]$, we have the diagram of fibrations

$$\begin{tikzcd}
 & \mathbb{C}^* \arrow[dl] \arrow[dr] & \\
\mathbb{C}^*\times\mathbb{C}^* \arrow[rr] \arrow[d, "\pi"] & & (\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{Z}_2 \arrow[d, "\tilde{\pi}"] \\
\mathbb{C}^* \arrow[rr] & & \mathbb{C}^*/\mathbb{Z}_2
\end{tikzcd}$$

so, by theorem 3.3.51, using that $e(\mathbb{C}^*)^+ = e(\mathbb{C}^*/\mathbb{Z}_2) = e(\mathbb{C}) = q$ and $e(\mathbb{C}^*)^- = e(\mathbb{C}^*) - e(\mathbb{C}^*)^+ = -1$, we have that

$$\boxed{e(\mathcal{M}_{Id}) = e\left((\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{Z}_2\right) = e(\mathbb{C}^*)^+e(\mathbb{C}^*)^+ + e(\mathbb{C}^*)^-e(\mathbb{C}^*)^- = q^2+1}$$

Finally, let us recover the mixed Hodge numbers of $\mathcal{M}_{Id} \cong (\mathbb{C}^*\times\mathbb{C}^*)/\mathbb{Z}_2$. Recall that, by the computations of remark 3.3.54 and the Künneth formula 3.3.42, we have that the only non trivial mixed Hodge numbers of $\mathbb{C}^*\times\mathbb{C}^*$ are

$$h_c^{4;2,2}(\mathbb{C}^*\times\mathbb{C}^*) = 1 \qquad h_c^{3;1,1}(\mathbb{C}^*\times\mathbb{C}^*) = 2 \qquad h_c^{2;0,0}(\mathbb{C}^*\times\mathbb{C}^*) = 1$$

Moreover, the action of $\mathbb{Z}_2$ in the cohomology of $\mathbb{C}^* \times \mathbb{C}^*$ give us a decomposition that preserves the mixed Hodge structures

$$H_c^*((\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2) \cong H_c^2(\mathbb{C}^* \times \mathbb{C}^*) \oplus H_c^4(\mathbb{C}^* \times \mathbb{C}^*)$$

and, therefore, the unique non vanishing mixed Hodge numbers of $\mathcal{M}_{Id} \cong (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$ are

$$h_c^{4;2,2}(\mathcal{M}_{Id}) = 1 \qquad h_c^{2;0,0}(\mathcal{M}_{Id}) = 1$$

Thus, the mixed Hodge polinomial of $\mathcal{M}_{Id}$ is

$$\boxed{H_c(\mathcal{M}_{Id})(u,v,t) = u^2v^2t^4 + t^2 = q^2t^4 + t^2}$$

4.2.2 The parabolic character variety $\mathcal{M}_{-Id}$

Recall that the parabolic $SL(2,\mathbb{C})$-character variety with one puncture and monodromy in the class of $-Id \in SL(2,\mathbb{C})$ is

$$\mathcal{M}_{-Id} = \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid AB = -BA\right\} /\!\!/ \, Stab(-Id) = X_{-Id} /\!\!/ \, SL(2,\mathbb{C})$$

since $Stab(-Id) = SL(2,\mathbb{C})$, because $-Id$ lives in the center of $SL(2,\mathbb{C})$. Recall that $SL(2,\mathbb{C})$ acts on X_{-Id} by simultaneous conjugation.

However, in subsection 4.1.4 we proved that the action of $SL(2,\mathbb{C})$ on X_{-Id} by conjugation is transitive, so the orbit space is a single point variety $X_{-Id}/SL(2,\mathbb{C}) = \{\star\}$. This space is obviously a good quotient for the action, so $X_{-Id} /\!\!/ \, SL(2,\mathbb{C}) \cong \{\star\}$ and, therefore

$$\boxed{\mathcal{M}_{-Id} \cong \{\star\}}$$

In particular, we can easily obtain all the algebraic and topological information of $\mathcal{M}_{-Id}$. As a compact Kähler manifold, its mixed Hodge structure is, in fact, a pure Hodge structure with unique non-vanishing Hodge number $h^{0;0,0}(\mathcal{M}_{-Id}) = 1$. Therefore, the mixed Hodge polynomial is

$$H_c(\mathcal{M}_{-Id})(u,v,t) = 1$$

4.2.3 The parabolic character variety $\mathcal{M}_{J_+}$

Recall that the parabolic $SL(2,\mathbb{C})$-character variety with monodromy around the puncture living in the conjugacy class of $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is

$$\mathcal{M}_{J_+} = \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid AB = J_+BA\right\} /\!\!/ \, Stab(J_+) = X_{J_+} /\!\!/ \, Stab(J_+)$$

where $Stab(J_+)$, the stabilizer of J_+ in $SL(2,\mathbb{C})$, acts by simultaneous conjugation. Recall, from the previous computations that

$$Stab(J_+) = \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C} \right\} \cong \mathbb{C}$$

In order to understand the conjugacy action of $Stab(J_+)$ on X_{J_+}, recall that in section 4.1.5 we obtained an explicit description via the variety

$$Y := \left\{ (x,a,y,b) \in (\mathbb{C}^*)^2 \times \mathbb{C}^2 \,|\, y(x(a^2-1)) + b(a(1-x^2)) = 1 \right\}$$

and the isomorphism $\varphi : Y \to X_{J_+}$

$$\begin{array}{ccc} Y & \overset{\varphi}{\longleftrightarrow} & X_{J_+} \\ (x,a,y,b) & \longmapsto & \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \right) \end{array}$$

Now, observe that, for $A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, B = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \in SL(2,\mathbb{C})$ and $P = \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \in Stab(J_+)$ for $\lambda \in \mathbb{C}$, we have

$$PAP^{-1} = \begin{pmatrix} a & b \pm \lambda(a^{-1} - a) \\ 0 & a^{-1} \end{pmatrix} \qquad PBP^{-1} = \begin{pmatrix} x & y \pm \lambda(x^{-1} - x) \\ 0 & x^{-1} \end{pmatrix}$$

Thus, if $P \cdot Aut(X_{J_+})$ is the morphism on X_{J_+} induced by $P \in Stab(J_+)$, and $\varphi^*(P\cdot) \in Aut(Y)$ is the induced automorphism (that is, $\varphi^*(P\cdot)(\overline{y}) := \varphi^{-1}(P \cdot (\varphi(\overline{y})))$ for $\overline{y} \in Y$) we have that

$$\begin{array}{cccc} \varphi^*(P\cdot) : & Y & \longrightarrow & Y \\ & (x,a,y,b) & \mapsto & \left(x, a, y \pm \lambda(x^{-1} - x), a \pm \lambda(a^{-1} - a)\right) \end{array}$$

In particular, recall that using the projection map $\pi : Y \to \mathbb{C}^* \times \mathbb{C}^* - \{(\pm 1, \pm 1)\}$ projected over coordinates (x,a), we have the algebraic line bundle

$$\mathbb{C} \to Y \to \mathbb{C}^* \times \mathbb{C}^* - \{(\pm 1, \pm 1)\}$$

so, for every $P \in Stab(J_+)$, $\varphi^*(P\cdot) : Y \to Y$ is a vector bundle map (in fact, it is an element of the gauge group of Y). In that case, if $L_{x,a} := \pi^{-1}(x,a) \subseteq Y$ is the fiber of π for any $(x,a) \in \mathbb{C}^* \times \mathbb{C}^* - \{(\pm 1, \pm 1)\}$ it transforms

$$\begin{array}{cccc} \varphi^*(P\cdot) : & L_{x,a} & \longrightarrow & L_{x,a} \\ & (y,b) & \mapsto & \left(y \pm \lambda(x^{-1} - x), a \pm \lambda(a^{-1} - a)\right) \end{array}$$

Therefore, vía φ, the action of $Stab(J_+)$ becomes, on Y, translation between the fibers. In particular, $Stab(J_+)$ is transitive in the fibers so $X_{J_+}/Stab(J_+)$ can be identified with the base variety $\mathbb{C}^* \times$

$\mathbb{C}^* - \{(\pm 1, \pm 1)\}$. Again, it can be proved that, in fact, this is a good quotient for the action, so, by uniqueness

$$X_{J_+} /\!\!/ Stab(J_+) \cong \mathbb{C}^* \times \mathbb{C}^* - \{(\pm 1, \pm 1)\}$$

and, therefore,

$$\boxed{\mathcal{M}_{J_+} = \mathbb{C}^* \times \mathbb{C}^* - \{(\pm 1, \pm 1)\}}$$

From this explicit description, we can compute all the desired algebraic and topological invariants. In fact, the Deligne-Hodge polynomial of $\mathcal{M}_{J_-}$ is

$$e(\mathcal{M}_{J_-}) = e(\mathbb{C}^* \times \mathbb{C}^*) - e(\{(\pm 1, \pm 1)\}) = (q-1)^2 - 4 = q^2 - 2q - 3$$

Furthermore, using a tipical Mayer-Vietoris type argument, it can be proved that its Poicaré polynomial is

$$P(X_{J_-})(t) = 4t^3 + t^2 + 2t + 1$$

and, since it is a smooth space, by Poincaré duality

$$P_c(X_{J_-})(t) = 4t^3 + t^2 + 2t + 1$$

However, this data is not enough for computing the Hodge numbers of this variety. To this end, we should use a more powerful tool as the long exact sequence of proposition 3.3.53. First of all, recall that the mixed Hodge structure of $\mathbb{C}^*$ (see example 3.3.54) has non-vanishing groups

$$H_c^{1;0,0}(\mathbb{C}^*) \cong \mathbb{C} \qquad H_c^{2;1,1}(\mathbb{C}^*) \cong \mathbb{C}$$

Therefore, by the Künneth formula for mixed Hodge structures (see theorem 3.3.42) we have that the mixed Hodge structure of $\mathbb{C}^* \times \mathbb{C}^*$ has non-vanishing groups

$$H_c^{2;0,0}(\mathbb{C}^* \times \mathbb{C}^*) \cong \mathbb{C} \qquad H_c^{3;1,1}(\mathbb{C}^* \times \mathbb{C}^*) \cong \mathbb{C}^2 \qquad H_c^{4;2,2}(\mathbb{C}^* \times \mathbb{C}^*) \cong \mathbb{C}$$

Now, we can remove the point $(1,1) \in \mathbb{C}^* \times \mathbb{C}^*$ using the considerations in example 3.3.55. Hence, for $(p,q) \neq (0,0)$ or for $(p,q) = (0,0)$ and $k \geq 2$ we have

$$H^{k;p,q}\left(\mathbb{C}^* \times \mathbb{C}^* - \{(1,1)\}\right) \cong H^{k;p,q}\left(\mathbb{C}^* \times \mathbb{C}^*\right)$$

and the other two groups must satisfy the long exact sequence

$$\begin{tikzcd}
 & & 0 \arrow[d, equal] & \mathbb{C} \arrow[d, equal] \\
0 \arrow[r] & H_c^{0;0,0}\left(\mathbb{C}^* \times \mathbb{C}^* - \{(1,1)\}\right) \arrow[r] & H_c^{0;0,0}\left(\mathbb{C}^* \times \mathbb{C}^*\right) \arrow[r] & H_c^{0;0,0}(\{(1,1)\}) \arrow[dll, rounded corners] \\
 & H_c^{1;0,0}\left(\mathbb{C}^* \times \mathbb{C}^* - \{(1,1)\}\right) \arrow[r] & H_c^{1;0,0}\left(\mathbb{C}^* \times \mathbb{C}^*\right) \arrow[r] \arrow[d, equal] & H_c^{1;0,0}(\{(1,1)\}) \arrow[d, equal] \\
 & & 0 & 0
\end{tikzcd}$$

so $H_c^{0;0,0}\left(\mathbb{C}^* \times \mathbb{C}^* - \{(1,1)\}\right) = 0$ and $H_c^{1;0,0}\left(\mathbb{C}^* \times \mathbb{C}^* - \{(1,1)\}\right) \cong \mathbb{C}$. Therefore, the mixed Hodge structure of $\mathbb{C}^* \times \mathbb{C}^* - \{(1,1)\}$ has non-vanishing groups

$$\begin{array}{ll}
H_c^{1;0,0}\left((\mathbb{C}^*)^2 - \{(1,1)\}\right) \cong \mathbb{C} & \qquad H_c^{2;0,0}\left((\mathbb{C}^*)^2 - \{(1,1)\}\right) \cong \mathbb{C} \\
H_c^{3;1,1}\left((\mathbb{C}^*)^2 - \{(1,1)\}\right) \cong \mathbb{C}^2 & \qquad H_c^{4;2,2}\left((\mathbb{C}^*)^2 - \{(1,1)\}\right) \cong \mathbb{C}
\end{array}$$

Thus, repeating the procedure in order to remove $(1,-1), (-1,1), (-1,-1) \in (\mathbb{C}^*)^2 - \{(1,1)\}$ we obtain that, for $\mathcal{M}_{J_+} \cong (\mathbb{C}^*)^2 - \{(\pm 1, \pm 1)\}$, the non-vanishing groups of its mixed Hodge structure are

$$\begin{array}{ll}
H_c^{1;0,0}\left(\mathcal{M}_{J_+}\right) \cong \mathbb{C}^4 & \qquad H_c^{2;0,0}\left(\mathcal{M}_{J_+}\right) \cong \mathbb{C} \\
H_c^{3;1,1}\left(\mathcal{M}_{J_+}\right) \cong \mathbb{C}^2 & \qquad H_c^{4;2,2}\left(\mathcal{M}_{J_+}\right) \cong \mathbb{C}
\end{array}$$

Therefore, summarizing, the mixed Hodge polynomial of $\mathcal{M}_{J_+}$ is

$$\boxed{H_c(\mathcal{M}_{J_+})(u,v,t) = u^2v^2t^4 + 2uvt^3 + t^2 + 4t = q^2t^4 + 2qt^3 + t^2 + 4t}$$

4.2.4 The parabolic character variety $\mathcal{M}_{J_-}$

Let us study the $SL(2,\mathbb{C})$-character variety

$$\mathcal{M}_{J_-} = \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid AB = J_-BA\right\} /\!\!/ \, Stab(J_-) = X_{J_-} /\!\!/ \, Stab(J_-)$$

where $Stab(J_-)$ acts by simultaneous conjugation. Moreover, since $\pm Id$ acts trivially on X_{J_-}, we have

$$\mathcal{M}_{J_-} = X_{J_-} /\!\!/ \, (Stab(J_-)/K)$$

Now, the key point is that $Stab(J_-)/K$ acts freely on X_{J_-}. Indeed, suppose that there exists a non trivial $P \in Stab(J_-)/\{\pm Id\}$ and $(A,B) \in X_{J_-}$ with $P(A,B)P^{-1} = Id$. Let us write

$$P = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\lambda \neq 0$. Since we have

$$PAP^{-1} = \begin{pmatrix} a+\lambda c & b-\lambda^2 c+\lambda(d-a) \\ c & d-\lambda c \end{pmatrix}$$

we obtain that $PAP^{-1} = A$ if and only if $a = d = \pm 1$ and $c = 0$, that is, if and only if $A \in U = Stab(J_+)$. Analogously, we should have $B \in U = Stab(J_+)$. Hence, since $[U,U] = Id$, we have that $[A,B] = Id$, which contradices $(A,B) \in X_{J_-}$.

Therefore, since $Stab(J_-)/K$ acts freely on X_{J_-}, its GIT quotient is just the usual quotient, so

$$\mathcal{M}_{J_-} \cong X_{J_-} /\!\!/ (Stab(J_-)/K) = \tfrac{X_{J_-}}{Stab(J_-)/K}$$

In particular, we have a principal bundle

$$Stab(J_-)/K \to X_{J_-} \to \mathcal{M}_{J_-}$$

which, automatically, is an E-fibration. Therefore, since $Stab(J_-)/K \cong \mathbb{C}/\mathbb{Z}_2 = \mathbb{C}$, we obtain

$$\boxed{e\left(\mathcal{M}_{J_-}\right) = \frac{e\left(X_{J_-}\right)}{e\left(Stab(J_-)/K\right)} = q^2 + 3q}$$

In order to compute its Hodge numbers, we need to identify explicitly this space. Using the notation of section 4.1.6, recall that, by the computed stratification, we have that

$$X_{J_-} = X_{J_-}^{\alpha} \sqcup X_{J_-}^{\beta} \sqcup \tilde{X}_{J_-}$$

and, since $\mathcal{M}_{J_-} = X_{J_-}/SL(2,\mathbb{C})$ and $\tilde{X}_{J_-}/SL(2,\mathbb{C}) \cong S$ we obtain

$$\mathcal{M}_{J_-} \cong S \sqcup X_{J_-}^{\alpha}/SL(2,\mathbb{C}) \sqcup X_{J_-}^{\beta}/SL(2,\mathbb{C})$$

For the study of S, recall that we have an stratification

$$S = \hat{S} - \hat{S}_\infty - S_0 = \hat{S}_{ND} \sqcup \hat{S}_D - \hat{S}_\infty - S_0$$

where $\hat{S} = \left\{ad = 1 + \lambda^2\right\}$ is the projective completion of S, $\hat{S}_{ND}$ and $\hat{S}_D$ are the non-degenerated and degenerated, respectively, fibers of the fibration $\pi : \hat{S} \to \mathbb{C}^*$ given by $\pi((a,d,\lambda)) = \lambda = \frac{\alpha}{\beta}$, $\hat{S}_\infty$ are the points at infinity of the conic bundle and S_0 are the removed points due to the constrain on S of $a+d \neq 0$.

The situation for $\hat{S}$ is as follows. Under the fibration $\pi : \hat{S} \to \mathbb{C}^*$ we have that the generic fiber is isomorphic to a projective conic, i.e. $\mathbb{P}^1$, with two degenerated fibers, corresponding to $\lambda = \pm i$ isomorphic to two copies of $\mathbb{P}^1$ intersecting in a point. Now, the points at infinity S_∞ are, for each fiber, two different points, so, removing those points, we obtain a fibration with generic fiber $\mathbb{P}^1 - \{2 \text{ points}\} \cong \mathbb{C}^*$. For the degenerated fibers, the description of section 4.1.6.2 shows that each of the points at infinity belongs to one copy of $\mathbb{P}^1$ and any of them is the intersection point. Therefore, we obtain that the degenerated fiber is isomorphic to $\mathbb{C} \cup \mathbb{C}$ intersecting in a point. Finally, for the removed points S_0 we observe that, the correspond to two different points on each generic fiber, but, for the degenerated fibers, it is just a point, the intersection point between the two copies of $\mathbb{C}$. Therefore, we have obtained a description of S as a fiber bundle over $\mathbb{C}^*$ with generic fiber isomorphic to $\mathbb{P}^1 - \{4 \text{ points}\}$ and two degenerated fiber isomorphic to $\mathbb{C} \sqcup \mathbb{C}$.

However, for $X_{J_-}^{\beta}/SL(2,\mathbb{C}) \cong \mathbb{C}^* \sqcup \mathbb{C}^*$, the situation is different. Taking the trace of this subspace in the conic bundle $\hat{S}$, we have that $X_{J_-}^{\beta}/SL(2,\mathbb{C})$ is exactly the points at infinity (since it correspond to taking $\lambda \to \infty$ or, equivalently, $x_0 \to 0$). Thus, filling these points, we have that $S \sqcup X_{J_-}^{\beta}/SL(2,\mathbb{C})$ is a fibration with generic fiber isomorphic to $\mathbb{P}^1 - \{2 \text{ points}\} \cong \mathbb{C}^*$ and two degenerated fibers isomorphic to $\mathbb{P}^1 \cup \mathbb{P}^1 - \{\text{intersection point}\}$, i.e. two parallel lines. Finally, the space $X_{J_-}^{\alpha}/SL(2,\mathbb{C}) \cong \mathbb{C}^* \sqcup \mathbb{C}^*$ can be considered as the missed fiber for $\alpha/\beta = \lambda = 0$, since it correspond to $\alpha = 0$.

Summarizing, we have obtained a description of $\mathcal{M}_{J_-}$ as follows. It is a fibration over $\mathbb{C}$ with generic fiber isomorphic to $\mathbb{P}^1 - \{2 \text{ points}\} \cong \mathbb{C}^*$. For the singular fibers, we have two degenerated fibers for $\lambda = \pm i$ isomorphic to $\mathbb{P}^1 \cup \mathbb{P}^1 - \{\text{intersection point}\}$, that is, two paralles lines. Finally, we also have a singular fiber over 0 isomorphic to $\mathbb{C}^* \sqcup \mathbb{C}^* \cong \mathbb{P}^1 \cup \mathbb{P}^1 - \{3 \text{ points}\}$.

Therefore, the space $\mathcal{M}_{J_-}$ can be build using the following algorithm:

(1) Start with the variety $M_1 = \mathbb{P}^1 \times \mathbb{C}$.

(2) For each on the fibers under $\{\pm, 0\}$, remove one point and add a copy of $\mathbb{P}^1$ replacing that point, obtaining the variety M_2. Equivalently, blow up three fibers.

(3) Remove a bisection of the fibration that intersects twice to each generic fibers and the singular fiber over 0 and once to the other two singular fibers. Obtain the variety M_3.

(4) Remove the intersection point in the fiber over 0.

Using these descriptions of $\mathcal{M}_{J_-}$ and a Mayer-Vietoris type argument for compactly supported cohomology, we obtain the Poincaré polynomial of $\mathcal{M}_{J_-}$ with compact support

$$P_c(\mathcal{M}_{J_-})(t) = t^4 + t^3 + 5t^2 + t$$

In order to computed the mixed Hodge numbers of $\mathcal{M}_{J_-}$, first observe that, by construction, $\mathcal{M}_{J_-}$ has balanced type, so $h_c^{k;p,q}(\mathcal{M}_{J_-}) = 0$ for $p \neq q$. Moreover, $\mathcal{M}_{J_-} - X^{\alpha}_{J_-}/SL(2,\mathbb{C}) = \hat{S} - S_0$, it is a smooth variety. Analogously, $\mathcal{M}_{J_-} - X^{\beta}_{J_-}/SL(2,\mathbb{C})$ is also smooth, so $\mathcal{M}_{J_-}$ is a smooth variety. Thus, it satifies $h_c^{k;p,q}(\mathcal{M}_{J_-}) = 0$ for $k < p+q$. In particular, since $H^4_c(\mathcal{M}_{J_-},\mathbb{C}) \cong \mathbb{C}$, it must be $h_c^{4;2,2}(\mathcal{M}_{J_-}) = 1$.

With this restrictions and the information of its Deligne-Hodge and Poincaré polynomial, we obtain that, the other possible non-vanishing mixed Hodge numbers for $\mathcal{M}_{J_-}$ are

$$h_c^{2;0,0}(\mathcal{M}_{J_-}) = 1 \qquad h_c^{2;1,1}(\mathcal{M}_{J_-}) = 4 \qquad h_c^{3;1,1}(\mathcal{M}_{J_-}) = 1$$

or the posibility

$$h_c^{2;0,0}(\mathcal{M}_{J_-}) = 2 \qquad h_c^{2;1,1}(\mathcal{M}_{J_-}) = 3 \qquad h_c^{3;0,0}(\mathcal{M}_{J_-}) = 1$$

In order to distinghish between the two possibilities, we are going to compute $h_c^{2;0,0}(\mathcal{M}_{J_-})$. First of all, using Küneth formula for mixed Hodge structures and the computations of section 3.3.3.3, we obtain for $M_1 = \mathbb{P}^1 \times \mathbb{C}$ that its non-vanishing mixed Hodge numbers are

$$h_c^{2;1,1}(M_1) = 1 \qquad h_c^{4;2,2}(M_1) = 1$$

Now, observe that, $h_c^{2;0,0}(M_1) = h_c^{2;0,0}(M_2)$. Indeed, recall that, when removing a point, by example 3.3.55 we have that $h_c^{2;0,0}(M_1 - \{\star\}) = h_c^{2;0,0}(M_1) = 0$. Now, if we want to add a copy of $\mathbb{P}^1$ on $M_1 - \{\star\}$, we obtain a long exact sequence

$$\begin{tikzcd}
{} \arrow[r] & H_c^{2;0,0}(M_1 - \{\star\}) \arrow[r] \arrow[d, equal] & H_c^{2;0,0}(M_1 - \{\star\} \sqcup \mathbb{P}^1) \arrow[r] & H_c^{2;0,0}(\mathbb{P}^1) \arrow[r] \arrow[d, equal] & {} \\
& 0 & & 0 &
\end{tikzcd}$$

so $h_c^{2;0,0}(M_1 - \{\star\} \sqcup \mathbb{P}^1) = h_c^{2;0,0}(M_1 - \{\star\}) = 0$. Doing this procedure three times, we have that $h_c^{2;0,0}(M_2) = 0$. Analogously, $h_c^{1;0,0}(M_2) = 0$

For the case of the bisection $C \subseteq M_2$, observe that, by description, $C \cong \mathbb{C}^*$. Therefore, for $M_2 - C$, we obtain a long exact sequence

$$\begin{tikzcd}
 & & 0 \arrow[d, equal] & \mathbb{C} \arrow[d, equal] \\
 & {} \arrow[r] & H_c^{1;0,0}(M_2) \arrow[r] & H_c^{1;0,0}(\mathbb{C}^*) \arrow[dll, rounded corners] \\
 & H_c^{2;0,0}(M_2 - C) \arrow[r] & H_c^{2;0,0}(M_2) \arrow[r] \arrow[d, equal] & {} \\
 & & 0 &
\end{tikzcd}$$

so $h_c^{2;0,0}(M_3) = h_c^{2;0,0}(M_2 - C) = 1$. Finally, since removing a point do not modify the mixed Hodge nomber $h_c^{2;0,0}$, we have that $h_c^{2;0,0}(\mathcal{M}_{J_-}) = h_c^{2;0,0}(M_3 - \{\star\}) = 1$.

Therefore, the true possibility is the first one conjectured and, thus, we obtain that the non-vanishing mixed Hodge numbers of $\mathcal{M}_{J_-}$ are

$$h_c^{1;0,0}(\mathcal{M}_{J_-}) = 1 \quad h_c^{2;0,0}(\mathcal{M}_{J_-}) = 1 \quad h_c^{2;1,1}(\mathcal{M}_{J_-}) = 4 \quad h_c^{3;1,1}(\mathcal{M}_{J_-}) = 1 \quad h_c^{4;2,2}(\mathcal{M}_{J_-}) = 1$$

Thus, the mixed Hodge polynomial of $\mathcal{M}_{J_-}$ is

$$\boxed{H_c(\mathcal{M}_{J_-})(u,v,t) = u^2v^2t^4 + uvt^3 + 4uvt^2 + t^2 + t = q^2t^4 + qt^3 + 4qt^2 + t^2 + 1}$$

Remark 4.2.1. Until the present day, this polynomial was unknown, so this is, in fact, a new result.

4.2.5 The parabolic character variety $\mathcal{M}_{D_\lambda}$

Let us fix $\lambda \in \mathbb{C}^* - \{\pm 1\}$. Recall that the parabolic $SL(2,\mathbb{C})$-character variety with one puncture and monodromy in the class of $D_\lambda \in SL(2,\mathbb{C})$ is

$$\mathcal{M}_{D_\lambda} = \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = D_\lambda\right\} /\!\!/ \, Stab(D_\lambda) = X_{D_\lambda} /\!\!/ \, Stab(D_\lambda)$$

where $Stab(D_\lambda)$ is, by remark 4.1.4

$$D := Stab(D_\lambda) = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \;\middle|\; \mu \in \mathbb{C}^* \right\} \cong \mathbb{C}^*$$

Observe that $Id, -Id \in D$ acts trivially on X_{D_λ}, so $X_{D_\lambda} /\!\!/ \, D = X_{D_\lambda} /\!\!/ \, D/K$ where $K = \{\pm Id\}$. But D/K is exactly the stabilizer of $D_\lambda \in PSL(2,\mathbb{C})$, which acts freely on X_{D_λ}. Therefore, the orbit space is just the GIT quotient, so

$$\mathcal{M}_{D_\lambda} = X_{D_\lambda} /\!\!/ (D/K) = X_{D_\lambda}/(D/K)$$

and, since $D/K \cong \mathbb{C}^*/\mathbb{Z}_2 \cong \mathbb{C}^*$ we have that

$$\boxed{e\left(\mathcal{M}_{D_\lambda}\right) = \frac{e\left(X_{D_\lambda}\right)}{e\left(D/K\right)} = q^2 + 4q + 1}$$

Moreover, we can also obtain the Hodge number of $\mathcal{M}_{D_\lambda}$. To this end, recall, that, by the results of section 2.3, we have that $\mathcal{M}_{D_\lambda}$ is homeomorphic to the moduli space of parabolic Higgs bundles over an elliptic curve with some fixed parabolic structure α, $\mathcal{M}^{\alpha}_{Dol}(X)$. In [8], its Poincaré polynomial with compact support is computed, obtaining

$$P_c(\mathcal{M}^P_{Higgs}(X,\lambda))(t) = P_c(\mathcal{M}_{D_\lambda})(t) = t^4 + 5t^2$$

Since the space $\mathcal{M}_{D_\lambda}$ is of balanced type, we have that $h_c^{k;p,q}(\mathcal{M}_{D_\lambda}) = 0$ for $p \neq q$. Moreover, since $\mathcal{M}_{D_\lambda}$ is smooth, $h_c^{k;p,p}(\mathcal{M}_{D_\lambda}) = 0$ for $p \geq k$. In addition, since $\mathcal{M}_{D_\lambda}$ is connected, $\sum_p h_c^{4;p,p}(\mathcal{M}_{D_\lambda}) = 1$ and, since $\mathcal{M}_{D_\lambda}$ is not compact, $h_c^{0;0,0}(\mathcal{M}_{D_\lambda}) = 0$.

