SYMPLECTIC GEOMETRY AND MOMENT MAPS WITH A VIEW TOWARDS MORSE-BOTT THEORY

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Contents

1. Symplectic Manifolds

Definition 1.1 (Symplectic Manifold)**.** Given a differentiable manifold M and $\omega \in \Omega^2(M)$, we say that (M, ω) is a **symplectic manifold** if ω is closed and non-degenerated, in the sense that, for all $p \in M$, $\varphi_\omega : T_pM \to T_p^*M$ given by $\varphi_\omega(X)(Y) = \omega(X, Y)$ is an isomorphism. In this case, ω is known as the **symplectic form**.

Remark 1.2. Equivalently $\omega \in \Omega^2(M)$ is non-degenerated if and only if, for all $p \in M$ and any base of T_pM, its matrix, as a bilinear form, is invertible. Using it, and that ω

is anti-symmetric, it follows that T_pM should be even dimensional. Thus, M is an even dimensional manifold.

Theorem 1.3 (Darboux)**.** *Let (M^{2n}, ω) be a symplectic manifold and let $p \in M$. There exists a chart arround p (U, φ), $\varphi = (q_1, \dots, q_n, p_1, \dots, p_n)$, such that the local expresion of ω in that chart, $\omega|_U$[1], is given by*

$$\omega|_U = \sum_{i=1}^{n} dq_i \wedge dp_i =: dq \wedge dp$$

This coordinates are called ***canonical coordinates*** *arround p.*

Definition 1.4. Given to symplectic manifolds (M, ω) and (M', ω'), a differentiable map $f : M \to M'$ is said to be a **symplectomorphism** if $f^*(\omega') = \omega$.

Example 1.5 (Euclidean space)**.** Let $\mathbb{R}^{2n}$ be the usual even-dimensional euclidean space. Then $\mathbb{R}^{2n}$ is a symplectic manifold with the 2-form, in the global chart $(q_1, \dots, q_n, p_1, \dots, p_n)$

$$\omega_0 = \sum_{i=1}^{n} dq_i \wedge dp_i$$

ω_0 is known as the **standard symplectic form** on $\mathbb{R}^{2n}$.

Remark 1.6. In that sense, the Darboux theorem can be reparaphased to say that every symplectic manifold is, locally, symplecto-isomorphic to $(\mathbb{R}^{2n}, \omega_0)$. That is why, usually, this area is called symplectic topology, more that symplectic geometry, because geometricaly (i.e., locally) all the symplectic manifold are equals and all the problems are global.

Example 1.7 (Cotangent Bundle)**.** This classical example of symplectic manifold, and the reason for living of this notion, due to its relationship with classical mechanics. Let Q (known in classical mechanics as the *states space*) be any differentiable manifold, not necessarily even-dimensional, and let us consider its cotangent bundle T^*Q (known as the *phase space*). $M = T^*N$ can be equipped with a 2-form to become a symplectic manifold.

[1]Rigorously, $\omega|_U := (\varphi^{-1})^*\omega$.

To this end, let us define, first, the 1-form $\eta \in \Omega^1(M)$, known as the *Liouville form* or the *canonical form*. Given $(q, \nu_q) \in M = T^*Q$, let us consider a vector $X \in T_{q,\nu_q}M$. Then, we define $\eta_{(q,\nu_q)}(X)$ as the result of applying ν_q to X, once taken to T_qQ. Explicitly, if $\pi : M = T^*Q \to Q$ is the bundle projection, then we define

$$\eta_{(q,\nu_q)}(X) = \nu_q((\pi_*)_{(q,\nu_q)}X)$$

Recall $(\pi_*)_{(q,\eta_q)} : T_{(q,\eta_q)}M \to T_qQ$. From this construction, we can consider the 2-form $\omega = -d\eta \in \Omega^2(M)$, which is trivially closed.

To check that ω is non-degenerate, let us write it in coordinates. If we take coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ in $M = T^*Q$, where the q_k are the coordinates in Q (the positions) and the p_k are the coordinates in T_q^*Q (the moments). Then, we have

$$\eta_{(q,p)}\left(\frac{\partial}{\partial q_k}\right) = p\left(\frac{\partial}{\partial q_k}\right) = p_i \qquad \eta_{(q,p)}\left(\frac{\partial}{\partial p_k}\right) = p\,(0) = 0$$

and, therefore

$$\eta_{(q,p)} = \sum_{k=1}^{n} p_k\, dq_k \qquad \omega = -d\eta = \sum dq_k \wedge dp_k$$

Hence, ω is non-degenerate, turning $M = (T^*Q, \omega)$ in a symplectic manifold

Example 1.8 (Kähler Manifolds)**.** Remember that a Kähler manifold is a riemannian almost-complex and symplectic manifold (M, g, J, ω) such that J integrable and it is a linear symplectomorphism (i.e. $J^*\omega = \omega$) and

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot)$$

Therefore, a Kähler manifold is, by definition, a symplectic manifold (with very much rigidity, in fact). In this context, usually the symplectic form is known as the **Kähler form**.

1.1. **Hamiltonians.**

Definition 1.9 (Contraction)**.** Let M be a differentiable manifold and let us take $X \in H^0(M, TM)$ a vector field. We define the **contraction operator** with respect to X, $\iota_X : \Omega^*(M) \to \Omega^{*-1}(M)$ such that

$$(\iota_X \eta)\,(Y_1, \dots, Y_{k-1}) = \eta(X, Y_1, \dots, Y_{k-1})$$

for every $\eta \in \Omega^k(M)$.

