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Sheaves in Geometry

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A different image came to me a few weeks ago.
The unknown thing to be known appeared to me
as some stretch of earth or hard marl, resisting
penetration . . . the sea advances insensibly in
silence, nothing seems to happen, nothing moves,
the water is so far off you hardly hear it . . . yet it
finally surrounds the resistant substance.

Alexander Grothendieck

Abstract

In this memoir we introduce the necessary notions of category theory to develop sheaf theory. Representability of functors, limits and adjunctions are covered in detail. After this, sheaf theory is presented together with its basic constructions. The study of sheaves with algebraic structures appears in section 4 and is applied in the theory of vector bundles. Section 5 is centered in the study of applications of sheaf theory, namely ringed spaces and local systems. We finish with two appendices showing some category theoretical properties of presheaf and sheaf categories and a functorial approach to algebraic geometry.

Contents

1 Introduction

This project arose as a consequence of my long-standing interest in category theory. Two years ago, I discovered category theory through the Handbooks of Categorical Algebra of Francis Borceux [2] and that point of view of mathematics automatically captivated me. Although I was not mature enough mathematically (I did not know what a sheaf was) I came across the notion of a topos, a categorified geometrical space. I felt very curious about this so I decided that one of my mathematical goals would be to understand what a topos is and the first step was sheaf theory.

In this memoir I first expose the basic results related to categories and functors. A special emphasis is given to the concept of representability and functor categories. These categories of functors are part of the natural process of categorification of mathematical theories. One of the revolutionary ideas of Alexander Grothendieck, is to replace a given topological space with its category of sheaves. On the other hand, representability lies at the heart of many constructions in category theory, for example, universal properties. Some attention is paid to cartesian closed categories and their internal logic since sheaf categories (of sets) are always cartesian closed.

Section 3 is devoted to the study of the basic results of sheaf theory and the change of base adjunction. After this preliminary work, sheaves with algebraic structures are introduced in the next section with special emphasis given to sheaves of $\mathcal{O}_X$-Modules. In section 4, I introduce the notion of a vector bundle and show that this concept is essentially equivalent to the idea of locally free sheaves. Finally the notion of Picard Group of a topological space is introduced as the group of locally free sheaves of rank 1, [6].

The last section shows some applications of the sheaf theoretic language. Locally ringed spaces are introduced in full generality together with the notions of models for these spaces. Since the extension of this work is limited I could not include any results in abelian categories or categories of complexes, however the study of local systems shown in section 4 has many interesting homological applications. Nevertheless, in this project the study of local systems is purely topological. We show that the monodromy of the local systems of a given topological space encodes the representations of the fundamental group.

Appendix I is devoted to categorical study of sheaf categories and presheaf categories. In this part of the memoir we prove completeness and cocompleteness of presheaf/sheaf categories. One central result is the fact that every presheaf is the colimit of representable presheaves which leads to a proof of cartesian closedness of presheaf categories. The last topic covered in this section are subobject classifiers which are of key importance in topos theory since they effectively handle the notion of subobjects in category theoretic terms. Appendix II shows a functorial approach to the theory of schemes and some conceptual results about locally ringed spaces. The key result of this section is to show that the functorial foundations of algebraic geometry are equivalent to geometric foundations via locally ringed spaces.

It is obvious that I could not say that all the proofs are mine, however I cannot refer to a specific source for finding them. I have been reading some of the titles of the bibliography (for example [2],[4]) for two years now and the proofs in this memoir shows my view of the ones presented there. The book of Tennison [6] can be thought as a complementary lecture to *Sheaves in Geometry and Logic* [4]. Due to my lack of knowledge of local systems I followed the development proposed by Achar in his notes [1] about perverse sheaves. However, I decided to give those notes a more topological flavour using the tools of covering spaces found in [5]. The last appendix is mostly inspired in [3].

2 An introduction to Category Theory

2.1 Generalities

We avoid all set theoretical issues assuming the existence of universes as in [2]. We will try to avoid this kind of discussion unless it is strictly necessary.

Definition 2.1.1. A category $\mathscr{C}$ is the following data:

- a class $|\mathscr{C}|$, whose elements will be called "objects of the category";
- for every pair A, B of objects a set $\mathrm{Hom}_{\mathscr{C}}(A, B) = \mathscr{C}(A, B)$, whose elements will be called "morphisms" or "arrows" from A to B;
- for every triple A, B, C of objects, a composition law

  $$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C)$$

  the composite of the pair (f, g) will be written $g \circ f$;
- for every object A, a morphism $\mathbb{1}_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$, called the identity on A.

These data are subject to the following axioms:

- *Associativity axiom*: given morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathscr{C}}(C, D)$ the following equality holds:

  $$h \circ (g \circ f) = (h \circ g) \circ f.$$

- *Identity axiom*: given morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$ the following equalities hold:

  $$\mathbb{1}_B \circ f = f, \quad g \circ \mathbb{1}_B = g.$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

Diagram 2.1

Most of our information will be summarized in commutativity conditions. An important example is Diagram 2.1, we say that this diagram commutes if and only if $f \circ g = h \circ k$.

Definition 2.1.2. A (covariant) functor F from a category $\mathscr{A}$ to a category $\mathscr{B}$ consists of the following:

- a mapping

  $$|\mathscr{A}| \longrightarrow |\mathscr{B}|$$

  between the classes of objects of $\mathscr{A}$ and $\mathscr{B}$; the image of $A \in |\mathscr{A}|$ is written $F(A)$ or FA;
- for every pair of objects A, A' of $\mathscr{A}$ a mapping

  $$\mathrm{Hom}_{\mathscr{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathscr{B}}(FA, FA')$$

  the image of $f \in \mathrm{Hom}_{\mathscr{A}}(A, A')$ is written $F(f)$ or Ff.

These data are subject to the following axioms:

- for every pair of morphisms $f \in \mathrm{Hom}_{\mathscr{A}}(A, A'), g \in \mathrm{Hom}_{\mathscr{A}}(A', A'')$

$$F(g \circ f) = F(g) \circ F(f);$$

- for every object $A \in |\mathscr{A}|$

$$F(\mathbb{1}_A) = \mathbb{1}_{FA}.$$

Remark 2.1.3. Pointwise composition of functors produces a new functor, however it is important to have in mind that if we feel tempted to define the category of categories and functors size problems appear. It is not true in general that $\mathrm{Hom}_{\mathsf{Cat}}(-,-)$ is a set.

Definition 2.1.4. A category $\mathscr{C}$ is called a small category if its class of objects constitutes a set.

Remark 2.1.5. Small categories and functors between them constitute a category.

Example 2.1.6. Some inmediate examples are:

- Set: The category of sets and mappings.
- Top: The category of topological spaces and continuous mappings.
- Gr: The category of groups and group homomorphisms.
- Rng: The category of rings with unit and ring homomorphisms.
- $\mathscr{R}$-Mod: The category of $\mathscr{R}$-Modules and module homomorphisms.
- Ab: The category of abelian groups and group homomorphisms.

Example 2.1.7 (Poset categories). A poset $(X, \leqslant)$ can be viewed as a category $\mathscr{X}$ whose objects are the elements of X; $\mathrm{Hom}_{\mathscr{X}}(x, y)$ is given by a singleton if $x \leqslant y$ and is empty otherwise. The composition law is given by the transitivity axiom and the existence of the identities is the reflexivity axiom.

Given a topological space X, we can construct a poset category, $\mathcal{O}(X)$. The objects are given by the open sets and the partial order is given by inclusion.

Example 2.1.8 (Monoid categories). A monoid $(M, \cdot)$ can be viewed as a category $\mathscr{M}$ with a single object $*$ and $M = \mathrm{Hom}_{\mathscr{M}}(*, *)$ as a set of morphisms; the composition law is just the multiplication of the monoid. A special case of this construction is the structure of a group.

Example 2.1.9 (Functor categories). Given two categories $\mathscr{A}, \mathscr{B}$ if the functor category (from $\mathscr{A}$ to $\mathscr{B}$) of functors and natural transformations between them is well defined we will denote it as $\mathsf{Fun}(\mathscr{A}, \mathscr{B}) = [\mathscr{A}, \mathscr{B}] = \mathscr{B}^{\mathscr{A}}$.

Example 2.1.10 (Slice categories). Given a category $\mathscr{C}$ and an object $C \in |\mathscr{C}|$ we can form the slice category $\mathscr{C}/C$ whose objects are arrows with codomain C. Given two objects $f\colon A \longrightarrow C$, $g\colon B \longrightarrow C$ an arrow between them is a morphism $h\colon A \longrightarrow B$ of $\mathscr{C}$ with the property $g \circ h = f$.

Definition 2.1.11 (Product category). Let $\mathscr{C}$,$\mathscr{D}$ be two categories. We can form the product category $\mathscr{C} \times \mathscr{D}$ with pairs (C, D) as objects such that $C \in |\mathscr{C}|$, $D \in |\mathscr{D}|$ and pairs of arrows as morphisms in each category. The composition is made componentwise and the identity of (C, D) equals $(\mathbb{1}_C, \mathbb{1}_D)$.

We provide some examples of functors.

Example 2.1.12. Given a category $\mathscr{C}$ and a fixed object $C \in |\mathscr{C}|$, we define the Hom functor as:

$$\mathrm{Hom}_{\mathscr{C}}(C,-)\colon \mathscr{C} \longrightarrow \mathsf{Set}$$

from $\mathscr{C}$ to the category of sets by first putting

$$\mathrm{Hom}_{\mathscr{C}}(C,-)(A) = \mathrm{Hom}_{\mathscr{C}}(C,A)$$

Now if $f\colon A \longrightarrow B$ is a morphism of $\mathscr{C}$, the corresponding mapping

$$\mathrm{Hom}_{\mathscr{C}}(C,f)\colon \mathrm{Hom}_{\mathscr{C}}(C,A) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(C,B)$$

is defined by the formula

$$\mathrm{Hom}_{\mathscr{C}}(C,f)(g) = f \circ g.$$

Example 2.1.13. Following the monoid construction we can see that given a functor:

$$F\colon M \longrightarrow \mathsf{Top}$$

We get a topological space $F(*)$ together with a set of continuous maps parametrized by a group, that is, a group action.

Definition 2.1.14 (Dual category). Given a category we set $\mathscr{C}^{\mathbf{op}}$ as the dual category of $\mathscr{C}$. Its objects are the same of $\mathscr{C}$ but all arrows are reversed.

Definition 2.1.15. A contravariant functor is a functor:

$$F\colon \mathscr{C}^{\mathbf{op}} \longrightarrow \mathscr{D}$$

That is a functor that reverses the direction of arrows.

Definition 2.1.16. The functors $\mathscr{F}: \mathscr{C}^{\mathbf{op}} \longrightarrow \mathsf{Set}$ are called presheaves. Whenever the category of presheaves is well defined we shall denote it $\mathsf{Psh}(\mathscr{C})$. In the case $\mathscr{C} = \mathcal{O}(X)$ we will denote the presheaf category as $\mathsf{Psh}(X)$.

Example 2.1.17 (Spectrum of a ring). There is a functor sending commutative rings with unit to topological spaces:

$$\begin{aligned} \mathrm{Spec}\colon \mathsf{Rng}^{\mathbf{op}} &\longrightarrow \mathsf{Top} \\ A &\longmapsto \mathrm{Spec}\, A \end{aligned}$$

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal of } A\}.$$

We endow this set with Zariski's topology declaring closed the sets:

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid I \subseteq \mathfrak{p},\ I \text{ ideal of } A\}.$$

Clearly $V(0) = \mathrm{Spec}\, A$, $V(1) = \emptyset$ and $\bigcap V(I_i) = V(\sum_i I_i)$. The action of Spec on a morphism $f\colon A \longrightarrow B$ is given by:

$$\begin{aligned} \mathrm{Spec}\, f\colon \mathrm{Spec}\, B &\longrightarrow \mathrm{Spec}\, A \\ \mathfrak{p} &\longmapsto f^{-1}(\mathfrak{p}) \end{aligned}.$$

The map is well defined, to finish we need to show that it is continuous. Let $V(I) \subset \mathrm{Spec}\, A$ we have:

$$\mathrm{Spec}(f)^{-1}(V(I)) = \{\mathfrak{p} \in \mathrm{Spec}\, B \mid f^{-1}(\mathfrak{p}) \in V(I)\} = V(f(I)^e)$$

$f(I)^e$ is the ideal generated by $f(I)$.

Example 2.1.18. For every $C \in |\mathscr{C}|$ we can define:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(-, C) \colon \mathscr{C}^{\mathbf{op}} &\longrightarrow \mathsf{Set} \\ A &\longmapsto \mathrm{Hom}_{\mathscr{C}}(A, C) \end{aligned}.$$

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

Diagram 2.2

Definition 2.1.19. Given two functors, $F, G \colon \mathscr{C} \longrightarrow \mathscr{D}$ a natural transformation denoted $F \overset{\eta}{\Longrightarrow} G$ is the following data:

- For every object $A \in |\mathscr{C}|$ a map $\eta_A \colon F(A) \longrightarrow G(A)$.
- For every map $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ a commutative square as in Diagram 2.2.

Definition 2.1.20. Two functors are naturally isomorphic if there exists a natural transformation $F \overset{\eta}{\Longrightarrow} G$ with all its components isomorphisms. We say that F is naturally isomorphic to G.

2.1.1 Arrows and functors: Properties.

Definition 2.1.21. An arrow $f \colon A \longrightarrow B$ is called an isomorphism if $\exists g \colon B \longrightarrow A$ such that $g \circ f = \mathbb{1}_A,\ \ f \circ g = \mathbb{1}_B$.

Definition 2.1.22. A monomorphism in a category $\mathscr{C}$ is an arrow $f \colon A \rightarrowtail B$ satisfying:

$$f \circ u = f \circ v \implies u = v.$$

Definition 2.1.23. An epimorphism in a category $\mathscr{C}$ is an arrow $f \colon A \twoheadrightarrow B$ satisfying:

$$u \circ f = v \circ f \implies u = v.$$

Example 2.1.24. In Set, monomorphisms and epimorphisms are injections and surjections respectively. We have an analogous situation in Gr.

Example 2.1.25. Let's consider the category whose objects are the pairs (X, x) where X is a connected topological space and $x \in X$ is a base point; in this category, a morphism is a continuous mapping preserving the base points. It is a basic result from algebraic topology that the circular helix $(\mathcal{H}, p)$ is a covering space of (S^1, q), given $f, g \colon (X, x) \longrightarrow (\mathcal{H}, p)$ such that $\pi \circ f = \pi \circ g$ we can regard $\pi \circ f$ as continuous map from (X, x) to (S^1, q) admitting a lifting f. The uniqueness property of the lifting [5] implies that π is a monomorphism but is far from being an injective map!

Example 2.1.26. In Rng, epimorphisms are not necessarily surjective ring homomorphisms. Consider for example:

$$i \colon \mathbb{Z} \hookrightarrow \mathbb{Q}$$

and two arrows $f, g \colon \mathbb{Q} \rightrightarrows A$ that agree on the integers. It is clear that i is not surjective. Since f, g agree on the integers and inverses are unique it follows that $f = g$.

Proposition 2.1.27. If $f \colon A \longrightarrow B$ is an isomorphism then f is a monomorphism and an epimorphism.

Proof. Let g denote the inverse of f. Then given $f \circ u = f \circ v$ composing with g we get $u = v$. The rest of the proof is obvious. □

Definition 2.1.28. Consider a functor $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ and for every pair of objects $A, A' \in |\mathscr{A}|$ the mapping

$$\begin{aligned} \mathrm{Hom}_{\mathscr{A}}(A, A') &\longrightarrow \mathrm{Hom}_{\mathscr{B}}(FA, FA') \\ f &\longrightarrow Ff. \end{aligned}$$

- The functor F is faithful when the abovementioned mappings are injective for every pair of objects.
- The functor F is full when the abovementioned mappings are surjective for every pair of objects.

Definition 2.1.29. A subcategory $\mathscr{B}$ of a category $\mathscr{A}$ consists of:

- a subclass $|\mathscr{B}| \subset |\mathscr{A}|$ of the class of objects,
- for every pair of objects $A, A' \in |\mathscr{A}|$ a subset $\mathrm{Hom}_{\mathscr{B}}(A, A') \subset \mathrm{Hom}_{\mathscr{A}}(A, A')$ which is closed under compositions and contains the identity in the case $A = A'$.

Remark 2.1.30. The definition of a subcategory gives rise to an inclusion functor $i \colon \mathscr{B} \longrightarrow \mathscr{A}$ which is faithful. A subcategory is called a full subcategory when the inclusion functor is also full.