Therefore, with this information and comparing $e\left(\mathcal{M}_{D_\lambda}\right)$ with $P_c(\mathcal{M}_{D_\lambda})$ we obtain that the only posibility for the Hodge numbers is that the non-vanishing numbers are

$$h_c^{2;0,0}(\mathcal{M}_{D_\lambda}) = 1 \quad h_c^{2;1,1}(\mathcal{M}_{D_\lambda}) = 4 \quad h_c^{4;2,2}(\mathcal{M}_{D_\lambda}) = 1$$

Thus, its mixed Hodge polynomial is

$$\boxed{H_c(\mathcal{M}_{D_\lambda})(u,v,t) = u^2v^2t^4 + 4uvt^2 + t^2 = q^2t^4 + 4qt^2 + t^2}$$

4.3 Moduli of $SL(2,\mathbb{C})$-Representations of Elliptic Curves with 2 marked points

In this section, we will extend our previous computations to the case of $SL(2,\mathbb{C})$-parabolic character varieties of an elliptic curve with two marked points. Again, this corresponds to the character variety of a compact Riemann surface of genus 1 (i.e. an algebraic curve of genus 1) with two removed points (the marked points), where the holonomy of the representation around the marked points is prescrived to live in fixed conjugacy classes.

Explicity, recall from section 2.2.5.1 that, fixed $SL(2,\mathbb{C})$-conjugacy classes $\mathcal{C}_1, \mathcal{C}_2 \subseteq SL(2,\mathbb{C})$ we define

$$\boxed{\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2} = \left\{ (A,B,C_1,C_2) \in SL(2,\mathbb{C})^4 \;\middle|\; \begin{array}{l} [A,B]C_1C_2 = Id \\ C_1 \in \mathcal{C}_1, \quad C_2 \in \mathcal{C}_2 \end{array} \right\} /\!\!/ \, SL(2,\mathbb{C})}$$

with $SL(2,\mathbb{C})$ acting by simultaneous conjugation.

Remark 4.3.1. The importance of these spaces was shown in section 2.3 in relation with other moduli spaces. In summary, in the case of holonomies of different diagonalizable type, the parabolic character variety $\mathcal{M}_{[D_\lambda],[D_\mu]}$ with $\lambda \neq \mu$, is homeomorphic to the moduli space of stable parabolic Higgs bundles of parabolic degree 0 and traceless Higgs field. This space, through Nahm transformation, is homeomorphic to the moduli space of doubly periodic instantons.

In order to study this spaces, analogously to what we did in section 4.2, for $SL(2,\mathbb{C})$-conjugacy classes $\mathcal{C}_1, \mathcal{C}_2 \subseteq SL(2,\mathbb{C})$, we define the auxiliar spaces

$$X_{\mathcal{C}_1,\mathcal{C}_2} := \left\{ (A,B,C_1,C_2) \in SL(2,\mathbb{C})^4 \;\middle|\; \begin{array}{l} [A,B]C_1C_2 = Id \\ C_1 \in \mathcal{C}_1, \quad C_2 \in \mathcal{C}_2 \end{array} \right\}$$

in the way that

$$\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2} = X_{\mathcal{C}_1,\mathcal{C}_2} /\!\!/ \, SL(2,\mathbb{C}) = X_{\mathcal{C}_1,\mathcal{C}_2} /\!\!/ \, PGL(2,\mathbb{C})$$

where the last identity follows from the fact that, given $P \in SL(2,\mathbb{C})$, the action of P on $X_{\mathcal{C}_1,\mathcal{C}_2}$ is equal to the action of $-P$ on $X_{\mathcal{C}_1,\mathcal{C}_2}$, so the action descends to the quotient $SL(2,\mathbb{C})/\left\{Id, -Id\right\} = PGL(2,\mathbb{C})$.

First of all, observe that there exists a strong symmetry between $\mathcal{C}_1$ and $\mathcal{C}_2$. Indeed, observe that the map

$$\begin{array}{cccc} \phi: & X_{\mathcal{C}_1,\mathcal{C}_2} & \longleftrightarrow & X_{\mathcal{C}_2,\mathcal{C}_1} \\ & (A,B,C_1,C_2) & \longmapsto & (B^{-1}, A^{-1}, C_2^{-1}, C_1^{-1}) \end{array}$$

is an isomorphism of algebraic varieties that commutes with the action of $SL(2,\mathbb{C})$ by simultaneous conjugation, so it descends to an algebraic isomorphism

$$\overline{\phi}: \mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2} = X_{\mathcal{C}_1,\mathcal{C}_2} /\!\!/ \, SL(2,\mathbb{C}) \longleftrightarrow \mathcal{M}_{\mathcal{C}_2,\mathcal{C}_1} = X_{\mathcal{C}_2,\mathcal{C}_1} /\!\!/ \, SL(2,\mathbb{C})$$

Recall that, in $SL(2,\mathbb{C})$, there are five conjugacy classes, determined by the matrices

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad -Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

so, a priori, using this symmetry between conjugacy clases, we have 16 different parabolic character varieties (observe that we have to consider as different the spaces $\mathcal{M}_{[D_\lambda],[D_\lambda]}$ and $\mathcal{M}_{[D_\lambda],[D_\mu]}$ with $\lambda \neq \mu$).

However, the case $\mathcal{C}_1 = [Id], [-Id]$ essentially reduces to the case of a single marked points. Indeed, observe that, since $[Id] = \{Id\}$ and $[-Id] = \{-Id\}$ we have

$$\begin{aligned}\mathcal{M}_{[\pm Id],\mathcal{C}} &= \left\{(A,B,\pm Id,C) \in SL(2,\mathbb{C})^4 \;\middle|\; \begin{array}{c}[A,B]C = \pm Id \\ C \in \mathcal{C}\end{array}\right\} /\!\!/ \, SL(2,\mathbb{C}) \\ &\cong \left\{(A,B,C) \in SL(2,\mathbb{C})^3 \;\middle|\; \begin{array}{c}[A,B]C = Id \\ C \in \mathcal{C}\end{array}\right\} /\!\!/ \, SL(2,\mathbb{C}) = \mathcal{M}_{\mathcal{C}}\end{aligned}$$

Therefore, essentially, we only have to study seven cases grouped in three families

- Holonomies of Jordan type: $\mathcal{M}_{[J_+],[J_+]}, \mathcal{M}_{[J_+],[J_-]}, \mathcal{M}_{[J_-],[J_-]}$.
- Holonomies of mixed type: $\mathcal{M}_{[J_+],[D_\lambda]}, \mathcal{M}_{[J_-],[D_\lambda]}$
- Holonomies of diagonalizable type: $\mathcal{M}_{[D_\lambda],[D_\lambda]}, \mathcal{M}_{[D_\lambda],[D_\mu]}$ with $\lambda \neq \mu$.

Finally, a very important tool that we will need in order to understand $\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2}$ are the auxiliar varieties

$$\boxed{Y^\xi_{\mathcal{C}_1} := \left\{(A,B,C) \in SL(2,\mathbb{C})^3 \;\middle|\; \begin{array}{c}[A,B]C = \xi \\ C \in \mathcal{C}_1\end{array}\right\}}$$

for any fixed $\xi \in \mathcal{C}_2$.

These spaces are very related to $X_{\mathcal{C}_1,\mathcal{C}_2}$. Suppose that there exists a subgroup $K \subseteq Stab(\xi) \subseteq SL(2,\mathbb{C})$ such that the action of K on $X_{\mathcal{C}_1,\mathcal{C}_2}$ by simultaneous conjugation is trivial and $SL(2,\mathbb{C})/K$ acts freely on $X_{\mathcal{C}_1,\mathcal{C}_2}$. Usually, it will be $K = \{Id, -Id\}$, so $SL(2,\mathbb{C})/K = PGL(2,\mathbb{C})$. By an argument similar to the one in section 4.1.2 we will show that considering $Y^\xi_{\mathcal{C}_1} \subseteq X_{\mathcal{C}_1,\mathcal{C}_2}$ via the inclusion $(A,B,C) \hookrightarrow (A,B,C,\xi)$, the hypothesis of proposition 3.3.47 hold for the varieties $Y^\xi_{\mathcal{C}_1} \subseteq X_{\mathcal{C}_1,\mathcal{C}_2}$.

- $SL(2,\mathbb{C})/K \cdot Y^\xi_{\mathcal{C}_1} = X_{\mathcal{C}_1,\mathcal{C}_2}$. Indeed, given $(A,B,C_1,C_2) \in X_{\mathcal{C}_1,\mathcal{C}_2}$, let $P \in SL(2,\mathbb{C})$ such that $PC_2P^{-1} = \xi$. Then, $y := (PAP^{-1}, PBP^{-1}, PC_1P^{-1}) \in Y^\xi_{\mathcal{C}_1}$ and $P^{-1}K \cdot y = (A,B,C_1,C_2)$.
- For all $y \in Y^\xi_{\mathcal{C}_1}$ we have that

  $$Stab(\xi)/K \cdot y = SL(2,\mathbb{C})/K \cdot y \cap Y^\xi_{\mathcal{C}_1}$$

  This is simply because, if $(A,B,C) \in Y^\xi_{\mathcal{C}_1}$ and $P \in SL(2,\mathbb{C})$ satisfies $P \cdot (A,B,C) \in Y^\xi_{\mathcal{C}_1}$ then it should satify

  $$P\xi P^{-1} = P[A,B]CP^{-1} = \xi$$

  so $P \in Stab(\xi)$.

Therefore, the algebraic groups $Stab(\xi)/K \subseteq SL(2,\mathbb{C})/K$ satisfies the hypotesis of proposition 3.3.47 for the varieties $Y^{\xi}_{\mathcal{C}_1} \subseteq X_{\mathcal{C}_1,\mathcal{C}_2}$ so we have a $Stab(\xi)/K$-principal bundle

$$Stab(\xi)/K \to SL(2,\mathbb{C})/K \times Y^{\xi}_{\mathcal{C}_1} \to X_{\mathcal{C}_1,\mathcal{C}_2}$$

Hence, it holds

$$e\left(X_{\mathcal{C}_1,\mathcal{C}_2}\right) = e\left(Y^{\xi}_{\mathcal{C}_1}\right) e\left(\frac{SL(2,\mathbb{C})/K}{Stab(\xi)/K}\right) = e\left(Y^{\xi}_{\mathcal{C}_1}\right) e\left(\frac{SL(2,\mathbb{C})}{Stab(\xi)}\right)$$

Remark 4.3.2. In some cases, it will be impossible to find such a $K \subseteq SL(2,\mathbb{C})$. However, we will find subsets $D^{\xi}_{\mathcal{C}_1} \subseteq Y^{\xi}_{\mathcal{C}_1}$ and $D_{\mathcal{C}_1,\mathcal{C}_2} \subseteq X_{\mathcal{C}_1,\mathcal{C}_2}$, called the set of reducibles, such that the action of $SL(2,\mathbb{C})/K$ on $X^*_{\mathcal{C}_1,\mathcal{C}_2} = X_{\mathcal{C}_1,\mathcal{C}_2} - D_{\mathcal{C}_1,\mathcal{C}_2}$ and the action of $Stab(\xi)/K$ on ${Y^{\xi}_{\mathcal{C}_1}}^* := Y^{\xi}_{\mathcal{C}_1} - D^{\xi}_{\mathcal{C}_1}$ are free actions. In that cases, the previous observation can be applied to relate the Deligne-Hodge polynomials of ${Y^{\xi}_{\mathcal{C}_1}}^* \subseteq X^*_{\mathcal{C}_1,\mathcal{C}_2}$ by

$$e\left(X^*_{\mathcal{C}_1,\mathcal{C}_2}\right) = e\left({Y^{\xi}_{\mathcal{C}_1}}^*\right) e\left(\frac{SL(2,\mathbb{C})}{Stab(\xi)}\right)$$

In the GIT quotient of $X_{\mathcal{C}_1,\mathcal{C}_2}$ by the action of $SL(2,\mathbb{C})$, we have that

$$X^*_{\mathcal{C}_1,\mathcal{C}_2} /\!\!/ SL(2,\mathbb{C}) = \frac{X^*_{\mathcal{C}_1,\mathcal{C}_2}}{SL(2,\mathbb{C})/K}$$

For the contribution of the irreducibles, we have to identify all the orbits that contains lower dimensional non-trivial stabilizers in their closure. In the jergue of GIT, these points are called the semi-stable points and this procedure is called the S-equivalence.

4.3.1 Representation varieties with holonomies of Jordan type

4.3.1.1 Deligne-Hodge polynomial of $Y^{J_+}_{[J_+]}$

We will compute the Deligne-Hodge polynomial of

$$Y^{J_+}_{[J_+]} := \left\{ (A,B,C) \in SL(2,\mathbb{C})^3 \;\middle|\; \begin{matrix} [A,B]C = J_+ \\ C \in [J_+] \end{matrix} \right\}$$

To this end, recall from remark 4.1.4 that

$$Stab(J_+) = U = \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C} \right\}$$

Therefore, for each $C \in Y_{[J_+]}^{J_+}$, there exists $P \in SL(2,\mathbb{C})$, defined up to product with U on the right, such that $C = PJ_+P^{-1}$. Thus, since $[A,B]C = J_+$ is equivalent to $[A,B][P,J_+] = Id$ we have

$$Y_{[J_+]}^{J_+} \cong \left\{(A,B,P) \in SL(2,\mathbb{C})^3 \,|\, [A,B][P,J_+] = Id\right\}/U$$

where U acts on P by right product. Let us write

$$P = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

with $xt - yz = 1$. Observe that, for all $Q = \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \in U$, we have that

$$PQ = \begin{pmatrix} \pm x & \lambda x \pm y \\ \pm z & \lambda z \pm t \end{pmatrix}$$

And, since

$$[P, J_+] = \begin{pmatrix} 1 - xz & -1 + x(x+z) \\ -z^2 & 1 + (x+z)z \end{pmatrix}$$

we have that $[P,J_+]$ only depends on the first column (x,z) of P, defined up to sign under the action of U. Therefore, if we define

$$R(x,z) := [P,J_+]^{-1} = \begin{pmatrix} 1 + (x+z)z & 1 - x(x+z) \\ z^2 & 1 - xz \end{pmatrix}$$

we can give the explicit description

$$Y_{[J_+]}^{J_+} \cong \left\{ \left((A,B), \begin{pmatrix} x \\ z \end{pmatrix} \right) \in SL(2,\mathbb{C})^2 \times \frac{\mathbb{C}^2 - \{0\}}{\mathbb{Z}_2} \,\middle|\, [A,B] = R(x,z) \right\}$$

where $\mathbb{Z}_2$ acts on $\mathbb{C}^2 - \{0\}$ by $-1 \cdot (x,z) = (-x,-z)$.

Using this description, we consider the stratification in terms of the trace map $t : Y_{[J_+]}^{J_+} \to \mathbb{C}$ given by $t(A,B,[x,z]) = \tilde{t}[x,z]$, where $\tilde{t} : \frac{\mathbb{C}^2 - \{0\}}{\mathbb{Z}_2} \to \mathbb{C}$ is

$$\tilde{t}[x,z] = tr(R(x,z)) = -2 - z^2$$

- $Z_{-2} := t^{-1}(-2) = Y_{[J_+]}^{J_+} \cap \{z = \pm 2i\}$: In this case, the only posibility is that $R(x,z) \sim J_-$. Consider the projection over (x,z) mapping $\pi : Z_{-2} \to (\mathbb{C} \times \{\pm 2i\})/\mathbb{Z}_2 \cong \mathbb{C}$ given by $\pi(A,B,[x,\pm 2i]) = [x, \pm 2i]$.

For understanding the fiber, let us fix $[x, \pm 2i] \in (\mathbb{C} \times \mathbb{Z}_2)/\mathbb{Z}_2$ and let us choose $Q \in SL(2,\mathbb{C})$ such that $QR(x,z)Q^{-1} = J_-$. Then we have that, algebraically

$$\pi^{-1}(x, \pm i) \cong \left\{ (A,B) \in SL(2,\mathbb{C})^2 \,\middle|\, Q[A,B]Q^{-1} = J_- \right\} = QX_{J_-}Q^{-1} \cong X_{J_-}$$

Moreover, this identification can be done algebraically in a Zariski neighbourhood of $[x, \pm 2i]$ so we have that the algebraic fiber bundle

$$X_{J_-} \to Z_{-2} \xrightarrow{\pi} \mathbb{C}$$

Thus, since $\mathbb{C}$ is irreducible, we have that, by remark 3.3.46, this is an E-fibration, so

$$e(Z_{-2}) = e(\mathbb{C})e(X_{J_-}) = q^4 + 3q^3$$

- $Z_2 := t^{-1}(2) = Y_{[J_+]}^{J_+} \cap \{z = 0\}$: In this case, we use the same fibration as before via the map $\pi : Z_2 \to \mathbb{C}^*/\mathbb{Z}_2$, $\pi(A, B, [x, 0]) = [x]$. However, we can find two diferent types of fibers:
  - $\pi^{-1}([\pm 1])$. In this situation, $R(x,z) = Id$. Thus, since it is an unique fiber, the contribution is $e(\pi^{-1}([\pm 1])) = e(X_{Id}) = q^4 + 4q^3 - q^2 - 4q$.
  - $\pi^{-1}([x])$ for $[x] \neq [\pm 1]$. In that cases, $R(x,z) \sim J_+$, so, by an analogous argument that the one for Z_{-2} we have an E-fibration

    $$X_{J_+} \to Z_2 - \pi^{-1}([\pm 1]) \to (\mathbb{C}^* - \{\pm 1\})/\mathbb{Z}_2$$

    Therefore, we have

    $$e\left(Z_2 - \pi^{-1}([\pm 1])\right) = e\left((\mathbb{C}^* - \{\pm 1\})/\mathbb{Z}_2\right) e\left(X_{J_+}\right) = (q-2)e\left(X_{J_+}\right) = q^4 - 4q^3 + q^2 + 6q$$

  Hence, putting together the Deligne-Hodge polynomials of the two strata we have that

  $$e(Z_2) = e(\pi^{-1}([\pm 1])) + e\left(Z_2 - \pi^{-1}([\pm 1])\right) = 2q^4 + 2q$$

- $\tilde{Z} := Y_{[J_+]}^{J_+} \cap \{z \neq 0, \pm 2i\}$: In this case, we have $t \neq \pm 2$, so fixed $(x,z) \in \mathbb{C} \times \mathbb{C} - \{0, \pm 2i\}$, we have $(A,B) \sim D_\lambda$ for some $\lambda \in \mathbb{C}^* - \{\pm 1\}$. Concretely, λ satifies $\lambda + \lambda^{-1} = t = 2 + z^2 \neq 2$. In order to explote this idea, let us consider the twisted projection $\pi : \tilde{Z} \to \mathbb{C}$ given by $\pi(A, B, [x,z]) = xz$. Observe that this map is well defined, though $[x,z]$ is defined up to sign. In order to compute the fiber of this map, the key fact is that $R(x,z)$ can be rewritten, for $z \neq 0$ and $v := xz$ as

  $$R(x,z) = \begin{pmatrix} 1 + (x+z)z & 1 - x(x+z) \\ z^2 & 1 - xz \end{pmatrix} = \begin{pmatrix} 1 + v + z^2 & 1 - v - \frac{v^2}{z^2} \\ z^2 & 1 - v \end{pmatrix} =: \tilde{R}(v, z^2)$$

and, since z^2 does not deppend on the sign of z we have that $w := z^2$ is well defined and takes values on $\mathbb{C} - \{0, 4\}$. Thus, fixing $v \in \mathbb{C}$ our fiber can be described as

$$\pi^{-1}(v) \cong \left\{ ((A,B), w) \in SL(2,\mathbb{C})^2 \times \mathbb{C} - \{0,4\} \,\middle|\, [A,B] = \tilde{R}(v,w) \right\}$$

with $t((A,B),w) = tr(\tilde{R}(v,w)) = 2 + w$. Moreover, fixing $w_0 \in \mathbb{C} - \{0, -4\}$ we have

$$\pi^{-1}(v)\Big|_{w=w_0} = \left\{ (A,B) \in SL(2,\mathbb{C})^2 \,|\, [A,B] = \tilde{R}(v,w_0) \sim D_\lambda \right\} \cong X_{D_\lambda}$$

for some $\lambda \in \mathbb{C}^* - \{\pm 2\}$. Hence, $\pi^{-1}(w)$ is the union of all the X_λ for $\lambda \in \mathbb{C}^* - \{\pm 2\}$, which is $X_D/\mathbb{Z}_2$. Thus, making this identification locally, we have a regular morphism $\pi : \tilde{Z} \to \mathbb{C}$ and a Zariski locally trivial fibration

$$X_D/\mathbb{Z}_2 \to \tilde{Z}_2 \to \mathbb{C}$$

so, this is an E-fibration and, in particular

$$e\left(\tilde{Z}\right) = e\left(\mathbb{C}\right) e\left(X_D/\mathbb{Z}_2\right) = q^5 - 2q^4 - 3q^3 + 3q^2 + q$$

Remark 4.3.3. Another possible approach could be using the double-fixing fibration $\pi' : \tilde{Z} \to \mathbb{C} \times \mathbb{C} - \{0, 4\}$ given by $\pi'(A, B, [x, z]) = (xz, z^2)$. In this case, we have that

$$\pi'^{-1}(v,w) = \left\{ (A,B) \in SL(2,\mathbb{C}) \,|\, [A,B] = \tilde{R}(v,w) \sim D_\lambda \right\} \cong X_{D_\lambda}$$

for any λ satisfying $\lambda + \lambda^{-1} = 2 + w$. Hence, we have a fibration

$$X_{D_\lambda} \to \tilde{Z} \to \mathbb{C} \times \mathbb{C} - \{0, 4\}$$

However, we have that $e(\tilde{Z}) \neq e(\mathbb{C})e(\mathbb{C} - \{0,4\})e(X_{D_\lambda})$ so this fibration cannot be an E-fibration. The reason is that this fibration is not locally trivial in the Zariski topology, since λ depends *quadratically* on w so the isomorphism $\pi'^{-1}(v,w) \cong X_{D_\lambda}$ uses square roots of w. Hence, locally, this is only a analytic mapping, and is not a regular map anymore.

The solution to this problem is to modify the fibration in order to deppend not on λ, but on the trace t, which has a linear dependece on w. For accomplishing this, we should pay passing through the quotient space $X_D/\mathbb{Z}_2$ and leaving w free, as done above.

Therefore, using this stratification of $Y_{[J_+]}^{J_+}$ we have

$$\boxed{e\left(Y_{[J_+]}^{J_+}\right) = e\left(Z_{-2}\right) + e\left(Z_2\right) + e(\tilde{Z}) = q^5 + q^4 + 3q^2 + 3q}$$

4.3.1.2 Deligne-Hodge polynomial of $Y_{[J_+]}^{J_-}$

Now, we focus in the variety

$$Y_{[J_+]}^{J_-} := \left\{ (A,B,C) \in SL(2,\mathbb{C})^3 \,\middle|\, \begin{matrix} [A,B]C = J_- \\ C \in [J_+] \end{matrix} \right\}$$

However, observe that, since $J_- \sim -J_+$ we have that, via simultaneous conjugation $Y_{[J_+]}^{J_-} \cong Y_{[J_+]}^{-J_+}$ so, instead of $Y_{[J_+]}^{J_-}$ we are going to study

$$Y_{[J_+]}^{-J_+} := \left\{ (A,B,C) \in SL(2,\mathbb{C})^3 \,\middle|\, \begin{matrix} [A,B]C = -J_+ \\ C \in [J_+] \end{matrix} \right\}$$

The analysis of this variety is very similar to the one of $Y_{[J_+]}^{J_+}$ in section 4.3.1.1. Again, recall that

$$Stab(J_+) = U = \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{C} \right\}$$

So, decomposing $C = PJ_+P^{-1}$ for $P \in SL(2,\mathbb{C})$, unique up to product with U on the right, we have

$$Y_{[J_+]}^{-J_+} \cong \left\{ (A,B,P) \in SL(2,\mathbb{C})^3 \,|\, [A,B][P,J_+] = -Id \right\} /U$$

where U acts on P by right product. More explicity, the same computations than in section 4.3.1.1 show the explicit description

$$Y_{[J_+]}^{-J_+} \cong \left\{ \left((A,B), \begin{pmatrix} x \\ z \end{pmatrix} \right) \in SL(2,\mathbb{C})^2 \times \frac{\mathbb{C}^2 - \{0\}}{\mathbb{Z}_2} \,\middle|\, [A,B] = R(x,z) \right\}$$

where $\mathbb{Z}_2$ acts on $\mathbb{C}^2 - \{0\}$ by $-1 \cdot (x,z) = (-x,-z)$ and

$$R(x,z) := -[P,J_+]^{-1} = \begin{pmatrix} -1-(x+z)z & -1+x(x+z) \\ -z^2 & -1+xz \end{pmatrix}$$

Analogously, we can stratify this space as before in terms of the trace map $t : Y_{[J_+]}^{-J_+} \to \mathbb{C}$ given by $t(A,B,[x,z]) = \tilde{t}[x,z]$, where $\tilde{t} : \frac{\mathbb{C}^2 - \{0\}}{\mathbb{Z}_2} \to \mathbb{C}$ is

$$\tilde{t}[x,z] = tr(R(x,z)) = -2 - z^2$$

- $Z_2 := t^{-1}(2) = Y_{[J_+]}^{-J_+} \cap \{z = \pm 2i\}$: In this case, the only posibility is that $R(x,z) \sim J_+$. Again, considering the projection over (x,z) mapping $\pi : Z_2 \to (\mathbb{C} \times \{\pm 2i\})/\mathbb{Z}_2 \cong \mathbb{C}$ given by $\pi(A,B,[x,\pm 2i]) = [x,\pm 2i]$ we have an E-fibration

$$X_{J_+} \to Z_2 \xrightarrow{\pi} \mathbb{C}$$

so, in terms of Deligne-Hodge polynomials

$$e(Z_2) = e(\mathbb{C})e(X_{J_+}) = q^4 - 2q^3 - 3q^2$$

- $Z_{-2} := t^{-1}(-2) = Y_{[J_+]}^{-J_+} \cap \{z = 0\}$: In this case, we again use the fibration $\pi : Z_{-2} \to \mathbb{C}^*/\mathbb{Z}_2$, $\pi(A, B, [x, 0]) = [x]$. A above, we can find two diferent types of fibers:
  - $\pi^{-1}([\pm 1])$. In this situation, $R(x, z) = -Id$. Thus, since it is an unique fiber, the contribution is $e(\pi^{-1}([\pm 1])) = e(X_{-Id}) = q^3 - q$.
  - $\pi^{-1}([x])$ for $[x] \neq [\pm 1]$. In that cases, $R(x, z) \sim J_-$, so we have an E-fibration

$$X_{J_-} \to Z_{-2} - \pi^{-1}([\pm 1]) \to (\mathbb{C}^* - \{\pm 1\})/\mathbb{Z}_2$$

  Therefore, we have

$$e\left(Z_{-2} - \pi^{-1}([\pm 1])\right) = e\left((\mathbb{C}^* - \{\pm 1\})/\mathbb{Z}_2\right) e\left(X_{J_-}\right) = (q - 2)e\left(X_{J_-}\right) = q^4 + q^3 - 6q^2$$

  Hence, putting together the Deligne-Hodge polynomials of the two strata we have that

$$e(Z_{-2}) = e(\pi^{-1}([\pm 1])) + e\left(Z_{-2} - \pi^{-1}([\pm 1])\right) = q^4 + 2q^3 - 6q^2 - q$$

- $\tilde{Z} := Y_{[J_+]}^{-J_+} \cap \{z \neq 0, \pm 2i\}$: In this case, we have $t \neq \pm 2$, so fixed $(x, z) \in \mathbb{C} \times \mathbb{C} - \{0, \pm 2i\}$, we have $(A, B) \sim D_\lambda$ for some $\lambda \in \mathbb{C}^* - \{\pm 1\}$. As in section 4.3.1.1, we take the twisted projection $\pi : \tilde{Z} \to \mathbb{C}$ given by $\pi(A, B, [x, z]) = xz$. Again, the key point is that, taking $v := xz$ we can write

$$R(x, z) = \begin{pmatrix} -1 - (x + z)z & -1 + x(x + z) \\ -z^2 & -1 + xz \end{pmatrix} = \begin{pmatrix} -1 - v - z^2 & v + \frac{v^2}{z^2} - 1 \\ -z^2 & v - 1 \end{pmatrix} =: \tilde{R}(v, z^2)$$

  and, since z^2 does not deppend on the sign of z we have that $w := z^2$ is well defined and takes values on $\mathbb{C} - \{0, 4\}$. Thus, fixing $v \in \mathbb{C}$ our fiber can be described as

$$\pi^{-1}(v) \cong \left\{ ((A, B), w) \in SL(2, \mathbb{C})^2 \times \mathbb{C} - \{0, 4\} \,\middle|\, [A, B] = \tilde{R}(v, w) \right\}$$

  with $t((A, B), w) = tr(\tilde{R}(v, w)) = 2 + w$. Using exactly the same argument than in 4.3.1.1, we have that $\pi^{-1}(v) \cong X_D/\mathbb{Z}_2$ for all $v \in \mathbb{C}$, and this identification can be done in a Zariski neighbourhood. Thus, we have and E-fibration

$$X_D/\mathbb{Z}_2 \to \tilde{Z}_2 \to \mathbb{C}$$

  so, in particular

$$e\left(\tilde{Z}_2\right) = e\left(\mathbb{C}\right) e\left(X_D/\mathbb{Z}_2\right) = q^5 - 2q^4 - 3q^3 + 3q^2 + q$$