Now, let us consider a symplectic manifold (M, ω) and let $H : M \to \mathbb{R}$ be any differentiable function. Let us consider the 1-form $dH \in \Omega^1(M)$. Cause ω is non-degenerated, $\varphi_\omega : H^0(M, TM) \to H^0(M, T^*M)$ is an isomorphism, so there exists an unique $X_H \in H^0(TM)$ such that $\varphi_\omega(X_H) = dH$. Hence, using the contraction operator we have that it holds

$$\iota_{X_H}\omega = dH$$

In this context, H is called a **hamiltonian** and X_H is the **hamiltonian vector field** asociated to $H : M \to \mathbb{R}$.

Let us write down the explicit equations for X_H. Let us take canonical coordinates (q, p) on M and suppose that is given by $X_H = \sum_k X_q^k \frac{\partial}{\partial q_k} + X_p^k \frac{\partial}{\partial p_k}$. Let $Y = \sum_k Y_q^k \frac{\partial}{\partial q_k} + Y_p^k \frac{\partial}{\partial p_k}$ be any other vector field, therefore, we have

$$\iota_{X_H}\omega\,(Y) = \omega(X_H, Y) = \sum_{k=1}^{n} X_q^k Y_p^k \omega\left(\frac{\partial}{\partial q_k}, \frac{\partial}{\partial p_k}\right) + X_p^k Y_q^k \omega\left(\frac{\partial}{\partial p_k}, \frac{\partial}{\partial q_k}\right) = X_q^k Y_p^k - X_p^k Y_q^k$$

so we have that

$$\iota_{X_H}\omega = -X_p^k dq_k + X_q^k dp_k$$

Hence, using that $dH = \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k$ we obtain

$$X_H = \sum_{k=1}^{n} \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q_k} - \frac{\partial H}{\partial q_k}\frac{\partial}{\partial p_k}$$

Moreover, if $\alpha : (-\epsilon, \epsilon) \to M$, $\alpha(t) = (q(t), p(t))$, is an integral curve of X_H, that is, $\alpha'(t) = (X_H)_{\alpha(t)}$ for all $-\epsilon < t < \epsilon$, then it should satisfy[2]

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

for $k = 1, \ldots, n$, which are the well known **Hamilton equations** of classical mechanics.

[2]Only for a moment, we use the Newton's notation for derivatives, with a dot, for recovering the classical form of this equations.

1.2. **The Poisson Bracket.**

Definition 1.10. A **Lie algebra**, $\mathfrak{g}$, is a $\mathbb{R}$-vector space with a bilinear antisymmetric map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, known as the **Lie bracket**, such that

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0$$

for all $x,y,z\in\mathfrak{g}$, which is called the **Jacobi identity**.

Definition 1.11. A **Poisson algebra**, $\mathcal{P}$, is a commutative $\mathbb{R}$-algebra with a bilinear antisymmetric map $\{\cdot,\cdot\} : \mathcal{P}\times\mathcal{P}\to\mathcal{P}$, known as the **Poisson bracket**, which is a Lie bracket satisfying

$$\{f,gh\}=g\,\{f,h\}+h\,\{f,g\}$$

for all $f,g,h\in\mathcal{P}$, which is called the **Leibniz rule**.

Definition 1.12. Let (M,ω) be a symplectic manifold. Over $C^\infty(M)$ define que Poisson algebra

$$\{f,g\}=\omega(X_f,X_g)$$

for $f,g\in C^\infty(M)$, where $X_f,X_g\in H^0(M,TM)$ are the hamiltonian vector fields associated to the hamiltonians f and g, respectively.

Remark 1.13. Using canonical coordinates (q,p) in M we can compute explicitly the Poisson braket. Indeed, cause $X_f=\sum_{k=1}^{n}\frac{\partial f}{\partial p_k}\frac{\partial}{\partial q_k}-\frac{\partial f}{\partial q_k}\frac{\partial}{\partial p_k}$ we have that

$$\{f,g\}=-\sum_{k=1}^{n}\omega\left(\frac{\partial f}{\partial p_k}\frac{\partial}{\partial q_k},\frac{\partial g}{\partial q_k}\frac{\partial}{\partial p_k}\right)+\omega\left(\frac{\partial f}{\partial q_k}\frac{\partial}{\partial p_k},\frac{\partial g}{\partial p_k}\frac{\partial}{\partial q_k}\right)=\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k}-\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}$$

Proposition 1.14. *In a symplectic manifold M, $(C^\infty(M),\{\cdot,\cdot\})$ is a Poisson algebra, and $\psi : C^\infty(M)\to H^0(M,TM)$ given by $\psi(f)=X_f$ is an anti-homomorphism of Lie algebras*[3].

Proposition 1.15. *Let $f,H\in C^\infty(M)$, then $\{f,H\}=0$ if and only if f is constant along the integral curves of X_H.*

[3]That is $\psi(\{f,g\})=-[\psi(f),\psi(g)]=-[X_f,X_g]$.

Corolary 1.16. *If H is any hamiltonian on the symplectic manifold (M, ω) then H is constant along the integral curves of X_H.*

Proof. It follows from the previous proposition because $\{H, H\} = 0$. □

2. Moment Maps

2.1. **Review of Lie Groups Theory.** Remember that a **Lie group** is a group G with a differentiable structure on it such that the map $G \times G \to G$, $(g, h) \mapsto gh^{-1}$, is differentiable. If G, G' are Lie groups, a **homomorphism of Lie groups** is a differentiable map $f : G \to G'$ that is also a homomorphism of groups.