Definition 2.1.31. A functor $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ is an equivalence of categories when there exists a functor $G \colon \mathscr{B} \longrightarrow \mathscr{A}$ such that $G \circ F \cong \mathbb{1}_{\mathscr{A}}$ and $F \circ G \cong \mathbb{1}_{\mathscr{B}}$

Theorem 2.1.32 (Characterization of equivalences). *A functor $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ is an equivalence of categories when:*

- *F is full and faithful.*
- *For every object $B \in |\mathscr{B}|$ there is some object $A \in |\mathscr{A}|$ such that $FA \cong B$. This condition recieves the name of being essentially surjective on objects.*

Proof. We need to construct the "inverse" of this functor. For every object $B \in |\mathscr{B}|$ we choose an object GB in $\mathscr{A}$ and an isomorphism η_B between $F(GB) \cong B$. Let $B \xrightarrow{f} C$, Gf is defined to be the only map in $\mathrm{Hom}_{\mathscr{A}}(GB, GC)$ such that $F(Gf) = \eta_C^{-1} \circ f \circ \eta_B$. This can be done since F is full and faithful. It follows from the definition that G is a functor and η is the desired natural isomorphism. □

2.1.2 Representable Functors and Yoneda Lemma.

Definition 2.1.33. A functor $F \colon \mathscr{C}^{\mathbf{op}} \longrightarrow \mathsf{Set}$, is called representable if F is naturally isomorphic to $\mathrm{Hom}_{\mathscr{C}}(-, A)$ for some object A of $\mathscr{C}$.

Proposition 2.1.34. Given an arbitrary category $\mathscr{C}$ there exists a embedding functor into the presheaf category, called the Yoneda embedding.

$$\begin{aligned} Y : \mathscr{C} &\longrightarrow \mathsf{Set}^{\mathscr{C}^{\mathbf{op}}} \\ A &\longmapsto \mathrm{Hom}_{\mathscr{C}}(-, A). \end{aligned}$$

This embedding induces an equivalence of categories between $\mathscr{C}$ and the subcategory of representable functors.

Proof. If we are given some map $f\colon A \longrightarrow B$ in $\mathscr{C}$, we can associate to it the natural transformation $\mathrm{Hom}_{\mathscr{C}}(-, f)$ defined objectwise as:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(C, f)\colon \mathrm{Hom}_{\mathscr{C}}(C, A) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(C, B) \\ g &\longmapsto f \circ g. \end{aligned}$$

We find that Y is clearly a functor. The rest of the proof follows from Proposition 2.1.35. □

$$\begin{array}{ccc} \mathrm{Hom}(C, A) & \xrightarrow{\theta_C} & F(C) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}(B, A) & \xrightarrow[\theta_B]{} & F(B) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}(A, A) & \xrightarrow{\eta_A} & F(A) \\ {\scriptstyle -\circ g}\downarrow & & \downarrow{\scriptstyle F(g)} \\ \mathrm{Hom}(B, A) & \xrightarrow[\eta_B]{} & F(B) \end{array}$$

Diagram 2.3.1 Diagram 2.3.2

Proposition 2.1.35 (Yoneda Lemma). Given an arbitrary category $\mathscr{C}$ and a presheaf $F\colon \mathscr{C}^{\mathbf{op}} \longrightarrow \mathsf{Set}$ there is a cannonical isomorphism,

$$\mathrm{Hom}_{[\mathscr{C}^{\mathbf{op}}, \mathsf{Set}]}\Big(\mathrm{Hom}_{\mathscr{C}}(-, A), F\Big) \cong F(A),$$

natural in A.

Proof. Given a natural transformation $\mathrm{Hom}_{\mathscr{C}}(-, A) \overset{\eta}{\Longrightarrow} F$ and its component η_A we can define:

$$\begin{aligned} \tau\colon \mathrm{Hom}_{[\mathscr{C}^{\mathbf{op}}, \mathsf{Set}]}\Big(\mathrm{Hom}_{\mathscr{C}}(-, A), F\Big) &\longrightarrow F(A) \\ \eta &\longmapsto \eta_A(\mathbb{1}_A) \end{aligned}$$

The construction of the inverse θ is slighty more delicate. We need to be able to produce a natural transformation from an element of a set. Choosing some $\alpha \in F(A)$ we proceed to define the family of maps:

$$\begin{aligned} \theta_{B,\alpha}\colon \mathrm{Hom}_{\mathsf{Set}}(B, A) &\longrightarrow F(B) \\ g &\longmapsto F(g)(\alpha) \end{aligned}$$

Given $f\colon B \longrightarrow C$ we need to show commutativity of Diagram 2.3.1. For every $h \in \mathrm{Hom}_{\mathscr{C}}(C, A)$, $F(f) \circ \theta_{C,\alpha}(h) = F(f) \circ F(h)(\alpha) = F(h \circ f)(\alpha) = \theta_{B,\alpha} \circ (- \circ f)(h)$.

We claim that θ, τ are mutual inverses. Indeed starting with $\alpha \in F(A)$

$$\tau \circ \theta = \theta_{A,\alpha}(\mathbb{1}_A) = F(\mathbb{1}_A)(\alpha) = \alpha.$$

Let $\alpha = \eta_A(\mathbb{1}_A)$, and θ_α the corresponding natural transformation. We want to see that $\theta_\alpha = \eta$

to prove this we pick some $g \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ and form Diagram 2.3.2. Then:

$$\theta_\alpha(g) = F(g)(\alpha) = F(g)(\eta_A(\mathbb{1}_A)) = F(g) \circ \eta_A(\mathbb{1}_A) = \eta_B \circ (- \circ g)(\mathbb{1}_A) = \eta_B(g).$$

We saw previously that Y is a functor, so it makes sense to consider a new functor:

$$\begin{aligned} N\colon \mathscr{C}^{\mathbf{op}} &\longrightarrow \mathsf{Set} \\ A &\longmapsto \mathrm{Hom}_{[\mathscr{C}^{\mathbf{op}},\mathsf{Set}]}\Big(\mathrm{Hom}_{\mathscr{C}}(-, A), F\Big) \end{aligned}$$

For $f\colon A \longrightarrow B$ we set:

$$\begin{aligned} N(f)\colon \mathrm{Hom}_{[\mathscr{C}^{\mathbf{op}},\mathsf{Set}]}\Big(\mathrm{Hom}_{\mathscr{C}}(-, B), F\Big) &\longrightarrow \mathrm{Hom}_{[\mathscr{C}^{\mathbf{op}},\mathsf{Set}]}\Big(\mathrm{Hom}_{\mathscr{C}}(-, A), F\Big) \\ \eta &\longmapsto \eta \circ \mathrm{Hom}_{\mathscr{C}}(-, f) \end{aligned}$$

Finally we are claiming that there exists a natural transformation (in fact a natural isomorphism):

$$\gamma\colon N \Longrightarrow F$$

Let $\eta \in \mathrm{Hom}_{[\mathscr{C}^{\mathbf{op}},\mathsf{Set}]}\Big(\mathrm{Hom}_{\mathscr{C}}(-, B), F\Big)$ and $\tau(\eta)$. We can see that:

$$F(f) \circ \tau(\eta) = F(f)(\eta_B(\mathbb{1}_B)) = \eta_A \circ (- \circ f)(\mathbb{1}_B) = \eta_A(f)$$

And composing in the other side of the commutative square yields:

$$\tau \circ N(f)(\eta) = \tau(\eta \circ \mathrm{Hom}_{\mathscr{C}}(-, f)) = (\eta \circ \mathrm{Hom}_{\mathscr{C}}(-, f))_A(\mathbb{1}_A) = \eta_A \circ (f \circ -)(\mathbb{1}_A) = \eta_A(f) \quad \square$$

Remark 2.1.36. If the domain category $\mathscr{C}$ is small then the presheaf category is locally small and we can prove that there exists a bijection natural in F. The reader is referred to [2].

Corollary 2.1.37. As we pointed out before, Y induces an equivalence of categories. It follows from the observation that $\mathrm{Hom}_{\mathscr{C}}(A, B) \cong \mathrm{Hom}_{[\mathscr{C}^{\mathbf{op}},\mathsf{Set}]}(Y(A), Y(B))$. Clearly given $f, g \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ it's easy to check that $\theta_f = \mathrm{Hom}_C(f, A)$ and the same for g so they cannot be mapped to the same natural transformation. Moreover, every natural transformation $\eta\colon Y(A) \Longrightarrow Y(B)$ can be mapped to $\eta_A(\mathbb{1}_A) \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ so our functor is faithful and full. Finally since every representable functor is isomorphic to some $Y(A)$ for some $A \in |\mathscr{C}|$ it follows that Y is essentially surjective on representable functors.

2.2 Limits and Colimits

Definition 2.2.1. Consider a small category $\mathscr{I}$ and an object $C \in \mathscr{C}$. The constant functor Δ_C sends every object $I \in |\mathscr{I}|$ to C and every morphism to $\mathbb{1}_C$.

Remark 2.2.2. Δ defines a functor $\mathscr{C} \longrightarrow \mathscr{C}^{\mathscr{I}}$ by setting $\Delta C = \Delta_C$.

Definition 2.2.3. Let $\mathscr{I}$ be a small category and $F\colon \mathscr{I} \longrightarrow \mathscr{C}$, a cone on F is a natural transformation $\Delta_C \Longrightarrow F$. $\mathscr{I}$ is called the index category.

Remark 2.2.4. For every cone we can consider its dual called a cocone. A cocone is a natural transformation $F \Longrightarrow \Delta_C$ where $F\colon \mathscr{I} \longrightarrow \mathscr{C}$.

Definition 2.2.5. A pair $(\lim FI, \theta)$ with $\lim F \in |\mathscr{C}|, \Delta_{\lim FI} \overset{\theta}{\Longrightarrow} F$ is called a limit if for every natural transformation $\Delta_C \overset{\eta}{\Longrightarrow} F$ there exists a unique natural transformation $\Delta_C \overset{\xi}{\Longrightarrow} \Delta_{\lim F}$ satisfying $\theta \circ \xi = \eta$. This last property is called the universal property of the limit. The components of θ will be denoted p_I.

Remark 2.2.6. Is clear from the universal property that the limit of F is unique up to isomorphism.

Remark 2.2.7. Dualizing this argument we get the notion of colimit

Remark 2.2.8. Since $\mathscr{I}$ is a small category, $[\mathscr{I}, \mathscr{C}]$ is a category. So it makes sense to consider the functor,

$$\mathrm{Hom}_{[\mathscr{I},\mathscr{C}]}(\Delta, F) \colon \mathscr{C}^{\mathbf{op}} \longrightarrow \mathsf{Set}$$

In this light, the existence of a limit is equivalent to the existence of a representing object for the functor.

Proposition 2.2.9. The functor $F \colon \mathscr{I} \longrightarrow \mathscr{C}$ has a limit if and only if $\mathrm{Hom}_{[\mathscr{I},\mathscr{C}]}(\Delta, F) \cong \mathrm{Hom}_{\mathscr{C}}(-, C)$.

Proof. $(\Rightarrow)$ This is clear since $\mathrm{Hom}_{[\mathscr{I},\mathscr{C}]}(\Delta, F) \cong \mathrm{Hom}_{[\mathscr{I},\mathscr{C}]}(\Delta, \Delta_C) \cong \mathrm{Hom}_{\mathscr{C}}(-, C)$.

$(\Leftarrow)$ Let α denote the natural isomorphism. Given $\mathbb{1}_C \in \mathrm{Hom}_{\mathscr{C}}(C, C)$ we can map it to a natural transformation $\alpha(\mathbb{1}_C) = \theta \in \mathrm{Hom}_{[\mathscr{I},\mathscr{C}]}(\Delta_C, F)$. Now given $\eta \in \mathrm{Hom}_{[\mathscr{I},\mathscr{C}]}(\Delta_D, F)$ it corresponds to an element $f \in \mathrm{Hom}_{\mathscr{C}}(D, C)$. We claim that $\eta = \theta \circ \Delta(f)$ but using naturality we can see that $\alpha(\theta \circ \Delta(f)) = f$ finishing the proof.

□

Definition 2.2.10. A category $\mathscr{C}$ is called complete (respect cocomplete) when every functor

$$F \colon \mathscr{I} \longrightarrow \mathscr{C}$$

with $\mathscr{I}$ a small category, has a limit.

A category $\mathscr{C}$ is called finitely complete (respect cocomplete) when every functor

$$F \colon \mathscr{I} \longrightarrow \mathscr{C}$$

with $\mathscr{I}$ a finite category, has a limit.

Example 2.2.11 (Limits in Set). Given $F \colon \mathscr{I} \longrightarrow \mathsf{Set}$ the description of the limit is given by

$$\lim I \in |\mathscr{I}| FI = \{(x_I)_{I \in \mathscr{I}} \mid x_I \in FI;\ \ \forall f \colon I \longrightarrow I' \ \ Ffx_I = x_{I'}\}$$

2.2.1 Important limits and colimits.

Diagram 2.4.1

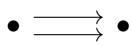

Diagram 2.4.2

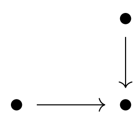

Diagram 2.4.3

Consider an index category $\mathscr{I}$, diagrams like Diagram 2.4.1 show some of the most common limits in category theory. Limits over these diagrams are called product, equalizer and pullback with their respective duals coproduct, coequalizer and pushout. Since this abstract definition may not sound very familiar to the reader we will show some examples of those limits in several categories. Limits have the advantage of defining in a element free way many common constructions in mathematics up to isomorphism.

Example 2.2.12 (Products). Products in category theory are adequate generalizations of the cartesian product. We list some incarnations of this concept

- The product in Set is the cartesian product together with projection maps
- The product in Gr is the product of groups together with projection homomorphisms
- The product in Top is the product of topological spaces endowed with product topology and projection maps

Example 2.2.13 (Coproducts). Coproducts in category theory are adequate generalizations of the disjoint union of sets, we list some incarnations of this concept

- The coproduct in Set is the disjoint union of sets together with inclusion maps
- The coproduct in Gr is the free product of groups together with inclusion homomorphisms
- The coproduct in Top is the disjoint union of topological spaces endowed with the disjoint union topology and injection maps
- The coproduct in $\mathsf{Psh}(X)$ is the pointwise coproduct together with the induced natural transformations

Example 2.2.14 (Equalizers). We will illustrate the behaviour of equalizers through their description in Set. Given two maps $f, g\colon A \rightrightarrows B$ the equalizer of this diagram is given by the set $C = \{a \in A \mid f(a) = g(a)\}$ together with the inclusion of map, i. The pair (C, i) is clearly a limit. Another useful application of this concept arises in the category Ab. Consider an equalizer diagram given by $f, 0\colon A \rightrightarrows B$, where 0 is the map sending every element to 0. Imitating the construction in Set we can form the group $\operatorname{Ker} f = \{a \in A \mid f(a) = 0\}$ together with the inclusion homomorphism.

Example 2.2.15 (Coequalizers). First we will give the description of coequalizers in Set. Let $f, g\colon A \longrightarrow B$, let "$\sim$" be the smallest equivalence relation such that $f(a) \sim g(a),\ \forall a \in A$. Then the pair $(B/\sim, \pi)$, with π the projection is the desired coequalizer. Indeed, given another pair (D, h) satisfying commutativity conditions we can see that h only depends on the class of equivalence of each $b \in B$ so factors uniquely through $B/\sim$. Algebraic analogues are quite simple. For example given the diagram $f, g\colon A \longrightarrow B$ in Ab the coequalizer is just $\operatorname{Coker}(f - g)$.

Example 2.2.16 (Pullbacks in Set). In the category Set of sets and mappings, given a pair of maps (f, g) in a pullback diagram their pullback is given by

$$\{(a, b) \mid a \in A,\ \ b \in B,\ \ f(a) = g(b)\}$$

The maps of this cone are the usual projections.

Example 2.2.17 (Pushouts in Gr). Since coproducts exists in Gr, given a pushout diagram $\varphi_1\colon H \longrightarrow G_1, \varphi_2\colon H \longrightarrow G_2$ we can construct the colimit in the following way. Take the coproduct of G_1, G_2, with cannonical injections i_1, i_2. In this group we consider the equivalence relation given by $g_1 \sim g_2$ if $\exists h \in H$ such that $i_1 \circ \varphi_1(h) = g_1$ and $i_2 \circ \varphi_2(h) = g_2$ and take the quotient by the subgroup generated by this equivalence relation. The result is the pushout of the diagram Diagram 2.4.3.