Hence, putting together all the strata of $Y_{[J_+]}^{-J_+}$ we have

$$\boxed{e\left(Y_{[J_+]}^{J_-}\right) = e\left(Y_{[J_+]}^{-J_+}\right) = e\left(Z_2\right) + e\left(Z_{-2}\right) + e(\tilde{Z}) = q^5 - 3q^3 - 6q^2}$$

4.3.2 Representation with holonomies of mixed type

4.3.2.1 Deligne-Hodge polynomial of $Y_{[J_+]}^{D_\lambda}$

In the mixed case, let us fix $\lambda \in \mathbb{C}^* - \{\pm 1\}$ and let us take $D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. We consider the auxiliar variety

$$Y_{[J_+]}^{D_\lambda} := \left\{ (A, B, C) \in SL(2,\mathbb{C})^3 \;\middle|\; \begin{matrix} [A,B]C = D_\lambda \\ C \in [J_+] \end{matrix} \right\}$$

Again, the analysis is very similar to the one of 4.3.1.1. Recall that

$$Stab(J_+) = U = \left\{ \begin{pmatrix} \pm 1 & \mu \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; \mu \in \mathbb{C} \right\}$$

So, decomposing $C = PJ_+P^{-1}$ for $P \in SL(2,\mathbb{C})$, unique up to product with U on the right, we have

$$Y_{[J_+]}^{D_\lambda} \cong \left\{ (A, B, P) \in SL(2,\mathbb{C})^3 \,|\, [A,B][P, J_+] = D_\lambda J_+^{-1} \right\} / U$$

where U acts on P by right product. As in 4.3.1.1, the commutator action of U on $[P, J_+]$ only depends on (x, z) up to sign, so computing again, we obtain the explicit description

$$Y_{[J_+]}^{D_\lambda} \cong \left\{ \left((A,B), \begin{pmatrix} x \\ z \end{pmatrix} \right) \in SL(2,\mathbb{C})^2 \times \frac{\mathbb{C}^2 - \{0\}}{\mathbb{Z}_2} \;\middle|\; [A,B] = R(x,z) \right\}$$

where $\mathbb{Z}_2$ acts on $\mathbb{C}^2 - \{0\}$ by $-1 \cdot (x, z) = (-x, -z)$ and

$$R(x,z) := D_\lambda J_+ [P, J_+]^{-1} = \begin{pmatrix} \lambda(1+xz) & -\lambda x^2 \\ \lambda^{-1} z^2 & \lambda^{-1}(1-xz) \end{pmatrix}$$

Analogously, we can stratify this space as before in terms of the trace map $t : Y_{[J_+]}^{D_\lambda} \to \mathbb{C}$ given by $t(A, B, [x,z]) = \tilde{t}[x, z]$, where $\tilde{t} : \frac{\mathbb{C}^2 - \{0\}}{\mathbb{Z}_2} \to \mathbb{C}$ is

$$\tilde{t}[x,z] = tr(R(x,z)) = \lambda + \lambda^{-1} + xz(\lambda - \lambda^{-1})$$

- $Z_2 := t^{-1}(2)$: In this case, since $\lambda + \lambda^{-1} \neq -2$ we should have $xz = \mu_0$ for some $\mu_0 \in \mathbb{C}^*$. Therefore, it is impossible to have $R(x,z) \sim Id$, so it must be $R(x,z) \sim J_+$. Hence, considering the projection over x mapping $\pi : Z_2 \to \mathbb{C}^*/\mathbb{Z}_2 \cong \mathbb{C}^*$ given by $\pi(A, B, [x, \frac{\mu_0}{x}]) = [x]$ we have an

E-fibration

$$X_{J_+} \to Z_2 \xrightarrow{\pi} \mathbb{C}^*$$

so, in terms of Deligne-Hodge polynomials

$$e(Z_2) = e(\mathbb{C}^*)e(X_{J_+}) = q^4 - 3q^3 - q^2 + 3q$$

- $Z_{-2} := t^{-1}(-2)$: This case is completely analogous to the Z_2 case with $R(x,z) \sim J_-$, so we have an E-fibration.

$$X_{J_-} \to Z_{-2} \to \mathbb{C}^*$$

Hence, the Deligne-Hodge polynomial of this stratum is

$$e(Z_{-2}) = e(\mathbb{C}^*)e(X_{J_-}) = q^4 + 2q^3 - 3q^2$$

- $Z_\lambda := t^{-1}(\lambda + \lambda^{-1}) = t^{-1}(tr\, D_\lambda)$: In that case, for fixed $(x,z) \in \mathbb{C}^2 - \{0\}/\mathbb{Z}_2$, we have $R(x,z) \sim D_\lambda$. For the possible values of (x,z), observe that, since $\tilde{t}[x,z] = \lambda + \lambda^{-1} + xz(\lambda - \lambda^{-1})$ we must have $xz = 0$. Thus, (x,z) runs over

$$\frac{\{xz=0\} - \{(0,0)\}}{\mathbb{Z}_2} = \frac{\{x=0, z \neq 0\}}{\mathbb{Z}_2} \sqcup \frac{\{z=0, x\neq 0\}}{\mathbb{Z}_2} \cong \mathbb{C}^*/\mathbb{Z}_2 \sqcup \mathbb{C}^*/\mathbb{Z}_2 \cong \mathbb{C}^* \sqcup \mathbb{C}^*$$

Hence, the projection over (x,z), $\pi : Z_\lambda \to \frac{\{xz=0\}-\{(0,0)\}}{\mathbb{Z}_2}$ given by $\pi(A,B,[x,z]) = [x,y]$ gives us a fibration with fiber

$$\pi^{-1}(x,z) \cong \left\{(A,B) \in SL(2,\mathbb{C})^2 \,|\, (A,B) = R(x,z) \sim D_\lambda\right\} \cong X_{D_\lambda}$$

Moreover, by the same reason than the other cases, this map is an E-fibration

$$X_{D_\lambda} \to Z_\lambda \to \mathbb{C}^* \sqcup \mathbb{C}^*$$

so the Deligne-Hodge polynomial of this stratum is

$$e(Z_{-2}) = 2e(\mathbb{C}^*)e(X_{D_\lambda}) = 2q^4 + 4q^3 - 12q^2 + 4q + 2$$

- $\tilde{Z} := t^{-1}(\mathbb{C} - \left\{\pm 2, \lambda + \lambda^{-1}\right\})$: In this case, fixed $(x,z) \in \tilde{t}^{-1}(\mathbb{C} - \left\{\pm 2, \lambda + \lambda^{-1}\right\})$ we have $x, z \neq 0$, and $(A,B) \sim D_\mu$ for some $\mu \in \mathbb{C}^* - \{\pm 1\}$. As in section 4.3.1.1, we take the twisted projection $\pi : \tilde{Z} \to \mathbb{C}^*$ given by $\pi(A,B,[x,z]) = xz$. Again, the key point is that, taking $v := xz$ we can write

$$R(x,z) = \begin{pmatrix} \lambda(1+xz) & -\lambda x^2 \\ \lambda^{-1}z^2 & \lambda^{-1}(1-xz) \end{pmatrix} = \begin{pmatrix} \lambda + \lambda v & -\lambda \frac{v^2}{z^2} \\ \lambda^{-1}z^2 & \lambda^{-1} - \lambda^{-1}v \end{pmatrix} =: \tilde{R}(v, z^2)$$

and, since z^2 does not deppend on the sign of z we have that $w := z^2$ is well defined an takes values on $\mathbb{C}^*$. Thus, taking the projection $\pi : \tilde{Z} \to \mathbb{C}^*$ given by $\pi(A, B, [x, z]) = z^2$, we have that

$$\pi^{-1}(w) = \left\{(A, B, v) \in SL(2, \mathbb{C})^2 \times \mathbb{C}^* - \{v_\pm\} \mid [A, B] = \tilde{R}(v, w)\right\}$$

with $tr(\tilde{R}(v, w)) = \lambda + \lambda^{-1} + v(\lambda - \lambda^{-1})$ and $v_\pm = \frac{-\lambda-\lambda^{-1}\pm 2}{\lambda-\lambda^{-1}}$. Moreover, fixing $v_0 \in \mathbb{C}^* - \{v_\pm\}$ we have

$$\pi^{-1}(w)\big|_{v=v_0} = \left\{(A, B) \in SL(2, \mathbb{C})^2 \,|\, [A, B] = \tilde{R}(v_0, w) \sim D_\mu\right\} \cong X_{D_\mu}$$

for some $\mu \in \mathbb{C}^* - \{\pm 2, \lambda\}$. Hence, $\pi^{-1}(w)$ is the union of all the X_μ for $\mu \in \mathbb{C}^* - \{\pm 2\}$, which is $X_D/\mathbb{Z}_2$, except X_{D_λ}. Hence,

$$\pi^{-1}(w) \cong X_D/\mathbb{Z}_2 - X_{D_\lambda}$$

so we have an E-fibration

$$X_D/\mathbb{Z}_2 - X_{D_\lambda} \to \tilde{Z} \xrightarrow{\pi} \mathbb{C}^*$$

Therefore, we have

$$e(\tilde{Z}) = e(\mathbb{C}^*)e(X_D/\mathbb{Z}_2 - X_{D_\lambda}) = e(\mathbb{C}^*)e(X_D/\mathbb{Z}_2) - e(\mathbb{C}^*)e(X_{D_\lambda}) = q^5 - 4q^4 - 3q^3 + 12q^2 - 4q - 2$$

Hence, putting together all the strata of $Y^{D_\lambda}_{[J_+]}$ we have

$$\boxed{e\left(Y^{D_\lambda}_{[J_+]}\right) = e\left(Z_2\right) + e\left(Z_{-2}\right) + e\left(Z_\lambda\right) + e(\tilde{Z}) = q^5 - 4q^2 + 3q}$$

Remark 4.3.4. In [46] there is an erratum while computing the Deligne-Hodge polynomial of $Y^{D_\lambda}_{[J_+]}$. The problem is that, in the corresponding stratum Z_λ, it is claimed that the contribution of the possible values of $(x, z) \in \mathbb{C}^2 - \{(0, 0)\}$ is $2q - 1$. However, as we shown above, the possible values runs over $\mathbb{C}^* \sqcup \mathbb{C}^*$, whose Deligne-Hodge polynomial is $2q - 2$.

4.3.3 Representation varieties with holonomies of diagonalizable type

First of all, let us fix $\lambda_1, \lambda_2 \in \mathbb{C}^* - \{\pm 1\}$, and let us take

$$D_{\lambda_1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \qquad D_{\lambda_2} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$$

We consider the auxiliar variety

$$Y^{D_{\lambda_2}}_{[D_{\lambda_1}]} := \left\{ (A, B, C) \in SL(2, \mathbb{C})^3 \;\middle|\; \begin{array}{c} [A, B]C = D_{\lambda_2} \\ C \in [D_{\lambda_1}] \end{array} \right\}$$

As always, let us remove the conjugation class. For this purpose, recall that

$$Stab(D_{\lambda_1}) = D = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \;\middle|\; \alpha \in \mathbb{C}^* \right\}$$

So, decomposing $C = PD_{\lambda_1}P^{-1}$ for $P \in SL(2,\mathbb{C})$, unique up to product with D on the right, we have that $Y^{D_{\lambda_2}}_{[D_{\lambda_1}]} \cong Z_{\lambda,\mu}$ where

$$\begin{aligned} Z_{\lambda,\mu} &= \left\{(A,B,P) \in SL(2,\mathbb{C})^3 \,|\, [A,B] = D_\mu [P, D_\lambda]^{-1}\right\}/D \\ &= \left\{ \left(A, B, \begin{pmatrix} x & y \\ z & t \end{pmatrix}\right) \in SL(2,\mathbb{C})^3 \,|\, [A,B] = \begin{pmatrix} \mu\left(xt - \lambda^2 yz\right) & \mu^{-1}xy\left(\lambda^2 - 1\right) \\ \mu tz\left(\lambda^{-2} - 1\right) & \mu^{-1}\left(xt - \lambda^{-2}yz\right) \end{pmatrix} \right\}/D \end{aligned}$$

D acts on P by right product, and we have defined $\lambda := \lambda_1$ and $\mu := \lambda_1^{-1}\lambda_2^{-1}$.

In order to study $Z_{\lambda,\mu}$, let us define the auxiliar variety

$$\boxed{\mathcal{P} := \left\{(P, M) \in SL(2,\mathbb{C})^2 \mid D_\mu [P, D_\lambda]^{-1} = M\right\}/D}$$

where D acts on $\mathcal{P}$ by right multiplication on P, i.e. $Q \cdot (P, M) = (PQ, M)$ for $P, M \in SL(2,\mathbb{C})$ and $Q \in D$. Moreover, let us define the morphism $\pi : Z_{\lambda,\mu} \to \mathcal{P}$ given by $\pi(A,B,P) = (P, [A,B])$. Observe that, in this case, the fiber is

$$\pi^{-1}(P, M) = \{(A,B) \,|\, [A,B] = M\}$$

which is isomorphic to the varieties X_ξ studied above, for $\xi = \pm Id, J_\pm, D_\alpha$ a Jordan canonical form. The especific class of ξ depends on the trace of M. Hence, the geometry of this fibration, and of $\mathcal{P}$ itself, strongly depends on the values of the traces of M.

In order to study this geometry, let us take the fibration $\zeta : \mathcal{P} \to \mathbb{C}$ given by $\zeta(P, M) = tr\,[P, D_\lambda]$ and $\rho : \mathcal{P} \to \mathbb{C}$ given by $\rho(P, M) = tr\, M$. Observe that ζ is well-defined since $tr\,[P, D_\lambda]$ is invariant under the action of D by right multiplication on P. Furthermore, remark that both fibrations are related by

$$\rho = \frac{\mu^2\lambda^2 - 1}{\mu(\lambda^2 - 1)}\zeta + \frac{(1-\mu^2)(1+\lambda^2)}{\mu(\lambda^2 - 1)}$$

Let us denote the fiber of ζ in $\mathcal{P}$ by $\mathcal{P}_t := \zeta^{-1}(t)$. The structure of this fibration is captured in the following proposition, whose complete proof can be found in [46].

Proposition 4.3.5. *Let us write*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$, *and let us consider the projection* $\varphi : \mathcal{P} \to \mathbb{C}^2$, *given by* $\varphi(P, M) = (b, c)$. *Then, there exists a regular morphism* $k : \mathbb{C} - \left\{2, \lambda^2 + \lambda^{-2}\right\} \to \mathbb{C}^*$ *such that the projection* $\varphi : \mathcal{P} \to \mathbb{C}^2$ *give* *us the isomorphisms in the following cases*

$$\mathcal{P}_2 \cong \{bc = 0\} \qquad \mathcal{P}_{\lambda^2+\lambda^{-2}} \cong \{bc = 0\} \qquad \mathcal{P}_t \cong \{bc = k(t)\} \cong \mathbb{C}^*$$

for $t \in \mathbb{C} - \left\{2, \lambda^2 + \lambda^{-2}\right\}$. *Observe that, in particular, the fibration in the non-degenerated locus* $\zeta : \mathcal{P}|_{t\neq 2,\lambda^2+\lambda^{-2}} \to \mathbb{C}$ *is trivial.*

Moreover, from this description, we have the Deligne-Hodge polynomials

$$e\left(\mathcal{P}_2\right) = e\left(\mathcal{P}_{\lambda^2+\lambda^{-2}}\right) = e\left(\{bc = 0\}\right) = 2q - 1 \qquad\qquad e\left(\mathcal{P}_t\right) = q - 1$$

for any $t \in \mathbb{C} - \left\{2, \lambda^2 + \lambda^{-2}\right\}$.

With this understanding of the fibration $\zeta : \mathcal{P} \to \mathbb{C}$, we can analyze the variety $Z_{\lambda,\mu}$. For this purpose, let us define the lifts $\tilde{\zeta} : Z_{\lambda,\mu} \to \mathbb{C}$ given by $\tilde{\zeta}(A, B, P) = tr\,[P, D_\lambda]$ and $\tilde{\rho} : Z_{\lambda,\mu} \to \mathbb{C}$ given by $\tilde{\zeta}(A, B, P) = tr\,[A, B]$. Observe that this fibrations are defined in order to obtain commutation of the following diagram.

$$\begin{tikzcd}
& Z_{\lambda,\mu} \arrow[dl, "\tilde{\zeta}"'] \arrow[dr, "\tilde{\rho}"] \arrow[dd, "\pi"] & \\
\mathbb{C} & & \mathbb{C} \\
& \mathcal{P} \arrow[ul, "\zeta"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

Hence, in particular, it also holds the relation between fibrations

$$\tilde{\rho} = \frac{\mu^2\lambda^2 - 1}{\mu(\lambda^2 - 1)}\tilde{\zeta} + \frac{(1 - \mu^2)(1 + \lambda^2)}{\mu(\lambda^2 - 1)}$$

For simplicity, let us define $f : \mathbb{C} \to \mathbb{C}$ by $f(t) = \frac{\mu^2\lambda^2-1}{\mu(\lambda^2-1)}t + \frac{(1-\mu^2)(1+\lambda^2)}{\mu(\lambda^2-1)}$, so $\rho = f(\zeta)$.

4.3.3.1 Deligne-Hodge polynomial of $Y_{[D_{\lambda_1}]}^{D_{\lambda_2}}$ with $\lambda_1 \neq \pm\lambda_2, \lambda_2^{-1}$

For the subsequent computations of this section, let us assume that $\lambda_1 \neq \pm\lambda_2, \lambda_2^{-1}$, which is equivalent to $\lambda^2\mu = \lambda_1\lambda_2^{-1} \neq \pm 1$. This condition will be dropped out in the following sections. Let us stratify $Z_{\lambda,\mu}$ according to the values of $\tilde{\zeta}$.

- $Z_{\lambda,\mu}^2 := \tilde{\zeta}^{-1}(2)$: In this case, the morphism $\pi : Z_{\lambda,\mu} \to \mathcal{P}$ restricts to a morphism $\pi : Z_{\lambda,\mu}^2 \to \mathcal{P}_2$. Moreover, since $\zeta \equiv 2$ on $\mathcal{P}_2$, then ρ is identically

$$\rho \equiv f(2) = \mu + \mu^{-1} \neq \pm 2$$

Therefore, the fiber of this fibration is, for all $(P,M) \in \mathcal{P}_2$

$$\pi^{-1}(P,M) = \{(A,B) \,|\, [A,B] = M\}$$

with $tr\, M = \rho(M) = \mu + \mu^{-1} \neq \pm 2$. Thus, $M \sim D_\mu$ which implies

$$\pi^{-1}(P,M) = \{(A,B) \,|\, [A,B] = M \sim D_\mu\} \cong X_{D_\mu}$$

Since this identification can be done locally, we have an E-fibration

$$X_{D_\mu} \to Z^2_{\lambda,\mu} \stackrel{\pi}{\to} \mathcal{P}_2$$

so, in particular, using proposition 4.3.5, we have

$$e\left(Z^2_{\lambda,\mu}\right) = e\left(\mathcal{P}_2\right) e\left(X_{D_\mu}\right) = 2q^4 + 5q^3 - 9q^2 + q + 1$$

- $Z^{\lambda^2+\lambda^{-2}}_{\lambda,\mu} := \tilde{\zeta}^{-1}(\lambda^2 + \lambda^{-2})$: In this case, the morphism $\pi : Z_{\lambda,\mu} \to \mathcal{P}$ restricts to a morphism $\pi : Z^{\lambda^2+\lambda^{-2}}_{\lambda,\mu} \to \mathcal{P}_{\lambda^2+\lambda^{-2}}$. Moreover, since $\zeta \equiv \lambda^2 + \lambda^{-2}$ on $\mathcal{P}_{\lambda^2+\lambda^{-2}}$, then ρ is identically

$$\rho \equiv f(\lambda^2 + \lambda^{-2}) = \mu\lambda^2 + \mu^{-1}\lambda^{-2} \neq \pm 2$$

since $\lambda^2\mu \pm 1$. Then, by the same argument than in the previous case, we have an E-fibration

$$X_{D_\mu} \to Z^{\lambda^2+\lambda^{-2}}_{\lambda,\mu} \stackrel{\pi}{\to} \mathcal{P}_{\lambda^2+\lambda^{-2}}$$

so, in particular, we obtain

$$e\left(Z^{\lambda^2+\lambda^{-2}}_{\lambda,\mu}\right) = e\left(\mathcal{P}_{\lambda^2+\lambda^{-2}}\right) e\left(X_{D_\mu}\right) = 2q^4 + 5q^3 - 9q^2 + q + 1$$

- $Z^{\alpha}_{\lambda,\mu} = \tilde{\zeta}^{-1}(f^{-1}(2))$: Let us define $\alpha = f^{-1}(2)$ so $\zeta \equiv \alpha$. Recall that, restricting, we have a morphism $\pi : Z^{\alpha}_{\lambda,\mu} \to \mathcal{P}_\alpha$. The fiber of this fibration is, for all $(P,M) \in \mathcal{P}_2$

$$\pi^{-1}(P,M) = \{(A,B) \,|\, [A,B] = M\}$$

with $tr\, M = \rho(M) = 2$. Observe that the fiber depends on if M could be Id or not. However, observe that, forall $(P,M) \in \mathcal{P}$, we have that $M \neq Id$. Indeed, if there exists $P \in SL(2,\mathbb{C})$ such that $(P, Id) \in \mathcal{P}$ then we must have $[P, D_\lambda] = D_\mu$, which is impossible.

Therefore, if $(P,M) \in \mathcal{P}_\alpha$, then $M \sim J_+$. Hence, the fiber of this fibration is

$$\pi^{-1}(P,M) = \{(A,B) \,|\, [A,B] = M \sim J_+\} \cong X_{J_+}$$

for all $P, M \in SL(2,\mathbb{C})$. Since this identifications can be done in a Zariski open set, we have an E-fibration

$$X_{J_+} \to Z^{\alpha}_{\lambda,\mu} \stackrel{\pi}{\to} \mathcal{P}_\alpha$$

which, in particular, means

$$e\left\{Z^{\alpha}_{\lambda,\mu}\right) = e\left(\mathcal{P}_\alpha\right) e\left(X_{J_+}\right) = q^4 - 3q^3 - q^2 + 3q$$

- $Z^{\beta}_{\lambda,\mu} = \tilde{\zeta}^{-1}(f^{-1}(-2))$: Let us define $\beta = f^{-1}(-2)$ so $\zeta \equiv \beta$. This case is analogous to the previous one except that, here $\zeta \equiv \beta$ if and only if $\rho \equiv -2$. Since, by an analogous argument than above, it cannot be $M = Id$ for any $(P, M) \in \mathcal{P}_\beta$. Therefore, the unique posibility is $M \sim J_-$ and, thus, the fiber is

$$\pi^{-1}(P, M) = \{(A, B) \,|\, [A, B] = M \sim J_-\} \cong X_{J_-}$$

for all $P, M \in SL(2,\mathbb{C})$. Since this identifications can be done in a Zariski open set, we have an E-fibration

$$X_{J_-} \to Z^{\beta}_{\lambda,\mu} \stackrel{\pi}{\to} \mathcal{P}_\beta$$

which, in particular, means

$$e\left(Z^{\beta}_{\lambda,\mu}\right) = e\left(\mathcal{P}_\beta\right) e\left(X_{J_-}\right) = q^4 + 2q^3 - 3q^2$$

- $\tilde{Z}_{\lambda,\mu} := \tilde{\zeta}^{-1}(\mathbb{C} - \left\{2, \lambda^2 + \lambda^{-2}, \alpha, \beta\right\})$: For the residual case let us denote $\tilde{\mathcal{P}} := \zeta^{-1}(\left\{2, \lambda^2 + \lambda^{-2}, \alpha, \beta\right\})$. Since $\zeta \in \mathbb{C} - \left\{2, \lambda^2 + \lambda^{-2}, \alpha, \beta\right\}$, then $\rho \in S$, where

$$S = \mathbb{C} - \left\{\pm 2, \mu + \mu^{-1}, \mu\lambda^2 + \mu^{-1}\lambda^{-2}\right\} = (\mathbb{C} - \{\pm 2\}) - \{2 \text{ points}\}$$

Let us define the auxiliar variety

$$W = \left\{(A, B, t) \in SL(2,\mathbb{C})^2 \times S \,|\, tr[A, B] = t\right\}$$

and let us define the morphism $\psi : \tilde{Z}_{\lambda,\mu} \to W$ by

$$\psi(A, B, P) = (A, B, tr\, D_\mu [P, D_\lambda]^{-1}) = (A, B, \tilde{\rho}(A, B, P))$$

Then, ψ has, for $(A, B, t) \in W$, the fiber

$$\begin{aligned}\psi^{-1}(A, B, t) &= \left\{P \in SL(2,\mathbb{C}) \,|\, tr\, D_\mu [P, D_\lambda]^{-1} = t\right\}/D \\ &= \{P \in SL(2,\mathbb{C}) \,|\, tr\,[P, D_\lambda] = f(t)\}/D \cong \mathcal{P}_{f(t)}\end{aligned}$$

where $f : S \to \mathbb{C}-\left\{2, \lambda^2+\lambda^{-2}, \alpha, \beta\right\}$ is given by $f(t) = \frac{t\mu(\lambda^2-1)+(\mu^2-1)(1+\lambda^2)}{\mu^2\lambda^2-1}$. Since the fibration $\zeta : \tilde{\mathcal{P}} \to S$ is trivial, then, in fact, we can do this isomorphism locally to obtain an E-fibration

$$\mathcal{P}_{t_0} \to \tilde{Z}_{\lambda,\mu} \overset{\psi}{\to} W$$

Now, observe that W is equal to the variety $X_D/\mathbb{Z}_2$, where we have removed the fibers corresponding to X_{D_μ} and $X_{D_{\lambda^2\mu}}$, so $e(W) = e(X_D/\mathbb{Z}_2) - 2e(X_{D_\lambda}) =$. Hence, since $\mathcal{P}_{t_0} \cong \mathbb{C}^*$, we have

$$e\left(\tilde{Z}_{\lambda,\mu}\right) = e\left(\mathcal{P}_{t_0}\right) e\left(W\right) = q^5 - 5q^4 - 5q^3 + 18q^2 - 6q - 3$$

Therefore, adding all the strata, we obtain

$$\boxed{e\left(Y^{D_{\lambda_2}}_{[D_{\lambda_1}]}\right) = e\left(Z^2_{\lambda,\mu}\right) + e\left(Z^{\lambda^2+\lambda^{-2}}_{\lambda,\mu}\right) + e\left(Z^{\alpha}_{\lambda,\mu}\right) + e\left(Z^{\beta}_{\lambda,\mu}\right) + e\left(\tilde{Z}_{\lambda,\mu}\right) = q^5 + q^4 + 4q^3 - 4q^2 - q - 1}$$

4.3.3.2 Deligne-Hodge polynomial of $Y^{D_{-\lambda}}_{[D_\lambda]}$

Analogous to the previous case, let us consider the auxiliar variety

$$Y^{D_{-\lambda}}_{[D_\lambda]} := \left\{ (A,B,C) \in SL(2,\mathbb{C})^3 \;\middle|\; \begin{array}{c} [A,B]C = D_{-\lambda} \\ C \in [D_\lambda] \end{array} \right\}$$

By the considerations in section 4.3.3, $Y^{D_{-\lambda}}_{[D_\lambda]} \cong Z_{\lambda,-\lambda^{-2}}$, which, in this case, reduces to

$$\begin{aligned} Z_{\lambda,-\lambda^{-2}} &= \left\{(A,B,P) \in SL(2,\mathbb{C})^3 \,|\, [A,B] = D_{-\lambda^{-2}}[P,D_\lambda]^{-1}\right\}/D \\ &= \left\{ \left(A, B, \begin{pmatrix} x & y \\ z & t \end{pmatrix}\right) \in SL(2,\mathbb{C})^3 \,|\, [A,B] = \begin{pmatrix} -\lambda^{-2}xt + yz & xy\left(\lambda^2 - \lambda^4\right) \\ tz\left(\lambda^{-2}-\lambda^{-4}\right) & -\lambda^2 xt + yz \end{pmatrix} \right\}/D \end{aligned}$$

where D acts on P by right product.