The **Lie algebra** of G, $\mathfrak{g}$, is $g = T_eG$, where e is the neutral element of G. In a Lie group we have, for every $g \in G$, two distiguished automorphisms, $L_g, R_g : G \to G$ given by $L_g(h) = gh$ and $R_g(h) = hg$. A vector field $X \in H^0(G, TG)$ is said to be **left invariant** if

$$(L_g)_* X_h = X_{gh}$$

for all $g, h \in G$. Observe that there is a linear isomorphism between $\mathfrak{g}$ and the $\mathbb{R}$-vector space of left invariant vector fields, $\mathcal{L}(G)$. Indeed, lets define $\psi : \mathfrak{g} \to \mathcal{L}(G)$ given by $\psi(\xi)(g) = (X^L_\xi)_g := (L_g)_*(\xi)$[4]. It is a $\mathbb{R}$-linear mapping, whose inverse is $\psi^{-1}(X) = X_e$. However, the point is that $\mathcal{L}(G)$ has a Lie bracket, the commutator of vector fields, that $\mathfrak{g}$ inherits by $[\xi_1, \xi_2] := [X_{\xi_1}, X_{\xi_2}]_e$. Hence, with this bracket, $\mathfrak{g}$ is a Lie algebra.

Moreover, we can improve this map to obtain another characterization or $\mathfrak{g}$. Recall that a **1-parameter subgroup** of G is a group homomorphism $\mathbb{R} \to G$, not necessarily injective. The 1-parameter subgroups of G can be identificated, 1 to 1, with the Lie algebra $\mathfrak{g}$. For this end, given $X \in H^0(G, TG)$, let ϕ_X be its flow, i.e. $\frac{d}{dt}\phi_X(t, x_0) = X_{\phi(t,x_0)}$. If X is left invariant, the flow is maximaly defined, that is $\phi_X : \mathbb{R} \times G \to G$.

[4]Do not confuse the subscript notation used for identify the left invariant vector field associated to $\xi \in T_eG$, that is, X^L_ξ, with the evaluation in $g \in G$ of a vector field, $X_g \in T_gG$.

Therefore, we can define

$$\begin{array}{ccc} \mathfrak{g} & \longleftrightarrow & \left\{ \begin{array}{c} \text{1-parameter} \\ \text{subgroups of } G \end{array} \right\} \\ \xi & \mapsto & \phi_{X_\xi^L}(\cdot, e) \end{array}$$

In particular, we obtain the map $exp : \mathfrak{g} \to G$ given by $exp(\xi) = \phi_{X_\xi}(1, e)$.

2.2. **Lie Group Actions and Representation.** Let G be a Lie group and let M be a differentiable manifold. Let $\Phi : G \times M \to M$ be a differentiable map and let us denote $\Phi_g = \Phi(g, \cdot)$. We will say that Φ is a (left) **differentiable action** of G over M if, for all $g, h \in G$

$$\Phi_h \circ \Phi_g = \Phi_{hg}$$

Observe that, in particular, every Φ_g is a diffeomorphism of M and $\Phi_e = id_M$.[5] Therefore, another alternative characterization of an action is as a differentiable map $\tilde{\Phi} : G \to Diff(M)$.

Another important fact about Lie group actions is that, to every $\xi \in \mathfrak{g}$, we can naturally associate it with a vector field on M, $X_\xi \in H^0(M, TM)$[6]. Remember that, associated to $\xi \in \mathfrak{g}$ is a vector field on G, X_ξ^L. Then, the map $\psi(t) = exp(tX_\xi^L) : \mathbb{R} \to G$ is its flow from $e \in G$. Therefore, we define

$$(X_\xi)_p = \left.\frac{d}{dt}\right|_{t=0} exp(tX_\xi^L) \cdot p$$

Sometimes, it will be necessary to restrict our attention to some subgroup of $Diff(M)$. In this sense, if our manifold is a symplectic manifold (M, ω), a group action Φ is said to **act by symplectomorphisms** if $\Phi_g : M \to M$ is a symplectomorphism for all $g \in G$.[7] Analogously, if (M, g) is a riemannian manifold, we say that Φ **acts by isometries** if $\Phi_g : M \to M$ is an isometry for all $g \in G$.

[5]In this context, usually it is denoted $g \cdot p := \Phi_g(p)$. Hence, the composition rule is $h \cdot (g \cdot p) = (hg) \cdot p$ for all $p \in M$ and $g, h \in G$.

[6]Again, do not confuse with the fancy subscript notation. While X_ξ^L is a vector field on the Lie group G, X_ξ is a vector field on the acted manifold M.

[7]Equivalently, if $\tilde{\Phi} : G \to Sympl(M, \omega)$.

In this setting, let us suppose that our manifold is a finite dimensional vector space, V, and that $\Phi : G \times V \to V$ acts by linear mappings, that is $\Phi_g : V \to V$ is linear for all $g \in G$. Then, Φ is called a **representation** of G on the vector space V. Of course, this is equivalent to give a homomorphism of Lie groups $\tilde{\Phi} : G \to GL(V)$.

Example 2.1. Let G be a subgroup of $GL(V)$ for some vector space V. Then we have a representation of G on V by $A \cdot v = A(v)$ for $A \in GL(V)$ and $v \in V$. Of course, the morphism $G = GL(V) \to GL(V)$ is simply the identity.