Proposition 2.2.18. An arrow $f\colon A \longrightarrow B$ is a monomorphism if and only if the pullback with itself is given by $(A, \mathbb{1}_A)$.

Proof. Obvious. $\square$

Example 2.2.19 (Initial and terminal objects). The limit over the empty diagram in a category $\mathscr{C}$ is called the initial object, the dual notion is the terminal object. Terminal and initial objects in Set are given by the singleton $\{*\}$ and the empty set $\emptyset$.

2.2.2 Interchange of limits

$$\begin{array}{ccccc}
\lim\limits_{D\in|\mathscr{D}|} F(C,D) & \xrightarrow{L(c_1)} & \lim\limits_{D\in|\mathscr{D}|} F(C',D) & \xrightarrow{L(c_2)} & \lim\limits_{D\in|\mathscr{D}|} F(C'',D) \\
\downarrow & & \downarrow & & \downarrow \\
F(C,D) & \longrightarrow & F(C',D) & \longrightarrow & F(C'',D)
\end{array}$$

Diagram 2.5.1

Consider a functor $F\colon \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{A}$ with $\mathscr{C}, \mathscr{D}$ small. For every $C \in |\mathscr{C}|$ there is a functor

$$F(C,-)\colon \mathscr{D} \longrightarrow \mathscr{A},\ D \longmapsto F(C,D)$$

whose action on a morphism $D \xrightarrow{d} D'$ is given by $F(\mathbb{1}_C, f)$. Similarly given an arrow $C \xrightarrow{c} C'$ we can define a natural transformation $F(C,-) \overset{\eta}{\Longrightarrow} F(C',-)$ with components $F(c, \mathbb{1}_D)$. Now suppose that $\lim\limits_{D\in|\mathscr{D}|} F(C,D)$, $\lim\limits_{D\in|\mathscr{D}|} F(C',D)$ exist. We are going to show that these limits form a $\mathscr{C}$-indexed diagram in $\mathscr{A}$. To see this consider the following family of maps

$$q'_D\colon \lim_{D\in|\mathscr{D}|} F(C,D) \xrightarrow{p_D} F(C,D) \xrightarrow{\eta_D} F(C',D).$$

We will check that we have just defined a cone on $\lim\limits_{D\in|\mathscr{D}|} F(C',D)$

$$F(\mathbb{1}_{C'}, d) \circ q'_D = F(\mathbb{1}_C, d) \circ F(c, \mathbb{1}_D) \circ p_D = F(c,d) \circ p_D = F(c, \mathbb{1}_{D'}) \circ F(\mathbb{1}_C, d) \circ p_D = {q'}_{D'}.$$

By basic property of limits we get a unique factorization map $\lim\limits_{D\in|\mathscr{D}|} F(C,D) \longrightarrow \lim\limits_{D\in|\mathscr{D}|} F(C',D)$. We will show that the following correspondence defines a functor

$$\begin{aligned} L\colon \mathscr{C} &\longrightarrow \mathscr{A} \\ C &\longrightarrow \lim_{D\in|\mathscr{D}|} F(C,D). \end{aligned}$$

Remark 2.2.20. If for every $C \in |\mathscr{C}|$ the limit $\lim\limits_{D\in|\mathscr{D}|} F(C,D)$ exists then L is a functor. This can be easily shown by commutativity of Diagram 2.5.1.

Definition 2.2.21 (Interchange property). The limit of L is denoted by $\lim\limits_{C\in|\mathscr{C}|}(\lim\limits_{D\in|\mathscr{D}|} F(C,D))$. We can define similarly L' using the functor $F(-,D)$. Then the interchange property is phrased:

$$\lim_{C\in|\mathscr{C}|}(\lim_{D\in|\mathscr{D}|} F(C,D)) \cong \lim_{D\in|\mathscr{D}|}(\lim_{C\in|\mathscr{C}|} F(C,D))$$

meaning that the cannonical morphisms λ, μ, connecting these limits are in fact isomorphisms.

We will only construct λ, since the construction of μ is analogous. Consider the following map

$$\lim_{C\in|\mathscr{C}|} LC \xrightarrow{p_C} \lim_{D\in|\mathscr{D}|} F(C,D) \xrightarrow{p_D} F(C,D)$$

and let $C \longrightarrow C'$. Then clearly $F(c,\mathbb{1}_D)\circ p_D \circ p_C = p'_D \circ \lim_{D\in|\mathscr{D}|} F(c,\mathbb{1}_D)\circ p_C = p'_D \circ p'_C$ where p'_D denotes the projection of the functor $F(C',-)$. This means that we have a cone on the functor $F(-,C)$ with the consequent factorization

$$\lim_{C\in|\mathscr{C}|} LC \xrightarrow{\lambda_D} \lim_{C\in|\mathscr{C}|} F(C,D)$$

satisfying $\overline{p}_C \circ \lambda_D = p_D \circ p_C$ with $\overline{p}_C$ the corresponding projection of $\lim\limits_{C\in|\mathscr{C}|} F(C,D)$. We are going to show that $\{\lambda_D\}_{D\in|\mathscr{D}|}$ constitutes a cone for $\lim\limits_{D\in|\mathscr{D}|}(\lim\limits_{C\in|\mathscr{C}|} F(C,D))$. Denote $\overline{p}'_C$ for the projection of $\lim\limits_{C\in|\mathscr{C}|} F(C,D')$. Given $D \xrightarrow{d} D'$

$$\overline{p}'_C \circ \lim_{C\in|\mathscr{C}|}(\mathbb{1}_C,d)\circ\lambda_D = F(\mathbb{1}_C,d)\circ\overline{p}_C\circ\lambda_D = F(\mathbb{1}_C,d)\circ p_D\circ p_C = \overline{p}'_C\circ\lambda_{D'}.$$

This implies that $F(\mathbb{1}_C,d)\circ\lambda_D = \lambda_{D'}$ by the uniqueness of the factorization. This implies the existence of λ and by a similar argument the existence of μ. The interchange property says these two arrows are inverses to each other. If $\mathscr{A}$ is complete then the fact that $p_D\circ p_C\circ\mu\circ\lambda = p_D\circ p_C$ implies $\mu\circ\lambda = \mathbb{1}$ so the interchange property holds.

2.2.3 Filtered colimits

Let $F\colon \mathscr{C}\times\mathscr{D} \longrightarrow \mathscr{A}$ as in the previous section. Assuming when necessary the existence of limits and colimits in $\mathscr{A}$ let

$$\lim_{D\in|\mathscr{D}|} F(C,D) \xrightarrow{p_D} F(C,D) \xrightarrow{s_C} \operatorname*{colim}_{C\in|\mathscr{C}|} F(C,D).$$

Following the arguments of the previous section is easy to see that these maps define a cocone inducing

$$\lambda_D\colon \operatorname*{colim}_{C\in|\mathscr{C}|}(\lim_{D\in|\mathscr{D}|} F(C,D)) \longrightarrow \operatorname*{colim}_{C\in|\mathscr{C}|} F(C,D).$$

Finally the family $\{\lambda_D\}_{D\in|\mathscr{D}|}$ is a family of morphisms that constitutes a cone obtaining the analogous λ to the one presented in subsubsection 2.2.2.

Definition 2.2.22. A category $\mathscr{C}$ is filtered when

- $\mathscr{C}$ is not empty.
- $\forall C_1, C_2 \in \mathscr{C} \quad \exists C_3 \in \mathscr{C} \quad \exists f\colon C_1 \longrightarrow C_3 \quad \exists g\colon C_2 \longrightarrow C_3.$
- $\forall C_1, C_2 \in \mathscr{C} \quad \forall f,g\colon C_1 \rightrightarrows C_2 \quad \exists C_3 \in \mathscr{C} \quad \exists h\colon C_2 \longrightarrow C_3, \quad h\circ f = h\circ g.$

Example 2.2.23. Let $\mathcal{O}(X)$, the neighbourhoods of a given point $x\in X$ form a filtered category.

The aim of this section is to show that filtered colimits commute with finite limits, to do this we are going to assume the existence of a cocone for every functor from a filtered category to Set. This is proved in Lemma 2.13.2 of [2].

Theorem 2.2.24. *Given a small filtered category $\mathscr{C}$, and a functor $F\colon \mathscr{C} \longrightarrow \mathsf{Set}$, then the* *limit of F exists and is given by*

$$L = \bigsqcup FC/\sim \qquad s_C\colon FC \longrightarrow L,\ x \longmapsto [x].$$

where $\sim$ denotes the equivalence relation defined as follows:

$$(x \in FC) \sim (x' \in FC') \iff \exists C'' \in |\mathscr{C}|,\ \exists f\colon C \longrightarrow C'' \quad \exists g\colon C' \longrightarrow C'' \quad Ff(x) = Fg(x').$$

Proof. First let us observe that $\sim$ is clearly reflexive and symmetric. To prove transitivity let $(x \in FC) \sim (x' \in FC')$, $(x' \in FC') \sim (x'' \in FC'')$. Let f_1, g_1, f_2, g_2 such that

$$FC \xrightarrow{Ff_1} FC_1,\ \ FC' \xrightarrow{Fg_1} FC_1,\ \ FC' \xrightarrow{Ff_2} FC_1,\ \ FC'' \xrightarrow{Fg_2} FC_2.$$

$$Ff_1(x) = Fg_1(x') \quad Ff_2(x') = Fg_2(x'').$$

The axioms of filtered categories implies that we can find some C_3 and morphisms f_3, g_3

$$FC_1 \xrightarrow{Ff_3} FC_3,\ \ FC_2 \xrightarrow{Fg_3} FC_3.$$

Using again the axiom of filtered categories we can find some $C_3 \xrightarrow{h} C_4$ such that $h \circ f_3 \circ g_1 = h \circ g_3 \circ f_2$. As a final comprobation we can see that:

$$Fh \circ Ff_3 \circ Ff_1(x) = F(h \circ f_3 \circ g_1)(x') = F(h \circ Fg_3 \circ f_2)(x') = Fh \circ Fg_3 \circ Fg_2(x'').$$

It is obvious that L is a cocone. The only thing left to prove is that L is a universal cocone. Given L' and compatible morphisms s'_C set

$$\begin{aligned} \alpha\colon L &\longrightarrow L' \\ [x_C] &\longmapsto s'_C(x). \end{aligned}$$

We left the comprobation to the reader that α is the desired factorization. $\square$

Theorem 2.2.25. *Consider a small filtered category $\mathscr{C}$ and a finite category $\mathscr{D}$. Given a functor $F\colon \mathscr{C} \times \mathscr{D} \longrightarrow \mathsf{Set}$ the following mixed interchange property holds:*

$$\operatorname*{colim}_{C \in |\mathscr{C}|}(\lim_{D \in |\mathscr{D}|} F(C, D)) \cong \lim_{D \in |\mathscr{D}|}(\operatorname*{colim}_{C \in |\mathscr{C}|} F(C, D)).$$

Proof. In this section we showed how filtered colimits look in Set. Also, remembering our description of limits in Set in Example 2.2.11 we have the following description of λ:

$$\lambda\colon [(x_D)_{D \in |\mathscr{D}|}] \longmapsto ([x_D])_{D \in |\mathscr{D}|}.$$

We are going to show that λ is bijective. Let $(x_D)_{D \in |\mathscr{D}|} \in F(C, D)$, $(y_D)_{D \in |\mathscr{D}|} \in F(C', D)$ and assume that $([x_D])_{D \in |\mathscr{D}|} = ([y_D])_{D \in |\mathscr{D}|}$ this implies $[x_D] = [y_D]$ for every index D. This means that we can find arrows

$$C \xrightarrow{f_D} C_D,\ \ C' \xrightarrow{g_D} C_D,\ \text{such that,} \quad F(f_D, \mathbb{1}_D)(x_D) = F(g_D, \mathbb{1}_D)(y_D).$$

We have arrived to a diagram in $\mathscr{C}$ formed by $(C, f_D), (C', g_D)$ indexed by the finite set $|\mathscr{D}|$, using the axioms of filtered categories we may form a cocone of this diagram C''. In particular we get

$$C \xrightarrow{f} C'', \;\; C' \xrightarrow{g} C'', \text{ such that, } F(f, \mathbb{1}_D)(x_D) = F(g, \mathbb{1}_D)(y_D) \;\; \forall D \in |\mathscr{D}.|$$

This precisely means $\lim D \in |\mathscr{D}| F(f, \mathbb{1})((x_D)_{D\in|\mathscr{D}|}) = \lim\limits_{D\in|\mathscr{D}|} F(g, \mathbb{1})((x_D)_{D\in|\mathscr{D}|})$, so both elements represent the same class thus λ is injective.

Let $([x_D])_{D\in|\mathscr{D}|}$ and pick representatives $(x_D) \in F(C_D, D)$. Now form a cocone given by $C_D \xrightarrow{f_D} C$, then each $F(f_D, \mathbb{1}_D)(x_D)$ is another family of representatives. Let $D \xrightarrow{d} D'$, then by the properties of limits in Set we know that $F(\mathbb{1}_D, d)(x_D) \sim x'_D$ thus we can identify $F(f_D, d)(x_D)$ with $F(f'_D, \mathbb{1}_D)(x_{D'})$. It is a consequence of this identification that there exists maps

$$g_d, h_d \colon C \rightrightarrows C_d \text{ such that } F(g_d \circ f_D, d)(x_D) = F(h_d \circ f_{D'}, \mathbb{1}_{D'})(x_{D'}).$$

Construct a cocone with the diagram given by $\{g_d, h_d\}$, finally arriving to

$$C \xrightarrow{k} C' \text{ such that } \;\; F(k \circ f_D, d)(x_D) = F(k \circ f_{D'}, \mathbb{1}_{D'})(x_{D'}).$$

As a final step we only need to set $[(F(k \circ f_D, \mathbb{1}_D))_{D\in|\mathscr{D}|}]$. This shows surjectivity. □

2.3 Adjunctions

$$\begin{array}{ccc} F & \xrightarrow{\eta * F} & FGF \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle F*\varepsilon} \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{G * \eta} & GFG \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \varepsilon*G} \\ & & G \end{array}$$

Diagram 2.6.1 Diagram 2.6.2

Definition 2.3.1. Consider two functors $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ and $G \colon \mathscr{B} \longrightarrow \mathscr{A}$. G is left adjoint to F (or F is right adjoint to G) if there exist natural transformations $\eta \colon \mathbb{1}_{\mathscr{B}} \Longrightarrow F \circ G$ and $\varepsilon \colon G \circ F \Longrightarrow \mathbb{1}_{\mathscr{A}}$ called unit and counit respectively such that

$$(F * \varepsilon) \circ (\eta * F) = \mathbb{1}_F, \quad (\varepsilon * G) \circ (G * \eta) = \mathbb{1}_G.$$

(see Diagram 2.6.1 and Diagram 2.6.2). These are called triangular identities.