The main difference between this case and the previous one is that, now

$$f(\lambda^2+\lambda^{-2}) = -2$$

so the strata $Z^{\lambda^2+\lambda^{-2}}_{\lambda,-\lambda^{-2}}$ and $Z^{\beta}_{\lambda,-\lambda^{-2}}$, with $\beta = f^{-1}(-2)$, coincides. Moreover, in this case, there exists $P \in SL(2,\mathbb{C})$ such that $[P, D_\lambda] = -Id$ so this strata $Z^{\beta}_{\lambda,-\lambda^{-2}}$ must be recomputed. Therefore, let us consider

$$Z^{\lambda^2+\lambda^{-2}}_{\lambda,-\lambda^{-2}} = \left\{(A,B,P) \in SL(2,\mathbb{C})^3 \,|\, [A,B] = D_{-\lambda^{-2}}[P,D_\lambda]^{-1},\;\; tr\,[A,B] = -2\right\}/D$$

As before, let us take the morphism $\pi : Z^{\lambda^2+\lambda^{-2}}_{\lambda,-\lambda^{-2}} \to \mathcal{P}_{\lambda^2+\lambda^{-2}} \cong \{bc = 0\}$, whose fiber is

$$\pi^{-1}(P,M) = \{(A,B) \,|\, [A,B] = M\}$$

for $M \in SL(2,\mathbb{C})$ with $tr\, M = -2$. Recall that the isomorphism $\varphi : \mathcal{P}_{\lambda^2+\lambda^{-2}} \leftrightarrow \{bc = 0\}$ is just the projection over (b,c), when $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Therefore, we obtain the degenerated fibers

- The degenerated fiber $Z_D := \pi^{-1}\varphi^{-1}(0,0)$. In that case, we have that $b = c = 0$, so $[A,B] = -Id$ and, thus, the fiber is X_{-Id}. Hence, its Deligne-Hodge polynomial is $e(Z_D) = e(X_{-Id})$.
- The non-degenerated locus $Z_{ND} := \pi^{-1}\left(\varphi^{-1}(\{bc = 0\} - \{(0,0)\})\right)$. In that case, since $b \neq 0$ or $c \neq 0$, we have that $[A,B] \neq -Id$, so, using that $tr\,[A,B] = -2$, it should be $[A,B] \sim J_-$. Hence, the fiber is
$$\pi^{-1}(P,M) = \{(A,B) \,|\, [A,B] = M \sim J_-\} \cong X_{J_-}$$
Therefore, we have an E-fibration
$$X_{J_-} \to Z_{ND} \to \{bc = 0\} - \{(0,0)\}$$
so we obtain the Deligne-Hodge polynomial
$$e(Z_{ND}) = e(\{bc = 0\} - \{(0,0)\})e(X_{J_-})$$

Therefore, putting together the fibers, we have

$$e(Z^{\lambda^2+\lambda^{-2}}_{\lambda,-\lambda^{-2}}) = e(Z_D) + e(Z_{ND}) = 2q^4 + 5q^3 - 6q^2 - q$$

Hence, since the others strata of $Z_{\lambda,-\lambda^{-2}}$ are the same than for the general case and remembering that the strata $Z^{\lambda^2+\lambda^{-2}}_{\lambda,-\lambda^{-2}}$ and $Z^{\beta}_{\lambda,-\lambda^{-2}}$ coincides, we obtain

$$\boxed{e\left(Y^{D_{-\lambda}}_{[D_\lambda]}\right) = e\left(Z^2_{\lambda,-\lambda^{-2}}\right) + e\left(Z^{\lambda^2+\lambda^{-2}}_{\lambda,-\lambda^{-2}}\right) + e\left(Z^{\alpha}_{\lambda,-\lambda^{-2}}\right) + e\left(\tilde{Z}_{\lambda,-\lambda^{-2}}\right) = q^5 + q^4 + 4q^3 - 4q^2 - q - 1}$$

Remark 4.3.6. This Deligne-Hodge polynomial agrees with the one of the general case $\lambda_1 \neq -\lambda_2$.

4.3.3.3 Deligne-Hodge polynomial of $Y^{D_{\lambda^{-1}}}_{[D_\lambda]}$

Now, let us consider the case $\lambda := \lambda_1 = \lambda_2^{-1}$. This corresponds to the auxiliar variety

$$Y^{D_{\lambda^{-1}}}_{[D_\lambda]} := \left\{ (A,B,C) \in SL(2,\mathbb{C})^3 \;\middle|\; \begin{array}{c} [A,B]C = D_{\lambda^{-1}} \\ C \in [D_{\lambda^{-1}}] \end{array} \right\}$$

Now, we have that $\mu = \lambda_1^{-1}\lambda_2^{-1} = \lambda^{-1}\lambda = 1$. Therefore, by the considerations in section 4.3.3, $Y_{[D_\lambda]}^{D_{\lambda^{-1}}} \cong Z_{\lambda,1}$, which, in this case, reduces to

$$\begin{aligned} Z_{\lambda,1} &= \left\{(A,B,P) \in SL(2,\mathbb{C})^3 \,|\, [A,B] = [P,D_\lambda]^{-1}\right\}/D \\ &= \left\{\left(A,B,\begin{pmatrix} x & y \\ z & t \end{pmatrix}\right) \in SL(2,\mathbb{C})^3 \,|\, [A,B] = \begin{pmatrix} xt - \lambda^2 yz & xy\left(\lambda^2-1\right) \\ tz\left(\lambda^{-2}-1\right) & xt - \lambda^{-2}yz \end{pmatrix}\right\}/D \end{aligned}$$

where D acts on P by right product.

In this case, the main difference between this case and general case of section 4.3.3.1, is that, now $\rho = \zeta$, or, equivalently $f = Id_{\mathbb{C}}$. Therefore, the strata $Z_{\lambda,1}^2 = \tilde{\zeta}^{-1}(2)$ and $Z_{\lambda,1}^\alpha$ with $\alpha = f^{-1}(2) = 2$ coincide. Moreover, in this case, there exists $P \in SL(2,\mathbb{C})$ such that $[P, D_\lambda] = Id$, so we have to recompute this strata.

As above, let us take the morphism $\pi : Z_{\lambda,1}^2 \to \mathcal{P}_2 \cong \{bc = 0\}$, whose fiber is

$$\pi^{-1}(P,M) = \{(A,B) \,|\, [A,B] = M\}$$

for $M \in SL(2,\mathbb{C})$ with $tr\, M = 2$. Recall that the isomorphism $\varphi : \mathcal{P}_2 \leftrightarrow \{bc = 0\}$ is just the projection over (b,c), when $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Therefore, we obtain the fibers:

- The degenerated fiber $Z_D := \pi^{-1}\varphi^{-1}(0,0)$. In that case, we have that $b = c = 0$, so, since $tr\,[A,B] = 2$, we have $[A,B] = Id$ and, thus, the fiber is X_{Id}. Hence, its Deligne-Hodge polynomial is $e(Z_D) = e(X_{Id})$.
- The non-degenerated locus $Z_{ND} := \pi^{-1}\left(\varphi^{-1}(\{bc = 0\} - \{(0,0)\})\right)$. In that case, since $b \neq 0$ or $c \neq 0$, we have that $[A,B] \neq Id$, so, using that $tr\,[A,B] = 2$, it should be $[A,B] \sim J_+$. Hence, the fiber is

  $$\pi^{-1}(P,M) = \{(A,B) \,|\, [A,B] = M \sim J_+\} \cong X_{J_+}$$

  Therefore, we have an E-fibration

  $$X_{J_+} \to Z_{ND} \to \{bc = 0\} - \{(0,0)\}$$

  so we obtain the Deligne-Hodge polynomial

  $$e(Z_{ND}) = e(\{bc = 0\} - \{(0,0)\})e(X_{J_+})$$

Therefore, putting together the fibers, we have

$$e(Z_{\lambda,1}^2) = e(Z_D) + e(Z_{ND}) = 3q^4 - 2q^3 - 3q^2 + 2q$$

Hence, since the others strata of $Z_{\lambda,1}$ are the same than for the general case and remembering that the strata $Z^1_{\lambda,-\lambda^{-2}}$ and $Z^{\alpha}_{\lambda,-\lambda^{-2}}$ coincides, we obtain

$$\boxed{e\left(Y^{D_{\lambda^{-1}}}_{[D_\lambda]}\right) = e\left(Z^2_{\lambda,1}\right) + e\left(Z^{\lambda^2+\lambda^{-2}}_{\lambda,1}\right) + e\left(Z^{\beta}_{\lambda,1}\right) + e\left(\tilde{Z}_{\lambda,1}\right) = q^5 + 2q^4 + 2q^3 - 3q^2 - q - 1}$$

Remark 4.3.7. This Deligne-Hodge polynomial does not agree with the one of the general case.

4.3.4 Deligne-Hodge polynomial of parabolic character varieties with two marked points

In the previous sections, we computed the Deligne-Hodge polynomials of some varieties $Y^{\xi}_{\mathcal{C}}$ for some cases of conjugacy classes $\mathcal{C} \subseteq SL(2,\mathbb{C})$ and $\xi \in SL(2,\mathbb{C})$. In this section, we will show how to reduce the other cases to the known ones. Moreover, analyzing the simultaneous conjugacy action of $SL(2,\mathbb{C})$ on the varieties $X_{\mathcal{C}_1,\mathcal{C}_2}$ we will compute the GIT quotient

$$\mathcal{M}_{\mathcal{C}_1,\mathcal{C}_2} = X_{\mathcal{C}_1,\mathcal{C}_2} /\!\!/ SL(2,\mathbb{C})$$

obtaining their Deligne-Hodge polynomials.

The main tool to deal with this problem will be the final considerations of section 4.3. Let us fix conjugacy classes $\mathcal{C}_1, \mathcal{C}_2 \subseteq SL(2,\mathbb{C})$ and $\xi \in \mathcal{C}_2$. Suppose that we can find a subgroup $K \subseteq Stab(\xi)$ such that the action of K on $X_{\mathcal{C}_1,\mathcal{C}_2}$ by simultaneous conjugation is trivial and $SL(2,\mathbb{C})/K$ acts freely on $X_{\mathcal{C}_1,\mathcal{C}_2}$. Usually, we will take $K = Center(SL(2,\mathbb{C})) = \{Id, -Id\}$, so $SL(2,\mathbb{C})/K = PGL(2,\mathbb{C})$.

In that cases, as shown in 4.3, we can apply proposition 3.3.47 for the varieties $Y^{\xi}_{\mathcal{C}_1} \subseteq X_{\mathcal{C}_1,\mathcal{C}_2}$ and the algebraic groups $Stab(\xi)/K \subseteq SL(2,\mathbb{C})/K$ so we will have

$$e\left(X_{\mathcal{C}_1,\mathcal{C}_2}\right) = e\left(Y^{\xi}_{\mathcal{C}_1}\right) e\left(\frac{SL(2,\mathbb{C})/K}{Stab(\xi)/K}\right) = e\left(Y^{\xi}_{\mathcal{C}_1}\right) e\left(\frac{SL(2,\mathbb{C})}{Stab(\xi)}\right)$$

For this purpose, observe that $SL(2,\mathbb{C})/K$ acts freely on $X_{\mathcal{C}_1,\mathcal{C}_2}$ if and only if the group $Stab(\xi)/K$ acts freely on $Y^{\xi}_{\mathcal{C}_1}$ for any $\xi \in \mathcal{C}_2$. Indeed, note that, since $C_2 \sim \xi$, there exists $Q \in SL(2,\mathbb{C})$ such that $C_2 = Q\xi Q^{-1}$, and, thus, $Q(A,B,C_1)Q^{-1} \in Y^{\xi}_{\mathcal{C}_1}$. Therefore, some non trivial $P \in SL(2,\mathbb{C})/K$ fixes $(A,B,C_1,C_2) \in X_{\mathcal{C}_1,\mathcal{C}_2}$ if and only if the non trivial element $Q^{-1}PQ \in Stab(\xi)/K$ fixes $Q(A,B,C_1)Q^{-1} \in Y^{\xi}_{\mathcal{C}_1}$ and this happens if and only if the action of $Stab(\xi)/K$ on $Y^{\xi}_{\mathcal{C}_1}$ is not free.

Moreover, let $\mathcal{D}^{\xi}_{\mathcal{C}_1}$ be the set of reducibles of $Y^{\xi}_{\mathcal{C}_1}$, that is, given $(A,B,C) \in Y^{\xi}_{\mathcal{C}_1}$, $(A,B,C) \in \mathcal{D}^{\xi}_{\mathcal{C}_1}$ if and only if there exists a non trivial $P \in Stab(\xi)/K$ such that $P(A,B,C)P^{-1} = (A,B,C)$. Analogously, let $\mathcal{D}_{\mathcal{C}_1,\mathcal{C}_2}$ be the set of reducibles of $X_{\mathcal{C}_1,\mathcal{C}_2}$. Then, by the previous computation, we have that, given $(A,B,C_1,C_2) \in X_{\mathcal{C}_1,\mathcal{C}_2}$ with $C_2 = Q\xi Q^{-1}$, $(A,B,C_1,C_2) \in \mathcal{D}_{\mathcal{C}_1,\mathcal{C}_2}$ if and only if

$(Q^{-1}AQ, Q^{-1}BQ, Q^{-1}C_1Q) \in \mathcal{D}^{\xi}_{\mathcal{C}_1}$. Therefore, we have

$$\mathcal{D}_{\mathcal{C}_1,\mathcal{C}_2} \cong \left\{(A,B,C,Q) \in Y^{\xi}_{\mathcal{C}_1} \times SL(2,\mathbb{C})/Stab(\xi) \,|\, Q^{-1}(A,B,C)Q \in \mathcal{D}^{\xi}_{\mathcal{C}_1}\right\}$$

4.3.4.1 Deligne-Hodge polynomial of $\mathcal{M}_{[J_+],[J_-]}$

Let us take $K = \{Id, -Id\}$, so $SL(2,\mathbb{C})/K \cong PGL(2,\mathbb{C})$. We will show that $PSL(2,\mathbb{C})$ acts freely on $X_{[J_+],[J_-]} = X_{[J_+],[-J_+]}$. As we explained before, this is equivalent to $Stab(J_+)/\{\pm Id\}$ acting freely on $Y^{-J_+}_{[J_+]}$. Suppose that there exists a non trivial $P \in Stab(J_+)/\{\pm Id\}$ and $(A,B,C) \in Y^{-J_+}_{[J_+]}$ with $P(A,B,C)P^{-1} = Id$. Let us write

$$P = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad\qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\lambda \neq 0$. Since we have

$$PAP^{-1} = \begin{pmatrix} a+\lambda c & b-\lambda^2 c+\lambda(d-a) \\ c & d-\lambda c \end{pmatrix}$$

we obtain that $PAP^{-1} = A$ if and only if $a = d = \pm 1$ and $c = 0$, that is, if and only if $A \in U = Stab(J_+)$. Analogously, we should have $B \in U = Stab(J_+)$. Hence, since $[U,U] = Id$, we have that $[A,B] = Id$, which is impossible by the stratification analysis of section 4.3.1.2.

Therefore, the action of $PSL(2,\mathbb{C})$ on $X_{[J_+],[J_-]}$ is free and, thus, we have

$$\boxed{e\left(X_{[J_+],[J_-]}\right) = e\left(X_{[J_+],[-J_+]}\right) = e\left(Y^{-J_+}_{[J_+]}\right)\frac{e\left(PGL(2,\mathbb{C})\right)}{e\left(Stab(J_-)\right)} = q^7 - 4q^5 - 6q^4 + 3q^3 + 6q^2}$$

Moreover, since we know that the action of $PGL(2,\mathbb{C})$ on $X_{[J_+],[J_-]}$ is free, its GIT quotient is just an usual quotient, so

$$X_{[J_+],[J_-]} /\!\!/ SL(2,\mathbb{C}) = X_{[J_+],[J_-]} /\!\!/ PSL(2,\mathbb{C}) = X_{[J_+],[J_-]}/PSL(2,\mathbb{C})$$

and, thus, we have

$$\boxed{\mathcal{M}_{[J_+],[J_-]} = X_{[J_+],[J_-]}/PSL(2,\mathbb{C})}$$

which, in particular, means

$$\boxed{e\left(\mathcal{M}_{[J_+],[J_-]}\right) = \frac{e\left(X_{[J_+],[J_-]}\right)}{e(PSL(2,\mathbb{C}))} = q^4 - 3q^2 - 6q.}$$

4.3.4.2 Deligne-Hodge polynomial of $\mathcal{M}_{[J_+],[J_+]}$

Again, we will take $K = \{\pm Id\}$, so $SL(2,\mathbb{C})/K \cong PGL(2,\mathbb{C})$. However, in contrast to section 4.3.4.1, in this case $PGL(2,\mathbb{C})$ does not act freely on $X_{[J_+],[J_+]}$, so we will found reducibles.

By the same argument than above, if $(A,B,C) \in \mathcal{D}_{[J_+]}^{J_+}$, then $A, B \in U = Stab(J_+)$, so, in particular, $[A,B] = Id$ and, thus, $C = J_+$. Therefore, we have the reducibles

$$\mathcal{D}_{[J_+]}^{J_+} = U \times U \times \{J_+\} \cong U^2$$

and, by the relation between $\mathcal{D}_{[J_+]}^{J_+}$ and $\mathcal{D}_{[J_+],[J_+]}$ we have

$$\begin{aligned}\mathcal{D}_{[J_+],[J_+]} &= \left\{(A,B,C,Q) \in Y_{[J_+]}^{J_+} \times SL(2,\mathbb{C})/Stab(J_+) \,|\, Q^{-1}(A,B,C)Q \in \mathcal{D}_{[J_+]}^{J_+}\right\} \\ &\cong \left\{(A,B,C,Q) \in Y_{[J_+]}^{J_+} \times SL(2,\mathbb{C})/Stab(J_+) \,|\, A, B \in QUQ^{-1}, C = QJ_+^{-1}Q^{-1}\right\}\end{aligned}$$

Let us denote $X_{[J_+],[J_+]}^* := X_{[J_+],[J_+]} - \mathcal{D}_{[J_+],[J_+]}$ and ${Y_{[J_+]}^{J_+}}^* = Y_{[J_+]}^{J_+} - \mathcal{D}_{[J_+]}^{J_+}$, the sets of non-reducible elements. Since $U \cong \mathbb{C} \sqcup \mathbb{C}$ and $\mathcal{D}_{[J_+]}^{J_+} \cong U^2$ we have that $e\left(\mathcal{D}_{[J_+]}^{J_+}\right) = e(U)^2 = 4e(\mathbb{C})^2 = 4q^2$ so

$$\boxed{e\left({Y_{[J_+]}^{J_+}}^*\right) = e\left(Y_{[J_+]}^{J_+}\right) - e\left(\mathcal{D}_{[J_+]}^{J_+}\right) = q^5 + q^4 - q^2 + 3q}$$

and, by remark 4.3.2 with $K = \{\pm Id\}$, we have

$$\boxed{e\left(X_{[J_+],[J_+]}^*\right) = e\left({Y_{[J_+]}^{J_+}}^*\right) e\left(\frac{PGL(2,\mathbb{C})}{Stab(J_+)}\right) = q^7 + q^6 - q^5 - 2q^4 + 3q^3 + q^2 - 3q.}$$

Finally, let us compute the Deligne-Hodge polynomial of $\mathcal{M}_{[J_+],[J_+]}$. For the non-reducible elements, we have that the action of $PSL(2,\mathbb{C})$ on $X_{[J_+],[J_+]}^*$ is free, so

$$X_{[J_+],[J_+]}^* /\!\!/ SL(2,\mathbb{C}) = X_{[J_+],[J_+]}^*/SL(2,\mathbb{C})$$

For the contribution of the reducibles, first of all, observe that, setwise

$$\mathcal{D}_{[J_+],[J_+]}/SL(2,\mathbb{C}) = \mathcal{D}_{[J_+]}^{J_+}/Stab(J_+)$$

so, passing to the GIT quotient, we have $\mathcal{D}_{[J_+],[J_+]} /\!\!/ SL(2,\mathbb{C}) = \mathcal{D}_{[J_+]}^{J_+} /\!\!/ Stab(J_+)$. Hence, it is enough to study

$$\mathcal{D}_{[J_+]}^{J_+} /\!\!/ Stab(J_+) \cong U^2 /\!\!/ Stab(J_+)$$

with $Stab(J_+)$ acting by simultaneous conjugation. Let us take $(A,B) \in U^2$, let us say

$$A = \begin{pmatrix} \epsilon & a \\ 0 & \epsilon \end{pmatrix} \qquad B = \begin{pmatrix} \delta & b \\ 0 & \delta \end{pmatrix}$$

with $\epsilon, \delta \in \{\pm 1\}$. Now, let us consider two sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq \mathbb{C}^*$ with $x_n \to \epsilon$ and $y_n \to \delta$. We consider the approximation matrices

$$A_n = \begin{pmatrix} x_n & a \\ 0 & x_n^{-1} \end{pmatrix} \qquad\qquad B_n = \begin{pmatrix} y_n & b \\ 0 & y_n^{-1} \end{pmatrix}$$

The condition for having $[A_n, B_n] = [A, B] = Id$ translates into

$$a(y_n - y_n^{-1}) = b(x_n - x_n^{-1})$$

We are going to prove that there exists a sequence of $\{P_n\}_{n=1}^{\infty} \subseteq Stab(J_+)$ such that

$$P_n \cdot (A_n, B_n) \to (\epsilon Id, \delta Id)$$

Once proven, we have obtained that, in the Zariski closure of each orbit by conjugation, we always find an element of the form $(\epsilon Id, \delta Id)$ for some $\epsilon, \delta \in \{\pm 1\}$. Hence, applying the S-identification, we have that

$$\mathcal{D}_{[J_+]}^{J_+} /\!\!/ Stab(J_+) \cong U^2 /\!\!/ Stab(J_+) \cong \{(\pm Id, \pm Id)\}$$

that is, four points, so $e\left(\mathcal{D}_{[J_+]}^{J_+} /\!\!/ Stab(J_+)\right) = 4$.

The selection of the appropiate P_n is just a computation. Let us take a generic matriz

$$P(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

We have that

$$P(\alpha) \cdot (A_n, B_n) = \left(\begin{pmatrix} x_n & a + \alpha(x_n - x_n^{-1}) \\ 0 & x_n^{-1} \end{pmatrix}, \begin{pmatrix} y_n & b + \alpha(y_n - y_n^{-1}) \\ 0 & y_n^{-1} \end{pmatrix} \right)$$

Hence, since $\frac{-a}{x_n - x_n^{-1}} = \frac{-b}{y_n - y_n^{-1}}$, taking $P_n = P\left(\frac{-a}{x_n - x_n^{-1}}\right)$ we obtain

$$P_n \cdot (A_n, B_n) = \left(\begin{pmatrix} x_n & 0 \\ 0 & x_n^{-1} \end{pmatrix}, \begin{pmatrix} y_n & 0 \\ 0 & y_n^{-1} \end{pmatrix} \right) \to (\epsilon Id, \delta Id)$$

as we wanted to show.

Therefore, have obtained that the contribution to the GIT quotient of the reducibles is just four points, so we have the Deligne-Hodge polynomial

$$\boxed{e\left(\mathcal{M}_{[J_+],[J_+]}\right) = e\left(X^*_{[J_+],[J_+]}/SL(2,\mathbb{C})\right) + e\left(\mathcal{D}_{[J_+],[J_+]} /\!\!/ SL(2,\mathbb{C})\right) = q^4 + q^3 - q + 7}$$

4.3.4.3 Deligne-Hodge polynomial of $\mathcal{M}_{[J_-],[J_-]}$

Observe that, since $-J_+$ is in the conjugacy class of J_- , we have that $-[J_+] = [J_-]$. Using this fact, we have the regular isomorphism

$$\begin{array}{ccc} X_{[J_+],[J_+]} & \leftrightarrow & X_{[J_-],[J_-]} \\ (A, B, C_1, C_2) & \longmapsto & (A, B, -C_1, -C_2) \end{array}$$

Moreover, this isomorphism respects the action of $SL(2, \mathbb{C})$ by simultaneous conjugation, so it descends to the GIT quotient. Thus,

$$\mathcal{M}_{[J_-],[J_-]} \cong \mathcal{M}_{[J_+],[J_+]}$$

so, in particular

$$\boxed{e\left(\mathcal{M}_{[J_-],[J_-]}\right) = e\left(\mathcal{M}_{[J_+],[J_+]}\right) = q^4 + q^3 - q + 7}$$

4.3.4.4 Deligne-Hodge polynomial of $\mathcal{M}_{[J_+],[D_\lambda]}$

This case is very similar to the reasoning for $\mathcal{M}_{[J_+],[J_-]}$ in 4.3.4.1. Indeed, taking $K = \{Id, -Id\}$, we have that $SL(2, \mathbb{C})/K \cong PSL(2, \mathbb{C})$ acts freely on $X_{[J_+],[D_\lambda]}$. To check this, observe that, by the same reason than in 4.3.4.1, if a non trivial $P \in PSL(2, \mathbb{C})$ fixes $(A, B, C_1, C_2) \in X_{[J_+],[D_\lambda]}$, then it would be $[A, B] = Id$, which is impossible by the analysis of section 4.3.2.1.

Hence, we have the Deligne-Hodge polynomial

$$\boxed{e\left(X_{[J_+],[D_\lambda]}\right) = e\left(Y_{[J_+]}^{D_\lambda}\right)\frac{e\left(PGL(2, \mathbb{C})\right)}{e\left(Stab(D_\lambda)\right)} = q^7 + q^6 - 4q^4 - q^3 + 3q^2}$$

Furthermore, again, since $PSL(2, \mathbb{C})$ acts freely on $X_{[J_+],[D_\lambda]}$, the GIT quotient is just a quotient, so

$$\mathcal{M}_{[J_+],[D_\lambda]} = X_{[J_+],[D_\lambda]} /\!\!/ SL(2, \mathbb{C}) = X_{[J_+],[D_\lambda]}/PSL(2, \mathbb{C})$$

which, in particular, means

$$\boxed{e\left(\mathcal{M}_{[J_+],[D_\lambda]}\right) = \frac{e\left(X_{[J_+],[D_\lambda]}\right)}{e(PSL(2, \mathbb{C}))} = q^4 + q^3 + q^2 - 3q}$$

Remark 4.3.8. As we mention in remark 4.3.4, in [46], there is an erratum while computing the Deligne-Hodge polynomial of $Y_{[J_+]}^{D_\lambda}$. Therefore, the corresponding Deligne-Hodge polynomials for $\mathcal{M}_{[J_+],[D_\lambda]}$ do not agree.

4.3.4.5 Deligne-Hodge polynomial of $\mathcal{M}_{[J_-],[D_\lambda]}$

Since $-[J_+] = [J_-]$ we have the algebraic isomorphism

$$\begin{array}{ccc} X_{[J_+],[D_\lambda]} & \leftrightarrow & X_{[J_-],[D_{-\lambda}]} \\ (A,B,C_1,C_2) & \longmapsto & (A,B,-C_1,-C_2) \end{array}$$

Thus, we have that

$$\mathcal{M}_{[J_-],[D_\lambda]} \cong \mathcal{M}_{[J_+],[D_{-\lambda}]}$$

so, in particular

$$\boxed{e\left(\mathcal{M}_{[J_-],[D_\lambda]}\right) = e\left(\mathcal{M}_{[J_+],[D_{-\lambda}]}\right) = q^4+q^3+q^2-3q}$$

4.3.4.6 Deligne-Hodge polynomial of $\mathcal{M}_{[D_{\lambda_1}],[D_{\lambda_2}]}$ with $\lambda_1 \neq \lambda_2, \lambda_2^{-1}$

Recall that, by the results of sections 4.3.3.1 and 4.3.3.2 in both cases we have that

$$e\left(Y_{[D_{\lambda_1}]}^{D_{\lambda_2}}\right) = q^5+q^4+4q^3-4q^2-q-1$$

Now, let us take $K = \{\pm ID\}$ so $SL(2,\mathbb{C})/K \cong PGL(2,\mathbb{C})$. We will show that $PSL(2,\mathbb{C})$ acts freely on $X_{[D_{\lambda_1}],[D_{\lambda_2}]}$. As we explained before, this is equivalent to $Stab(D_{\lambda_2})/\{\pm Id\}$ acting freely on $Y_{[D_{\lambda_1}]}^{D_{\lambda_2}}$. Suppose that there exists a non trivial $P \in Stab(D_{\lambda_2})/\{\pm Id\}$ and $(A,B,C) \in Y_{[D_{\lambda_1}]}^{D_{\lambda_2}}$ with $P(A,B,C)P^{-1} = Id$. Let us suppose that $P = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ for some $\alpha \in \mathbb{C} - \{\pm 1\}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have that

$$PAP^{-1} = \begin{pmatrix} a & \alpha b \\ \alpha^{-1}c & d \end{pmatrix}$$

so it should be $b = c = 0$, that is $A \in D$. Analogously, $B \in D$, so, since $[D,D] = Id$, it must be $[A,B] = Id$. However, the stratifications of sections 4.3.3.1 and 4.3.3.2 show that this is impossible.