Example 2.2 (Spin representation)**.** Some of the most important representations are the representations of $SU(2)$, known as *spin representations*. Given $k \in \mathbb{N}$, let us consider the space of homogenious complex polynomials in two variables of degree $2k$, that is $V = (\mathbb{C}[x, y])_{2k}$. Then, we have that $SU(2)$ acts on V by preevaluation, that is

$$(A \cdot P)(v) = P(A^{-1}(v))$$

for $A \in SU(2)$, $P \in V = (\mathbb{C}[x, y])_{2k}$ and $v \in \mathbb{C}^2$. This is a representantion, known as the spin-$\frac{k}{2}$ representation of $SU(2)$. It can be also shown that, for the spin-1 representation, we have that $V = (\mathbb{C}[x, y])_2$ is three dimensional and the representation is, in fact, a double cover $SU(2) \to SO(3)$, which is known as the fundamental representation.

Example 2.3 (Dual representation)**.** Suppose that we have a representation $\tilde{\Phi} : G \to GL(V)$, then we can naturally induce a representation $\tilde{\Phi}^* : G \to GL(V^*)$, known as the **dual representation**. This is defined, for $g \in G$, $\omega \in V^*$ and $v \in V$ by

$$(g \cdot \omega)(v) = \omega(g^{-1} \cdot v)$$

Note that the inverse is necessary if we want that $h \cdot (g \cdot \omega) = (hg) \cdot \omega$ holds.

Example 2.4 (Adjoint representation)**.** Given a Lie group G, we always have a representation of G on $\mathfrak{g}$, known as the **adjoint representation** of G. To this end, observe that, for all $g \in G$, we have the conjugation automorphism $c_g : G \to G$ given by $c_g(h) = ghg^{-1}$. Then, cause e is in the normalizer of G, its differential $(c_g)_{*_e} : T_eG = \mathfrak{g} \to T_eG = \mathfrak{g}$ is an automorphism of $\mathfrak{g}$. Thus, we define the adjoint representation $Ad : G \to GL(\mathfrak{g})$ by

$$Ad_g = (c_g)_{*_e}$$

As expected, the dual representation of $Ad : G \to GL(\mathfrak{g})$ is called the **co-adjoint representation**, $Ad^* : G \to GL(\mathfrak{g}^*)$.

Moreover, we can go a step forward and derivate the Ad map to obtain the map $Ad_{*e} : T_eG = \mathfrak{g} \to T_{id}\,GL(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g})$. Hence, we have obtain a map $ad = Ad_{*e} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, known as the **adjoint representation** of the Lie algebra $\mathfrak{g}$. In fact, it can be shown that

$$\xi \cdot \eta = ad_\xi(\eta) = [\xi, \eta]$$

where $\xi, \eta \in \mathfrak{g}$. As a final remark, observe that this is not a representation of $\mathfrak{g}$ as a Lie group, as we have introduced, because the maps $ad_\xi : \mathfrak{g} \to \mathfrak{g}$ are not isomorphisms.

2.3. **Moment Maps.** Let (M, ω) be a symplectic manifold and let G be a group acting on M by symplectomorphisms, $\Phi : G \times M \to M$. let us take a map

$$\mu : M \to \mathfrak{g}^*$$

and, for any $\xi \in \mathfrak{g}$, let us define $\mu^\xi(p) := \mu(p)(\xi)$, that is

$$\mu^\xi : M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{e_\xi} \mathbb{R}$$

where $e_\xi : \mathfrak{g}^* \to \mathbb{R}$ is the evaluation on ξ map. We will say that μ is a **moment map**, and Φ a **hamiltonian action** if it satisfies:

- *(Hamiltonian)* For all $\xi \in \mathfrak{g}$, μ^ξ is hamiltonian with hamilton vector field X_ξ, that is

$$d\mu^\xi = \iota_{X_\xi}\omega$$

- *(Equivariant)* μ is equivariant for the action $\Phi : G \times M \to M$ on M and the coadjoint action $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$, that is

$$\mu \circ \Phi_g = Ad^*_g \circ \mu$$

  for all $g \in G$.

2.3.1. *Comoment Maps.* If G is a connected Lie group, a moment map can be defined using another map, known as a comoment map.

Definition 2.5. Let (M, ω) be a symplectic manifold and let G be a group acting on M by symplectomorphisms, $\Phi : G \times M \to M$. let us take a map

$$\mu^* : \mathfrak{g} \to C^\infty(M)$$

μ^* is called a **comoment map** for the action Φ if it satisfies:

- *(Hamiltonian)* For every $\xi \in \mathfrak{g}$, $\mu^*(\xi) : M \to \mathbb{R}$ is hamiltonian with hamilton vector field X_ξ, that is
$$d(\mu^*(\xi)) = \iota_{X_\xi}\omega$$
- *(Equivariant)* μ^* is a Lie algebra homomorphism between $\mathfrak{g}$ and $C^\infty(M)$ with the Poisson bracket, that is
$$\mu^*([\xi_1, \xi_2]) = \{\mu^*(\xi_1), \mu^*(\xi_2)\}$$
for every $\xi_1, \xi_2 \in \mathfrak{g}$.