Remark 2.3.2. An adjunction can be thought as a weaker version of an equivalence of categories. The categories involved will have different features but their similarities will play a crucial role in many constructions.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}}(GB, A) & \xrightarrow{\theta_{A,B}} & \mathrm{Hom}_{\mathscr{B}}(B, FA) \\ {\scriptstyle a\circ -}\downarrow & & \downarrow{\scriptstyle Fa\circ -} \\ \mathrm{Hom}_{\mathscr{A}}(GB, C) & \xrightarrow{\theta_{C,B}} & \mathrm{Hom}_{\mathscr{B}}(B, FC) \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\eta_B} & FGB \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle FG(b)} \\ C & \xrightarrow{\eta_C} & FGC \end{array}$$

Diagram 2.7.1 Diagram 2.7.2

Theorem 2.3.3. *The following statements are equivalent*

1. *G is left adjoint to F*

2. For every $A \in |\mathscr{A}|$, $B \in |\mathscr{B}|$ there exists a natural bijection in A and in B

$$\theta_{A,B}: \ \operatorname{Hom}_{\mathscr{A}}(GB, A) \cong \operatorname{Hom}_{\mathscr{B}}(B, FA).$$

Proof. (1) ⇒ (2) Given $f \in \operatorname{Hom}_{\mathscr{A}}(GB, A)$ and $g \in \operatorname{Hom}_{\mathscr{B}}(B, FA)$ set

$$\theta_{A,B}(f) = Ff \circ \eta_B, \quad \tau_{A,B}(g) = \varepsilon_A \circ Gg.$$

It is a consequence of ε, η being natural transformations that,

$$\varepsilon_A \circ GF(f) = f \circ \varepsilon_{GB}, \quad \eta_{FA} \circ g = FG(g) \circ \eta_B,$$

also is a consequence of triangular identities that,

$$\varepsilon_{GB} \circ G(\eta_B) = \mathbb{1}_{GB}, \quad F(\varepsilon_A) \circ \eta_{FA} = \mathbb{1}_{FA}.$$

Then we can verify the following,

$$\tau_{A,B}(\theta_{A,B} f) = \varepsilon_A \circ GF(f) \circ G(\eta_B) = f \circ \varepsilon_{GB} \circ G(\eta_B) = f, \quad \theta_{A,B} \circ \tau_{A,B}(g) = g.$$

To check naturality in A (in B is completely analogous) let $A \xrightarrow{a} C$ we will check commutativity of Diagram 2.7.1

$$Fa \circ \theta_{A,B}(f) = Fa \circ Ff \circ \eta_B = F(a \circ f)\eta_B.$$

(2) ⇒ (1) Given $B \in |\mathscr{B}|$ we set $\eta_B = \theta_{GB,B}(\mathbb{1}_{GB})$. Let $B \xrightarrow{b} C$, and form the square of Diagram 2.7.2 it is a consequence of the naturality of the bijection that

$$FG(b) \circ \eta_B = \theta_{GC,B}(Gb) = b \circ \theta_{GC,C}(\mathbb{1}_{GC}).$$

We leave the analogous comprobations to the reader. □

Corollary 2.3.4. Adjoints are unique up to unique isomorphism

Proof. Let G, H be left adjoints to F, together with their natural Hom-set bijections $\theta, \overline{\theta}$ for every $B \in |B|$ set

$$\eta_B : GB \longrightarrow HB, \quad \eta_B = \theta^{-1}_{HB,B} \circ \overline{\theta}_{HB,B}(\mathbb{1}_{HB}).$$

Naturality of both bijections implies naturality of η, moreover each η_B has an inverse given by

$$\eta_B^{-1} = \overline{\theta}^{-1}_{GB,B} \circ \theta_{GB,B}(\mathbb{1}_{GB}).$$

It is clear from the construction that the isomorphism is unique. □

Theorem 2.3.5. *Given a diagram $H : \mathscr{I} \longrightarrow \mathscr{A}$ with a limit $L \in |\mathscr{A}|$ if $F : \mathscr{A} \longrightarrow \mathscr{B}$ is right adjoint to G then $F(L)$ is a limit in $\mathscr{B}$.*

Proof. Let $\Delta_{FL} \Rightarrow F \circ H$ be a cone. Suppose another cone $\Delta_B \Rightarrow F \circ H$, then every map $B \longrightarrow F \circ H(I)$ corresponds to some map $GB \longrightarrow H(I)$ that gives rise to a cone due to the naturality of the bijection $\theta_{A,B}$. Then the universal property of L induces a factorization map $GB \longrightarrow L$. The image of the factorization map via θ can be shown to be a factorization

$B \longrightarrow F(L)$ using the naturality of the bijection. Finally it is clear that this map is the only map with this property showing that $F(L)$ is a limit. $\square$

Example 2.3.6 (Free and forgetful functors). Let $G \in |\mathsf{Gr}|$ the forgetful functor $U\colon \mathsf{Gr} \longrightarrow \mathsf{Set}$ that maps every group to its underlying set and every homomorphism to the underlying mapping. This functor has a left adjoint F, the free functor. F maps every set A to the free group generated by the elements of A. Clearly given $A \xrightarrow{f} B$ we have

$$\begin{aligned} Ff\colon FA &\longrightarrow FB \\ a_1 * a_2 * \cdots * a_n &\longmapsto f(a_1) * f(a_2) * \cdots * f(a_n). \end{aligned}$$

Construction of the unit is given by the fact that we can map each element $a \in A$ to the one letter word "a". Similarly it easy to set

$$\begin{aligned} \varepsilon_G\colon FUG &\longrightarrow G \\ g_1 * g_2 * \cdots * g_n &\longmapsto g_1 \cdot g_2 \cdots g_n. \end{aligned}$$

2.3.1 Cartesian closed categories

Definition 2.3.7. Let $\mathscr{C}$ be a category with finite products and a terminal object denoted by 1. $\mathscr{C}$ is called cartesian closed if for every $C \in |\mathscr{C}|$ the functor

$$- \times X\colon \mathscr{C} \longrightarrow \mathscr{C},\ Y \longmapsto Y \times X$$

has a right adjoint denoted $\underline{\mathrm{Hom}}_{\mathscr{C}}(X, -) = -^X$ called the exponential functor or the internal $\underline{\mathrm{Hom}}$.

Remark 2.3.8. Cartesian closed categories have many important applications in logic. The reason of this is that the internal $\underline{\mathrm{Hom}}$ can efficiently imitate the Hom-set in the internal language of the category. The most basic example of cartesian closed category is Set.

Proposition 2.3.9. $\mathrm{Hom}_{\mathscr{C}}(1, Y^X)$ can be identified with the set of maps from X to Y.

Proof. First let's observe that in Set maps from the terminal object to a set can be identified with the elements of this set. Our problem can be reduced to show that $\mathrm{Hom}_{\mathscr{C}}(X, Y) \cong \mathrm{Hom}_{\mathscr{C}}(1 \times X, Y)$. Given $f \in \mathrm{Hom}_{\mathscr{C}}(1 \times X, Y)$ take X and let $\alpha = t \times \mathbb{1}_X$ be the map induced by $\mathbb{1}_X$ and the terminal map then $f \circ \alpha \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Finally given $g \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, take $g \circ p_X$ with p_X the projection of $1 \times X$. Then $f \circ (t \times \mathbb{1}_X) \circ p_X = f \circ (t \times p_X) = f \circ \mathbb{1} = f$ the other comprobation being obvious. $\square$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(Z, Y^X) & \xrightarrow{\ \theta\ } & \mathrm{Hom}_{\mathscr{C}}(Z \times X, Y) \\ {\scriptstyle -\circ f}\uparrow & & \uparrow{\scriptstyle -\circ(f \times \mathbb{1}_X)} \\ \mathrm{Hom}_{\mathscr{C}}(Y^X, Y^X) & \xleftarrow[\theta^{-1}]{} & \mathrm{Hom}_{\mathscr{C}}(Y^X \times X, Y) \end{array}$$

Diagram 2.8.1

Remark 2.3.10 (Monoidal Categories). The notion of the internal $\underline{\mathrm{Hom}}$ goes beyond cartesian closed categories. For example, in the category of $\mathscr{R}$-Modules is clear that we can endow the Hom-set with the structure of a module through pointwise addition, however this $\underline{\mathrm{Hom}}$ is not adjoint to the categorical product but to the tensor product of modules. It is enough to observe that the terminal object coincides with the initial object in $\mathscr{R}$-Mod so $\mathrm{Hom}_{\mathscr{R}}(1, B^C) = \{*\}$. The general theory can be understood through closed monoidal categories.

Definition 2.3.11 (Evaluation map). Under the cannonical bijections of the adjunction the image of $\mathbb{1}_{B^C} \in \mathrm{Hom}_{\mathscr{C}}(B^C, B^C)$ is called the evaluation map:

$$\mathrm{ev}_{B,C} \colon B^C \times C \longrightarrow B.$$

Proposition 2.3.12. The cannonical bijection satisfies:

$$\begin{aligned} \theta \colon \mathrm{Hom}_{\mathscr{C}}(Z, Y^X) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(Z \times X, Y) \\ f &\longmapsto \mathrm{ev}_{X,Y} \circ([f] \times \mathbb{1}_X). \end{aligned}$$

Proof. The proof is just a consequence of the naturality of the Diagram 2.8.1. It is useful to observe that given $[f] \in \mathrm{Hom}_{\mathscr{C}}(1, Y^X)$ coming from $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ then

$$(1 \times X \xrightarrow{p_X} X \xrightarrow{f} Y) = (1 \times X \xrightarrow{[f]\times 1_X} Y^X \times X \xrightarrow{ev_{X,Y}} Y).$$ □

Proposition 2.3.13. Consider a map $1 \xrightarrow{[f]} Y^X$ and a generalized point $1 \xrightarrow{x} X$ then the composite map

$$(1 \xrightarrow{[f]\times x} Y^X \times X \xrightarrow{ev_{X,Y}} Y) = (1 \xrightarrow{fx} Y).$$

Proof.

$$(1 \xrightarrow{[f]\times x} Y^X \times X \xrightarrow{ev_{X,Y}} Y) = (1 \longrightarrow 1 \times 1 \xrightarrow{t\times x} 1 \times X \xrightarrow{[f]\times 1_X} Y^X \times X \xrightarrow{ev_{X,Y}} Y) \cong$$

$$\cong (1 \longrightarrow 1 \times 1 \xrightarrow{t\times x} 1 \times X \xrightarrow{p_X} X \xrightarrow{f} Y) = (1 \xrightarrow{fx} Y)$$ □

Definition 2.3.14 (Internal composition). The map $c_{X,Y,Z} \in \mathrm{Hom}_{\mathscr{C}}(Z^Y \times Y^X, Z^X)$, called the internal composition is the image through the cannonical bijection of:

$$Z^Y \times Y^X \times X \xrightarrow{1\times ev_{X,Y}} Z^Y \times Y \xrightarrow{ev_{Y,Z}} Z.$$

Proposition 2.3.15. Given $[f] \in \mathrm{Hom}_{\mathscr{C}}(1, Y^X), [g] \in \mathrm{Hom}_{\mathscr{C}}(1, Z^Y)$ then

$$(1 \xrightarrow{[g]\times[f]} Z^Y \times Y^X \xrightarrow{c_{X,Y,Z}} Z^X) \cong 1 \xrightarrow{[g\circ f]} Z^X.$$

Proof. Using the naturality of the cannonical bijection it can be shown that the following maps coincide:

$$1 \times X \xrightarrow{([g]\times[f])\times 1_X} Z^Y \times Y^X \times X \xrightarrow{c_{X,Y,Z}\times 1_X} Z^X \times X \xrightarrow{ev_{X,Z}} Z$$

$$1 \times X \xrightarrow{([g]\times[f])\times 1_X} Z^Y \times Y^X \times X \xrightarrow{1_{Z^Y}\times ev_{X,Y}} Z^Y \times Y \xrightarrow{ev_{Y,Z}} Z$$

$$1 \times X \xrightarrow{[g]\times f} Z^Y \times Y \xrightarrow{ev_{Y,Z}} Z = 1 \times X \xrightarrow{t\times f} 1 \times Y \xrightarrow{[g]\times 1_Y} Z^Y \times Y \xrightarrow{ev_{Y,Z}} Z$$

$$1 \times X \xrightarrow{t\times f} 1 \times Y \xrightarrow{g\circ p_Y} Z.$$ □

3 Sheaves: The language of geometry

3.1 First Definitions

In this section all maps between topological spaces are continuous otherwise stated.

Definition 3.1.1. Let X be a topological space and $\mathsf{Set}^{\mathcal{O}^{\mathrm{op}}(X)}$ the presheaf category over X. An object in this category is called a sheaf if for every open $U \subset X$ and every open cover $\{U_i\}_{i \in I}$ of U the following diagram is an equalizer in Set.

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \underset{s}{\overset{t}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

The first map is the map induced by the restriction maps from U to U_i. The other maps are defined as follow:

For every every pair $U_i \cap U_j \neq \emptyset$ we consider $\mathscr{F}(U_i)$ and its restriction map. Clearly it induces one of the maps in the LHS, the other map is defined doing the same but with $\mathscr{F}(U_j)$ and its restriction.

This definition seems rather abstract and not quite connected with the geometry of X but we are going to see that from this simple diagram we can obtain a more down to earth defintion of sheaves.

Remark 3.1.2. If we have a map from the one point set to $\prod_i \mathscr{F}(U_i)$ satisfying commutativity the unique factorization can be interpreted as:

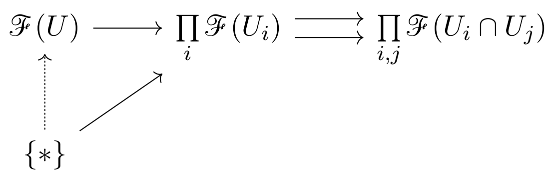

Diagram 3.1

> Given a family of sections in the open cover $\{s_i\}_I$ (that's what we get when we map one point set) with the property that for every $U_i \cap U_j \neq \emptyset$, $s_i|_{U_j} = s_j|_{U_i}$ (the map commutes) there exists a unique section $s \in \mathscr{F}(U)$ (that's the universal property) such that $s|_{U_i} = s_i$.

Remark 3.1.3. If we choose $U = \emptyset$ we can cover it with $\{U_i\}_I$ and $U_i = \emptyset$, $I = \emptyset$. After some thought we can see that this being an equalizer means that $\mathscr{F}(\emptyset) = \{*\}$.

Definition 3.1.4. As an alternative and more classical definition we can define a sheaf as a presheaf satisfying:

Axiom I: Given $U \subset X$ open and a open cover $\{U_i\}_I$ of U if we have two sections $s, t \in \mathscr{F}$ such that they coincide when restricted to U_i then $s = t$.

Axiom II: In the conditions of Axiom I if for every U_i we have sections s_i such that they agree on intersections then exists a unique section $s \in \mathscr{F}(U)$ with $s|_{U_i} = s_i$.

Definition 3.1.5. A morphism of sheaves is just a natural transformation of presheaves that gives $\mathsf{Sh}(X)$ the structure of a full subcategory of $\mathcal{O}^{\mathbf{op}}(X)$.

Example 3.1.6. Fix a field $\mathbb{K}$. For every $U \subset X$ we can define a sheaf as follows:

$$\mathscr{F}(U) = \{f : U \longrightarrow \mathbb{K} \mid f \text{ is continuous.}\}$$

Example 3.1.7. Let M be a smooth manifold. The so called structure sheaf of M is given by:

$$\mathcal{O}_M(U) = \{f : U \longrightarrow \mathbb{R} \mid f \text{ is differentiable.}\}$$

Example 3.1.8. A very important example is the so called skyscrapper sheaf. Given $x \in X$ and a set S define:

$$\mathrm{Sky}_x^S(V) = \begin{cases} S & x \in V \\ * & \text{otherwise} \end{cases}$$

The restriction maps are given by the identity if $x \in U \subset V$ or the terminal map in the other cases.

Proposition 3.1.9. The skyscrapper sheaf construction gives rise to a functor:

$$\begin{aligned} \mathrm{Sky}_x \colon \mathsf{Set} &\longrightarrow \mathsf{Sh}(X) \\ S &\longmapsto \mathrm{Sky}_x^S . \end{aligned}$$

Proof. If we have $S_1 \xrightarrow{f} S_2$ in Set we can define a natural transformation between the skyscrapper sheaves. If $x \in U$ the map is $\mathrm{Sky}_x^{S_1}(U) = S_1 \xrightarrow{f} S_2 = \mathrm{Sky}_x^{S_2}(U)$ if $x \notin U$ the map is the identity between the singletons. Commutativity is due to the presence of lots of terminal morphisms and identities. □

Proposition 3.1.10. For each $x \in X$ we can define a functor called **stalk at** x:

$$\begin{aligned} \mathrm{Stalk}_x \colon \mathcal{O}^{\mathbf{op}}(X) &\longrightarrow \mathsf{Set} \\ \mathscr{F} &\longmapsto \mathscr{F}_x = \operatorname*{colim}_{x \in U} \mathscr{F} . \end{aligned}$$

Proof. Given a natural transformation $\mathscr{F} \Longrightarrow \mathscr{G}$ the universal property of colimits gives us an induced map $\mathscr{F}_x \longrightarrow \mathscr{G}_x$. Uniqueness of this factorization shows that if the natural transformation is the identity the induced map is the identity. □

3.1.1 Sheaves on a basis

Often we encounter sheaves that are only defined on a basis of our topological space. We would like to extend these sheaves to the whole space. Fortunately there's a canonical way to do so.

Definition 3.1.11. Consider a basis $\mathcal{B}$ of X and $\mathcal{O}_{\mathcal{B}}(X) \hookrightarrow \mathcal{O}(X)$ denote the full subcategory consisting of open sets $U \in \mathcal{B}$. A sheaf on a basis $\mathcal{B}$ is just a sheaf in this subcategory.