Therefore, we have that $Stab(D_{\lambda_2})/\{\pm Id\} \cong \mathbb{C}^*/\mathbb{Z}_2 \cong \mathbb{C}^*$ acts freely on $X_{[D_{\lambda_1}],[D_{\lambda_2}]}$. Thus, we have

$$\boxed{e\left(X_{[D_{\lambda_1}],[D_{\lambda_2}]}\right) = e\left(Y_{[D_{\lambda_1}]}^{D_{\lambda_2}}\right)\frac{e\left(PGL(2,\mathbb{C})\right)}{e\left(Stab(D_{\lambda_2})\right)} = q^7+2q^6+5q^5-5q^3-2q^2-q}$$

Moreover, since we know that the action of $PGL(2,\mathbb{C})$ on $X_{[D_{\lambda_1}],[D_{\lambda_2}]}$ is free, its GIT quotient is just an usual quotient, so

$$\mathcal{M}_{[D_{\lambda_1}],[D_{\lambda_2}]} = X_{[D_{\lambda_1}],[D_{\lambda_2}]} /\!\!/ SL(2,\mathbb{C}) = X_{[D_{\lambda_1}],[D_{\lambda_2}]} /\!\!/ PGL(2,\mathbb{C})$$

and, thus, we have

$$\boxed{\mathcal{M}_{[D_{\lambda_1}],[D_{\lambda_2}]} = X_{[D_{\lambda_1}],[D_{\lambda_2}]}/PGL(2,\mathbb{C})}$$

which, in particular, means

$$\boxed{e\left(\mathcal{M}_{[D_{\lambda_1}],[D_{\lambda_2}]}\right) = \frac{e\left(X_{[D_{\lambda_1}],[D_{\lambda_2}]}\right)}{e(PGL(2,\mathbb{C}))} = q^4 + 2q^3 + 6q^2 + 2q + 1}$$

4.3.4.7 Deligne-Hodge polynomial of $\mathcal{M}_{[D_\lambda],[D_{\lambda^{-1}}]}$

In this case, in contrast to the general case 4.3.4.6, we will find reducibles. Let us take $K = \{\pm Id\}$, so $SL(2,\mathbb{C})/K = PGL(2,\mathbb{C})$. Let $\mathcal{D}_{[D_\lambda],[D_{\lambda^{-1}}]}$ and $\mathcal{D}_{[D_\lambda]}^{D_{\lambda^{-1}}}$ be the set of reducibles of $X_{[D_\lambda],[D_{\lambda^{-1}}]}$ and $Y_{[D_\lambda]}^{D_{\lambda^{-1}}}$, under the action of $PGL(2,\mathbb{C})$ and $Stab(D_{\lambda^{-1}})/K$ respectively. Recall that, since K acts trivially

$$\mathcal{M}_{[D_\lambda],[D_{\lambda^{-1}}]} = X_{[D_\lambda],[D_{\lambda^{-1}}]} /\!\!/ SL(2,\mathbb{C}) = X_{[D_\lambda],[D_{\lambda^{-1}}]} /\!\!/ PGL(2,\mathbb{C}) = Y_{[D_\lambda]}^{D_{\lambda^{-1}}} /\!\!/ Stab(D_{\lambda^{-1}})/K$$

So it is enough to compute $Y_{[D_\lambda]}^{D_{\lambda^{-1}}} /\!\!/ Stab(D_{\lambda^{-1}})/K$. To this end, recall that the variety $Y_{[D_\lambda]}^{D_{\lambda^{-1}}}$ is

$$Y_{[D_\lambda]}^{D_{\lambda^{-1}}} = \left\{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B]C = D_{\lambda^{-1}}\right\}$$

By the same argument than above, if $(A,B,C) \in \mathcal{D}_{[D_\lambda]}^{D_{\lambda^{-1}}}$, then $A, B \in D = Stab(D_\lambda)$, so, in particular, $[A,B] = Id$ and, thus, $C = D_{\lambda^{-1}}$. Therefore, we have the reducibles

$$\mathcal{D}_{[D_\lambda]}^{D_{\lambda^{-1}}} = D \times D \times \{D_{\lambda^{-1}}\} \cong D^2$$

In order to implement S-equivalence for computing the GIT quotient, let us study the set of elements of $Y_{[D_\lambda]}^{D_{\lambda^{-1}}}$ whose orbit, under the action of $Stab(D)/K$, contains a reducible element in its closure. Let us take $(A,B,C) \in Y_{[D_\lambda]}^{D_{\lambda^{-1}}}$ and $P_n \in Stab(D)/K$ a sequence. Let us write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \qquad P_n = \begin{pmatrix} \alpha_n & 0 \\ 0 & \alpha_n^{-1} \end{pmatrix}$$

so we have

$$A_n := P_n A P_n^{-1} = \begin{pmatrix} a & \alpha_n b \\ \alpha_n^{-1} c & d \end{pmatrix} \qquad B := P_n B P_n^{-1} = \begin{pmatrix} x & \alpha_n y \\ \alpha_n^{-1} z & t \end{pmatrix}$$

Therefore, (A_n, B_n, C_n) converges to some element of $\mathcal{D}_{[D_\lambda]}^{D_{\lambda^{-1}}} = D \times D \times \{D_{\lambda^{-1}}\}$ if an only if A and B are simultaneous upper triangular and $\alpha_n \to 0$ or if A and B are simultaneus lower triangular and $\alpha_n \to \infty$.

In this way, if $\overline{\mathcal{D}}_{[D_\lambda]}^{D_{\lambda^{-1}}}$ is the set of elements of $Y_{[D_\lambda]}^{D_{\lambda^{-1}}}$ whose Zariski closure of its $Stab(D)/K$ orbit contains some element of $\mathcal{D}_{[D_\lambda]}^{D_{\lambda^{-1}}} = D \times D \times \{D_{\lambda^{-1}}\}$, we have just prove that

$$\overline{\mathcal{D}}_{[D_\lambda]}{}^{D_{\lambda^{-1}}} = \left\{ \left(\begin{pmatrix} \mu_1 & a \\ 0 & \mu_1^{-1} \end{pmatrix}, \begin{pmatrix} \mu_2 & b \\ 0 & \mu_2^{-1} \end{pmatrix} \right) \middle| \ (\mu_1, \mu_2, a, b) \in (\mathbb{C}^*)^2 \times \mathbb{C} \right\}$$
$$\cup \left\{ \left(\begin{pmatrix} \mu_1 & 0 \\ a & \mu_1^{-1} \end{pmatrix}, \begin{pmatrix} \mu_2 & 0 \\ b & \mu_2^{-1} \end{pmatrix} \right) \middle| \ (\mu_1, \mu_2, a, b) \in (\mathbb{C}^*)^2 \times \mathbb{C} \right\} \cong (\mathbb{C}^*)^4 \times \mathbb{Z}_2 \sqcup (\mathbb{C}^*)^3 \times \mathbb{Z}_4 \sqcup (\mathbb{C}^*)^2$$

Hence, we have that

$$e\left(\overline{\mathcal{D}}_{[D_\lambda]}^{D_{\lambda^{-1}}}\right) = 2e\left(\mathbb{C}^*\right)^4 + 4e\left(\mathbb{C}^*\right)^3 + e\left(\mathbb{C}^*\right)^2 = (q-1)^2(2q^2-1)$$

Let us denote $Y_{[D_\lambda]}^{D_{\lambda^{-1}}{}^*} = Y_{[D_\lambda]}^{D_{\lambda^{-1}}} - \overline{\mathcal{D}}_{[D_\lambda]}^{D_{\lambda^{-1}}}$. Since the orbits of elements of $Y_{[D_\lambda]}^{D_{\lambda^{-1}}{}^*}$ do not contain any reducible element in its closure, we have that the GIT quotient is just the usual quotient

$$Y_{[D_\lambda]}^{D_{\lambda^{-1}}{}^*} /\!\!/ \, Stab(D_{\lambda^{-1}})/K = Y_{[D_\lambda]}^{D_{\lambda^{-1}}{}^*} / (Stab(D_{\lambda^{-1}})/K)$$

so, since $Stab(D_{\lambda^{-1}})/K \cong \mathbb{C}^*/\mathbb{Z}_2 \cong \mathbb{C}^*$, we obtain

$$e\left(Y_{[D_\lambda]}^{D_{\lambda^{-1}}{}^*} /\!\!/ \, (Stab(D_{\lambda^{-1}})/K)\right) = \frac{e\left(Y_{[D_\lambda]}^{D_{\lambda^{-1}}{}^*}\right)}{e(Stab(D_{\lambda^{-1}})/K)} = \frac{e\left(Y_{[D_\lambda]}^{D_{\lambda^{-1}}}\right) - e\left(\overline{\mathcal{D}}_{[D_\lambda]}^{D_{\lambda^{-1}}}\right)}{e(\mathbb{C}^*)}$$
$$= q^4 + q^3 + 6q^2 + 5q - 1$$

Moreover, by the previous argument, we have that, realizing S-equivalence on $\overline{\mathcal{D}}_{[D_\lambda]}^{D_{\lambda^{-1}}}$ under the action of $(Stab(D_{\lambda^{-1}})/K)$, we have that the GIT quotient is

$$\overline{\mathcal{D}}_{[D_\lambda]}^{D_{\lambda^{-1}}} /\!\!/ \, (Stab(D_{\lambda^{-1}})/K) = \mathcal{D}_{[D_\lambda]}^{D_{\lambda^{-1}}}$$

so we obtain the other piece

$$e\left(\overline{\mathcal{D}}_{[D_\lambda]}^{D_{\lambda^{-1}}} /\!\!/ \, (Stab(D_{\lambda^{-1}})/K)\right) = e\left(\mathcal{D}_{[D_\lambda]}^{D_{\lambda^{-1}}}\right) = e(\mathbb{C}^*)^2 = q^2 - 2q + 1$$

Therefore, finally, adding the two contributions, we obtain

$$\boxed{e\left(\mathcal{M}_{[D_\lambda],[D_{\lambda^{-1}}]}\right) = e\left(Y_{[D_\lambda]}^{D_{\lambda^{-1}}{}^*} /\!\!/ \, (D/K)\right) + e\left(\overline{\mathcal{D}}_{[D_\lambda]}^{D_{\lambda^{-1}}} /\!\!/ \, (D/K)\right) = q^4 + q^3 + 7q^2 + 3q}$$

4.3.4.8 Deligne-Hodge polynomial of $\mathcal{M}_{[D_\lambda],[D_\lambda]}$

This case is analogous to the previous case of section 4.3.4.7. Indeed, observe that, since $D_\lambda = P_0 D_{\lambda^{-1}} P_0^{-1}$ with

$$P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence, we have the algebraic isomorphism

$$\begin{array}{ccc} Y^{D_\lambda}_{[D_\lambda]} & \longleftrightarrow & Y^{D_{\lambda^{-1}}}_{[D_\lambda]} \\ (A,B,C) & \longmapsto & (P_0AP_0^{-1}, P_0BP_0^{-1}, P_0CP_0^{-1}) \end{array}$$

Thus, we have that

$$\mathcal{M}_{[D_\lambda],[D_\lambda]} \cong \mathcal{M}_{[D_\lambda],[D_{\lambda^{-1}}]}$$

so, in particular

$$\boxed{e\left(\mathcal{M}_{[D_\lambda],[D_\lambda]}\right) = e\left(\mathcal{M}_{[D_\lambda],[D_{\lambda^{-1}}]}\right) = q^4 + q^3 + 7q^2 + 3q}$$

Appendix A

Review of Complex Geometry

Kähler manifolds are a special class of complex manifolds that have particularly good analytical and algebraic properties. The key point is that they are manifolds that compatibilize three structures, giving a strong rigidity to the geometry.

A.1 Complex and Almost Complex Manifolds

Recall that a complex manifold M of complex dimension n is a differentiable manifold of real dimension $2n$ whose changes of charts are biholomorphic maps. Given any $\mathbb{R}$-vector space (possibly infinite dimensional) V, we will denote $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ to its complexification. In particular, $\Omega^k_{\mathbb{C}}(M)$ is the space of complexified k-forms.

Definition A.1.1. Let M^{2n} be a differentiable manifold. A section J of $End(TM)$ with $J^2 = -1$ is called an **almost complex structure** on M. We will say that J is **integrable** if there exists a complex structure on M such that, for every holomorphic chart $\varphi : U \subset M \to \mathbb{C}^n$, $\varphi_* \circ J = i\varphi_*$.

Remark A.1.2. In a complex manifold M, we can always define an almost complex structure in the following way. Let $(z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n)$ be holomorphic coordinates around some $p \in M$. We define J locally satisfying

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k} \qquad\qquad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}$$

By Cauchy-Riemann ecuations, J is well-defined and, thus, it defines almost complex structure.

Given two almost complex manifold (M, J) and (M', J') and a map $f : M \to M'$, we say that f is (J, J')-holomorphic if $f_* \circ J = J' \circ f_*$. In this contex the integrability condition is equivalent to the existence of a complex atlas whose charts are (J,i)-holomorphic, where i, seen as an automorphism of $\mathbb{C}^n$, is the standar almost complex structure of $\mathbb{C}^n$[1].

In this case, Cauchy-Riemann ecuations simply say that, if M and M' are complex and J, J' are their almost complex structures associated, then a map is holomorphic if and only if is (J,J')-holomorphic.

Therefore, given an almost complex manifold (M^{2n},J), using the minimal polinomial of J, we see that J is diagonalizable with eigenvalues i and $-i$. Let us denote $T^{1,0}M$ and $T^{0,1}M$ the eigenspaces of J of eigenvalues i and $-i$ on $T_{\mathbb{C}}M$, respectively, and we define (p,q)-forms, $\Omega^{p,q}(M)$ as

$$\Omega^{p,q}(M) := \bigwedge^p T^{1,0}(M)^* \otimes \bigwedge^q T^{0,1}(M)^* \subset \Omega^{p+q}_{\mathbb{C}}(M)$$

However, if M is a complex manifold and J is its associated almost complex structure, we can give an effective criterion to identify (p,q)-forms. Indeed, if in a chart (U,φ) we have the coordinate vector basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$, then, defining

$$\frac{\partial}{\partial z_i} := \frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i} \qquad\qquad \frac{\partial}{\partial \overline{z}_i} := \frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}$$

we have that $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \dots, \frac{\partial}{\partial \overline{z}_n}$ is also a basis. Moreover, we have that

$$T^{1,0}(M) = \langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \rangle \qquad\qquad T^{0,1}(M) = \langle \frac{\partial}{\partial \overline{z}_1}, \dots, \frac{\partial}{\partial \overline{z}_n} \rangle$$

Now, if $dz_1, \dots, dz_n, d\overline{z}_1, \dots, d\overline{z}_n$ is the dual basis[2] of $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \dots, \frac{\partial}{\partial \overline{z}_n}$, then we have the explicit description

$$dz_i = dx_i + idy_i \qquad\qquad d\overline{z}_i = dx_i - idy_i$$

Hence, given a form $\omega \in \Omega^k_{\mathbb{C}}(M)$, we have that $\omega \in \Omega^{p,q}(M)$ if and only if, locally, ω can be written

$$\omega|_U = \sum_{\substack{i_1<i_2\dots<i_p \\ j_1<j_2\dots<j_q}} a_{i_1,\dots,i_p,j_1,\dots,j_q}\, dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \dots \wedge d\overline{z}_{j_q}$$

A.1.1 Dolbeault Cohomology

Given a differentiable manifold M and an almost complex structure $J \in \Gamma(Aut(TM))$, the precise conditions to for J been integrable are given by the **Newlander-Niremberg theorem**, which states that J is integrable if and only if the Nijenhuis tensor

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

[1]Moreover, in an almost complex manifold (M,J), a map $f: M \to \mathbb{C}$ (J,i)-holomorphic is called a J-holomorphic map. Analogously, a map $f: \mathbb{C} \to M$ (i,J)-holomorphic is called a pseudo-holomorphic curve. In general, an almost complex manifold does not have any no constant J-holomorphic maps, but it has a lot of pseudo-holomorphic curves.

[2]Observe that dz_i has two possible interpretations. On one hand, in the way we have just defined; and, on the other hand, as the exterior derivative of the coordinate function z_k. However, both definitions agree.

vanish identically.

However, the vanishing of the Nijenhuis tensor is equivalent to the decomposition of the exterior derivative, restricted to (p,q)-forms, $d : \Omega^{p,q}(M) \to \Omega^{p+q+1}_{\mathbb{C}}(M)$ into two operators $d = \partial + \overline{\partial}$, with $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$, known as the **anti-Dolbeault** and **Dolbeault** operators, respectively.

Let us consider a complex manifold M, such that its exterior derivative decomposes as $d = \partial + \overline{\partial}$. Since $d^2 = (\partial + \overline{\partial}) = 0$, we have

$$\partial^2 = \overline{\partial}^2 = 0 \qquad\qquad \partial\overline{\partial} + \overline{\partial}\partial = 0$$

which, in particular, means that, for all $p \geq 0$, the complex

$$\Omega^{p,0}(M) \xrightarrow{\overline{\partial}} \Omega^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,q}(M) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(M) \xrightarrow{\overline{\partial}} \cdots$$

is a co-chain complex, since $\overline{\partial}^2 = 0$. In this way, the **Dolbeault cohomology**, $H^{p,q}(M)$, is precisely the cohomology of this complex, that is

$$H^{p,q}(M) = \frac{\ker \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q}(M)}{\mathrm{Im}\ \overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M)}$$

Remark A.1.3. It is very useful to understand the Dolbeault cohomology as a sheaf cohomology. Let us consider the sheaf $\mathbf{\Omega}^p_M$ of holomorphic p-forms on M, that is, if $\mathbf{\Omega}_X$ is the canonical sheaf [3] of M, then $\mathbf{\Omega}^p_M = \bigwedge^p \mathbf{\Omega}_M$. Equivalently, this sheaf is isomorphic to $\ker \overline{\partial} : \Omega^{p,0}(M) \to \Omega^{p,1}(M)$, that is, for all open set $U \subset M$, we have

$$\mathbf{\Omega}^p_M(U) = \left\{\omega \in \Omega^{p,0}(U) \ \mid \ \overline{\partial}\,\omega = 0\right\}$$

However, by the $\overline{\partial}$-lemma (see, for example, [37]) given an open set $U \subset M$ and $\omega \in \Omega^{p,q}(U)$ with $\overline{\partial}\,\omega = 0$, there exists a neighbourhood $V \subset U$ and $\eta \in \Omega^{p,q-1}(V)$ such that $\overline{\partial}\,\eta = \omega|_V$. Therefore, the co-chain complex

$$\mathbf{\Omega}^p_M \xrightarrow{\overline{\partial}} \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,q} \xrightarrow{\overline{\partial}} \Omega^{p,q+1} \xrightarrow{\overline{\partial}} \cdots$$

is a resolution of $\mathbf{\Omega}^p_M$. But, now, since $\Omega^{p,q}$ is a fine sheaf for all $q \geq 0$, in particular this resolution is acyclic and, thus, it can be used to compute the derived functors. Therefore, we have

$$H^q(M, \mathbf{\Omega}^p_M) = R^q\Gamma\,(\mathbf{\Omega}^p_M) = H^q(\Gamma(\Omega^{p,*})) = H^q(\Omega^{p,*}(M)) = H^{p,q}(M)$$

and, thus, $H^{p,q}(M) = H^q(M, \mathbf{\Omega}^p_M)$.

[3] That is, the locally free sheaf associated to the canonical line bundle.

A.2 Symplectic Manifolds

Another important structure that gives rigidity to our manifolds is the symplectic structure. The motivation for this notion arise from the formalization and geometrization of classical mechanichs, looking for an invariant formulation of its principles. For an introduction to the relation between classical mechanics and symplectic geometry, see, for example, [1] or [5].

Definition A.2.1. Let M be a differentiable manifold and $\omega \in \Omega^2(M)$ a non-degenerated 2-form, i.e., such that ω^n never vanish[4]. If ω is closed, then we say that (M, ω) is a **symplectic manifold**. A map $f : (M, \omega) \to (M', \omega')$ between symplectic manifolds is called a **simplectomorfism** if $f^*\omega' = \omega$.

Remark A.2.2. Since ω^n is closed (because it is a top-dimensional form), it defines a cohomology class $[\omega^n] \in H^{2n}(M)$. Moreover, since ω is non-degenerated, ω^n never vanish and, thus, $\int_M \omega^n \neq 0$, so $[\omega^n] \neq 0$.

In fact, this trick can be repeated for all the even forms. Observe that, if $\omega^k = d\eta$ for some $0 \leq k \leq n$ y $\eta \in \Omega^{2k-1}(M)$, then it will hold

$$\int_M \omega^n = \int_M d(\eta \wedge \omega^{n-k}) = \int_{\partial M} \eta \wedge \omega^{n-k} = 0$$

Thus, $0 \neq [\omega^k] \in \Omega^{2k}(M)$, which in particular means that the even Betti numbers of a symplectic manifold never vanish.

A very important property of symplectic manifolds is that, locally, they are all equal, justifying the name symplectic topology instead of symplectic geometry. The proof can be found in [5].

Proposition A.2.3 (Darboux). *Given a symplectic manifold (M^{2n}, ω) and $p \in M$, there exists a neighbourhood of p, $U \subset M$, such that*

$$\omega|_U = \sum_{k=1}^{n} dp_k \wedge dq_k$$

In particular, every symplectic manifolds of the same dimension are, locally, simplectoisomorphic.

Example A.2.4. The most important example of symplectic manifold, at least for classical mechanichs, is the cotangent bundle. Indeed, let Q be a differentiable manifold (which, in this context, is usually called the configuration space) and let $M = T^*Q$ its cotangent bundle (which, in this context, is usually called the phase space).

In order to become M a symplectic manifold, let us define the 1-form $\nu \in \Omega^1(M)$, known as the *Liouville form* or *canonical form.* Given a point $(q, \eta_q) \in M = T * Q$, let us consider a vector

[4] Equivalently, the map $T_pM \to T_p^*M$ given by $X \mapsto \omega(X, \cdot)$ is an isomorphism for all $p \in M$. If ω is non-degenerated, then M must be even-dimensional.

$X \in T_{q,\eta_q}M$. Then, we define $\nu_{(q,\eta_q)}(X)$ as the result of applying η_q t X, once taken to T_qQ. Explicitly, if $\pi : M = T^*Q \to Q$ is the bundle projection, then we define

$$\nu_{(q,\eta_q)}(X) = \eta_q((\pi_*)_{(q,\eta_q)}X)$$

because, recall that $(\pi_*)_{(q,\eta_q)} : T_{(q,\eta_q)}M \to T_qQ$. From this construction, let us consider the 2-form $\omega = d\nu \in \Omega^2(M)$. Observe that ω es trivially closed.

In order to check that ω is non-degenerated, let us write it down in coordinates. If we take coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ on M, where q_k are coordinates in Q (the positions) and p_k are coordinates in T_q^*Q (the momentum), then, we have

$$\nu_{(q,p)}\left(\frac{\partial}{\partial q_k}\right) = p\left(\frac{\partial}{\partial q_k}\right) = p_i \qquad \nu_{(q,p)}\left(\frac{\partial}{\partial p_k}\right) = p\,(0) = 0$$

and, thus

$$\nu_{(q,p)} = \sum_{k=1}^{n} p_k\, dq_k \qquad \omega = d\nu = \sum dp_k \wedge dq_k$$

In consecuence, ω is non-degenerated and, thus, (T^*Q, ω) is a symplectic manifold.

A.3 Kähler Manifolds

Once defined almost complex structures and symplectic structures, joining them to a riemannian metric, we can obtain the Kähler manifolds, one of the most important categories in complex geometry. Roughly speaking, a Kähler manifold is a differentiable manifold, that is also, at the same time, complex, symplectic and riemannian, in the way that the three structures are compatibles. This conditions give to Kähler manifolds a strong rigidity that is crucial for complex geometry.

Definition A.3.1. A complex, riemannian and symplecit manifold, (M, g, J, ω) is called a **Kähler manifold** if J is a linear symplectomorphism (i.e. $J^*\omega = \omega$) and

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

In this case, the symplectic form ω is usually called the **Kähler form**.

Remark A.3.2. Since ω is J-invariante, in a Kähler manifold, it is also the riemannian metric g. Moreover, defining

$$h(X, Y) := g(X, Y) + i\omega(X, Y)$$

we have that h is and hermitian metric in $T_{\mathbb{C}}M$, called the **Kähler metic**.

Remark A.3.3. Using the relations between metrics, anti-symmetric mappings and almost complex structures (which is called the rule *two of three*), we can derive different versions of this definition.

One of the most common in the literature is to say that a Kähler manifold is an almost complex riemannian manifold (M, g, J) such that J is integrable, g is J-invariant and $\omega := g(J\cdot, \cdot)$ is closed.

Using this characterization, we obtain that every complex submanifold of a Kähler manifold is Kähler. Indeed, if $i : N \hookrightarrow M$ is a complex submanifold of a Kähler manifold (M, g_M), then i^*g_M is a riemannian metric in N such that $\omega_N := i^*g_M(J\cdot, \cdot)$ is cerrada, since $d\omega_N = di^*g_M(J\cdot, \cdot) = i^*dg_M(J\cdot, \cdot) = 0$, so, N is Kähler.

Remark A.3.4. Since M is complex and $\omega \in \Omega^2_{\mathbb{C}}(M) = \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$ is a 2-form, locally it has the form

$$\omega = \sum_{i<j} a_{i,j} dz_i \wedge dz_j + \sum_{i<j} b_{i,j} dz_i \wedge d\overline{z}_j + \sum_{i<j} c_{i,j} d\overline{z}_i \wedge d\overline{z}_j$$

Now, since J is an almost complex structure, we have that $dz_k \circ J = idz_k$ y $d\overline{z}_k \circ J = -id\overline{z}_k$ for $k = 1, \ldots, n$. Therefore, since $J^*\omega = \omega$ we have

$$\omega = J^*\omega = \sum_{i<j} i^2 a_{i,j} dz_i \wedge dz_j + \sum_{i<j} i(-i) b_{i,j} dz_i \wedge d\overline{z}_j + \sum_{i<j} (-i)^2 c_{i,j} d\overline{z}_i \wedge d\overline{z}_j$$

so $a_{i,j} = c_{i,j} = 0$. In consecuence, $\omega \in \Omega^{1,1}(M)$. In particular, taking $h_{i,j} := -2ib_{i,j}$, locally we have

$$\omega = \frac{i}{2} \sum_{i<j} h_{i,j} dz_i \wedge d\overline{z}_j$$

with $H = (h_{i,j})_{i,j=1}^n$ an hermitian matrix. Using that ω is non-degenerated, we have that H is invertible; and, since g is positive defined, H is positive defined, so H defines an hermitian form. Playing with the symmetries of this structures, we see that H is, in fact, the matrix of the hermitian metric h defined on A.3.2.

One of the most important properties of Kähler metric is that they are euclidean up to order 2, which endows the manifold with a great rigidity and, as we will see, allow us to analyze the operator algebra defined on it.

Theorem A.3.5. *Let M be a Kähler manifold and let h be its Kähler metric. For all $p \in M$ there exists holomorphic coordinates in a neighbourhood of p, (U, φ) such that $\varphi(p) = 0$ and, if $H(z)$ is the matrix of $h_{\varphi^{-1}(z)}$ in the coordinate basis, we have that*

$$H(z) = I_{2n} + O(\|z\|^2)$$

It is said that, the metric h oscules to order 2.

Remark A.3.6. Using that the exterior derivative only requires one derivative, we obtain that the reciprocal also holds, that is, if M is a complex manifold and h is and hermitian metric, then h oscules to order 2 for each $p \in M$ if and only if M is Kähler.

Example A.3.7 (Complex space)**.** The standard hermitian metric of $\mathbb{C}^n$ oscules, trivially, to order 2 in each point and, thus, $\mathbb{C}^n$ is a Kähler manifold. Moreover, given a lattice $\Gamma \in \mathbb{C}^n$, the complex n-torus $\mathbb{C}^n/\Gamma$ is a Kähler manifold, by passing to the quotient the hermitian metric of $\mathbb{C}^n$, which is trivially Γ-invariant.

Example A.3.8 (Projective space)**.** $\mathbb{P}^n_{\mathbb{C}}$ is a Kähler manifold with the known Fubini-Study metric, whose Kähler form is, locally

$$\omega_z = \frac{i}{2}\partial\overline{\partial}\log\left(\|z\|^2 + 1\right)$$

in particular, every projective complex manifold, or smooth complex variety is Kähler.

Example A.3.9 (Riemann surface)**.** Let us take a compact oriented Riemann surface X. Let us endow X with any riemannian metric g and, given $v \in T_xX$ not null, we define Jv as the unique vector such that $\left\{\frac{v}{\|v\|}, \frac{Jv}{\|v\|}\right\}$ is a positive oriented orthonormal basis. Since J is an integrable almost complex structure and $\omega(\cdot,\cdot) := g(J\cdot,\cdot)$ is a closed non-degenerated 2-form, with this structure X becomes a Kähler manifold.

A.4 A Panoramic View of GAGA

Complex geometry is strongly related with algebra, since the rigidity of holomorphicity is so strong that forces complex manifolds to behave as algebraic objects. For illustrating this idea, recall that, since every holomorphic function is analytic, it is, roughly speaking, an *infinite complex polynomial*. In this sence, holomorphic functions (or even meromorphic ones) behaves as complex polynomials, as in the case of Liouville's theorem, the identity principle of analytic continuation or Picard's theorem. Therefore, if we had some sort of *finiteness property*, as nÃ¶etherianity or compactness, we could even assert that meromorphic functions are, in fact, simply quotients of complex polynomials. In this case, complex manifolds would be indistinguishable of algebraic varieties.

A very precise way of make this ideas possible is via a theory that links algebraic geometry and analytic geometry, named GAGA theory (from the french *Géométrie Algébrique et Géométrie Analytique a fundational article by Serre [66]*). Here, we are going to discuss two of the mains theorems of this theory named Chow's theorem of algebraicity and Kodaira's embedding theorem.