Proposition 2.6. *Let G be a connected Lie group and M a symplectic manifold acted by G by symplectomorphisms. If $\mu : M \to \mathfrak{g}^*$ is a moment map, then $\mu^* : \mathfrak{g} \to C^\infty(M)$ given by*

$$\mu^*(\xi)(p) := \mu(p)(\xi) = \mu^\xi(p)$$

is a comoment map. Reciprocally, if $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a comoment map, then $\mu : M \to \mathfrak{g}^*$ given by

$$\mu(p)(\xi) := \mu^*(\xi)(p)$$

is a moment map.

2.3.2. *Real Moment Maps.* Suppose that $G = \mathbb{R}$. In that case, we have $\mathfrak{r} \cong \mathfrak{r}^* \cong \mathbb{R}$. In fact, $exp : \mathfrak{r} \to \mathbb{R}$ is an isomorphism. Taking $\xi = exp^{-1}(1)$, it is a generator of $\mathfrak{r}$, so $e_\xi : \mathfrak{r}^* \to \mathbb{R}$ is an isomorphism. Using it, we have an isomorphism of $\mathbb{R}$-modules $\psi : C^\infty(M, \mathfrak{r}^*) \to C^\infty(M)$ given by $\psi(\mu) = \mu^\xi := e_\xi \circ \mu$.

Suppose now that $\mu : M \to \mathfrak{r}^*$ is a moment map for the action of $\mathbb{R}$ in M, $\Phi : \mathbb{R}\times M \to M$. Then, since ξ is a generator of $\mathfrak{r}$, it is enought to satisfy for all $\lambda \in \mathbb{R}$

$$\mu^{\lambda\xi} = e_{\lambda\xi} \circ \mu = \lambda e_\xi \circ \mu = \lambda \mu^\xi$$

Hence, is μ is hamiltonian if and only if $\psi(\mu) = \mu^\xi : M \to \mathbb{R}$ satisfies

$$d\mu^\xi = \iota_{X_\xi}\omega$$

However, observe that

$$(X_\xi)_p = \left.\frac{d}{dt}\right|_{t=0} (exp(t\xi)\cdot p) = \left.\frac{d}{dt}\right|_{t=0} (t\cdot p) = (X_\Phi)_p$$

the vector field of the action $\Phi : \mathbb{R}\times M \to M$. Therefore, μ is hamiltonian if and only if the hamilton vector field of μ^ξ is X_Φ.

For the equivariant condition, observe that, cause $\mathbb{R}$ is an abelian Lie group, we have that all the conjugations $c_x = id_{\mathbb{R}}$ for all $x \in \mathbb{R}$. Hence, the adjoint representation $Ad : \mathbb{R} \to GL(\mathfrak{r})$ is $Ad(x) = (c_x)_{*0} = (id_{\mathbb{R}})_{*0} = id_{\mathfrak{r}}$. Therefore $Ad^*(x) = id_{\mathfrak{r}^*}$ for all $x \in \mathbb{R}$. Hence, the equivariance condition means, for all $x \in \mathbb{R}$

$$\mu \circ \Phi_x = Ad^*_x \circ \mu = \mu$$

that is $\mu(x\cdot p) = \mu(p)$ for all $p \in M$ and $x \in \mathbb{R}$ or, equivalently, μ^ξ is constant in the orbits of $\mathbb{R}$. But this always holds because, by corolary 1.16, μ^ξ is constant in the integral curves of X_ξ, which are exactly the orbits of $\mathbb{R}$.

Summarizing

Proposition 2.7. *Let (M,ω) be a symplectic manifold and $\Phi : \mathbb{R}\times M \to M$ an action by symplectomorphisms. let us define $X_\Phi(p) = \frac{d}{dt}|_{t=0}\Phi_t(p)$. Then, for every $\xi \in \mathfrak{r}$, there is a biyection*

$$\varphi : \{\textit{Moment maps for } \Phi\} \to \{\textit{Hamiltonians of } X_\Phi\}$$

given by $\varphi(\mu) = e_\xi \circ \mu$.

2.3.3. *Cyclic Moment Maps.* Using the exponential map $exp : \mathfrak{u}(1) \cong i\mathbb{R} \to U(1)$, an action $\Phi : U(1) \times M \to M$ by symplectomorphisms is the same than an action $\tilde{\Phi} : \mathbb{R} \times M \to M$ with $\tilde{\Phi}_{2\pi} = id_M$. Therefore, using the previous characterization, we have

Proposition 2.8. *Let (M, ω) be a symplectic manifold and $\Phi : U(1) \times M \to M$ a cyclic action by symplectomorphisms. Let $\tilde{\Phi} : \mathbb{R} \times M \to M$ the underlying periodic real action and define $X_{\tilde{\Phi}}(p) = \frac{d}{dt}|_{t=0}\tilde{\Phi}_t(p)$. Then, for every $\xi \in \mathfrak{r}$, there is a biyection*

$$\varphi : \{\textit{Moment maps for } \Phi\} \to \{\textit{Hamiltonians of } X_{\tilde{\Phi}}\}$$

given by $\varphi(\mu) = e_\xi \circ \mu$.

Remark 2.9. Reciprocally, given a vector field $X \in H^0(M, TM)$ it defines a moment map for some $\mathbb{R}$-action on M if and only if X is a hamiltonian vector field, that is, if $X = d\mu$ for some $\mu \in C^\infty(M)$. For example, this can be always done if $H^1(M) = 0$, so this kind of manifolds are plenty of moment maps. Of course, if the flow of X is periodic (i.e. its orbits are closed) then the moment map found can be understood as a $U(1)$-moment map.