Theorem 3.1.12 (Sheaf extension). *A sheaf $\mathscr{F}_{\mathcal{B}}$ on a basis $\mathcal{B}$ can be extended uniquely (up to isomorphism) to a sheaf $\mathscr{F}$,such that for every $U \in \mathcal{B}$, $\mathscr{F}(U) = \mathscr{F}_{\mathcal{B}}(U)$.*

Proof. Let $\mathcal{B} = \{V_i\}_{i \in I}$ we define the sheaf on each open subset U as follows

$$\mathscr{F}(U) = \lim_{V_i \subset U} \mathscr{F}_{\mathcal{B}}.$$

Namely, we take the limit over the diagram of basic opens contained in U. If $V \subset U$, every basic open contained in V is contained in U so the universal property of the limit induces a unique factorization $\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$. Also, following this definition we can see that $\mathscr{F}(V_i) \cong \mathscr{F}_{\mathcal{B}}(V_i)$ since $\mathscr{F}_{\mathcal{B}}(V_i)$ satisfies the universal property. The only things that are left to check are the sheaf axioms. Consider an open cover of U given by $\{U_j\}_{j\in J}$ and a family of compatible sections $\{t_j\}$. Refining this open cover with the basis we get another open cover U_{ij}, with a new family of compatible sections t_{ij}. Considering the singleton $\{*\} \in |\mathsf{Set}|$, we can find functions:

$$\begin{aligned} f_{ij}\colon \{*\} &\longrightarrow \mathscr{F}_{\mathcal{B}}(U_{ij}) \\ * &\longmapsto t_{ij}. \end{aligned}$$

satisfying commutativity conditions. The universal property of the limit induces a map $\{*\} \xrightarrow{f} \mathscr{F}(U)$ giving the desired section. $\square$

3.2 Understanding sheaves via stalks

Proposition 3.2.1. Let $U \subset X$, for every $x \in U$ we have a map:

$$\begin{aligned} \mathrm{germ}_x\colon \mathscr{F}(U) &\longrightarrow \mathscr{F}_x \\ s &\longmapsto \mathrm{germ}_x\, s. \end{aligned}$$

This map induces a monomorphism:

$$\mathscr{F}(U) \longrightarrow \prod_{x\in U} \mathscr{F}_x.$$

Proof. Given two sections s_1, s_2 such that their germs coincide in each point of U it's clear from the definition of germ that for every point in U there exists a neighbourhood U_x such that $s_1|_{U_x} = s_2|_{U_x}$. Proceeding in this fashion we find an open cover $\{U_x\}$ of U where the sections coincide by uniqueness of the gluing $s_1 = s_2$. $\square$

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\eta^i_U} & \mathscr{G}(U) \\ \downarrow & & \downarrow i \\ \prod_{x\in U} \mathscr{F}_x & \longrightarrow & \prod_{x\in U} \mathscr{G}_x \end{array}$$

Diagram 3.2

Proposition 3.2.2. Let η^1, η^2 be two natural transformations between a presheaf $\mathscr{F}$ and a sheaf $\mathscr{G}$. If the induced maps in the stalks are the same then both natural transformations coincide.

Proof. For every component of a natural transformation we have the square given by Diagram 3.2, the column maps are induced by the inclusions of the colimit. We see that since the maps on the stalks are the same $i(\eta^1_U(s)) = i(\eta^2_U(s))$ but we showed that i is a monomorphism so $\eta^1_U(s) = \eta^2_U(s)$ and the natural transformations coincide on each open set. $\square$

Proposition 3.2.3. Let $\mathscr{F} \overset{\eta}{\Longrightarrow} \mathscr{G}$ be a natural transformation of sheaves then η is a monomorphism if and only if the induced maps on stalks are monomorphisms.

Proof. ($\Rightarrow$) This is a consequence of Theorem 2.2.25.

($\Leftarrow$) Suppose a diagram of the form:

$$\mathscr{H} \underset{\theta}{\overset{\varepsilon}{\rightrightarrows}} \mathscr{G} \Longrightarrow \mathscr{F}$$

It is clear that the product of a family of monomorphisms is still a monomorphism implying that the induced maps in the stalk ε_x, θ_x coincide.

$$\prod_{x \in U} \mathscr{H}_x \rightrightarrows \prod_{x \in U} \mathscr{G}_x \longrightarrow \prod_{x \in U} \mathscr{F}_x$$

The result follows from Proposition 3.2.2. $\square$

Proposition 3.2.4. Let $\mathscr{F} \overset{\eta}{\Longrightarrow} \mathscr{G}$ be a natural transformation of sheaves then η is a epimorphism if and only if the induced maps on stalks are epimorphisms.

Proof. ($\Rightarrow$) We will see in Corollary 3.4.10 that the Stalk_x functor is left adjoint to Sky_x so it preserves colimits.

($\Leftarrow$) Suppose that we have natural transformations such that this diagram commutes:

$$\mathscr{F} \overset{\eta}{\Longrightarrow} \mathscr{G} \underset{\theta}{\overset{\varepsilon}{\rightrightarrows}} \mathscr{H}$$

Let's fix $U \subset X$ and $x \in U$, then applying Stalk_x to this diagram we see that under our hypothesis $\theta_x = \varepsilon_x$. That means that there's an open set U_i such that $\theta_{U_i} = \varepsilon_{U_i}$. It's clear that we can find an open cover $\{U_i\}$ of U such that the natural transformations coincide on this the cover. Now we can see that given $s \in \mathscr{F}(U)$, $\theta_U(s) = \varepsilon_U(s)$ because they agree when restricted to the open cover concluding that $\theta = \varepsilon$. $\square$

3.3 Étalé Space: An equivalence between sheaves and bundles

Definition 3.3.1. A bundle over a topological space X is an object in the category Top/X.

We could argue that this definition is indeed not very restrictive, we are only asking for a continuous map $Y \overset{\pi}{\longrightarrow} X$.

Remark 3.3.2. This definition allows us to define some tautological bundles. The empty set is an example of this behaviour. Usually other conditions are imposed, for example, local trivializations. We'll discuss this further in this memoir.

Definition 3.3.3. A bundle is said to be etale if the following conditions holds:
$\forall p \in Y$ exists a neighbourhood of $p \in U$ such that $\pi(U)$ is open in X and $\pi|_U$ is an homeomorphism.

Definition 3.3.4. Let Y be a bundle and given $U \subset X$ open let $\Gamma_Y(U) = \{s \colon U \longrightarrow Y \mid \pi \circ s = \mathbb{1}_X\}$ the sections of U.

Proposition 3.3.5. $\Gamma \colon \mathcal{O}^{\mathbf{op}}(X) \longrightarrow \mathsf{Set}$ is a sheaf called the *sheaf of cross sections.*

Proof. Functoriality is clear since we can asociate to every $U \subset V$ the corresponding restriction morphism. Suppose now a family of compatible sections on a open cover $\{U_i\}_{i\in I}$ of U it's obvious that we can define a section in U setting $s(x) = s_i(x) \ \forall x \in U_i$. The section is indeed continuous, given $V \subset Y$ open $s^{-1}(V) = \bigcup s_i^{-1}(V)$. To finish this proof is enough to see that $\forall x \in U$ we have $\pi \circ s(x) = \pi \circ s_i(x) = \mathbb{1}_X$. □

Remark 3.3.6. Sections of an étalé bundle are open maps. Indeed, let $U \subset X$ and $s\colon U \longrightarrow Y$. For every $y \in s(U)$ we consider a trivializing open V^y. And $s^{-1}(V^y) = U^y$ which is open in U. Clearly $\pi|_{V^y}(s(U^y)) = U^y$ so $s(U^y)$ must be open.

Proposition 3.3.7. Taking the sheaf of cross sections can be considered as a functor from the category of bundles to the category of sheaves on X:

$$\Gamma\colon \mathsf{Top}/X \longrightarrow \mathsf{Sh}(X).$$

Proof. For every object in Top/X we have its corresponding sheaf. It's a trivial observation to see that given a map in Top/X between two bundles $f\colon Y \longrightarrow W$ we can find a natural transformation between both sheaves sending each section $s\colon U \longrightarrow Y$ to $f \circ s\colon U \longrightarrow W$. It's also an inmediate fact that $\pi_W \circ f \circ s = \pi_Y \circ s = \mathbb{1}_X$ and this construction is compatible with the restriction maps of X. □

In this section we prove the equivalence between sheaves on a topological space and étalé bundles. Every presheaf has an associate bundle known as étalé space. Moreover, given a bundle we can associate its sheaf of cross sections. Following in this fashion given a presheaf we can construct the best possible sheaf associated with this presheaf in a process called *sheafification* which satifies a crucial universal property.

Proposition 3.3.8. There is functor sending presheaves on X to bundles over X:

$$\Lambda\colon \mathsf{Set}^{\mathcal{O}^{\mathbf{op}}(X)} \longrightarrow \mathsf{Top}/X = \mathsf{Bund}(X).$$

Proof. Let $\mathscr{F} : \mathcal{O}^{\mathbf{op}}(X) \longrightarrow \mathsf{Set}$. Let's consider for all $x \in X$ the stalk $\mathscr{F}_x$. We define $\Lambda_{\mathscr{F}} = \bigsqcup_{x\in X} \mathscr{F}_x$. Note this useful description

$$\Lambda_{\mathscr{F}} := \{\operatorname{germ}_x s \mid x \in X,\ s \in \mathscr{F}(U) \text{ with } U \text{ a neighbourhood of } x\}$$

Our first observation is that given $s \in \Lambda_{\mathscr{F}}$ since s lives in just one $\mathscr{F}_x$ we can map it to x. Therefore, we have a morphism $\Lambda_{\mathscr{F}} \xrightarrow{\pi} X$. Now we proceed to give $\Lambda_{\mathscr{F}}$ a topology that will make π a continuous map. Every section of the sheaf determines a function $\dot{s}\colon U \longrightarrow \Lambda_{\mathscr{F}}$ given by $\dot{s}(x) = \operatorname{germ}_x s$. We declare every subset of the form $\dot{s}(U) = s_U$ open and we topologize $\Lambda_{\mathscr{F}}$ with this basis. Let s_U a basic open of $\Lambda_{\mathscr{F}}$ and $\dot{v}\colon W \longrightarrow \Lambda_{\mathscr{F}}$. It's easy to see that if $W \cap U = \emptyset$ then $\dot{v}^{-1}(s_U) = \emptyset$. Therefore given W such that $W \cap U \neq \emptyset$ we have:

$$\dot{v}^{-1}(s_U) = \{x \in U \cap W \mid \operatorname{germ}_x v = \operatorname{germ}_x s.\}$$

For every point in the preimage we have as a consequence of the definition of germ_x that $\exists A$ open in X such that $A \subset U \cap W$ with $s|_A = v|_A$. It follows that $\dot{s}$ are continuous. In a similar way it's easy to check that for every open set U in X, $\pi^{-1}(U) = \{\operatorname{germ}_x s \mid s \in \mathscr{F}(U), x \in U\}$ which obviously can be expressed as an union of basic open sets . Let's note that $\pi \circ \dot{s} = \mathbb{1}_X$. It's been shown that $\Lambda_{\mathscr{F}} \xrightarrow{\pi} X$ is a bundle, now it's only left to check that this construction is functorial. Given two presheaves $\mathscr{F}, \mathscr{G}$ and $\eta \colon \mathscr{F} \Longrightarrow \mathscr{G}$ a natural transformation, we know from previous results that it induces a map between stalks for all $x \in X$ denoted η_x. This family of maps between stalks induces another map $\dot{\eta} \colon \Lambda_{\mathscr{F}} \longrightarrow \Lambda_{\mathscr{G}}$.

We want to show that $\dot{\eta}$ is a continuous map between$(\Lambda_{\mathscr{F}}, \pi_1)$ and $(\Lambda_{\mathscr{G}}, \pi_2)$ satisfying $\pi_2 \circ \dot{\eta} = \pi_1$. Let $\operatorname{germ}_x v \in \Lambda_{\mathscr{F}}$, since η is the disjoint union of the maps η_x we know that $\eta(\operatorname{germ}_x v) = \eta_x(\operatorname{germ}_x v) = \operatorname{germ}_x w$. To check continuity let's take a basic open set $w_U \subset \Lambda_{\mathscr{G}}$. Using $\eta_U^{-1}(w) \subset \mathscr{F}(U)$ we consider $\bigcup s_U$ with $s \in \eta_U^{-1}(w)$ which equals $\eta^{-1}(w_U)$. □

Remark 3.3.9. The bundle obtained with this procedure is an étalé bundle. It's enough to observe that given $\operatorname{germ}_x s$ with $s \in \mathscr{F}(U)$ we can take s_U satisfying local homeomorphism condition.

Proposition 3.3.10. If $\mathscr{F}$ is a sheaf $\mathscr{F} \cong \Gamma\Lambda\mathscr{F}$

Proof. To prove this isomorphism we need to construct a natural isomorphism, we'll do this component wise:

$$\begin{aligned} \eta_U \colon \mathscr{F}(U) &\longrightarrow \Gamma\Lambda\mathscr{F}(U) \\ s &\longrightarrow \dot{s}. \end{aligned}$$

Given $s, t \in \mathscr{F}(U)$ such that $\dot{s} = \dot{t}$ we have $\operatorname{germ}_x s = \operatorname{germ}_x t$ for every $x \in U$. This means that we can find a neighbourhood V_x such that $s|_{V_x} = t|_{V_x}$. In this way we can get an open cover $\{V_x\}$ such that s, t coincide in the intersections. By Axiom I it follows that $s = t$. Moreover, η_U is a monomorphism. We continue the proof in order to show that our natural transformation is an epimorphism. Since we're working in Set we will have an isomorphism. Given $h \in \Gamma\Lambda\mathscr{F}(U)$ it's clear from the definition that for every $x \in U$, $h(x) = \operatorname{germ}_x s^x$ for some section s^x defined in U_x. Considering the open set $\dot{s}^x(U_x)$ and its inverse image $h^{-1}(\dot{s}^x(U_x)) = V_x$ we get an open set contained in U satisfying $h|_{V_x} = s^x|_{V_x}$. Repeating this process with every x in $U \subset X$ we find an open cover with sections $s^x \in \mathscr{F}(U_x)$ with the following property:

$$\text{For every } U_x \cap U_y \neq \emptyset \text{ then } h|_{U_x \cap U_y} = \dot{s}^x|_{U_x \cap U_y} = \dot{s}^y|_{U_x \cap U_y}$$

Since $\dot{s}^x$ matchs with $\dot{s}^y$ in $U_x \cap U_y$ it means that if we consider $s^x \in \mathscr{F}(U_x)$ and $s^y \in \mathscr{F}(U_y)$ they have the same germ for every point in the intersection. Recalling the results from previous sections we see that the existence of monomorphism as in Proposition 3.2.1 implies that $s^x|_{U_x \cap U_y} = s^y|_{U_x \cap U_y}$.

We defined a family of sections in a cover which are compatible in the intersections of the elements of the cover, so Axiom II tells us that exists some $s \in \mathscr{F}(U)$ such that $\dot{v} = h$. We omit the proof of the compatibility of η_U with the restrictions since it's clear. □

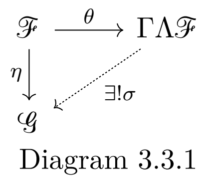

Diagram 3.3.1

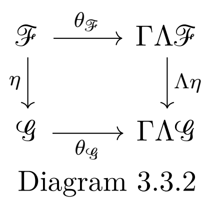

Diagram 3.3.2

Proposition 3.3.11. The composition of functors given by $\Gamma\Lambda \colon \mathsf{Set}^{\mathcal{O}^{\mathbf{op}}(X)} \longrightarrow \mathsf{Sh}(X)$ has the following universal property:

Given a natural transformation between a presheaf $\mathscr{F}$ and a sheaf $\mathscr{G}$ there's a unique natural transformation making Diagram 3.3.1 commutative

Proof. Using the previous proposition we know that $\Gamma\Lambda\mathscr{G} \cong \mathscr{G}$. So we can define Diagram 3.3.2 where $\theta_{\mathscr{G}}$ is an isomorphism allowing us to define $\sigma = \theta_{\mathscr{G}}^{-1} \circ \Lambda\eta$ satisfying commutativity conditions. It's left to check that the morphism is indeed unique.

Claim I: Let $\mathscr{F}$ be a presheaf in X and let $\sigma, \tau \colon \Gamma\Lambda\mathscr{F} \Longrightarrow \mathscr{G}$ where $\mathscr{G}$ is a sheaf. If $\sigma \circ \theta = \tau \circ \theta$ then $\sigma = \tau$.