A.4.1 Analytic spaces and Chow's theorem

We are going to work, exclusively, in the complex framework, so, for all the algebraic definitions, we are going to suppose that the base field is $\mathbb{C}$. The affine n-dimensional space over $\mathbb{C}$ will be denoted by $\mathbb{A}^n$, while the complex projective n-space will be denoted $\mathbb{P}^n$. Recall that an affine variety $X \subseteq \mathbb{A}^n$ is a subset of $\mathbb{C}^n$ that is closed in the Zariski topology of $\mathring{A}^n$, that is, is the common zeros of a (finite) set of polynomial in $\mathbb{C}[x_1, \dots, x_n]$. A quasi-affine variety is a open set (in the Zariski topology) of an affine variety.

Analogously, a projective variety $X \subseteq \mathbb{P}^n$ is a subset of $\mathbb{P}^n$ that is closed in the Zariski topology of $\mathbb{P}^n$, that is, is the common zeros of a (finite) set of homogeneous polynomial in $\mathbb{C}[x_0, x_1, \ldots, x_n]$. A quasi-projective variety is a open set (in the Zariski topology) of an affine variety. A variety, without any adjetive, will denote any affine, quasi-affine, projective and quasi-projective variety.

Copying this basic definitions of algebraic geometry, we can work with the usual analytic topology of $\mathbb{C}^n$ and define analytic spaces. For the following, $\mathcal{O}_{\mathbb{C}^n}$ and $\mathcal{O}_{\mathbb{P}^n}$ will denote the sheaf of rings of holomorphic functions in $\mathbb{C}^n$ and $\mathbb{P}^n$, respectively

Definition A.4.1. Let $U \subseteq \mathbb{C}^n$ be an open set (in the analytic topology). A subset $S \subseteq U$ is called a **analytic subset** of U if S is the zero locus of holomorphic functions in U, i.e., if there exists $f_1, \ldots, f_m \in \mathcal{O}_{\mathbb{C}^n}(U)$ such that

$$S = \{x \in U \mid f_1(x) = \ldots = f_m(x) = 0\}$$

Definition A.4.2. Let $U \subseteq \mathbb{C}^n$ be an open set. A subset $Z \subset U$ is called a **analytic affine variety** if, for all $p \in U$, there exists a neighbourhood $V \subseteq U$ of p such that $Z \cap V$ is an analytic subset.

In this case, we define the sheaf of ideals $\mathcal{I}_Z$ given by $\mathcal{I}_Z(V) := \{f \in \mathcal{O}_{\mathbb{C}^n}(V) \mid f \equiv 0 \text{ in } Z\}$. In this way, any analytic variety Z can be endowed with a sheaf, known as the **structure analytic sheaf**, $\mathcal{O}_Z$, via

$$\mathcal{O}_Z = i^{-1}\left(\frac{\mathcal{O}_{\mathbb{C}^n}|_U}{\mathcal{I}_Z}\right)$$

where $i : Z \hookrightarrow U$ is the inclusion. With this sheaf of rings, it can be shown that $(Z, \mathcal{O}_Z)$ is a ringed space.

Moreover, if $Z' \subseteq Z$, we will say that Z' is an **analytic affine subvariety** of Z if, for every $z \in Z$ there exists a neighbourhood V of z such that $Z' \cap V$ is an analytic subspace.

Remark A.4.3. In a first sight, this definitions could result a little baffling and look similar. However, as an example of the differences, observe that the analytic subsets $S \subseteq U$ are necessarilly closed in U with the subspace analytic topology of U but the analytic varieties do not have to, as shown in the following example.

Example A.4.4. Let us consider the polydisc $D_\epsilon = \{x \in \mathbb{C}^n \mid |x_i| < \epsilon_i\}$ where $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in (0, \infty)^n$. Note that D_ϵ is not an analytic subset of $\mathbb{C}^n$ as it cannot be the zero locus of a finite set of holomorphic functions (since it is open). However, D_ϵ is an analytic subset of itself, becoming an analytic variety.

Example A.4.5. With the same argument, every open set of $\mathbb{C}^n$ is an analytic affine variety.

Remark A.4.6. A easy way of understanding this definition is by recursion. First of all, we decreet that the open sets of $\mathbb{C}^n$ are affine analytic varieties. The subsets $Z' \subseteq Z$ of an affine analytic variety that are, localy, analytic subsets are analytic affine subvarieties of Z. An analytic affine variety is a analytic affine subvariety of $\mathbb{C}^n$.

Definition A.4.7. A locally ringed space $(X, \mathcal{O}_X)$ is called an **analytic space** or **analytic variety** if, for all $x \in X$ there exists an open set $U \subseteq X$, an analytic affine variety $(Z, \mathcal{O}_Z)$ and an isomorphism of locally ringed spaces[5]

$$(\varphi, \varphi^\sharp) : (U, \mathcal{O}_X|_U) \to (Z, \mathcal{O}_Z)$$

Moreover, given two analytic spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ with $Y \subseteq X$, we say that Y is an **analytic subvariety** of X if, for all $x \in X$, there exists a chart $\varphi : U \to Z \subseteq \mathbb{C}^n$, such that $\varphi(Y \cap U, \mathcal{O}_Y|_{Y\cap U})$ is an analytic affine subvariety of Z.

Example A.4.8. Since the open set of $\mathbb{C}^n$ are analytic affine varieties, we have that the complex manifolds are analytic spaces. Furthermore, if M is a complex manifold such that there exists an analytic embedding $M \hookrightarrow \mathbb{P}^N$ for some $N > 0$, then M is an analytic subvariety of $\mathbb{P}^N$.

Remark A.4.9. Recall that, if M is a compact complex manifold, we cannot have an analytic embedding $f : M \hookrightarrow \mathbb{C}^N$ for any $N > 0$. Indeed, taking the projection over any axis of $\mathbb{C}^N$ $f_i = \pi_i \circ f : M \hookrightarrow \mathbb{C}^N \to \mathbb{C}$ for $i = 1, \dots, N$, by the maximum principle, f_i must be constant. Hence, f must be constant, contradicting that f is an embedding.

The definition of smoothness in analytic space is exactly the same that the one for algebraic schemes (or varieties).

Definition A.4.10. Let $(X, \mathcal{O}_X)$ be an analytic space and let $x \in X$. We will say that X is **smooth** in x if $\mathcal{O}_{Xx}$ is a regular ring, that is, if $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{Xx}$ where $\mathfrak{m}_x$ is the unique maximal ideal of $\mathcal{O}_{Xx}$. X is smooth if it is smooth in each of its points.

One of the main theorems of analytic spaces, that allows us to catch a glimpse of the interplay between complex geometry and algebraic geometry is the following theorem, whose proof can be found in [64].

Theorem A.4.11 (Chow)**.** *Every closed analytic subvariety of* $\mathbb{P}^n_{\mathbb{C}}$ *is an algebraic set.*

Remark A.4.12. Due to this theorem, an analytic subvariety of $\mathbb{P}^n_{\mathbb{C}}$ is called a **analytic projective variety**.

Moreover, this theorem can easyly be extended to the case of morphisims remembering the following characterization of a regular map.

Proposition A.4.13. *Given two algebraic varieties X e Y, a map $f : X \to Y$ is regular if and only if its graph $\Gamma(f) \subset X \times Y$ is a closed algebraic subvariety of $X \times Y$. Analogously, a map $f : X \to Y$ between analytic spaces is analytic if and only if its graph $\Gamma(f) \subset X \times Y$ is a closed analytic suvariety of $X \times Y$.*

Corollary A.4.14. *Given two closed projective analytic varieties, $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ and $Y \subseteq \mathbb{P}^m_{\mathbb{C}}$, every analytic map $f : X \to Y$ is regular (i.e. algebraic).*

[5] Recall that a morphism of locally ringed spaces $(\varphi, \varphi^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continous map $f : X \to Y$ with a morphism of sheaves $\varphi^\sharp : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ such that, for any $x \in X$, the induced map in stalks $\varphi^\sharp_x : \mathcal{O}_{Y f(x)} \to \mathcal{O}_{Xx}$ is a homomorphism of local rings, i.e., the inverse image of the maximal ideal of $\mathcal{O}_{Xx}$ is the maximal ideal of $\mathcal{O}_{Y f(x)}$.

A.4.2 Kodaira's embedding theorem

Let M be a complex compact manifold and let $\underline{\mathbb{Z}}$ the constant sheaf $\mathbb{Z}$ and $\underline{\mathbb{C}}$ the constant sheaf $\mathbb{C}$ over it. Using the sheaf morphism $\varphi : \underline{\mathbb{Z}} \to \underline{\mathbb{C}}$ we have induced a homomorphism on homology $\varphi^* : H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{C})$.[6] In this way, the homology class of some $\omega \in \Omega^*_{\mathbb{C}}(X)$ is said to be **integral** if $[\omega] \in \mathrm{Im}\, \varphi^*$.

Remark A.4.15. Using the de Rham pairing, it can be shown that $\omega \in \Omega^k_{\mathbb{C}}(X)$ is integral if and only if

$$\int_{[Y]} \omega \in \mathbb{Z}$$

for all $[Y] \in H^k(X, \mathbb{Z})$, the homology class of a k-cycle in X.

Definition A.4.16. Let M be a compact complex manifold. M is called a **Hodge manifold** if there exists a Kähler structure on M such that its Kähler form is integral. In that case, such Kähler form is called **Hodge form**.

Example A.4.17. In a Riemann surface, every almost-complex structure is integrable and every 2-form is close, so every Riemann surface X is a Kähler manifold. If $\omega \in \Omega^2(X)$ is a Kähler form, then, redefining $\tilde{\omega} = \frac{1}{\int_X \omega}\omega$ we have that $\tilde{\omega}$ induces a Kähler structure on X and

$$\int_X \tilde{\omega} = 1$$

so ω integral. Therefore, every Riemann surface is a Hodge manifold.

The main issue of Hodge manifolds is the closeness of this property with the ability of being analytically embedded on $\mathbb{P}^N$ for N large enough.

Definition A.4.18. Let M be a compact complex manifold. M is called an **analytic projective manifold** if there exists a holomorphic embedding $M \hookrightarrow \mathbb{P}^N$ for some N large enough.

Example A.4.19. Let $\omega \in \Omega^2(M)$ be the Kähler form of $\mathbb{P}^n$ for the Fubini-Studi metric. It can be shown (see [76]) that ω is the Chern class of some line bundle over $\mathbb{P}^n$, called the universal bundle. With this, we obtain that ω is integral, so $\mathbb{P}^n$ is a Hodge manifold. Moreover, for any analitic submanifold $M \subseteq \mathbb{P}^n$, restrinctring the Fubini-Studi metric, we can induce a Kähler structure on M whose Kähler form is integral. Therefore, every analytic projective submanifold is a Hodge manifold.

With this example, we have obtain that every analytic projective manifold is a Hodge manifold. The following theorem, whose proof can be found, for example, in [76], states the converse statement.

[6] Strictly speaking, we have an induced homomorphism on homology of sheaves $\varphi^* : H^*(M, \underline{\mathbb{Z}}) \to H^*(M, \underline{\mathbb{C}})$. However, in a complex manifold, the sheaf $\Omega^*_{\mathbb{C}}$ is a fine sheaf so $\underline{\mathbb{C}} \to \Omega^*_{\mathbb{C}}$ is an acyclic resolution. Hence, by the de Rham-Weil theorem, this resolution computes the sheaf cohomology of $\underline{\mathbb{C}}$, so $H^*(M, \underline{\mathbb{C}}) \cong H^*(\Omega^*_{\mathbb{C}})(M) \cong H^*_{dR}(M, \mathbb{C})$. Analogous considerations are valid with the sheaf of cochains C^*, which is a soft sheaf, in order to relate $H^*(M, \underline{\mathbb{Z}})$ with $H^*(M, \mathbb{Z})$.

Theorem A.4.20 (Kodaira Embedding Theorem)**.** *Let M be a compact complex manifold. Then, M is an analytic projective manifold if and only if M is a Hodge manifold.*

By example A.4.8, we have that every analytic projective manifold is an analytic projective variety. Hence, using Chow's theorem, we have

Corollary A.4.21. *Every Hodge manifold is a projective algebraic variety.*

Moreover, since, by example A.4.17 every Riemann surface is a Hodge manifold, we have

Corollary A.4.22. *Every Riemann surface is a projective algebraic variety.*

Appendix B

Hodge Decomposition Theorem

B.1 Metrics on Differential Forms

First of all, let's discuss simply the case of a vector space, in a purely linear algebra setting. Let's consider a euclidean vector space V, finite dimensional, with inner product $\langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{R}$. From this product, we can induce, naturally, another one in $\bigwedge^k V$, the spaces of k-alterned vectors, by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle_{\bigwedge^k V} = \det \langle v_i, w_j \rangle_V$$

Thus, using this construction, the space $\bigwedge^k V$ becomes, naturally, an euclidean space.

Furthermore, using again the inner product, we clearly have an isomorphism[1] between V and V^* given by

$$\begin{array}{rccc} \cdot^\flat : & V & \longrightarrow & V^* \\ & v & \longmapsto & \langle v, \cdot \rangle_V \end{array}$$

and its inverse is usually denoted $\cdot^\sharp := (\cdot^\flat)^{-1}$ Therefore, using this isomorphism, we can define, naturally, a inner product in V^* by

$$\langle \omega, \eta \rangle_{V^*} = \langle \omega^\sharp, \eta^\sharp \rangle_V$$

Hence, putting all together, a inner product in a finite dimensional vector space V induces, in a natural way, a inner product in its space of k-forms, that is

$$\langle \omega_1 \wedge \cdots \wedge \omega_k, \eta_1 \wedge \cdots \wedge \eta_k \rangle_{\bigwedge^k V^*} = \det \langle \omega_i^\sharp, \eta_j^\sharp \rangle_V$$

Remark B.1.1. It is a simple computation to note that this inner product is characterized by the property that, if $e_1, \ldots, e_n$ forms an ortonormal base of V, then $\left\{ e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \right\}$ forms an ortonormal base of $\bigwedge^k V^*$, where $e_k^* = \phi(e_k)$.

[1] However, this easy result becomes harder in the Hilbert spaces setting (it is known as the Riesz lemma) and, in fact, is no longer true for Banach spaces in general.

Now, we can make this definitions global and extend them for riemannian manifolds. Let M be a riemannian manifold with metric g, then, doing this operations fiberwise on T^*M, we create an inner product on $\Omega^k(M)$ given by

$$g_p^k\left(\omega_{1p} \wedge \cdots \wedge \omega_{kp}, \eta_{1p} \wedge \cdots \wedge \eta_{kp}\right) = \det g_p\left(\phi_p^{-1}\omega_{ip}, \phi_p^{-1}\eta_{j_p}\right)$$

Remark B.1.2. An extremely dark (but exact) way of defining what we have just described is to say that g^k is a section of the subbundle of the bilinear, symmetric and positive definite forms on $\bigwedge^k T^*M \otimes_{\mathbb{R}} \bigwedge^k T^*M$.

Remark B.1.3. For the same reason that remark B.1.1, if $\omega_{1p}, \ldots, \omega_{np}$ is a ortonormal basis of T_p^*M, with respect to the product on 1-forms, then $\omega_{i_1 p} \wedge \cdots \wedge \omega_{i_k p}$ is an orthogonal basis of $\Omega_p^k(M)$ for all $p \in M$.

B.1.1 The L^2 Product

Let (M, g) be a compact riemannian manifold of dimension n, from whose riemannian metric we have an induced inner product g^k on $\Omega^k(M)$ for all $k \geq 0$. Suppose that M is orientable, with volume form[2], Ω.

As a real vector space, Ω^k is an infinite dimensional vector space, on which we can define an inner product.

Definition B.1.4. Let $\Omega^k(M)$ be the space of k-differential forms on M. Then, we can define an inner product on $\Omega^k(M)$ writen $\langle\cdot,\cdot\rangle_{L^2}$, known as the L^2 **metric** on $\Omega^k(M)$, given by

$$\langle\omega, \eta\rangle_{L^2} = \int_M g^k(\omega, \eta)\,\Omega$$

Proposition B.1.5. *The L^2 metric is an inner product on $\Omega^*(M)$.*

Proof. The bilineality, symmetry and positivity are obvious from the fact that g^k is a inner product. For showing that this metric is positive defined, suppose that $\omega \in \Omega^*(M)$ satisfies $\langle\omega,\omega\rangle_{L^2} = 0$, so $\int_M g^k(\omega,\omega)\,\Omega = 0$.

However, cause Ω is always not null, this integral is null if and only if $g_p^k(\omega_p, \omega_p) = 0$ for almost every $p \in M$. Nevertheless, cause $g_p^k(\omega_p, \omega_p)$ is continous in p, it is almost everywhere null if and only if it is everywhere null. But, again using that g_p^k is an inner product for all p, $g_p^k(\omega_p, \omega_p) = 0$ for all p if and only if $\omega = 0$, as we wanted to show. ■

[2]For those who don't remember what is that, or who have never seen it before, the volume form Ω is the unique never-null n-form such that $\Omega_p(e_1, \ldots, e_n) = 1$ for every $e_1, \ldots, e_n$ ortonormal base of T_pM positive oriented. It can be constructed form a rescaling from the never-null n-form that defines the orientation of M.

Remark B.1.6. The space $\Omega^k(M)$ is not complete with the topology induced by the L^2 metric[3], so it has the structure of a pre-Hilbert space.

Remark B.1.7. The L^2 product can be extended to the graded ring $\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$, by stating that two forms of different degree are always orthogonal.

Example B.1.8. Let's compute the L^2 product of two differentiable functions $f, h \in C^\infty(M) = \Omega^0(M)$. By definition, we have

$$\langle f, h\rangle_{L^2} = \int_M g^0(f,h)\,\Omega = \int_M fh\,\Omega = \int_M fh\,dg$$

where dg is the measure asociated to the volume form Ω determined by the riemannian metric g. Hence, using this measure, the L^2 product on functions coincides with the classical L^2 product used in analysis.

The L^2 metric admites a cleaner expresion in terms of an operator between forms known as the Hodge star operator.

B.1.2 The Hodge Star Operator

The Hodge Star operator is a linear operator over the space of differential forms that highlights the duality between the high and low degree forms. First of all, observe that for all $p \in M$

$$\dim_{\mathbb{R}} \Omega^k_p(M) = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} = \dim_{\mathbb{R}} \Omega^{n-k}_p(M)$$

And, hence, $\Omega^k_p(M)$ y $\Omega^{n-k}_p(M)$ are isomorphic. However, as always in that cases, those isomorphism are not canonical, they are defined *ad hoc* using basis arbitrary chosen, so the cannot be fitted together to form a global operator on the manifold.

Nevertheless, the choosing of the riemannian metric allows us to make this isomorphism canonical. Let (M, g) be a riemannian manifold of dimension n, with volume form $\Omega \in \Omega^n(M)$. Given vector bundles E, F, let $Hom(E,F)$ be the bundle of linear transformations between them, that is $Hom(E,F) = E^* \otimes F$. Then, we can define the linear applications

$$\begin{array}{rccc} \phi_1: & \Omega^k(M) & \to & Hom(\Omega^k(M), \Omega^n(M)) \\ & \omega & \mapsto & g^k(\cdot, \omega)\ \Omega \end{array} \quad \begin{array}{rccc} \phi_2: & \Omega^{n-k}(M) & \to & Hom(\Omega^k(M), \Omega^n(M)) \\ & \omega & \mapsto & \cdot \wedge \omega \end{array}$$

It easy to see that both are isomorphisms, so we have an isomorphism $\phi_2^{-1} \circ \phi_1 : \Omega^k_p(M) \to \Omega^{n-k}_p(M)$, known as the Hodge Star operator.

[3]For example, has we shall see, this metric is, over $\Omega^0(M)$, the usual L^2 product, which is not complete.

Definition B.1.9. Let (M, g) be a compact oriented riemannian manifold of dimension n, with volume form Ω. Then, given $\omega \in \Omega_p^k(M)$, we define the **Hodge Star** of ω, denoted by $\star\omega$, as the unique $(n-k)$-form such that

$$g^k(\eta, \omega)\,\Omega = \eta \wedge \star\omega$$

for all $\eta \in \Omega^k(M)$.

Remark B.1.10. As we promised, using the Hodge Star operator, the L^2 metric has the simpler apparience of

$$\langle \eta, \omega \rangle_{L^2} = \int_M \eta \wedge \star\omega$$

However, the previous definition of the Hodge Star is unuseful for effective computing, so we need the next proposition, whose proof can be found in 3.1.2.

Proposition B.1.11 (Computation of the Hodge Star Operator)**.** *Let (M, g) be a compact oriented riemannian manifold of dimension n and let $p \in M$. Let $\omega_1, \dots, \omega_n$ be a positively oriented orthonormal base of T_p^*M with respect to the induced inner product on 1-forms. Then, over k-forms, the Hodge Star operator can be computed as*

$$\star\left(\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}\right) = \mathit{sign}(\sigma) \cdot \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-k}}$$

where $\sigma = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & n \\ i_1 & i_2 & \cdots & i_k & j_1 & j_2 & \cdots & j_{n-k} \end{pmatrix}$ *is a permutation of* $\{1, \dots, n\}$*.*

Remark B.1.12. From this characterization for the Hodge Star, is very simply to observe that $\star^{-1} = (-1)^{k(n-k)}\star$, so $\star\star = (-1)^{k(n-k)}$.

B.2 The Laplace-Beltrami Operator

The laplacian operator is one of the most important linear operators in functional analysis. Indeed, its kernel, known as the harmonic functions, has very rigid properties, related with the properties of the complex functions.

In this sense, it is logical that there exists a generalization of this operator to the context of differential manifolds. This is, in fact, the Laplace-Beltrami operator, one of the most important operators in differantial and complex geometry and the begining of a vaste and rich theory known as Hodge Theory. Unfortunately, the generalization is not obvious, and requieres the concept of adjoint operator.

B.2.1 Adjointness and self-adjointness

First of all, we will begin with the simplest case of an adjoint operator, which, as we will see, its insufficient for out purposes.

Definition B.2.1. Let X, Y be (real or complex) Banach spaces, $S \subset X$ a linear subspace and let $T : S \to Y$ be a linear operator. We will say that T is **bounded over** S if

$$\|T\| := \sup_{\omega \in S} \frac{\|T(\omega)\|_Y}{\|\omega\|_X} < \infty$$

Moreover, if $S = X$, then we will simply say that T is **bounded**.

Remark B.2.2. It is an standard fact that T is bounded if and only if T is continous (See, for example [63]).

A very important property of the bounded operators is that they can be extended, with continuity and in an unique way, to the clausure of its domain.

Theorem B.2.3 (Extension theorem)**.** *Let X, Y be Banach spaces and $S \subset X$ a linear dense subspace. If $T : S \to Y$ is bounded over S, then there exists an unique bounded extension $\tilde{T} : X \to Y$.*

Proof. Let $x \in X$, since S is dense in X, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset S$ such that $x_n \to x$. Then, we define $\tilde{T}(x) = \lim_{n\to\infty} T(x_n)$.

For showing that $\tilde{T}$ is well defined, suppose that $\{y_n\}_{n=1}^{\infty} \subset S$ is another sequence converging to x. Then, cause T is bounded over S, we have

$$\|T(x_n) - T(y_n)\| = \|T(x_n - y_n)\| \leq \|T\|\|x_n - y_n\| \overset{n\to\infty}{\to} 0$$

so $\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} T(y_n)$ and $\tilde{T}$ is well defined. The uniqueness follows from the fact that any continous map is uniquely defined by its image on a dense subset. ■

In this setting of bounded operators, the notion of adjointness can be easily defined.

Definition B.2.4. Let H be a (separable, real or complex) Hilbert space, and let $T : H \to H$ be a bounded operator. If there exists a bounded operator $T^* : H \to H$ such that, for all $\omega, \eta \in H$

$$\langle \eta, T(\omega) \rangle = \langle T^*(\eta), \omega \rangle$$

we will say that T^* is the **adjoint operator** of T. Moreover, if $T = T^*$, we will say that T is **self-adjoint**.

Example B.2.5 (Fourier Transform)**.** Let $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ be the Schwartz class, i.e., the class of rapidly decreasing functions, in the sense that

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta \in \mathbb{N}^n\}$$

where $\|f\|_{\alpha,\beta}$ are the semi-norms $\|f\|_{\alpha,\beta} = \|x^\alpha \partial^\beta f\|_\infty$ for each multiindices $\alpha, \beta \in \mathbb{N}^n$.

Its easy to check that the Fourier Transform is well defined over the Schwartz class and map it to itself, $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Furthermore, the Fourier Transform is an isometry in the L^2 norm (i.e. $\|F\| = 1$) by the Plancherel theorem.

Moreover, cause $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, by the extension theorem B.2.3, there exists an unique extension $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. However, by the Parseval formula, we have

$$\langle \hat{f}, g\rangle_2 = \int_{\mathbb{R}^n} \hat{f} g \, d\mu = \int_{\mathbb{R}^n} f \hat{g} \, d\mu = \langle f, \hat{g}\rangle_2$$

for all $f, g \in L^2(\mathbb{R}^n)$, so the Fourier Transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a self-adjoint operator over $L^2(\mathbb{R}^n)$. For proofs for these claims, see, for example, [23].

However, the life is not so easy and many of the important operators are not bounded. One of the most important examples, taken from physics, is the position operator.

Example B.2.6 (Position operator)**.** Let x be the operator *multiply by* x, i.e., $(xf)(x) := xf(x)$ (this operator is known, in quantum mechanics, as the position operator). First of all, note that x is well defined, in the L^2 norm, over the subset

$$S = \left\{ f \in L^2(\mathbb{R}) \mid \int_{-\infty}^{\infty} x^2 f^2 < \infty \right\}$$

so we can define $x : S \to L^2(\mathbb{R})$.

However, x is not a bounded operator. For this end, let's define $f_\epsilon(x) = \frac{1}{x^{1+\epsilon}} \chi_{[1,\infty)}$ and observe that $f_\epsilon \in S$ for $\epsilon > \frac{1}{2}$. Then, by simple computation, we have that

$$\|f_\epsilon\|_2 = \int_1^{\infty} \frac{1}{x^{2+2\epsilon}} = \frac{1}{1+2\epsilon} \qquad \|xf_\epsilon\|_2 = \int_1^{\infty} \frac{1}{x^{2\epsilon}} = \frac{1}{2\epsilon - 1}$$

Hence $\frac{\|xf_\epsilon\|_2}{\|f_\epsilon\|_2} = \frac{1+2\epsilon}{2\epsilon-1} \xrightarrow{\epsilon \to 1/2} \infty$, and x is not bounded.

Very related with this position operator is the derivation operator, that is central for our purposes.

Example B.2.7 (Derivation)**.** Let's take the derivation operator ∂ that can be defined, for example, in the Schwartz class, $\mathcal{S}(\mathbb{R})$, so that $\partial : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is given by $\partial f(x) = f'(x)$.

Let's define de momentum operator $p = \frac{1}{2\pi i}\partial$ (once again, this name comes from quantum mechanics). Note that ∂ is bounded in $L^2(\mathbb{R})$ if and only if p in bounded in $L^2(\mathbb{R})$ and, since the Fourier Transform is an bounded automorphism, this is bounded if and only if $\mathcal{F} \circ p$ is bounded. However, for $f \in \mathcal{S}(\mathbb{R})$, by the properties of the Fourier Transform, we have

$$(\mathcal{F} \circ p)(f)(\xi) = \frac{1}{2\pi i} \widehat{\partial f}(\xi) = \xi \, f(\xi)$$

so $\mathcal{F} \circ p = x$ over $\mathcal{S}(\mathbb{R})$, that is not a bounded operator. Therefore, ∂ is not a bounded operator.

For this reason it is necessary to improve our previous definition in order to consider the case of unbounded operators.

Let H be a Hilbert space and let T be a linear operator over H. As we have just see, if T is not bounded, it can be defined over a proper subset of H and it may not admit any extension to H. So, let $\mathcal{D}(T)$ be it domain of definition, that we will suppose that is a linear subspace of H, dense in H (in other case, restrict H to $\overline{\mathcal{D}(T)}$), so that $T : \mathcal{D}(T) \to H$.

Definition B.2.8. Given a densily defined linear operator $T : \mathcal{D}(T) \to H$, we define the **domain of the formal adjoint operator** of T as

$$\mathcal{D}(T^*) = \{\eta \in H \mid \exists \tilde{\eta} \in H \ \forall \omega \in \mathcal{D}(T) : \langle \eta, T(\omega) \rangle = \langle \tilde{\eta}, \omega \rangle\}$$

and we will define the **formal adjoint operator** $T^* : \mathcal{D}(T^*) \to H$ by $T^*(\eta) = \tilde{\eta}$.

Remark B.2.9. The adjoint operator is well defined because $\mathcal{D}(T)$ is dense in H.

Remark B.2.10. Decoding the language, we have that $Ker\, T^* = \mathrm{Im}\, T^{\perp}$. In particular, $Ker\, T^*$ is closed.

In general, $\mathcal{D}(T^*)$ is not dense, (or even not null!) so there is no obvious relation between $\mathcal{D}(T)$ and $\mathcal{D}(T^*)$. However, a very important class of operators has this two linear subspaces coincident.