3. Morse-Bott Theory

In this section, let M be a differentiable manifold of dimension n.

Definition 3.1. Let $f : M \to \mathbb{R}$ be a differentiable function. We will say that $p \in M$ is a **critical point** of f if $df_p = 0$. Given a critical point $p \in M$ of f, we define the **Hessian of f in** p as the bilinear symmetric map $Hf_p : T_pM \times T_pM \to \mathbb{R}$ given by $Hf_p(v_p, w_p) = w_p(\tilde{v}(f))$, where $\tilde{v}$ is any vector field that extend v in a neighborhood of p.

If the Hessian Hf_p is non-degenerated (i.e. the linear map $T_pM \to T_p^*M$, $v \mapsto Hf_p(v, \cdot)$ is an isomorphism), we will say that p is a **non-degenerated** critical point of f. The dimension of the maximal vector subspace of T_pM in which Hf_p is negative defined is called the **index** of f in p, and is denoted by $\lambda(p)$.

Remark 3.2. Recall that Hf_p is symmetric for a critical point p because $Hf_p(v,w) - Hf_p(w,v) = w_p(\tilde{v}(f)) - v_p(\tilde{w}(f)) = [\tilde{w},\tilde{v}]_p f = 0$.

Proposition 3.3. *If $p \in M$ is a critical point of f, then the matrix of Hf_p in the basis $\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$ is $\left(\left.\frac{\partial^2 f}{\partial x_j \partial x_i}\right|_p\right)_{i,j=1}^n$.*

Proof. Is a simple check

$$Hf_p\left(\left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p\right) = \left.\frac{\partial}{\partial x_j}\right|_p\left(\frac{\partial}{\partial x_i} f\right) = \left.\frac{\partial^2 f}{\partial x_j \partial x_i}\right|_p$$

□

Remark 3.4. From this proposition, we have that Hf_p does not depends on the extension $\tilde{v}$ chosen. Moreover, p is a non-degenerated point of f if and only if

$$\det\left(\left.\frac{\partial^2 f}{\partial x_j \partial x_i}\right|_p\right)_{i,j=1}^n \neq 0$$

Maybe the most important analytic property of this functions is that, locally, the are defined as a quadratic form.

Lemma 3.5 (Morse)**.** *Let $f : M \to \mathbb{R}$ be a differentiable mapping with $p \in M$ a non-degenerated critical point. Then, there exists a chart (U,φ) with $\varphi(p) = (0,\dots,0)$ such that*

$$f \circ \varphi^{-1}(x_1,\dots,x_m) = f(p) + -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots x_m^2$$

where λ is the index of f in p.

Definition 3.6. Let M be a differentiable manifold, possibly with boundary. A differentiable function $f : M \to \mathbb{R}$ is said to be a **Morse function** if all its critical points are interior (i.e. belong to $M - \partial M$) and non-degenerated.

Remark 3.7. Thanks to the Morse lemma, all the critical points of a Morse function are isolated. In particular, if M is a closed manifold (compact and with boundary) then it has a finite number of critical points.

3.0.4. *Reeb's theorem.* A very useful theorem that can be proved using the theory of Morse functions is the Reeb's theorem. Only for mention one of its most popular applications, it is intensively used when we want to prove that certain differentiable manifold is an exotic spheres.

The main idea behind this theorem is the following simple example.

Example 3.8. Let $S^n \subset \mathbb{R}^{n+1}$ be the n-sphere, as seen embbeded on $\mathbb{R}^{n+1}$. Let $f : S^n \to \mathbb{R}$ be the last coordinate function, that is $f(x) = x_n$. Then f has two critical values, -1 and 1, its minimum and maximum, that is easy to check that they are non-degenerate. Therefore, S^n has a Morse function with only two critical points.

The key is that this property characterizes the sphere topologicaly.

Theorem 3.9 (Reeb)**.** *Let M^n be a closed manifold. Suppose that there exists a Morse function $f : M \to \mathbb{R}$ with only two critical points, then M^n is homeomorphic to S^n.*

Remark 3.10. The existence of exotic spheres implies that we cannot improve the Reeb theorem in order to assure that M is diffeomorphic to S^n.

3.1. **Morse inequalities.** One of the most important theorems in the theory of Morse functions is that it is possible to estimate the topological information of M only using the information of the critical points of a Morse function $f : M \to \mathbb{R}$.

Definition 3.11. let us define an order in the space of integer polynomials in one variable, $\mathbb{Z}[t]$, saying that $P \preceq Q$ if there exists $R \in \mathbb{Z}[t]$, with all its coefficients non-negative, such that

$$Q(t) = P(t) + (1+t)R$$

with this definition, $(\mathbb{Z}[t], \preceq)$ is a (non-strict) partial order.

Definition 3.12. Let M^n be a differentiable manifold[8]. We define the **Poincare polynomial** of M, $P_M \in \mathbb{Z}[t]$ as

$$P_M(t) = \sum_{k=0}^{n} b^k(M) t^k$$

[8]Of course, we can aweaken this definition to topological spaces using the singular cohomology instead of the de Rham Cohomology.

where $b^k(M) = \dim_{\mathbb{R}} H^k(M, \mathbb{R})$ is the k-th **Betti number**.