Given $h \in \Gamma\Lambda\mathscr{F}(U)$, by the proof of the previous proposition we know that there are some open sets U_x for every point in U with the property that $h|_{U_x} = \dot{s}^x = \theta(s^x)$ with $s^x \in \mathscr{F}(U_x)$. Observing that fact that $\sigma(h) \in \mathscr{G}(U)$ verifies that:

$$\sigma|_{U_x}(h) = \sigma(h|_{U_x}) = (\sigma \circ \theta)(s^x) = \tau|_{U_x}(h).$$

Therefore,[1] we have found a cover of U such that $\sigma|_{U_x}(h) = \tau|_{U_x}(h)$. Since $\mathscr{G}$ is a sheaf it follows that $\sigma(h) = \tau(h)$. Since the choice of h was arbitrary we conclude that both natural transformations are the same. □

To ease our notation we'll call the sheafification of $\mathscr{F}$ as $\mathscr{F}^+$.

Example 3.3.12. The constant presheaf with section M, $\mathscr{A}_M$ is defined in the following way:

$$\mathscr{A}_M(U) = M \text{ with trivial restriction maps.}$$

It's clear that the stalks of this presheaf at each point are M. We'll compute now the sheafification of M, the so called constant sheaf which plays a central role in geometrical theories, for example in cohomology. Note that $\Lambda_{\mathscr{A}} = \bigsqcup_{x\in X} M \cong M \times X$. Moreover, each basic open in our topology looks like $s_U = \{m\} \times U$.

Using a well known argument it's clear that for every $h \in \mathscr{A}_M^+(U)$ and $x \in U$ we can find $x \in V_x \subset U$ such that $h(V_x) = \{m\} \times V_x$. So we can identify this new sheaf with a sheaf of locally constant functions with values in M. Composing h with a projection we can get the desired locally constant function and given a locally constant function f we can associate to it the section $f(x) = (m, x)$.

Remark 3.3.13. There is another useful description of $\mathscr{F}^+$.

$$\mathscr{F}^+(\text{U}) = \{\text{germ}_p\, s \text{ with p} \in U\text{, such that there exists an open neighbourhood V of p}$$

$$\text{contained in U and } t \in \mathscr{F}(V) \text{ with } \text{germ}_p\, s = \text{germ}_p\, t \text{ for all q} \in V\}$$

This gives us the interpretation of the sections of the sheafification as the set of compatible germs.

Corollary 3.3.14. The inclusion $\mathsf{Sh}(X) \hookrightarrow \mathsf{Set}^{\mathcal{O}^{\mathbf{op}}(X)} = \mathsf{Psh}(X)$ has a left adjoint given by $\Gamma \circ \Lambda$. This gives $\mathsf{Sh}(X)$ the structure of a reflective subcategory of $\mathsf{Set}^{\mathcal{O}^{\mathbf{op}}(X)}$

Proof. We'll send each morphism between $\mathscr{F}$ and $\mathscr{G}$ seen as presheaves to the universal morphism

[1] There is a small abuse of notation since we haven't specified the components of the natural transformations (for example σ_U)

of Proposition 3.3.11.

$$f\colon \operatorname{Hom}_{\mathsf{Psh}(X)}(\mathscr{F}, i(\mathscr{G})) \longrightarrow \operatorname{Hom}_{\mathsf{Sh}(X)}(\mathscr{F}^+, \mathscr{G}),\ \eta \longmapsto \sigma$$

Injectivity is clear. Furthermore, given $\xi \in \operatorname{Hom}_{\mathsf{Sh}(X)}(\mathscr{F}^+, \mathscr{G})$ we have that $\theta \circ \xi$ is mapped to ξ. Naturality is a consequence of our proposition. □

Corollary 3.3.15. The inclusion of presheaves into $\mathsf{Sh}(X)$ preserves limits, as an example we can see that a monomorphism of sheaves in the same as a monomorphism of presheaves.

Theorem 3.3.16. *There is an adjunction given by:*

$$\mathsf{Top}/X = \mathsf{Bund}(X) \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathsf{Set}^{\mathcal{O}^{\mathrm{op}}(X)}$$

This adjunction gives rise to an equivalence of categories when restricted to:

$$\mathsf{\acute{E}tale}(X) \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathsf{Sh}(X)$$

Proof. We will omit the proof of the first part of the theorem, the interested reader is refered to [4]. We showed previousy that $\mathscr{F} \cong \Gamma\Lambda\mathscr{F}$. What's left to prove is that given an étalé bundle Y, $\Lambda\Gamma Y \cong Y$. Note that if Y is not étalé $\Lambda\Gamma Y \cong Y$ is much bigger than Y.

It's clear that Γ_Y has as stalks the germs of sections of Γ. Let's consider the following map:

$$\begin{aligned} \varepsilon_Y\colon \Lambda\Gamma_Y &\longrightarrow Y \\ \operatorname{germ}_x h &\longmapsto h(x) \end{aligned}$$

Of course this map is well defined, if we take another representant $\operatorname{germ}_x v$ of the equivalence class they should agree in a open set so $v(x) = h(x)$. In order to show that this function is continuous we consider an open set in $W \subset Y$. Observing that $\varepsilon_Y^{-1}(W) = \{\operatorname{germ}_x h \mid x \in U^x, h \in \Gamma_Y(U^x) \text{ such that } h(x) \in W\}$ we can argue in the following way:

Given $\operatorname{germ}_x h \in \varepsilon_Y^{-1}(W)$ we can find a section $h \in \Gamma_Y(U^x)$ in some neighbourhood of $x \in X$ such that $h(x) \in W$. We know that h is a continuous section so $h^{-1}(W) \subset U^x$. Let's call $h^{-1}(W) = V^x$ so that $\dot{h}|_{V^x}(V^x)$ is an open subset of $\Lambda\Gamma_Y$ contained in $\varepsilon_Y^{-1}(W)$. As a final observation we note that $\pi_1 \operatorname{germ}_x h = \pi_2 h(x)$. Let's construct an inverse to this morphism:

$$\begin{aligned} \theta_{\Lambda\Gamma_Y}\colon Y &\longrightarrow \Lambda\Gamma_Y \\ y &\longmapsto \operatorname{germ}_x s \end{aligned}$$

With $s(x) = y$.

Claim I: This correspondence is independent of the choice of section.

Assume s_1, s_2 two sections such that $s_1(x) = s_2(x) = y \in Y$. We can assume without loss of generality (WLOG) that $s_1, s_2 \in \Gamma_Y(\pi(U))$ with U an open neighbourhood of y satisfying the local homeomorphism condition. We want to find an open subset $V \subset \pi(U)$ such that $\forall x \in V$, $s_i(x) \in U$. In that case since π_U is injective $\pi_U(s_1(x)) = \pi_U(s_2(x)) \implies s_1(x) = s_2(x)$.

Let $s_i^{-1}(U) = V_i$ clearly $V_1 \cap V_2 = V$ is an open neighbourhood of x satisfying that $\forall z \in V \quad s_i(z) \in U$. As a final remark note that since $s_1|_V = s_2|_V$ we get $\operatorname{germ}_x s_1 = \operatorname{germ}_x s_2$. To prove continuity we take a basic open set $\dot{s}(U) \subset \Lambda\Gamma_Y(U)$. Then $\theta_{\Lambda\Gamma_Y}^{-1}(s_U) = \{y \in Y \mid \exists x \in U \text{ with } s(x) = y\}$. Let $s(x) = y$ using the previous remark we can construct an open set

$W \subset U$ such that $s(W) \subset V$ with V a trivializing open. That is $\forall z \in s(W) \quad \exists x \in U$ such that $s(x) = z$, of course, $s(W)$ is an open set of Y. Naturality follows easily from this construction. Moreover, it is clear that these two natural transformations are mutual inverses finishing the proof. □

3.4 Change of base: The functors f^*, f_*

To continue developing the theory and to make it geometrically interesting it's necesary to have a notion of change of base space. We prove in this section that for every $f\colon X \longrightarrow Y$ we have two adjoint functors:

$$\mathsf{Sh}(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Sh}(Y)$$

Definition 3.4.1. Given a map $f\colon X \longrightarrow Y$ it induces a functor f_* called the direct image under f.

$$f_*\colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(Y).$$

The functor is defined on sheaves : $f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$. We omit the proof of the fact that $f_*\mathscr{F}$ is indeed a sheaf since it's obvious. Given a natural transformation $\mathscr{F}$,$\mathscr{H}$ between functors on X it's clear that f_* induces a natural transformation $f_*\mathscr{F} \Longrightarrow f_*\mathscr{H}$

Remark 3.4.2. Let $V \subset Y$, $y \in V$ and $s \in f_*\mathscr{F}(V)$ then the stalk at y is given by:

$$f_*\mathscr{F}_y = \operatorname*{colim}_{y \in U} f_*\mathscr{F} = \operatorname*{colim}_{f^{-1}(y) \subset f^{-1}(W)} \mathscr{F}$$

Lemma 3.4.3. *Given a pullback diagram in* Top*:*

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow[f]{} & Y \end{array}$$

With E étale over Y then f^*E is étale over X.

Proof. We know that pullbacks in Top are of the form $E = \{(x, e) \mid \pi(e) = f(x)\}$ topologizing E with the product topology. The maps are simple projections. Now, let's take $e \in E$ such that $\pi(x) = f(x)$ and consider an open set $U \subset E$ mapped homeomorphically to $\pi(U) = V$. Clearly $f^{-1}(V)$ is a neighbourhood of x so we can consider $f^{-1}(V) \times U \subset X \times E$. It's clear that this open set when restricted to the pullback it's an open neighbourhood of (x, e) in f^*E so we get a local homeomorphism onto $f^{-1}(V)$. □

Corollary 3.4.4. $f\colon X \longrightarrow Y$ induces a functor:

$$f^*\colon \mathsf{\acute{E}tale}(Y) \longrightarrow \mathsf{\acute{E}tale}(X).$$

Definition 3.4.5. The inverse image of a sheaf $\mathscr{F}$ is constructed as:

$$\mathsf{Sh}(Y) \xrightarrow{\Lambda} \mathsf{\acute{E}tale}(Y) \xrightarrow{f^*} \mathsf{\acute{E}tale}(X) \xrightarrow{\Gamma} \mathsf{Sh}(X)$$

Proposition 3.4.6. Given continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(g \circ f)^* \cong f^* \circ g^*$.

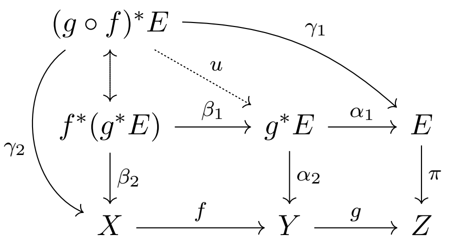

Diagram 3.4.1

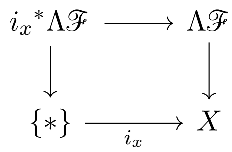

Diagram 3.4.2

Proof. This result is a basic consequence of pullback properties so the proof given will not make any references to sheaves. We will be using Diagram 3.4.1 throughout the proof.

We consider γ_1 and $f \circ \gamma_2$. And we can see that $\pi \circ \gamma_1 = g \circ f \circ \gamma_2$ so we get the universal arrow $(g \circ f)^*E \xrightarrow{u} g^*E$ with the property that $\alpha_1 \circ u = \gamma_1$ and $\alpha_2 \circ u = f \circ \gamma_2$. Repeating the same process with u and γ_2 we get an universal arrow from $(g \circ f)^*E$ to $f^*(g^*E)$. Finally considering $\alpha_1 \circ \beta_1$ and $g \circ f \circ \beta_2$ we find another universal arrow from $f^*(g^*E)$ to $(g \circ f)^*E$. It is straightfoward to check that we've just defined an isomorphism. $\square$

Remark 3.4.7. Each section $t \in f^*\mathscr{F}(U)$ satisfies $t(x) = (x, \operatorname{germ}_{f(x)} t), \; x \in f^{-1}(V)$ for some $V \subset Y$. Given a section $s \in \mathscr{F}(U)$ we can form a section of the inverse image

$$\begin{aligned} t_s \colon f^{-1}(U) &\longrightarrow \Lambda f^*\mathscr{F} \\ x &\longmapsto (x, \operatorname{germ}_x s) \end{aligned}$$

Moreover, since each section is an open map the opens $t_s(U)$ cover $\Lambda f^*\mathscr{F}$. With this point of view it is clear that every function $k \colon \Lambda f^*\mathscr{F} \longrightarrow X$ is continuous if and only if each $k \circ t_s$ is continuous for every s.

Remark 3.4.8. There is another useful description of the inverse image given by the sheafification of

$$f^*\mathscr{F} : U \longrightarrow \operatorname*{colim}_{f(U) \subset V} \mathscr{F}.$$

Theorem 3.4.9. *The functor f^* is left adjoint to f_**

Proof. Invoking Theorem 3.3.16 we can see that $\operatorname{Hom}_{\mathsf{Sh}(X)}(f^*\mathscr{F}, \mathscr{G}) \cong \operatorname{Hom}_{\mathsf{Étale}(X)}(\Lambda f^*\mathscr{F}, \Lambda\mathscr{G})$. We are going to show that there exists a natural bijection in both arguments between:

$$\operatorname{Hom}_{\mathsf{Étale}(X)}(\Lambda f^*\mathscr{F}, \Lambda\mathscr{G}) \cong \operatorname{Hom}_{\mathsf{Sh}(Y)}(\mathscr{F}, \Gamma\Lambda f_*\mathscr{G}).$$

Throughout the proof we will use the identification between a sheaf and the sheaf of sections of its étalé bundle. Let $k \in \operatorname{Hom}_{\mathsf{Étale}(X)}(\Lambda f^*\mathscr{F}, \Lambda\mathscr{G})$ and construct

$$\begin{aligned} \mathscr{F}(U) &\longrightarrow f_*\Gamma\Lambda\mathscr{G}(U) \\ s &\longmapsto k \circ t_s. \end{aligned}$$

This correspondence defines a natural transformation and it is natural in both variables, $\mathscr{F}, \mathscr{G}$. To find an inverse given $\eta \in \operatorname{Hom}_{\mathsf{Sh}(Y)}(\mathscr{F}, \Gamma\Lambda f_*\mathscr{G})$ set

$$\begin{aligned} \Lambda f^*\mathscr{F} &\xrightarrow{\overline{\eta}} \Lambda\mathscr{G} \\ (p, \operatorname{germ}_{f(p)} s) &\longmapsto \eta_U(s)(p). \end{aligned}$$

This map does not depend on the representative of $\text{germ}_{f(p)}\, s$ chosen. To see that $\overline{\eta}$ is continuous we will check that each $\overline{\eta} \circ t_s$ is continuous. This is clear from the description given below.

$$\begin{aligned} \overline{\eta} \circ t_s \colon f^{-1}(V) &\longrightarrow \Lambda\mathscr{G} \\ p &\longmapsto \eta_U(s)(p). \end{aligned}$$

Finally we need to check that both correspondences are inverse to each other. Start with a bundle map k, and denote its associated natural transformation η_k, we need to show that $\overline{\eta}_k = k$. But clearly

$$\overline{\eta}_k(p, \text{germ}_{f(p)}\, s) = \eta_k(s)(p) = k \circ t_s(p) = k(p, \text{germ}_{f(p)}\, s).$$

Given $\eta \in \text{Hom}_{\mathsf{Sh}(Y)}(\mathscr{F}, \Gamma\Lambda f_*\mathscr{G})$ denote by ε the natural transformation induced by $\overline{\eta}$ then let $s \in \mathscr{F}(V)$

$$\varepsilon_V(s)(p) = \overline{\eta} \circ t_s(p) = \overline{\eta}(p, \text{germ}_{f(p)}\, s) = \eta_V(s)(p).$$

Naturality of bijections is clear from the construction so the proof is finished. □

Corollary 3.4.10. The functor Stalk_x is left adjoint to Sky_x.

Proof. There is a natural equivalence of categories between sheaves on the one point space and the category Set. If we consider the map:

$$\begin{aligned} i_x \colon \{*\} &\longrightarrow X \\ * &\longmapsto x. \end{aligned}$$

In this light we can identify the skyscrapper sheaf at x with the direct image i_{x*}. The only delicate point is to remember that $\mathscr{F}(\emptyset) = \{*\}$.

Also applying the inverse image construction to i_x we can form diagram Diagram 3.4.2. It is a consequence of commutativity that $i_x{}^*\Lambda\mathscr{F} \cong \mathscr{F}_x$ so $i_x{}^*\mathscr{F} \cong \text{Stalk}_x \mathscr{F}$ □

Corollary 3.4.11. The stalk of $f^*\mathscr{F}$ at $x \in X$ is given by $i_x{}^* f^*\mathscr{F} \cong (f \circ i_x)^*\mathscr{F} \cong i_{f(x)}{}^*\mathscr{F} \cong \mathscr{F}_{f(x)}$.