Definition B.2.11. A linear operator $T : \mathcal{D}(T) \to H$ is called **symmetric** if, for all $\omega, \eta \in \mathcal{D}(T)$

$$\langle \eta, T(\omega) \rangle = \langle T(\eta), \omega \rangle$$

Moreover, a linear operator $T : \mathcal{D}(T) \to H$ is called **self-adjoint** if T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$.

Remark B.2.12. For any symmetric operator, we have that $\mathcal{D}(T) \subset \mathcal{D}(T^*)$. The property of being self-adjoint requieres that the contention becomes an equality.

A very important property of the self-adjoint operators is that they cannot be extended.

Proposition B.2.13. *A self-adjoint operator is maximaly defined, in the sense that it does not admit any symmetric extension.*

Proof. Observe that, in general if T, R are two linear operators such that $\mathcal{D}(T) \subset \mathcal{D}(R)$, then $\mathcal{D}(R^*) \subset \mathcal{D}(T^*)$. Hence, if T is a self-adjoint operator and $\tilde{T}$ is any symmetric extension, then we have

$$\mathcal{D}(T) \subset \mathcal{D}(\tilde{T}) \subset \mathcal{D}(\tilde{T^*}) \subset \mathcal{D}(T^*) = \mathcal{D}(T)$$

Hence, every contention is an equality and $\mathcal{D}(\tilde{T}) = \mathcal{D}(T)$. ■

Example B.2.14 (Multiplication operator)**.** Suppose that $A : \mathbb{R}^n \to \mathbb{C}$ is a measurable function. Let's consider the set $\mathcal{D}(A) = \left\{ f \in L^2(\mathbb{R}^n) \,|\, A(\cdot)f(\cdot) \in L^2(\mathbb{R}^n) \right\}$, we can define the *multiplication by* A operator, given, for $f \in \mathcal{D}(A)$ by

$$A\,f(x) = A(x)f(x)$$

It is easy to see that $\mathcal{D}(A)$ is dense subset of $L^2(\mathbb{R}^n)$, so A is a densely defined linear operator.

Let's compute the adjoint of A, A^*. First of all, observe that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$, because, for every $f, g \in \mathcal{D}(A)$, we have

$$\langle g, Af\rangle_{L^2(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} g(x) \left(\overline{A(x)f(x)}\right) dx = \int\limits_{\mathbb{R}^n} \left(\overline{A(x)}g(x)\right) \overline{f(x)}\, dx = \langle \overline{A}g, f\rangle_{L^2(\mathbb{R}^n)}$$

where $\overline{z}$ denote the conjugate of $z \in \mathbb{C}$. Hence, we have found that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and, furthermore, $A^* = \overline{A}$ on $\mathcal{D}(A)$.

In fact, $\mathcal{D}(A^*) = \mathcal{D}(A)$ but, for showing this, we need to recompute. Suppose that $g \in \mathcal{D}(A^*)$, then there exists $\tilde{g} \in L^2(\mathbb{R}^n)$ such that, for all $f \in \mathcal{D}(A)$ we have

$$\langle g, Af\rangle_{L^2(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} g(x) \left(\overline{A(x)f(x)}\right) dx = \int\limits_{\mathbb{R}^n} \tilde{g}(x)\overline{f(x)}\, dx = \langle \tilde{g}, f\rangle_{L^2(\mathbb{R}^n)}$$

so we have, for all $f \in \mathcal{D}(A)$,

$$0 = \int\limits_{\mathbb{R}^n} \left(\overline{A(x)}g(x) - \tilde{g}(x)\right) \overline{f(x)}\, dx = \langle \overline{A}g, f\rangle_{L^2(\mathbb{R}^n)}$$

Therefore, since $\mathcal{D}(A)$ is dense in $L^2(\mathbb{R}^n)$, then it should be $\overline{A(x)}g(x) = \tilde{g}(x)$. Hence, $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $A^* = \overline{A}$. In particular, if A is real (i.e., $\mathrm{Im}(A) = A$), then A is self-adjoint. For example, the position operator, as defined above, is self-adjoint.

B.2.2 Hodge Decomposition Theorem

Now, we will apply this theory of adjoints operators to the most important operator in differential geometry.

Definition B.2.15. Let M be a differentiable manifold and let $\Omega^*(M)$ be the space of differential forms. The **exterior differential** is the unique linear operator $d : \Omega^*(M) \to \Omega^{*+1}(M)$ such that

- $df(X) = X(f)$ for all $f \in C^\infty(M) = \Omega^0(M)$ and $X \in T_pM$ for $p \in M$.
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega}\omega \wedge d\eta$

Remark B.2.16. Remember, from the basic courses of differential geometry, that, if in a local chart, a k-form $\omega \in \Omega^k(M)$ is given by $\omega = \sum\limits_{i_1 < i_2 < \ldots < i_k} a_{i_1 \ldots i_k}\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ then its exterior differential is given by

$$d\omega = \sum_{i_1<i_2<\ldots<i_k} \sum_{j=1}^{n} \frac{\partial a_{i_1\ldots i_k}}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

Its easy to see that $d : \Omega^*(M) \to \Omega^*(M)$ is not a bounded operator, because it is very related with the derivation operator which, as we saw above, it is not bounded. However, using the Hodge star operator, we can define a formal adjoint operator over its domain of definition in a riemannian manifold. The proof can be found in 3.1.6.

Proposition B.2.17. *Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior differential over a compact oriented riemannian manifold (M, g). Then, the linear operator $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$ given by*

$$d^* = (-1)^{n(k+1)+1} \star d \star$$

is the formal adjoint of d over $\Omega^*(M)$ with respect to the L^2 inner product.

Definition B.2.18. Let (M, g) be a compact oriented riemannian manifold with exterior differential $d : \Omega^*(M) \to \Omega^{*+1}(M)$, whose formal adjoint operator, with respect to the L^2 norm, is $d^* : \Omega^*(M) \to \Omega^{*-1}(M)$. The **Laplace-Beltrami operator**, $\Delta : \Omega^k(M) \to \Omega^k(M)$, is given by

$$\Delta = d\, d^* + d^*\, d$$

Moreover, a differential form $\omega \in \Omega^*(M)$ is said **harmonic** if $\Delta\omega = 0$.

As we saw in 3.1.2 the Laplace-Beltrami operator has the following properties.

Proposition B.2.19. *Let (M, g) be and differentiable compact oriented riemannian manifold and let $\Delta : \Omega^*(M) \to \Omega^*(M)$ be the Laplace-Beltrami operator.*

- *Δ is symmetric with respect to the L^2 product, that is*

  $$\langle \Delta\omega, \eta \rangle_{L^2} = \langle \omega, \Delta\eta \rangle_{L^2}$$

  for all $\omega, \eta \in \Omega^*(M)$.

- *A differential form $\omega \in \Omega^*(M)$ is harmonic if and only if $d\omega = 0$ and $d^*\omega = 0$.*

The most important result in the Hodge Theory is the theorem known as the Hodge Decomposition, that allows us to have a better understanding of the space of differentiable forms. As we saw in 3.1.3, this insight becomes very useful for topological and geometric considerations.

Theorem B.2.20 (Hodge Decomposition)**.** *Let (M, g) be a compact oriented riemannian manifold of dimension n, with Laplace-Beltrami operator $\Delta : \Omega^*(M) \to \Omega^*(M)$. Then, for each $0 \leq k \leq n$, $\mathcal{H}^k(M)$ is finite dimensional and we have the split*

$$\Omega^k(M) = \Delta\, \Omega^k(M) \oplus \mathcal{H}^k(M)$$

where $\mathcal{H}^k(M)$ is the space of harmonic differentiable k-forms. Furthermore, this decomposition is orthogonal with respect to the L^2 norm.

Corollary B.2.21. *For each $0 \leq k \leq n$ we have the orthogonal decomposition*

$$\Omega^k(M) = d\,\Omega^{k-1}(M) \oplus d^*\,\Omega^{k+1}(M) \oplus \mathcal{H}^k(M)$$

B.3 Proof of the Hodge Decomposition Theorem

In order to prove the Hodge decomposition, we have to develop a complete framework of Hilbert spaces and norms that allows us to use functional analytical methods.

This approach is so powerful that, without any explicit computation, only by linear algebra (but linear algebra in an infinite dimensional vector space) we will obtain a slightly version of the Hodge decompostion, known as it weak version. Once we have obtained the weak version, we will study the regularity of the solutions and, with that, we will derive the classical strong version.

B.3.1 Sobolev Spaces

The most important spaces that we are going to use are the Sobolev spaces. Roughly speaking, we are going to weaken the notion of derivative and, with that, we will define the (k, p)-Sobolev space as the space of function in L^p with its k-th first weak derivatives in L^p. Hence, in this spaces, we are authorized to derivate in an almost formal way, without any mention of regularity.

B.3.1.1 Weak derivatives

First of all, suppose that we have a function $f \in C_c^\infty(\mathbb{R}^n)$. Given a multiindex $\alpha \in \mathbb{N}^n$, we will denote $\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n} f$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let's take any function $\phi \in C_c^\infty(\mathbb{R}^n)$ (this functions are called *test functions* in this setting). Then, integrating by parts several times and using the compactness of the support of ϕ (that kills the boundary term) we have

$$\int\limits_{\mathbb{R}^n} \phi\,\partial^\alpha f\,d\mu = (-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} \partial^\alpha \phi\,f\,d\mu$$

Althought the left-hand-side requieres that f is differentiable, the right-hand-side has a completely perfect sense even for non-differentiable f. Furthermore, this formula characterizes the derivative, because, if g is another function satisfing this property, then, for all $\phi \in C_c^\infty(\mathbb{R}^n)$

$$\int\limits_{\mathbb{R}^n} \phi\,\partial^\alpha f = (-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} \partial^\alpha \phi\,f\,d\mu = \int\limits_{\mathbb{R}^n} \phi\,\partial^\alpha f \Rightarrow \int\limits_{\mathbb{R}^n} \phi\,(\partial^\alpha f - g) = \langle \phi, \partial^\alpha f - g \rangle_{L^2(\mathbb{R}^n)} = 0$$

But $C_c^\infty(\mathbb{R}^n)$ is dense in L^2, so it should be $g = \partial^\alpha f$ almost everywhere. Therefore, we can generalize the notion of derivative to a function satisfying this property.

Definition B.3.1. Let $U \subset \mathbb{R}^n$ be a open set, $f \in L^1_{loc}(U)$ and $\alpha \in \mathbb{N}^n$ a multiindex, we will say that a function $g \in L^1_{loc}(U)$ is the α-**weak derivative** of f if, for all $\phi \in C_c^\infty(U)$ we have

$$\int_U \phi \, g \, d\mu = (-1)^{|\alpha|} \int_U \partial^\alpha \phi \, f \, d\mu$$

in that case, we will denote $g = \partial^\alpha f$.

Remark B.3.2. By the same argument that in the motivation, the weak derivate of a function is unique almost everywhere. Furthermore, for the same reason, the classical derivative of a differentiable function is also its weak derivate (what justifies the abuse of notation).

Example B.3.3. Let's define $f(x) = x$ if $0 < x \leq 1$ and $f(x) = 1$ if $x > 1$. Observe that f is not derivable at $x = 1$, but, however, a very simple computation shows that the function $g = \chi_{(0,1]}$ is its first weak derivate. Therefore, the notion of weak derivative, indeed, improve the usual notion of derivate.

Example B.3.4. Not every function in L^1_{loc} has a weak derivative. For example, let's take the Heviside step function in $[-1, 1]$, given by $H = \chi_{[-1,0]}$. Suppose that $g \in L^1_{loc}([-1, 1])$ would be its weak derivative[4], then, for all $\phi \in C_c^\infty(\mathbb{R}^n)$, we will have

$$\int_{-1}^{1} \phi \, g \, d\mu = -\int_{-1}^{1} \phi' \, H \, d\mu = -\int_{-1}^{0} \phi' \, d\mu = \phi(1)$$

Let's take a sequence $\phi_n \in C_c^\infty$ such that $0 \leq \phi_n \leq 1$, $\phi_n(0) = 1$ for all $n \in \mathbb{N}$ and $\phi_n(x) \to 0$ for all $x \neq 0$. Then, by the dominated converge theorem, we must have

$$1 = \lim_{n\to\infty} \phi_n(0) = \lim_{n\to\infty} \int_{-1}^{1} \phi_n \, g \, d\mu = \int_{-1}^{1} \lim_{n\to\infty} \phi_n \, g \, d\mu = 0$$

B.3.1.2 Sobolev spaces in the euclidean space

With this notion of weak derivate, we can make precise the sentence, *weak derivative in* L^p. The spaces that arise of this construction are known as Sobolev spaces.

Definition B.3.5. Let $U \subset \mathbb{R}^n$ be an open set, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the $(k, p)-$**Sobolev space**, $W^{k,p}(U)$, to be the space functions $f \in L^p(U)$ such that, for all $|\alpha| \leq k$, $\partial^\alpha f$ exists and $\partial^\alpha f \in L^p(U)$. In that space, we define the (k, p)-**Sobolev norm** given by

$$\|f\|_{W^{k,p}(U)} = \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p(U)}$$

[4] As we will see after, we can improve the notion weak derivative, in the setting of distributions, and obtain that, in this general concept, the weak derivate of H is the Dirac-delta operator, $\delta : C_c^\infty(U) \to \mathbb{R}$ given by $\delta(\phi) = \phi(0)$.

Remark B.3.6. Using the elementary properties of powers, its easy to note that $\|\cdot\|_{W^{k,p}(U)}$ is actually a norm, with which $W^{k,p}(U)$ becomes a normed space.

In fact, we have even more (for proofs, see, for example, [22])

Theorem B.3.7. *For $1 \leq p < \infty$, $W^{k,p}(U)$ is a Banach space. Moreover, for $p = 2$, the space $W^{k,2}(U)$, usually denoted $H^k(U)$, is a Hilbert space with inner product*

$$\langle f, g\rangle_{H^k(U)} := \sum_{|\alpha|\leq k} \langle \partial^\alpha f, \partial^\alpha g\rangle_{L^2(U)}$$

Furthermore, $C^\infty(U)$ (without compact support!) is dense in $W^{k,p}(U)$.

Remark B.3.8. It can be seen that, in the previous theorem, the hypotesis that the functions do not need to have compact support is crucial. In fact, the space $C_c^\infty(U)$ is not dense in $W^{k,p}(U)$, so its clausure, with respect to the (k,p)-Sobolev norm, is a proper closed linear subspace, denoted by $W_0^{k,p}(U)$.

The most important theorem, from the theory of Sobolev spaces, that we will need is the following. Again, the proof can be found in [22].

Definition B.3.9. Let $U \subset \mathbb{R}^n$ be bounded open set. Given $1 \leq p < \infty$, we call the **Sobolev conjugate** of p over M to the unique p^* such that

$$\frac{1}{p} = \frac{1}{p^*} + \frac{1}{n}$$

Theorem B.3.10 (Rellich–Kondrachov)**.** *Let $U \subset \mathbb{R}^n$ be an open bounded set of $\mathbb{R}^n$ with C^1 boundary. Let $1 \leq p < \infty$ and let p^* its Sobolev conjugate. Suppose that $1 \leq q < p^*$ if $p < n$ or $1 \leq q < \infty$ if $p \geq n$. Then $W_0^{1,p}(U)$ is compactly embedded in $L^q(U)$ that is*

- *$W_0^{1,p}(U) \hookrightarrow L^q(U)$ and the inclusion is continous.*
- *If $\{\omega_n\}_{n=1}^\infty$ is a bounded sequence in $W^{1,p}(U)$, then there exists a subsequence convergent in $L^q(U)$[5].*

Only for completeness, one of the most important inequalities in the theory of Sobolev spaces is the known as Poincaré inequality. We will not need it here, but we include its statement for completeness. The proof can be found, for example, in [22].

Theorem B.3.11 (Poincaré inequality)**.** *Let $U \subset \mathbb{R}^n$ be a <u>bounded</u> connected open set. Let $1 \leq p \leq \infty$, then, there exists $C > 0$, depending only on p, n and U, such that*

$$\|f\|_{L^p(U)} \leq C\|\nabla f\|_{L^2(U)}$$

[5] In the topological jergue, this can be reparaphrased saying that every bounded sequence in $W^{1,p}(U)$ is precompact in $L^q(U)$

for all $f \in W^{1,p}$ *with* $\int_U f = 0$.

B.3.1.3 Sobolev spaces in manifolds

Now, we will extend this notions to the context of differentiable manifolds.

To this end, we can follow two different approach. On one hand, we can use the spaces created in the previous section and, using the charts of the manifold, pasting them to arrive to a global definitions. On the other hand, analogously to what we have done in the euclidean case, we can define normed spaces globally, reflecting the properties of the functions that we want to uses.

Firstly, we will follow the later approach and define the needed Sobolev spaces in a non-constructive way. For a explicit construction of this spaces, and others, using pasting techniques, see section B.3.1.4.

Definition B.3.12. Let M be a compact oriented riemannian manifold. The space of L^2**-differential forms**, $L^2_\Omega(M)$ is the closure of $\Omega^*(M)$ with respect to the L^2 inner product.

Definition B.3.13. Let M be a compact oriented riemannian manifold. Given $\omega, \eta \in \Omega^*(M)$, let's define the H^1_Ω-inner product

$$\langle \omega, \eta \rangle_{H^1(M)} := \langle \omega, \eta \rangle_{L^2} + \langle d\omega, d\eta \rangle_{L^2} + \langle d^*\omega, d^*\eta \rangle_{L^2}$$

Then, we define the 1-**Sobolev space**, $H^1_\Omega(M)$, as the closure of $\Omega^*(M)$ with respect to the $H^1(M)$-norm.

Using the explicit description of the Sobolev spaces in section B.3.1.4, it can be shown that the classical results about Sobolev spaces extends to the manifolds framework. In particular, we have the following extension of the Rellich-Kondrachov theorem B.3.10 to manifolds.

Corollary B.3.14. *On a compact oriented riemannian manifold,* $H^1(M)$ *is compactly embedded in* $L^2(M)$.

B.3.1.4 Constructive definition of Sobolev spaces on manifolds

First of all, a warning: This section is optional, and a little masochistic. In this section, we will contruct explicitly the Sobolev spaces of differential forms on a compact manifold. For a perfect understanding of this constructions, the reader should be confortable with the notion of vector bundles and, preferably, with the notion of sheaves. For example, see [30].

Through this section, M will be a compact oriented differentiable manifold.

Definition B.3.15. Let $1 \leq p \leq \infty$ and $k \geq 0$, we define space of $W^{k,p}$-functions over M, $W^{k,p}(M)$, to be the set of functions $f : M \to \mathbb{R}$ such that, for every $x \in M$, there exists a chart (U, φ), with $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$, arround x, such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R} \in W^{k,p}(\varphi(U))$.

Moreover, given two differentiable manifolds M, N and a map $F : M \to N$, we say that $F \in W^{k,p}(M, N)$ if, for every $f \in W^{k,p}(N)$, we have that $f \circ F : M \to \mathbb{R} \in W^{k,p}(M)$.

Definition B.3.16. Let $V \subset \mathbb{R}^n$ be an open set. We define the space of $W^{k,p}$-diferentials over V, $W^{k,p}_\Omega(V)$, to be the space

$$W^{k,p}_\Omega(V) = \Omega^*(V) \otimes_{\mathbb{R}} W^{k,p}(V)$$

where $\otimes_{\mathbb{R}}$ denote the tensor product of $\mathbb{R}$-módules. Furthermore, given a differentiable map $F : V \subset \mathbb{R}^n \to V' \subset \mathbb{R}^m$, we define the **pullback** $F^* : W^{k,p}_\Omega(V') \to W^{k,p}_\Omega(V)$ that, in the basic elements $\omega \otimes f \in \Omega^*(V') \otimes_{\mathbb{R}} W^{k,p}(V')$ is given by

$$F^*(\omega \otimes f) = F^*(\omega) \otimes (f \circ F)$$

Let M be a compact differentiable manifold and let (U_i, φ_i), $i = 1, \ldots, m$, be an atlas for M, with $\varphi_i : U_i \to V_i \subset \mathbb{R}^n$

Definition B.3.17 (Pasting form)**.** We define the space of $W^{k,p}$**-diferentials** over M, $L^p_\Omega(M)$, to be the space

$$W^{k,p}_\Omega(M) = \frac{\sqcup_{i=1}^n W^{k,p}_\Omega(V_i)}{\sim} = \frac{\cup_{i=1}^n \{i\} \times W^{k,p}_\Omega(V_i)}{\sim}$$

where $\sim$ is the equivalence relation given by $(i, \omega_i) \sim (j, \omega_j)$ if and only if

$$\omega_i|_{\varphi_i(U_i \cap U_j)} = (\varphi_j \circ \varphi_i^{-1})^*(\omega_j|_{\varphi_j(U_i \cap U_j)})$$

If $\omega \in L^p_\Omega(M)$, we denote by $\omega|_{U_i}$ its i-th part.

Remark B.3.18. Using the fact that the change of charts are C^∞, is possible to show that this definition does not depend on the atlas (U_i, φ_i) chosen.

Remark B.3.19. As in the case of differentiable forms, the space $W^{k,p}_\Omega(M)$ is a $W^{k,p}(M)$-module, with a grading inheritated from the usual grading on $\Omega^*(M) = \bigoplus_{k=1}^{n} \Omega^k(M)$. We will denote its k-part as $W^{k,p}_\Omega(M)^k$.

Remark B.3.20. Playing with the definitions, it is easy to show that $\omega \in L^p_\Omega(M)^k$ if and only if, for every $i = 1, \ldots, m$, the i-th component of ω is given by a form

$$\omega|_{U_i} = \sum_{i_1 < i_2 \ldots < i_k} f_{i_1 \ldots i_k} dx_{i_1} \wedge dx_{i_k}$$

with $f_{i_1 \ldots i_k} \in L^p(V_i)$. Usually, this is the definition that appears in the classical textbooks, but, unfornunatelly, this is not completely satisfactory, cause the universal set $L^p_\Omega(M)$ is not defined. That is the reason why we need to define locally a L^p-form and, then, paste them together.

Remark B.3.21. As in the euclidean case, usually the space $W^{0,p}_\Omega(M)$ is denoted by $L^p_\Omega(M)$ and is known as the space of L^p-differentials. Moreover, the space $W^{k,2}_\Omega(M)$ is usually denoted $H^k_\Omega(M)$.

Remark B.3.22. For those who have studied Algebraic Geometry, remember that a sheaf over a topological space X is a contravariant functor $\mathcal{F} : \mathbf{Op}_X \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of Abelian groups and $\mathbf{Op}_X$ is the category whose objects are the open sets in X, and whose morphism are the inclusion maps. Then, the theory of sheaves shows that there is a natural equivalence between locally free sheaf of modules and vector bundles.

In fact, we can easily define the sheaf of L^p-functions on M, which, for every open set $U \subset M$, asigns the abelian group $L^p(U)$, as defined in B.3.15. Then, using the sheaf of differential forms $\Omega^* : U \mapsto \Omega^*(U)$, then we consider the L^p-module sheaf

$$L^p_\Omega := \Omega^* \otimes_{\mathbb{R}} L^p$$

given by $\Omega^* \otimes_{\mathbb{R}} L^p(U_i) := \Omega^*(U_i) \otimes_{\mathbb{R}} L^p(U_i)$ on the covering U_i that forms the atlas on M, and pasted together. This is a locally free sheaf of L^p-modules, so, it is related with a vector bundle (in the L^p category), whose sections are, in fact, $L^p_\Omega(M)$. Strictely speaking, this is the most formal construction of the space of L^p-forms. For more information and possible constructions, see [30].

Once we have define the set, we can define over it a norm

Definition B.3.23. Let $W^{k,p}_\Omega(M)$ be the space of $W^{k,p}$-differentials over M. We endow it with a metric, given by

$$\|\omega\|_{W^{k,p}_\Omega(M)} = \sum_{i=1}^{n} \|\omega\|_{W^{k,p}_\Omega(V_i)}$$

where $\|\cdot\|_{W^{k,p}_\Omega(V_i)}$ is the norm in $\Omega^*(V_i) \oplus W^{k,p}(V_i)$, that, if $\omega|_{V_i} = \sum\limits_{i_1 < i_2 \ldots < i_k} f_{i_1 \ldots i_k} dx_{i_1} \wedge dx_{i_k}$ then we have

$$\|\omega|_{V_i}\|_{W^{k,p}_\Omega(V_i)} = \sum_{i_1 < i_2 \ldots < i_k} \|f_{i_1 \ldots i_k}\|_{W^{k,p}(V_i)}$$

Using the euclidean properties of $W^{k,p}(V')$, it can be shown

Proposition B.3.24. *For $1 \leq p < \infty$, $W^{k,p}_\Omega(M)$ is a Banach space. In fact, for $p = 2$, $H^k_\Omega(M)$ it is a Hilbert space.*

Proposition B.3.25. *For $1 \leq p < \infty$, $\Omega^*(M)$ is a dense subspace of $W^{k,p}_\Omega(M)$.*

Remark B.3.26. Observe that, since M is compact, the set of forms with compact support is the whole space, so the space ${W_0^{k,p}}_\Omega(M)$ is meaningless.

As we promised, in the special case of $p = 2$, we recover the previous definition

Theorem B.3.27. *Let M be a compact oriented riemannian manifold. $L^2_\Omega(M)$, as defined in this section, is isomorphic to the closure of $\Omega^*(M)$ with respect to the L^2-norm defined in B.1.4. Furthermore, the H^1-norm given in this section, is equivalent to the norm defined in B.3.13.*

Sketch of the proof. It is enought to compute this norm locally, in terms of the Christoffel symbols, and, refining the covering of M, suppose that $|g_{i,j} - \delta_{i,j}| < \epsilon$ and $|\Gamma^i_{j,k}| < \epsilon$ for ϵ small. For details, see [42]. ■

B.3.2 The Weak Version of the Hodge Decomposition

B.3.2.1 Weak solutions

Before the final theorem, we will proof a restricted version of the decomposition in the larger class of the L^2 forms on M. Furthermore, as we will see, most of this study can be done only in terms of Hilbert spaces and dense subsets, so we will follow a slightly more general setting. Nevertheless, I have not found this (easy) generalization in any analysis or PDE's textbook, so, sadly, the terminology of this section is not standard.

Definition B.3.28. Let H be a (separable, real or complex) Hilbert space and $S \subset H$ be a dense subset. A linear operator $L : S \to H$ is called **smooth** if $\mathcal{D}(L^*) \supset S$, i.e., the formal adjoint $L^* : S \to H$ is well defined.

Example B.3.29. If $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is the Schwartz class, then the Fourier Transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is symmetric, so, in particular, it is smooth.

Example B.3.30. As the standard textbooks in PDE's shows, any elliptic operator defined on a bounded open set of $\mathbb{R}^n$ is smooth operator.

In this general setting, we can weaken the notion of solution to the Poisson equation $L\,\omega = \eta$ of a smooth linear operator.

For motivate this concept, let's consider the case of good solutions. Suppose that we have a smooth operator $L : S \to H$ defined over a dense linear subespace S (we can think, for example, that S is the space $C_c^\infty(\mathbb{R}^n)$) of a Hilbert space H (for example $L^2(\mathbb{R}^n)$). Suppose that $\omega, \eta \in S$ satisfies $L\,\omega = \eta$. Then, using the formal adjoint $L^* : S \to H$ we have that, for all $\phi \in S$

$$\langle \phi, \eta \rangle_H = \langle \phi, L\,\omega \rangle_H = \langle L^*\,\phi, \omega \rangle_H$$

However, even if this computation is done in the differentiable class, the right hand and the left hand side of this equality make sense even for non-differentiable one. These are known as weak solutions.

Definition B.3.31. Let H be a Hilbert space, $S \subset H$ dense and $L : S \to H$ a smooth operator. Given $\omega, \eta \in H$ we will say that $L\,\omega = \eta$ in **weak sense** if, for all $\phi \in S$

$$\langle \phi, \eta \rangle_H = \langle L^*\,\phi, \omega \rangle_H$$

In contrast, the previous notion of solution will be called solution in **strong sense**. We will denote by $L(H)$ the space of weak solutions of the Poisson equation, i.e. $\omega \in L(H)$ if there exists $\eta \in H$ such that $L\,\omega = \eta$ in weak sense.

Remark B.3.32. The discussion above shows that, if $\omega, \eta \in S$, then $L\,\omega = \eta$ in weak sense, if and only if, $L\,\omega = \eta$ in strong sense.

Remark B.3.33 (Interpretation in terms of distributions). Given $S \subset H$ dense we will define the space of **distributions** with base S to be the dual space S^*. Observe that H is canonicaly isomorphic to H^* by $\varphi : H \to H^*$ given by $\varphi(\omega)(\phi) = \langle \phi, \omega \rangle_H$. Using this isomorphism, H can be viewed as a linear subspace of the space of distribution S^*.

Suppose that we have a smooth operator $L : S \to H$. Then, we can extend it to $\tilde{L} : H^* \to S^*$. Given $\omega \in H \cong H^*$ we define

$$\tilde{L}(\varphi(\omega))(\phi) := \varphi(\omega)(L^*\phi) = \langle L^*\phi, \omega \rangle_H$$

for all $\phi \in S$. In this sense, given $\omega, \eta \in H$, saying that $L\,\omega = \eta$ in weak sense is the same that $\tilde{L}(\varphi(\omega)) = \varphi(\eta)$ because

$$\langle \phi, \eta \rangle_H = \varphi(\eta)(\phi) = \tilde{L}(\varphi(\omega))(\phi) = \langle L^*\phi, \omega \rangle_H$$

So, a weak solution is no more that a solution in the space of distributions.