Definition 3.13. Let M be a differentiable manifold and let $f : M \to \mathbb{R}$ be a Morse function. We define the **Morse polynomial** of f, $M_f \in \mathbb{Z}[t]$ as

$$M_f(t) = \sum_{p \in Crit(f)} t^{\lambda(p)}$$

Theorem 3.14 (Morse inequality)**.** *Let M be a differentiable compact manifold and $f : M \to \mathbb{R}$ a Morse function. We have*

$$M_f \preceq P_M$$

Corolary 3.15 (Weak Morse inequality)**.** *Let M^n be a differentiable compact manifold and let $f : M \to \mathbb{R}$ be a Morse function. Then, for all $0 \leq k \leq n$*

$$\mathcal{C}_k(f) \leq b^k(M)$$

where $\mathcal{C}_k(f)$ denote the set of critical points of f of index k.

Proof. Check the k-th coefficient of M_f and P_M and compare it using the Morse inequility. □

Corolary 3.16 (Strong Morse inequality)**.** *Let M be a differentiable compact manifold and let $f : M \to \mathbb{R}$ be a Morse function. Then, it holds*

$$\sum_{k=0}^{\infty} (-1)^k \mathcal{C}_{m-k}(f) \leq \sum_{k=0}^{\infty} (-1)^k b^{m-k}(M)$$

for all $m \geq 0$.

Definition 3.17. Given a differentiable manifold M and a Morse function $f : M \to \mathbb{R}$ on it, we will say that f is a **perfect Morse function** if $M_f = P_M$, or, equivalently

$$\mathcal{C}_k(f) = b^k(M)$$

3.2. **Morse-Bott Functions.** As we have just seen, Morse theory can be used to obtain topological information of a differentiable manifold. However, there are cases when the Morse function are too restrictive, and we cannot find a suitable Morse function. To solve this problem, it is possible to use a generalization of Morse functions, known as Morse-Bott functions.

The key idea of this functions is that they are not restricted to have isolated critical points, as the Morse function are. Instead, we can have some submanifolds of critical points, and, in order to eliminate pathological behaviours, we should impose some kind of non-degeneration property to the Morse-Bott function.

Let M be any differentiable manifold and let $i : S \hookrightarrow M$ be a submanifold of M, with tangent bundle TS. The **normal bundle** of S in M, $\nu^M S$ is the rank $\dim M - \dim S$ vector bundle $\nu^M S = TM|_S/TS \to S$[9]. Let $i_0 : S \hookrightarrow \nu^M S$ be the zero section, which is a embedding of S in $\nu^M S$.

To understand this name, let us take any riemannian metric g on M. Using this metric, for every $s \in S$ we obtain an orthogonal splitting

$$T_s M = T_s S \oplus (T_s S)^\perp$$

But, by linear algebra, there is an isomorphism $\nu^M_s S = T_s M/T_s S \cong (T_s S)^\perp$. Therefore, $\nu^M S$ can be seen as the vector bundle of the orthogonal complement of TS with respect any metric. A very important fact of the normal bundle is the following theorem.

Theorem 3.18 (Tubular neighborhood theorem)**.** *Let M be any differentiable manifold and $i : S \hookrightarrow M$ a submanifold with normal bundle $\nu^M S \to S$. There exist $U_0 \subset \nu^M S$ neighborhood of the zero section, $U \subset M$ neighborhood of S and a diffeomorphism $\varphi : U_0 \to U$ such that the following diagram commutes*

$$\begin{array}{ccc} \nu^M S \supset U_0 & \xrightarrow{\ \varphi\ } & U \subset M \\ & {\scriptstyle i_0}\nwarrow \quad \nearrow{\scriptstyle i} & \\ & S & \end{array}$$

[9]Recall that $TM|_S$ is a shorthand for $i^*(TM)$, the pullback of vector bundles.

Definition 3.19. Let M be a differentiable manifold and $f : M \to \mathbb{R}$ a differentiable function. A compact connected submanifold $S \subset M$ is said to be a **critical submanifold** if $S \subset Crit(f)$ and $Hf|_{\nu^M S}$ is non-degenerate, in the sense that $Hf_s(X, \cdot) : T_sM \to T_s^*M$ is not an isomorphism if and only if $X \in T_sS$.

Definition 3.20. Let M be a differentiable manifold and $f : M \to \mathbb{R}$ a differentiable function. We will say that f is a **Morse-Bott** function if its critical points are a finite disjoint union of critical submanifolds.

If $S \subset M$ is critical submanifold of $f : M \to \mathbb{R}$, then for all $s \in S$ the bilinear form $Hf_s : T_sM \times T_sM \to \mathbb{R}$ descends to a non-degenerate bilinear form on $\nu_s^M S = \frac{T_sM}{T_sS}$

$$\overline{Hf}_s : \nu_s^M S \times \nu_s^M S \to \mathbb{R}$$

For $s \in S$, let $\lambda(s)$ be the index of $\overline{Hf}_s$, that is, the dimension of the maximal subspace in which it is negative definite. Observe that the condition of negative definiteness is an open condition[10] so $s \mapsto \lambda(s)$ is a continous function $S \to \mathbb{N}$. Cause $\mathbb{N}$ is discrete, a continous function is locally constant, so, using that S is connected, we have that $\lambda(s)$ is constant. let us denote this common value as $\lambda(S)$, the **index** of f in S.

Definition 3.21. Let M be a differentiable function and $f : M \to \mathbb{R}$ a Morse-Bott function on it. Let $Crit(f) = \bigsqcup_{k=1}^{m} S_k$ with S_k critical submanifolds of f. The **Morse-Bott** polynomial of f, $M_f \in \mathbb{Z}[t]$, is

$$M_f(t) = \sum_{k=1}^{m} t^{\lambda(S_k)} P_{S_k}(t)$$

Remark 3.22. Obviously, every Morse function is a Morse-Bott function and each critical point is a critical submanifold of dimension 0. Hence, since $P_\star(t) = 1$, the Morse and the Morse-Bott polynomial of f coincide.