4 Sheaves with algebraic structures

4.1 Sheaves of abelian groups.

Definition 4.1.1. Given $\mathscr{F}$ in $\mathsf{Ab}^{\mathcal{O}^{\mathbf{op}}(X)}$ we say that $\mathscr{F}$ is a sheaf of abelian groups if the following diagram[2] is an exact sequence in Ab:

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \xrightarrow{t-s} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

The last map is well define since Hom in Ab has the structure of an abelian group. This category will be denoted as $\mathsf{Sh}^{\mathsf{Ab}}(X)$

Proposition 4.1.2. Filtered colimits exist in Ab, hence we can consider an abelian version of the Stalk_x functor namely:

$$\begin{aligned} \text{Stalk}_x \colon \mathsf{Ab}^{\mathcal{O}^{\mathbf{op}}(X)} &\longrightarrow \mathsf{Ab} \\ \mathscr{F} &\longmapsto \mathscr{F}_x. \end{aligned}$$

[2]This definition can be generalized to any abelian category.

Proof. We start given $\mathscr{F}_x$ the structure of an abelian group. Given $\text{germ}_x\, s_1$ and $\text{germ}_x\, s_2$ we define the sum as the germ of $s_1|_{U\cap V} + s_2|_{U\cap V}$ which is clearly well defined. Also the 0 element is defined as the germ of the 0 element. It's a consequence of the abelian group structure defined of the stalk that the injection maps are homomorphisms. □

Corollary 4.1.3. Consider the étalé space construction. Then $\pi^{-1}(x) = \mathscr{F}_x$ has the structure of an abelian group, furthermore since for every $x \in X$ every section verifies $s_i(x) \in \mathscr{F}_x$ we can define addition pointwise giving $\Gamma\Lambda\mathscr{F}$ the structure of an abelian sheaf.

Remark 4.1.4. All constructions of the previous sections regarding sheafification and change of base functors still apply to sheaves of abelian groups. In particular all theorems about adjunctions hold.

Definition 4.1.5. Given $\mathscr{F} \overset{\eta}{\Longrightarrow} \mathscr{G}$ the kernel presheaf is defined pointwise as:

$$\text{Ker}_\eta(U) = \text{Ker}_{\eta_U} \longrightarrow \mathscr{F}(U) \xrightarrow{\eta_U} \mathscr{G}(U).$$

Following previous remarks the kernel presheaf coincides with the kernel sheaf.

Definition 4.1.6. The presheaf cokernel in general doesn't coincide with the sheaf cokernel but since sheafification preserves colimits we can define the presheaf cokernel as:

$$\mathscr{F} \overset{\eta}{\Longrightarrow} \mathscr{G} \Longrightarrow \Gamma\Lambda\text{Coker}_\eta.$$

4.2 $\mathcal{O}_X$-Modules

In this sections all rings are commutative with unity otherwise stated.

Definition 4.2.1. Let $\mathcal{O}_X$ a sheaf of rings on X. Given $\mathscr{F}$ a sheaf of abelian groups we say that $\mathscr{F}$ has the structure of a $\mathcal{O}_X$-Module if and only if for every $V \subset U \subset X$ the following diagram commutes and the rows define a module structure on $\mathscr{F}(U_i)$:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathscr{F}(V) & \longrightarrow & \mathscr{F}(V) \end{array}$$

This definition can be thought as a natural transformation $\mathcal{O}_X \times \mathscr{F} \Longrightarrow \mathscr{F}$ with some extra axioms.

Definition 4.2.2. A morphism of $\mathcal{O}_X$-Modules $\mathscr{F} \overset{\eta}{\Longrightarrow} \mathscr{G}$ is a sheaf morphism with each component η_U an $\mathcal{O}_X(U)$-Module morphism. We will denote the category of $\mathcal{O}_X$-Modules by $\mathcal{O}_X$-Mod.

Proposition 4.2.3. The stalk of $\mathscr{F}$ at p has a natural structure of $\mathcal{O}_{X,p}$ module.

Proof. If we forget some structure and we regard $\mathcal{O}_X$ as an abelian group it's clear that since we have for every open set a map $\mathcal{O}_X(U) \times \mathscr{F}(U) \longrightarrow \mathscr{F}(U)$ it induces a bilinear map:

$$\mathcal{O}_{X,p} \times \mathscr{F}_p \longrightarrow \mathscr{F}_p$$

That means that some of the module axioms are automatically satisfied, in particular:

$$a(x+y) = ax + ay$$
$$(a+b)x = ax + bx$$

For the multiplication we just define $a_p \operatorname{germ}_p s = \operatorname{germ}_p as$ in some neighbourhood in which we can take the operation. The rest of the axioms follow easily from this definition. □

Definition 4.2.4. Given two sheaves of $\mathcal{O}_X$-Modules $\mathscr{F}, \mathscr{G}$, we can construct the tensor product presheaf as follows

$$\mathscr{F} \otimes \mathscr{G} : U \longmapsto \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U).$$

We will drop subscripts when it is clear form the context. Let $V \subset U$ and $t \otimes s \in (\mathscr{F} \otimes \mathscr{G})(U)$ setting $(s \otimes_{\mathcal{O}_X(U)} t)|_V = s|_V \otimes_{\mathcal{O}_X(V)} t|_V$ it is clear that we have just defined a presheaf. Using sheafification we can define the tensor product of sheaves.

4.2.1 Locally free sheaves and vector bundles

Definition 4.2.5. A sheaf $\mathscr{F}$ of $\mathcal{O}_X$-Modules is called locally free if for every $x \in X$ there exists some $x \in U$, open such that $\mathscr{F}|_U \cong \mathcal{O}_X^n|_U$.

Definition 4.2.6. Every locally free sheaf defines a locally constant function given by:

$$\begin{aligned} \operatorname{rank} \mathscr{F} : X &\longrightarrow \mathbb{N} \\ x &\longmapsto n \end{aligned}$$

$n \in \mathbb{N}$ is given by the isomorphism $\mathscr{F}|_U \cong \mathcal{O}_X^n|_U$.

Remark 4.2.7. If X is connected $\operatorname{rank} \mathscr{F}$ is a constant function. We will say that $\mathscr{F}$ is a sheaf of rank $= n$.

Definition 4.2.8 (Vector Bundle). Let $E \stackrel{\pi}{\twoheadrightarrow} X$ be a bundle over X. E is called a vector bundle of rank n if the following conditions hold

- $\forall x \in X,\ \pi^{-1}(x) \cong \mathbb{K}^n$ for some vector space.
- There exists a cover $\{U\}_{i \in I}$ of X such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{K}^n$. This isomorphism is a morphism in Top/U_i and is linear at each point.

The category of vector bundles over X wil be denoted $\mathsf{VBund}(X)$.

Remark 4.2.9. Suppose that X is a topological space together with a sheaf $\mathcal{O}_X$ of continuous functions into $\mathbb{K}$. And consider the sheaf of sections of the bundle E in Definition 4.2.8 denoted by Γ. Clearly Γ is an $\mathcal{O}_X$-Module via pointwise multiplication of functions. Moreover, given an open set in the cover U_i

$$\Gamma(U_i) = \{s : U_i \longrightarrow U_i \times \mathbb{K}^n\} \cong \mathcal{O}_X^n(U_i).$$

Definition 4.2.10 (Transition functions). Let $\mathscr{F}$ be a locally free sheaf on X and two opens U_i, U_j with non empty intersection such that $\mathscr{F}|_{U_k} \cong \mathcal{O}_X^n|_{U_k},\ k = i, j$. Denote $U_i \cap U_j = U_{ij}$, and each isomorphism by g_k. Then

$$g_j|_{U_{ij}} \circ g_i^{-1}|_{U_{ij}} : \mathcal{O}_X^n|_{U_{ij}} \longrightarrow \mathcal{O}_X^n|_{U_{ij}}$$

is a natural isomorphism whose U_{ij} component is given by an element $M_{ij} \in \mathrm{GL}(n, \mathcal{O}_X(U_{ij}))$. We can define now a family of maps

$$\begin{aligned} p_{ij} \colon U_{ij} &\longrightarrow \mathrm{GL}(n, \mathbb{K}^n) \\ p &\longmapsto M_{ij}(p) \end{aligned} \quad \text{called transition functions.}$$

$$\begin{array}{ccccc} \pi^{-1}(U_{ij}) & \xrightarrow{\chi_i|_{ij}} & U_{ij} \times \mathbb{K}^n & \xrightarrow{\overline{\chi_i}^{-1}|_{ij}} & \pi^{-1}(U_{ij}) \\ \| & & \downarrow p_{ij} & & \| \\ \pi^{-1}(U_{ij}) & \xrightarrow{\chi_j|_{ij}} & U_{ij} \times \mathbb{K}^n & \xrightarrow{\overline{\chi_j}^{-1}|_{ij}} & \pi^{-1}(U_{ij}) \end{array}$$

Diagram 4.1.1

$$\begin{array}{ccccc} \mathcal{O}_X^n(U_{ij}) & \xrightarrow{\sim} & \mathscr{F}(U_{ij}) & \xrightarrow{\sim} & \mathcal{O}_X^n(U_{ij}) \\ \downarrow \xi_i|_{U_{ij}} & & \downarrow \eta_{U_{ij}} & & \downarrow \xi_j|_{U_{ij}} \\ \mathcal{O}_X^m(U_{ij}) & \xrightarrow{\sim} & \mathscr{G}(U_{ij}) & \xrightarrow{\sim} & \mathcal{O}_X^m(U_{ij}) \end{array}$$

Diagram 4.1.2

Proposition 4.2.11 (Cocycle condition). In the conditions of Definition 4.2.10 consider a triple intersection $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$. Then $M_{jk} \circ M_{ij} = M_{ik} \in \mathrm{GL}(n, \mathcal{O}_X(U_{ijk}))$.

Proof. In the following composition of isomorphisms we will use the abuse of notation $g_i = g_i|_{U_{ijk}}$ then

$$\mathcal{O}_X^n|_{U_{ijk}} \xRightarrow{g_i^{-1}} \mathscr{F}|_{U_{ijk}} \xRightarrow{g_j} \mathcal{O}_X^n|_{U_{ijk}} \xRightarrow{g_j^{-1}} \mathscr{F}|_{U_{ijk}} \xRightarrow{g_k} \mathcal{O}_X^n|_{U_{ijk}}$$ □

Theorem 4.2.12 (Reconstruction of a bundle). *Given the following data*

- *A connected topological space X together with a cover $\{U_i\}_{i\in I}$*
- *For every $U_{ij} \neq \emptyset$ a transition function p_{ij} as in Definition 4.2.10 satisfying the cocycle condition*

We can recover an unique (up to isomorphism) vector bundle $E \xrightarrow{\pi} X$.

Proof. Let

$$\overline{E} = \bigsqcup_{i\in I} U_i \times \mathbb{K}^n$$

we will think about the elements of $\overline{E}$ as triples (i, u, k) such that $i \in I, u \in U_i$ and $k \in \mathbb{K}^n$. Declare two elements equivalent $(i_1, u_1, k_1) \sim (i_2, u_2, k_2)$ if and only if $u_1 = u_2,\ k_2 = p_{12}(u_1)k_1$. Since M_{ii} is the identity map it follows that "$\sim$" is an equivalence relation and we can consider the quotient space

$$\begin{aligned} E &\xrightarrow{\pi} X \\ [(i, u, k)] &\longmapsto u \end{aligned}$$

It is clear that $\pi^{-1}(U_i) \cong U_i \times \mathbb{K}^n$ (denote this isomorphism by χ_i) since $U_i \times \mathbb{K}^n \rightarrowtail E$ is a monomorphism due to $p_{ii} = \mathrm{Id}$. To construct the vector space structure in each fiber we set:

$$(i, u, k_i) + (j, u, k_j) = (i, u, k_i) + (i, u, p_{ji}(u)k_j) = (i, u, k_i + p_{ji}(u)k_j)$$

Finally to see that this construction is unique up to isomorphism, consider another bundle $\overline{E} \xrightarrow{\overline{\pi}} X$ with the same trivializing opens and transition functions. Fix the family $\{\overline{\chi_i}^{-1} \circ \chi_i\}_{i\in I}$ this is a family of isomorphism on a cover of E that can be glued to a bundle isomorphism since Diagram 4.1.1 commutes. □

Proposition 4.2.13. Given a locally free sheaf we can construct a bundle as in Theorem 4.2.12. This construction is functorial.

Proof. The first part is clear we only need to form the transition functions of the sheaf and then apply Theorem 4.2.12. Let $\mathscr{F}, \mathscr{G}$ be two locally free sheaves of rank m, n respectively together with a natural transformation $\mathscr{F} \xrightarrow{\eta} \mathscr{G}$. It is clear that we can form a cover $\{U_i\}_{i \in I}$ that trivializes both sheaves. Let $g_i, \overline{g}_i$ denote the U_i component of the natural isomorphisms of the trivialization and set $\xi_i = \overline{g}_i \circ \eta_{U_i} \circ g_i^{-1}$ finally

$$\begin{aligned} f_i \colon U_i \times \mathbb{K}^n &\longrightarrow U_i \times \mathbb{K}^m \\ (u, k) &\longmapsto (u, \xi_i(u)k). \end{aligned}$$

Commutativity of Diagram 4.1.2 shows that

$$\xi_j \circ p_{ij} = \overline{p}_{ij} \circ \xi_i \text{ with } p_{ij}, \overline{p}_{ij} \text{ the corresponding transition functions}$$

So $\{f_i\}_{i \in I}$ is a family of functions that can be glued to a bundle morphism. □

Theorem 4.2.14. *Let $(X, \mathcal{O}_X)$ be a connected topological space together with a sheaf of continuous functions into $\mathbb{K}$. Then the category of locally free sheaves on X is equivalent to the category of vector bundles with fiber $\mathbb{K}$.*

Proof. Starting with the sheaf of sections of a vector bundle E, it is easy to verify that the transition functions of the sheaf coincide with the transition functions of the bundle. Theorem 4.2.12 shows that the bundle obtained from the sheaf of sections is isomorphic to E.

For the other direction if we start with a sheaf $\mathscr{F}$ and construct its associated bundle $E_{\mathscr{F}}$ we will construct a natural isomorphism. Let $U \subset X$ and fix a cover of U by trivializing opens $\{U_i\}_{i \in I}$. Given $f \in \mathscr{F}(U)$ consider $\{f|_{U_i}\}_{i \in I}$ a compatible family of sections. Let $f_i = g_i(f|_{U_i})$ it is easy to see that

$$f_i \colon U_i \longrightarrow \mathbb{K}^n \text{ such that } f_i|_{U_{ij}} = p_{ij} f_j|_{U_{ij}}.$$

This condition given by the transition functions allows us to define a family of gluable sections.

$$\begin{aligned} s_i \colon U_i &\longrightarrow U_i \times \mathbb{K}^n \\ u &\longmapsto (u, f_i(u)). \end{aligned}$$

The comprobation that this natural transformation induces an isomorphism on stalks is left to the reader. □

Theorem 4.2.15. *Let X be a connected topological space, the subcategory of locally free sheaves of rank 1 has a group structure recieving the name of Picard Group of X, $\mathsf{Pic}(X)$.*

Proof. First we note that the map induced by the multiplication of $\mathcal{O}_X$ is an isomorphism of presheaves between

$$\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X.$$

It is a well known fact that sheafification is an exact functor, so we get a sheaf isomorphism. Now let $\mathscr{F}, \mathscr{G}$ be locally free sheaves of rank 1. It is a quick consequence of the definition of product that given a trivializing open for both sheaves

$$\mathscr{F}|_U \times \mathscr{G}|_U \cong (\mathscr{F} \times G)|_U.$$

so $(\mathscr{F} \otimes \mathscr{G})|_U \cong \mathcal{O}_X|_U \otimes \mathcal{O}_X|_U \cong \mathcal{O}_X|_U$ and the previous discussion applies. A similar argument shows that $\mathscr{F} \otimes \mathcal{O}_X$ is locally isomorphic to $\mathscr{F}$.