Furthermore, we can generalize this even more (however, we will not need this here). Suppose that our smooth operator maps S to itself, that is $L(S) \subset S$. Then, we can extend L to the whole space of distributions, $\hat{L} : S^* \to S^*$ by

$$\hat{L}(l)(\phi) := l(L^*\phi)$$

for $l \in S^*$ and $\phi \in S$.

Example B.3.34. The dual space of $C_c^\infty(\mathbb{R}^n)$ (which, in this context, is denoted by $\mathcal{D}$) with respect to the L^2 norm, $\mathcal{D}' := (C_c^\infty(\mathbb{R}^n))^*$ is known as the space of distributions over $\mathbb{R}^n$. A easy example of a distribution that is not an usual L^2 function is the Dirac delta distribution $\delta : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ given by $\delta(f) = f(0)$.

In this space, we can extend the derivative operator $\partial_\alpha : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ to the space $\mathcal{D}'$ by

$$\partial_\alpha(l)(\phi) = (-1)^{|\alpha|} l(\partial_\alpha(\phi))$$

for $l \in \mathcal{D}'$ and $\phi \in C_c^\infty(\mathbb{R}^n)$. In this setting, we can reinterpret the notion of weak derivate as follows. Given $f \in L^2(\mathbb{R}^n)$, we say that $\partial_\alpha(\varphi(f))$ is the derivative of f in a distributional sense. If the distributional derivative lies in $L^2(\mathbb{R}^n)^* \subset \mathcal{D}'$, say $\varphi(g) = \partial_\alpha(\varphi(f))$, then we say that g is the weak derivative of f. In fact, using this definition we have that, for all $\phi \in C_c^\infty(\mathbb{R}^n$.

$$\int_{\mathbb{R}^n} \phi\, g\, d\mu = \varphi(g)(\phi) = \partial_\alpha(\varphi(f))(\phi) = (-1)^{|\alpha|} \varphi(f)(\partial_\alpha(\phi)) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial_\alpha \phi\, f\, d\mu$$

That is, precisely, the definition of weak derivative. Hence, both definitions coincide.

Example B.3.35. Let $\mathcal{S}$ be the Schwartz class on $\mathbb{R}^n$. It is easy to check that $\mathcal{S}$ is a linear subspace that contains $C_c^\infty(\mathbb{R}^n)$, so it is dense in $L^2(\mathbb{R}^n)$. Let $\mathcal{S}^*$ be its dual, this space is known as the **space of temperated distributions**. This distributions have a very important property, the Fourier Transform can be defined there.

Indeed, the most important porperty of the Schwartz class is that the Fourier Transform maps it to itself $\mathcal{F} : \mathcal{S} \to \mathcal{S}$, so, by the previous remark, we can extend $\hat{\mathcal{F}} : S^* \to S^*$ by

$$\hat{\mathcal{F}}(l)(\phi) = l(\mathcal{F}(\phi))$$

for every $l \in S^*$ and $\phi \in \mathcal{S}$.

B.3.2.2 Generalized elliptic operators

The key part of the theory of elliptic operators is that they can be solved in weak sense in a purely functional analitic way.

Definition B.3.36. Let H be a (separable, real or complex) Hilbert space and $S \subset H$ be a dense subset. A smooth linear operator $L : S \to H$ is called **generalized elliptic operator** if there exists a closed linear subspace $C_L \subset H$ (the **control region**) and $C > 0$ such that $\|\phi\| \leq C\|L^* \phi\|$ for all $\phi \in C_L$. Furthermore, if $Ker\, L$ is finite dimensional, we will say that L is **finite**.

Theorem B.3.37 (à la Malgrange-Ehrenpreis)**.** *Let H be a Hilbert space over the field $\mathbb{K}$, $S \subset H$ be a dense subset and $L : S \to H$ a generalized elliptic operator with control region C_L, such that $(Ker\, L^*)^\perp \subset C_L$. Then, there exists a bounded linear operator*

$$K : C_L \to H$$

such that, for all $\eta \in H$, $K(\eta)$ is the weak solution of the Poisson problem with data η, that is

$$L\, K(\eta) = \eta$$

in weak sense.

Proof. Let's define, over $S \cap C_L$, the inner product

$$\langle \phi, \psi \rangle_L := \langle L^* \phi, L^* \psi \rangle_H$$

for all $\phi, \psi \in S \cap C_L$, and let $S_L = (S \cap C_L, \langle \cdot, \cdot \rangle_L)$ be this prehilbert space. Let W be the clausure of S_L. Cause, trivially, $L^* : S_L \to H$ is a bounded operator on S_L, it has an extension to W, that we will keep calling it L.

Let's fix $\eta \in C_L$, and define the functional $l_\eta : S_L \to \mathbb{K}$ by

$$l_\eta(\phi) = \langle \phi, \eta \rangle_H$$

Let's check that l_η is bounded. Let $\phi \in S_L$, cause the estimate $\|\phi\|_H \leq C\|L^* \phi\|_H$ holds is this subspace, we have

$$|l_\eta(\phi)| = |\langle \phi, \eta \rangle_H| \leq \|\phi\|_H \|\eta\|_H \leq C\|L^* \phi\|_H \|\eta\|_H = C\|\eta\|_H \|\phi\|_L$$

Hence l_η is bounded, so it can be extended to a bounded linear operator $l_\eta : W \to \mathbb{K}$ with $\|l_\eta\| \leq C\|\eta\|_H$. By the Riesz Lemma, there exists $\omega_\eta \in W$ such that

$$l_\eta(\alpha) = \langle \alpha, \omega_\eta \rangle_L = \langle L^*\alpha, L^*\omega_\eta \rangle_H$$

for all $\alpha \in W$. Moreover, $\|\omega_\eta\|_L = \|l_\eta\| \leq C\|\eta\|_H$

Let's define $K(\eta) = \pi(L^*\omega_\eta)$, where $\pi : H \to \overline{L^*(S)}$ is the orthogonal projection to the clausure of the image of L in the H-norm. In this case, by construction of $K(\eta)$ we have

$$\langle \phi, \eta \rangle_H = l_\eta(\phi) = \langle L^*\phi, K(\eta) \rangle_H \tag{B.1}$$

for all $\phi \in S_L = S \cap C_L$.

Furthermore, this formula also holds for $\phi \in S \cap C_L^\perp$. By hypothesis, $(Ker\, L^*)^\perp \subset C_L$, so $C_L^\perp \subset Ker\, L^*$ (recall that $Ker\, L^*$ is closed, remark B.2.10). Hence, if $\phi \in S \cap C_L^\perp$, then $\phi \in Ker\, L^*$ and, therefore, the right hand side is null. But the left hand size is also null due to the fact that $\phi \in C_L^\perp$ and $\eta \in C_L$, so the identity ($B.1$) holds.

Hence, the equality ($B.1$) holds for elements in $S \cap C_L$ and $S \cap C_L^\perp$ and, thus, for the union of both spaces, that is, S. Therefore, ($B.1$) holds for all $\phi \in S$ or, in the languaje of weak solutions, $K(\eta)$ is a weak solution of $L\,\omega = \eta$.

Furthermore, K is linear. Trivially, $K(\lambda\eta) = \lambda K(\eta)$ for all $\lambda \in \mathbb{K}$. For the distibution with the sum, observe that, for all $\phi \in S$, we have

$$\begin{aligned}\langle L^*\phi, K(\omega + \eta) \rangle_H &= l_{\omega+\eta}(\phi) = \langle \phi, \omega \rangle_H + \langle \phi, \eta \rangle_H = l_\omega(\phi) + l_\eta(\phi) \\ &= \langle L^*\phi, K(\omega) \rangle_H + \langle L^*\phi, K(\omega) \rangle_H = \langle L^*\phi, K(\omega) + K(\eta) \rangle_H\end{aligned}$$

Therefore, $K(\omega) + K(\eta) - K(\omega + \eta) \in (L^*(S))^\perp$. However, by construction, $K(\omega) + K(\eta) - K(\omega + \eta) \in \overline{L^*(S)}$, so it must be $K(\omega) + K(\eta) - K(\omega + \eta) = 0$.

Finally, for the boundedness of K, observe that, by the application of the Riesz lemma that we did above, we have

$$\|K(\eta)\|_H = \|L^*\omega_\eta\|_H = \|\omega_\eta\|_L \leq C\|\eta\|_H$$

so K is bounded, as we wanted to show. ■

Remark B.3.38. The conclusion of this theorem is usually said that L has a bounded right inverse on C_L.

Remark B.3.39. Remember that any elliptic operator defined on a bounded open set, $U \subset \mathbb{R}^n$ has control region equal to $C_c^\infty(U)$. But this class is dense in $L^2(U)$, so we obtain the classical version of this theorem with $K : L^2(U) \to L^2(U)$.

Corollary B.3.40 (Hodge Decompostion, weak sense)**.** *Let H be a Hilbert space, $S \subset H$ be a dense subset and $L : S \to H$ a symmetric finite generalized elliptic operator with $Ker\, L^\perp \subset C_L$, where C_L is the control region of L. Then, we have the orthogonal decomposition*

$$H = L(H) \oplus Ker\, L$$

Proof. Since $Ker\, L$ is finite dimensional, it is a closed linear space so, by the orthogonal projection theorem, we have

$$H = (Ker\, L)^\perp \oplus Ker\, L$$

Therefore, it remains to prove that $(Ker\, L)^\perp = L(H)$. For $(Ker\, L)^\perp \supset L(H)$, suppose that $\eta \in L(H)$, say $L\,\omega = \eta$ in weak sense for some $\omega \in H$, then for all $\phi \in Ker\, L$ we obtain

$$\langle \phi, \eta\rangle_H = \langle L\,\phi, \omega\rangle_H = 0$$

so η is orthogonal to $Ker\, L$, as we want.

The other way is a consequence of the Malgrange-Ehrenpreis theorem. Note that, in general $Ker\, L \subset Ker\, L^*$, so the hypothesis of the Malgrange-Ehrenpreis theorem hold. Therefore, the right-inverse operator $K : Ker\, L^\perp \to H$ gives, for every $\eta \in Ker\, L^\perp$ a weak solution of $L\,\omega = \eta$, as we want. ■

B.3.2.3 Return to the Laplace-Beltrami operator

Using the Rellich-Kondrachov theorem, we can prove that the Laplace-Beltrami operator is a finite generalized elliptic operator.

Proposition B.3.41. *Let (M, g) be a compact oriented riemannian manifold and let $\mathcal{H}^*(M)$ be its space of harmonic differential forms. There exists a constant $C > 0$ such that, for every $\omega \in \mathcal{H}^*(M)^\perp$, we have*

$$\|\omega\|_{L^2} \leq C\|\Delta\,\omega\|_{L^2}$$

Proof. Otherwise, there would exists a sequence $\{\omega_n\}_{n=1}^{\infty}$ with $\|\omega_n\|_{L^2} = 1$ and satisfiying $\|\Delta\,\omega_n\|_{L^2} \to 0$. In particular, we have

$$\|d\omega_n\|_{L^2} + \|d^*\omega_n\|_{L^2} = \langle \Delta\,\omega, \omega\rangle_{L^2} \leq \|\Delta\,\omega_n\|_{L^2}\,\|\omega_n\|_{L^2} = \|\Delta\,\omega_n\| \to 0$$

so $\|d\omega_n\|, \|d^*\omega_n\| \to 0$. Hence, their H^1-norm $\|\omega_n\|_{H^1} = \|\omega_n\|_{L^2} + \|d\omega_n\|_{L^2} + \|d^*\omega_n\|_{L^2}$ is bounded.

Therefore, by the Rellich-Kondrachov theorem, there exists a subsequence convergent in L^2, so, replacing the sequence by its subsequence, we can suppose without loss of generality that $\{\omega_n\}_{n=1}^{\infty}$ converges in the L^2-norm.

Let ω be its limit in the L^2 norm. Observe that, cause $\Delta\,\omega \to 0$, for all $\phi \in \Omega^*(M)$ we have

$$0 = \lim_{n\to\infty} \langle \Delta\,\omega_n, \phi\rangle_{L^2} = \lim_{n\to\infty} \langle \omega_n, \Delta\,\phi\rangle_{L^2} = \lim_{n\to\infty} \langle \omega, \Delta\,\phi\rangle_{L^2}$$

so $\Delta\,\omega = 0$ in weak sense. But, by the Weyl's lemma B.3.51 below, a weak harmonic function is, in fact, a strong harmonic function, so $\omega \in \mathcal{H}^*(M)$. Hence, cause $\omega_n \in \mathcal{H}^*(M)^{\perp}$, it must be $\langle \omega, \omega_n\rangle_{L^2} = 0$, so, taking limits, $\|\omega\|_{L^2} = 0$. Therefore, $\lim \omega_n = \omega = 0$, which is impossible since $\|\omega_n\| = 1$. ■

Proposition B.3.42. *Let (M, g) be a compact oriented riemannian manifold of dimension n. Then, the space of harmonic forms, $\mathcal{H}^*(M)$, is finite dimensional.*

Proof. Suppose that $\mathcal{H}^*(M)$ is infinite dimensional. Hence, it should exist an infinite orthonormal set $\{\omega_n\}_{n=1}^{\infty} \subset \mathcal{H}^*(M)$ such that $\langle \omega_i, \omega_j\rangle_{L^2} = \delta_{ij}$. In particular, cause the ω_n are harmonic, they have $d\omega_n = d^*\omega_n = 0$ for all $n \in \mathbb{N}$. Therefore, we have

$$\|\omega_n\|_{H^1} = \|\omega_n\|_{L^2} + \|d\omega_n\|_{L^2} + \|d^*\omega_n\|_{L^2} = \|\omega_n\|_{L^2} \leq 1$$

Hence, $\{\omega_n\}_{n=1}^{\infty}$ is a bounded sequence in the Sobolev space H^1, so, by the Rellich-Kondrachov theorem, there exists a convergent subsequence in the L^2-norm. But, this is impossible, cause this subsequence remains being an orthonormal set in the L^2-metric (and, therefore, they are not a Cauchy sequence). ■

Corollary B.3.43. *In a compact oriented riemannian manifold, the Hodge-Laplace operator is a symmetric finite generalized elliptic operator with control region $C_\Delta = \mathcal{H}^*(M)^{\perp}$.*

Therefore, we have just prove

Corollary B.3.44. *Let (M, g) be a compact oriented riemannian manifold of dimension n. Then, for each $0 \leq k \leq n$, we have the orthogonal decomposition*

$$L^2_\Omega(M)^k = \Delta\, L^2_\Omega(M)^k \oplus \mathcal{H}^k(M)$$

B.3.3 Regularity of the Weak Solutions

As we have see, we can solve the Laplace-Beltrami equation $\Delta\,\omega = \eta$ in a weak sense in a purely algebraic way, only using functional analysis. However, if the original data η is smooth, it is expected that a weak solution ω is also smooth.

This is the realm of the regularity theory of elliptic operators, a very difficult task that requires a lot of work. We will discuss completely the case of weak harmonic forms, and we will sketch the proof for the general case.

B.3.3.1 Weak harmonic forms

The fundamental tool in this work is the lemma known as the Weyl's lemma that treats the case of an harmonic function over $\mathbb{R}^n$. To reach this result, we need a very important property of the harmonic functions, known as the mean value property.

Proposition B.3.45 (Mean value property)**.** *Suppose that $u \in C^2(\mathbb{R}^n)$ is harmonic, then it satisfies the **mean value property** for spheres, that is, for every $x \in \mathbb{R}^n$ and every $R > 0$ we have*

$$u(x) = \frac{1}{|\partial B(x,R)|} \int_{\partial B(x,R)} u(y)\, dS(y)$$

Proof. Let's fix $x_0 \in \mathbb{R}^n$ and define $\Phi : (0,\infty) \to \mathbb{R}$ by

$$\Phi(r) = \frac{1}{|\partial B(x_0,r)|} \int\limits_{\partial B(x_0,r)} u(y)\, dS(y) = C \int\limits_{\partial B(0,1)} u(x_0 + rz)\, dS(z)$$

Cause $u \in C^2$, Φ is differentiable and, by the dominated convergence theorem we have

$$\Phi'(r) = C \int\limits_{\partial B(0,1)} \langle \nabla\, u(x_0 + rz), z\rangle\, dS(z) = C \int\limits_{\partial B(x_0,r)} \left\langle \nabla\, u(y), \frac{y - x_0}{r} \right\rangle dS(y)$$

But $\frac{y-x_0}{r}$ is the unit outward normal vector to the sphere $B(x_0,r)$ in $y \in B(x_0,r)$, so

$$\Phi'(r) = C \int\limits_{\partial B(x_0,r)} \frac{\partial u}{\partial \nu}(y)\, dS(y) = C' \int\limits_{B(x_0,r)} \Delta\, u(y)\, dy = 0 \tag{B.2}$$

where the second equality follows from the Stokes' theorem. Therefore, Φ is constant, so

$$\Phi(R) = \lim_{r\to 0} \Phi(r) = \lim_{r\to 0} \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y) = u(x_0)$$

as we wanted to show. ∎

Remark B.3.46. Observe that, $u \in C^0(\mathbb{R}^n)$ satisfies the mean value property for spheres if and only if (using the co-area formula) it satisfies the mean value property for balls, that is

$$u(x) = \frac{1}{|B(x,R)|} \int_{B(x,R)} u(y)\, dy$$

for all $x \in \mathbb{R}^n$ and $R > 0$.

For the converse property, we need the following lemma.

Proposition B.3.47. *Let $u \in C^0(\mathbb{R}^n)$ be a continous function that satisfies the mean value property. Then, for all radial functions $\phi \in C_c^0(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi = 1$, we have $u \star \phi = u$.*

Proof. The proof is only a clever computation using the co-area formula. Let $B(0,R)$ be the support of ϕ, then, for all $x_0 \in \mathbb{R}^n$, we have

$$\begin{aligned} u \star \phi\,(x_0) &= \int_{\mathbb{R}^n} u(x_0 - y)\phi(y)\, dy = \int_{B(0,R)} u(x_0 - y)\phi(y)\, dy \\ &= \int_0^R \int\limits_{\partial B(0,R)} u(x_0 - z)\phi(r)\, dS(z)\, dr = \int_0^R \phi(r) \left(\int\limits_{\partial B(x_0,R)} u(z)\, dS(z) \right) dr \end{aligned}$$

Now, because u satisfies the mean value property we have

$$\begin{aligned} u \star \phi\,(x_0) &= \int_0^R \phi(r) \left(\int\limits_{\partial B(x_0,R)} u(z)\, dS(z) \right) dr = u(x_0)\, |\partial B(x_0,R)| \int_0^R \phi(r)\, dr \\ &= u(x_0) \int_0^R \int\limits_{\partial B(x_0,R)} \phi(r)\, dS\, dr = u(x_0) \int_{\mathbb{R}^n} \phi\, d\mu = u(x_0) \end{aligned}$$

■

Corollary B.3.48 (Smoothness of harmonic functions)**.** *If $u \in C^0(\mathbb{R}^n)$ satisfies the mean value property, then $u \in C^\infty(\mathbb{R}^n)$ and u is harmonic in strong sense.*

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be the standart radial mollifier. Observe that, by the properties of the convolution, $u \star \eta$ is C^∞. But, by the previous lemma, we have $u \star \eta = u$, so u is also infinitely differentiable.

Therefore, we can assume that $u \in C^\infty(\mathbb{R}^n)$. In that case, suppose that u is not harmonic. Then, there must exists $x_0 \in \mathbb{R}^n$ and $R > 0$ such that $|\Delta\, u(x)| > 0$ for all $x \in B(x_0,R)$. Without loss of generality, we can assume $\Delta\, u > 0$ in $B(x_0,R)$.

Then, using the function Φ defined above, we have that $\Phi' = 0$, since u satisfies the mean value property. However, using the formula B.2, we have

$$\Phi'(R) = C' \int_{B(x_0,R)} \Delta\, u(y)\, dy > 0$$

which is a contradiction. ■

Corollary B.3.49. *If $u \in C^2(\mathbb{R}^n)$ is harmonic, then $u \in C^\infty(\mathbb{R}^n)$.*

Lemma B.3.50 (Weyl)**.** *Let $u \in L^1_{loc}(\mathbb{R}^n)$ be a weak harmonic function, that is, suppose that*

$$\langle u, \Delta\phi\rangle_{L^2} = \int_{\mathbb{R}^n} u\, \Delta\phi\, d\mu = 0$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$. Then there exists a C^∞ harmonic function $\tilde{u}$ such that $u = \tilde{u}$ almost everywhere.

Proof. Given $r > 0$, let η_r be the radial standard mollifier with $\operatorname{supp} \eta_r = B(0, r)$. Observe that $u \star \eta_r$ is weakly harmonic. Indeed, for every $\phi \in C_c^\infty(\mathbb{R}^n)$, using Fubini's theorem, we have

$$\begin{aligned}\langle \Delta\, \phi, u \star \eta_r\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y - rx)\eta(x)\, \Delta\phi(y)\, dxdy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u(y - rx)\, \Delta\phi(y)\, dy \right) \eta(x)\, dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u(z)\, \Delta\phi(z + rx)\, dy \right) \eta(x)\, dx = \int_{\mathbb{R}^n} \langle \Delta(\phi \circ \tau_{rx}), u\, \rangle \eta(x)\, dx = 0\end{aligned}$$

where $\tau_t(s) = s + t$ and the last equality follows from the fact that u is weakly harmonic. Therefore, $u \star \eta_r$ is a smooth weak harmonic function, so it is also a strong harmonic function. In particular, it satisfies the mean value property.

Observe that, in this case, we have that, for every $r, s > 0$

$$u \star \eta_r = u \star \eta_s$$

To this end, observe that, cause $u \star \eta_r$ is continous and satisfies the mean value property, then, by lemma B.3.47 $(u \star \eta_r) \star \eta_s = u \star \eta_r$. However, convolution is commutative, so we have $(u \star \eta_r) \star \eta_s = (u \star \eta_s) \star \eta_r = u \star \eta_s$, by the same reason, and the equality follows.

Let's fix $s > 0$ to any value. By the properties of the approximation to the identity,

$$u \star \eta_s = u \star \eta_r \overset{r \to 0}{\to} u$$

almost everywhere, when $r \to 0$. Therefore, $u \star \eta_s = u$ almost everywhere, so u can be modified in a null set for getting u continous. Moreover, $u \in C^0(\mathbb{R}^n)$ satisfies the mean value property, so by B.3.48, $u \in C^\infty(\mathbb{R}^n)$, u is strong harmonic. ■

Furthermore, computing locally, we can extend this result to differentiable manifolds.

Theorem B.3.51 (Weyl for manifolds)**.** *Let (M,g) be a compact oriented riemannian manifold. Suppose that $\omega \in L^2_\Omega(M)$ is weak harmonic, i.e. $\langle \omega, \Delta\, \phi \rangle_{L^2} = 0$ for all $\phi \in \Omega^*(M)$. Then $\omega \in \Omega^*(M)$ and $\Delta\, \omega = 0$ in strong sense.*

Proof. Let $x \in M$ and let $\phi : U \to \mathbb{R}^n$ be a local chart arround $x \in U$. Suppose that, locally in this chart, ω is written as $\omega = \sum_{i_1 < \ldots < i_j} a_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Let fix a multiindex $i_1 < \cdots < i_k$ and let $j_1 < \cdots < j_{n-k}$ be its complementary index, i.e., the unique multiindex with such that $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\}$. It can be shown that there exists a dense subset $D \subset L^2(\mathbb{R}^n)$ such that, for every $f \in D$ there exists $\phi_f \in \Omega^*(M)$ with support in U such that, locally

$$\star \Delta\, \phi = (\Delta f)\, dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}}$$

Therefore, cause ω is a weak harmonic function, it must satisfy

$$\begin{aligned}
0 = \langle \omega, \Delta\, \phi_f \rangle_{L^2} &= \int_M \omega \wedge \star \phi_f = \int_U \omega \wedge \star \phi_f \\
&= \int_{\mathbb{R}^n} (a_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge \big((\Delta f)\, dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}} \big) \\
&= \pm \int_{\mathbb{R}^n} a_{i_1 \cdots i_k} \Delta\, f\, dx_1 \wedge \cdots dx_n = \pm \int_{\mathbb{R}^n} a_{i_1 \cdots i_k} \Delta\, f\, d\mu = \pm \langle a_{i_1 \cdots i_k}, \Delta\, f \rangle_{L^2(\mathbb{R}^n)}
\end{aligned}$$

Hence, every coefficient $a_{i_1 \cdots i_k}$ is weak harmonic in the usual sense of $\mathbb{R}^n$ in a neighbourhood of x. But, by the Weyl's lemma for $\mathbb{R}^n$ above, then $a_{i_1 \cdots i_k} \in C^\infty(\mathbb{R}^n)$ and, therefore, cause the differentiability is only a local question, ω is differentiable. But every smooth weak harmonic form is also a harmonic form in strong sense (by Stokes' theorem) so ω is harmonic in strong sense, as we wanted to prove. ∎

Remark B.3.52. Observe that, if Δ would be self-adjoint, then the proof of the Weyl's lemma will be markedly easier. Indeed, in that case, using the remark B.2.10 with $\Delta = \Delta^*$, we would have $Ker\, \Delta = \operatorname{Im} \Delta^\perp$, which, decoding the language, would mean that every weak harmonic form is, in fact, harmonic.

B.3.3.2 General weak solutions

As we saw in the previous section, we can obtain the regularity of a weak harmonic form using the mean value property, which characterize the harmonic functions. However, for the general Poisson equation, $\Delta\, \omega = \eta$, this tool is no longer available.

Therefore, for obtaining the regularity of solutions of the Poisson equation, we need to invoke deeper results from functional analysis. Indeed, the fundamental theorem that we need is the following, whose proof, and all of this sections, can be found in [22].

Theorem B.3.53 (Sobolev's lemma)**.** *Let $U \subset \mathbb{R}^n$ be a bounded domain and let $k, s \in \mathbb{N}$ such that $s > n/2 + k$. Then, $H_0^s(U)$ is embedded into $C^k(\overline{\Omega})$.*

Furthermore, we need to improve some estimations to show that a solution of the Poisson equation, with a regular data (in some sense), is more regular that the data. This can be realized replacing derivatives by diference quotiens, as in the classical theory of elliptic operators.

Theorem B.3.54 (Weak regularity of Poisson equation)**.** *Let $U \subset \mathbb{R}^n$ be a bounded domain and let $u \in L^2(U)$ be a solution of $\Delta u = f$ in weak sense. Then, for every $s > 0$, if $f \in H^s(U)$ then $u \in H^{s+2}(V)$ for every V compactly contained in U.*

Corollary B.3.55 (C^∞ regularity of Poisson equation)**.** *Let (M, g) be a compact oriented riemannian manifold. Suppose that $\eta \in \Omega^*(M)$ and let $\omega \in L^2_\Omega(M)$ such that $\Delta \omega = \eta$ in weak sense. Then $\omega \in \Omega^*(M)$ and $\Delta \omega = \eta$ in strong sense.*

Proof. As in the proof of B.3.51, it is enought to prove this for the euclidean case an functions for the Poisson equation $\Delta u = f$. By the weak regularity, $u \in H^s(U)$ for all $s > 0$ and U small enought, so, by the Sobolev's lemma, $u \in C^k(U)$ for all $k > 0$ and, therefore, $u \in C^\infty(U)$. ■

Corollary B.3.56 (Hodge decompostion, strong sense)**.** *Let (M, g) be a compact oriented riemannian manifold of dimension n, with Laplace-Beltrami operator $\Delta : \Omega^*(M) \to \Omega^*(M)$. Then, for each $0 \leq k \leq n$, $\mathcal{H}^k(M)$ is finite dimensional and we have the split*

$$\Omega^k(M) = \Delta\, \Omega^k(M) \oplus \mathcal{H}^k(M)$$

Furthermore, this decomposition is orthogonal with respect to the L^2 norm.

Proof. The orthogonality is obvious, once proven the decomposition. By the weak Hodge decomposition B.3.44, we have

$$L^2_\Omega(M)^k = \Delta\, L^2_\Omega(M)^k \oplus \mathcal{H}^k(M)$$

Let $\eta \in \Omega^*(M)$. By the weak decomposition, we have $\eta = \eta_1 + \eta_2$, where $\eta_2 \in \mathcal{H}^k(M)$ and $\eta_1 \in \Delta\, L^2_\Omega(M)^k$, say $\Delta \omega = \eta_1$ in weak sense, for some $\omega \in L^2_\Omega(M)$.

Cause the elements of $\mathcal{H}^k(M)$ are smooth, we have that $\eta_1 = \eta - \eta_2 \in \Omega^*(M)$. Therefore, by corolary B.3.55, $\omega \in \Omega^*(M)$ and $\eta_1 = \Delta \omega$ in strong sense. Hence, we have that $\eta = \eta_1 + \eta_2 = \Delta \omega + \eta_2 \in \Delta\, \Omega^*(M) \oplus \mathcal{H}^*(M)$, as we wanted to show. ■

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