Analogous to the Morse inequality for Morse functions, using spectral sequences we can prove.

[10]Using Silvester criterion, it is essencially a condition on the sign of some minors of the matrix of $\overline{Hf}$, which is an open condition.

Theorem 3.23 (Morse-Bott inequality)**.** *Let M be a differentiable compact manifold and $f : M \to \mathbb{R}$ a Morse-Bott function. We have*

$$M_f \preceq P_M$$

Definition 3.24. A Morse-Bott function $f : M \to \mathbb{R}$ is called **perfect** if $M_f = P_M$.

3.3. **Moment Maps as Morse-Bott functions.** Suppose that we have a Lie group G with a hamiltonian action on a symplectic manifold (M, ω) and let us consider its moment map $\mu : G \to \mathfrak{g}$. Given any $\xi \in \mathfrak{g}$, we can consider $\mu^\xi : M \to \mathbb{R}$. As we will see, for toric groups under rather general conditions, this maps are going to be perfect Morse-Bott functions.

First of all, let us identificate the critical points of μ^ξ in terms of the action.

Proposition 3.25. *Let G be a Lie group with a hamiltonian action on a symplectic manifold M with moment map $\mu : M \to \mathfrak{g}^*$. For every $\xi \in \mathfrak{g}$ the critical points of $\mu^\xi : M \to \mathbb{R}$ are exactly the fixed points of M under the action of G.*

Proof. Let $\xi \in \mathfrak{g}$ and let $p \in M$ be a fixed point of $\mu^\xi : M \to \mathbb{R}$. Then, since the action is hamiltonian, it means that

$$0 = d\mu^\xi{}_p = \iota_{X_{\xi_p}} \omega_p$$

but, as ω_p is non-degenerated, it is only possible if $X_{\xi_p} = 0$. But $X_{\xi_p} = 0$ if and only if p is a fixed point under the action of G. □

let us now focus on hamiltonian actions of $\mathbb{T}^r = \overbrace{S^1 \times \cdots \times S^1}^{r \text{ times}}$ over a symplectic manifold M.

Theorem 3.26. *Let M be a symplectic manifold and suppose that $\mathbb{T}^r$ acts as a hamiltonian action on M with moment map $\mu : M \to \mathfrak{t}$. Then, for every $\xi \in \mathfrak{t}$, the function $\mu^\xi : M \to \mathbb{R}$ is a Morse-Bott function. Furthermore, the critical submanifolds of μ^ξ are symplectic with all its indices even.*

Corolary 3.27. *The moment map of a $U(1)$-hamiltonian action on a symplectic manifold is a Morse-Bott function with all critical submanifolds symplectic and with even index.*

Moreover, we can go a step forward and use $U(1)$-moment maps to understand completely the topology of the symplectic manifold. Indeed, if the manifold is compact, it can be shown that we can compute its handlebody decomposition[11].

However, if the manifold is not compact (like the moduli space of Higgs bundles), we have some obstructions for understanding its topology, very related with the compacness property. As the following theorem claims, the only obtruction is the boundedness and properness of this map.

For having some intuition, we can follow this program if the moment map can see the manifold as *it would be compact*. More precisely, understanding the topology of the manifold is equivalent to compute a handlebody decomposition of the manifold. But, for this end, we need that the Morse-Bott would be bounded below, for having a beginning of the handlebody, and proper, for pasting correctly the handles.

Theorem 3.28. *Suppose that $U(1)$ has a hamiltonian action on a symplectic manifold M with a <u>proper and bounded below</u> moment map $\mu : M \to \mathbb{R}$. Then μ is a perfect Morse-Bott function, that is*

$$P_M(t) = \sum_{k=1}^{m} t^{\lambda(S_k)} P_{S_k}(t)$$

where $S_1, \ldots, S_m$ are the connected components of the set of fixed points of M under de action of $U(1)$. Furthermore, all the indices are even and the S_k are symplectic manifolds.

The proof of this theorem is the result of successive improvements. Maybe, the fundational paper, and in fact the main line of argument, is [6]. However, this paper is old and the notions introducted are old-fashioned. For example, the notion of moment map is not introduced, thought central in the argument, so all the computations should

[11] Essentially, a decomposition like the CW-complex decomposition, but in the differentiable category.

be done by hand. Even more important, in this paper the inicial hypothesis is that M is a Kähler manifold. However, as some people has pointed out, this hypothesis is not crucial, cause it can be used a compatible almost complex structure, which always exists, and complete the computations.

Therefore, we should look for more modern references. Two classical textbooks about Morse Theory are [11] and [13]. While the later is focused in Morse homology and the algebraic and analytic issues, the later is focused in Morse, Morse-Bott and Flöer theory, and its relations with symplectic topology. However, these books are not enought for our purposes, cause everywhere, they require the compacness of the symplectic manifold.

For avoiding this hypothesis, we should refer to [12]. There are some lecture notes in which the theory is developed using the handlebody decomposition created from a Morse-Bott function. Therefore, the proofs are simpler and clearer, based on geometrical properties, and not assuming that the manifold is compact. In turn, this notes are based on [2], where the theory of toric actions is developed with lot of details.

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