The last part is a little bit more delicate, we need to find an inverse for this operation. Consider $\underline{\mathrm{Hom}}_{\mathcal{O}_X\text{-Mod}}(\mathscr{F}, \mathcal{O}_X)$ and U a trivializing open for $\mathscr{F}$. Then

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X\text{-Mod}}(\mathscr{F}, \mathcal{O}_X)(U) = \{\eta\colon \mathscr{F}|_U \longrightarrow \mathcal{O}_X|_U\} \cong \{\eta\colon \mathcal{O}_X|_U \to \mathcal{O}_X|_U\} \cong \mathcal{O}_X|_U$$

Taking $\underline{\mathrm{Hom}}_{\mathcal{O}_X\text{-Mod}}(\mathscr{F}, \mathcal{O}_X)\otimes\mathscr{F}$ for every open set $V \subset X$ we can construct natural transformation given by:

$$\begin{aligned} \xi_V\colon (\underline{\mathrm{Hom}}_{\mathcal{O}_X\text{-Mod}}(\mathscr{F}, \mathcal{O}_X) \otimes \mathscr{F})(V) &\longrightarrow \mathcal{O}_X(V) \\ \eta \otimes t &\longmapsto \eta(t) \end{aligned}$$

Clearly this map when restricted to a trivializing open induces an isomorphism so it must induce an isomorphism on stalks hence this is a natural isomorphism of sheaves. □

5 Applications

5.1 Ringed Spaces

Usually some geometrical categories are not well behaved with respect to the categorical point of view. That is one of the main reasons to substitute categories of manifolds or varieties by (locally) ringed spaces modelled on some particular structure.

Definition 5.1.1. A locally ringed space $(X, \mathcal{O}_X)$ is a topological space together with a sheaf $\mathcal{O}_X$ of rings (or algebras) playing the role of the sheaf of "good" functions on each open subset of X.

Example 5.1.2. Consider a smooth manifold $(M, \mathcal{O}_M)$ and the sheaf of $\mathcal{C}^\infty$ functions. We are going to use this example to find the right definition of morphism of ringed spaces. Suppose $(N, \mathcal{O}_N)$ another smooth manifold and a $\mathcal{C}^\infty$ map $M \xrightarrow{f} N$. Clearly we can see that given $g \in \mathcal{O}_N(U)$ the assignation $g \circ f \in \mathcal{O}_M(f^{-1}(U)) = f_*\mathcal{O}_M(U)$ is a natural transformation.

Definition 5.1.3. A morphism between ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the following data:

- A continuous map $f\colon X \longrightarrow Y$ between the underlying topological spaces
- A natural transformation $f^\sharp\colon \mathcal{O}_Y \Longrightarrow f_*\mathcal{O}_X$

We will denote this pair $(f, f^\sharp)$.

Example 5.1.4. Following the previous example we see that the stalk, $\mathcal{O}_{M,p}$ is given by the set of germs of $\mathcal{C}^\infty$ functions at p. Considering the ideal $\mathfrak{m}_p = \{\mathrm{germ}_p\, f \mid f(p) = 0\}$, it is clear that every element not in $\mathfrak{m}_p$ is a unit. That tell us that $\mathfrak{m}_p$ is a local ring, i.e it has only one maximal ideal. The evaluation morphism induces an isomorphism $\mathcal{O}_{M,p}/\mathfrak{m}_p \cong \mathbb{R}$. So this maximal ideal seems like a good option for generalizing the idea of evaluation of functions. We have in mind that our structure sheaf is a generalization of the sheaf of "good" functions on X. Moreover, given $M \xrightarrow{f} N$ and $p \in M$ we can consider the induced map $\mathcal{O}_{Y,f(p)} \xrightarrow{f^\sharp_{f(p)}} f_*\mathcal{O}_{X,f(p)} \longrightarrow \mathcal{O}_{X,x}$ which clearly sends every function vanishing on $f(p)$ to a function vanishing on p.

Definition 5.1.5. A locally ringed space $(X, \mathcal{O}_X)$ is a ringed space with the property that every stalk $\mathcal{O}_{X,p}$ is a local ring. A morphism of locally ringed spaces is a morphism of ringed spaces such that for every $p \in X$ the induced map on stalks satisfies:

$$f^\sharp_{f(p)}\colon \mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p},\ \mathfrak{m}^Y_{f(p)} \longmapsto f^\sharp_{f(p)}(\mathfrak{m}^Y_{f(p)}) \subset \mathfrak{m}^X_p.$$

Example 5.1.6 (Spectrum of a Ring). As we saw before there is a functor sending commutative rings to topological spaces. Actually there is a way of improving this functor changing its codomain to the category of locally ringed spaces. Consider the spectrum of a ring $\operatorname{Spec} A$, it is an easy fact that $V(f)^c = D(f)$ with $f \in A$ is a basis for $\operatorname{Spec} A$. We set:

$$\mathcal{O}_A(D(f)) = A_f \text{ with } A_f \text{ the localization of } A \text{ in the multiplicative subset of powers of } f.$$

Given $D(f) \subset D(g)$ we want to find a restriction map $p|_{D(f)} \colon A_g \longrightarrow A_f$. We only need to find a map $\overline{p} \colon A \longrightarrow A_f$ sending g to a unity in A_f. The universal property of the localization will induce the desired map.

Choosing $\overline{p}$ to be the canonical inclusion of A in A_g we claim that $\overline{p}(g)$ is a unity. Suppose for contradiction that g is contained in some maximal ideal $\mathfrak{m}$, then $\overline{p}^{-1}(\mathfrak{m})$ is a prime ideal of A containing g. We need to note that since $\overline{p}(f)$ is a unity it cannot be contained in $\mathfrak{m}$. It is also straightfoward to see that $f \notin \overline{p}^{-1}(\mathfrak{m})$ and this a contradiction since $D(f) \subset D(g)$. The universal property of the localization guarantees that we have just defined a presheaf on a basis of $\operatorname{Spec} A$.

Proposition 5.1.7. The presheaf $\mathcal{O}_A$ is a sheaf on a basis of $\operatorname{Spec} A$.

Proof. To check sheaf axioms we need to remember some basic commutative algebra facts:

- There is a bijection between $\operatorname{Spec} A_f$ and $D(f)$.
- $\operatorname{Spec} A$ is quasicompact i.e for every cover we can substract a finite subcover.
- $D(f_i) \cap D(f_j) = D(f_i f_j)$ and $\bigcup_{i \in I} D(f_i) = D(\sum_{i \in I} f_i)$.

We start with $f = 1$ so that $D(f) = \operatorname{Spec} A$ and a cover $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$. It is enough to check this particular case since otherwise $D(f)$ is equivalent to $\operatorname{Spec} A_f$ and we can adapt the argument. Suppose that we have $g, h \in A$ such that $g - h = 0$ in each A_{f_i}. That means that for every f_i exists some N_i such that $f_i^{N_i}(g - h) = 0$. Since $\operatorname{Spec} A$ is quasicompact we can assume that the cover is finite. Given the expansion of the unity given by $1 = \sum_i a_i f_i$ we know that there exists a natural number N such that $\big(\sum_{i=1} a_i f_i\big)^N$ annihilates $(f - g)$ implying that $f = g$. This shows Axiom I hold.

For the second sheaf axiom, we start with compatible sections $g_i \in A_{f_i}$. Compatibility means that exists some N_{ij} such that $(f_i f_j)^{N_{ij}}(g_i - g_j) = 0$, also for every g_i there is some N_i such that $f_i^{N_i} g_i$ is the image of some $h_i \in A$ via the cannonical map. Since the cover is finite there exists some N that works for every g_i. We know that (1) can be generated by $\{f_i^N \mid i \in I\}$. Let $1 = \sum_i e_i f_i^N$, we are claiming that $g = \sum_i e_i f_i^N g_i$. As a final comprobation we can see that

$$f_j^N g = \sum_i e_i f_i^N f_j^N g_i = \sum_i e_i f_i^N f_j^N g_j = f_j^N g_j \sum_i e_i f_i^N = f_j^N g_j. \qquad \square$$

Proposition 5.1.8. The pair $(\operatorname{Spec} A, \mathcal{O}_A)$ forms a locally ringed space called an affine scheme. In other words,

$$\begin{aligned} \operatorname{Spec} \colon \mathsf{Rng}^{\mathbf{op}} &\longrightarrow \mathsf{AffSch} \\ A &\longrightarrow (\operatorname{Spec} A, \mathcal{O}_A). \end{aligned}$$

More precisely $\mathcal{O}_{A,\mathfrak{p}} \cong A_{\mathfrak{p}}$.

Proof. We are going to show that $A_{\mathfrak{p}}$ satisfies the universal property of the stalk. Every $D(f_i)$ such that $\mathfrak{p} \in D(f_i)$ is by definition an prime ideal not containing f_i so every f_i maps to a unit in $A_{\mathfrak{p}}$. We get this way maps $A_{f_i} \xrightarrow{i_{f_i}} A_{\mathfrak{p}}$ satisfying commutativity conditions. Suppose

$\varphi_i \colon A_{f_i} \longrightarrow B$ one can easily check that setting $\varphi \colon A_{\mathfrak{p}} \longrightarrow B$, $\varphi(\frac{a}{b}) = \varphi(i_i(\frac{a_i}{f_i^n})) = \varphi_i(\frac{a_i}{f_i^n})$ is the necessary factorization for the universal property.

We need to show how the functor Spec induces a morphism of locally ringed spaces. We already showed that given $A \xrightarrow{f} B$ a ring morphism, $\operatorname{Spec}(f) \colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is a continuous map. Let $D(h) \subset \operatorname{Spec} A$ it is clear that $\operatorname{Spec}(f)^{-1}(D(h)) = D(f(h))$ and f induces a ring map:

$$\eta_{D(h)} \colon A_h \longrightarrow B_{f(h)}.$$

Compatible with restrictions and giving rise to the desired natural transformation. It is straightfoward to check that this map is an homomorphism of local rings. □

Remark 5.1.9. We will show at the end of this project that $\mathsf{Rng}^{\mathbf{op}}$ is equivalent to the category AffSch of affine schemes. The observation that $\mathsf{Rng}^{\mathbf{op}}$ is essentially a geometric category is one of the key points of Algebraic Geometry. From this point of view, the study of classical algebraic structures can be understood as the algebraic realization of a geometrical theory and vice versa.

Example 5.1.10. In order to finish our analogy with smooth manifolds we need to translate the local homeomorphism condition into sheaf theoretical language. We recall that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is a locally ringed space. Let $U \subset M$, such that $U \cong \mathbb{R}^n$, and denote the isomorphism by f. Then it is clear that $\mathcal{O}_M|_U \cong \mathcal{O}_{\mathbb{R}^n}$. So $(M, \mathcal{O}_M)$ is a locally ringed space which is locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$. $\mathbb{R}^n$ will be called the modelling space of M.

Definition 5.1.11. We say that a locally ringed space $(X, \mathcal{O}_X)$ is modelled on $(A, \mathcal{O}_A)$ if for every point $x \in X$ there exists a neighbourhood $x \in U$ such that $(U, \mathcal{O}_X|_U) \cong (A, \mathcal{O}_A)$ as locally ringed spaces.

Definition 5.1.12. Let $\mathfrak{M}$ denote a class of models. Many geometrical theories can be regarded as theories that study locally ringed spaces based on a certain class of models. Here we list some examples.

- Putting $\mathfrak{M} = \{(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \mid \mathcal{O}_{\mathbb{R}^n}$ is the sheaf of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n\}$ we obtain smooth manifolds.
- Putting $\mathfrak{M} = \{(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) | O_{\mathbb{C}}$ is the sheaf of holomorphic functions on $\mathbb{C}\}$ we obtain Riemann surfaces.
- Putting $\mathfrak{M} = \{(\operatorname{Spec} A, \mathcal{O}_A) \mid \mathcal{O}_A$ is the structure sheaf of the spectrum of a ring$\}$ we obtain schemes.

5.2 Local Systems

All topological spaces considered in this section are connected, path connected and semilocally simply connected. Under this conditions the existence of an universal covering is guaranteed. We refer to [5] for the theory of covering spaces.

Definition 5.2.1. Let X be a topological space, a local system $\mathcal{L}$ on X is a sheaf such that for every $x \in X$ there exists some $U \subset X$ open and an isomorphism $\mathcal{L}|_U \cong \mathscr{A}|_U$ with $\mathscr{A}$ a constant sheaf.

Proposition 5.2.2. Given a trivializing open subset of X then $\forall x_1, x_2 \in U, \mathcal{L}_{x_1} \cong \mathcal{L}_{x_2}$.

Proof. We may assume that U is connected. Then clearly $\mathcal{L}|_U \cong \mathscr{A}|_U$. In this neighbourhood $\mathcal{L}(U)$ looks like the sheaf of locally constant functions on U, but since U is connected locally constant functions are constant. Clearly given $x_1 \in U$ $\mathrm{germ}_{x_1} f$ can be extended to a section $f \in \mathcal{L}(U)$ so $\mathcal{L}_{x_1} \cong \mathcal{L}(U) \cong \mathcal{L}_{x_2}$. ☐

Remark 5.2.3. Given $x_1, x_2, x_3 \in U$ then $\mathcal{L}_{x_1} \cong \mathcal{L}_{x_3}$ coincides with $\mathcal{L}_{x_1} \cong \mathcal{L}_{x_2} \cong \mathcal{L}_{x_3}$

Proposition 5.2.4. A sheaf $\mathcal{L}$ is constant if and only if its corresponding étalé space is of the form

$$\Lambda\mathcal{L} = \bigsqcup_{i \in I} V_i \quad \text{such that } \pi|_{V_i} \colon V_i \longrightarrow X \text{ is an homeomorphism}$$

Proof. $(\Rightarrow)$ Let $\mathrm{germ}_x f \in \Lambda\mathcal{L}$. Since X is connected we know that a representative of this germ section $f \in \mathcal{L}(X)$. Now we can consider the basic open set:

$$\dot{f}(X) = \{\mathrm{germ}_x f \mid x \in X\}$$

Clearly the projection restricted to this open set is an homeomorphism onto its image. We claim that $V_i = \dot{f}_i(X)$ for some $f_i \in \mathcal{L}(X)$. Suppose $\mathrm{germ}_x g \in V_i \cap V_j$. Then we can find neighborhoods U_i and U_j of x such that

$$g|_{U_i} = f_i \text{ and } g|_{U_j} = f_j \text{ with } f_i \neq f_j$$

However, since we can find a connected open set contained in the intersection it follows that $f_i = f_j$ finding a contradiction. So $\Lambda\mathcal{L} = \bigsqcup_{i \in I} V_i$.

$(\Leftarrow)$ We have already proven that the sheaf of sections of this bundle is isomorphic to $\mathcal{L}$. Let $U \subset X$, clearly $\pi^{-1}(U) = \bigsqcup_{i \in I} W_i$ with the restriction of π an homeomorphism. It follows from this fact that the sections of this sheaf satisfy $s(U) \subset W_i$ and can be regardad as locally constant functions with values in the index set I. That proves that $\mathcal{L}$ is a constant sheaf.

☐

$$\begin{array}{ccc} \Lambda\mathcal{L}|_U & \rightarrowtail & \Lambda\mathcal{L} \\ \downarrow & & \downarrow \\ U & \xhookrightarrow{i} & X \end{array} \qquad \begin{array}{ccc} \overline{Y} & \xrightarrow{\overline{f}} & \overline{X} \\ \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

Diagram 5.1.1 Diagram 5.1.2

Proposition 5.2.5. The étalé space of a local system $\mathcal{L}$ is a covering space of X.

Proof. Suppose U a connected trivializing open for $\mathcal{L}$ then $\pi^{-1}(U)$ is given by Diagram 5.1.1, but this is basically the étalé space of $\mathcal{L}|_U = i^*\mathcal{L}$ and the previous proposition tell us that this is the disjoint union of open sets homeomorphic to U. ☐

Remark 5.2.6. It is obvious from the construction that we have to allow disconnected covering spaces in our definition.

Corollary 5.2.7. A local system $\mathcal{L}$ is a constant sheaf if and only if has a global section.

Proof. $(\Rightarrow)$ Obvious

$(\Leftarrow)$ Consider the étalé space $\Lambda\mathcal{L}$ and a global section $X \xrightarrow{s} \Lambda\mathcal{L}$. Then each loop in X can be lifted to a loop in the covering space. The basic theory of covering spaces shows that either $\Lambda\mathcal{L}$ is disconnected or $\Lambda\mathcal{L}$ is homeomorphic to X so $\mathcal{L}$ must be a constant sheaf.

□

Proposition 5.2.8. The sheaf of sections of a covering space is locally constant.

Proof. Given a trivializing open we can apply the proof of Proposition 5.2.4. □

Proposition 5.2.9. The pullback of a covering space is a covering space.

Proof. Following Diagram 5.1.2 let $U \subset X$ such that $\pi^{-1}(U) = \bigsqcup_{i\in I} U_i$ and $f^{-1}(U) \subset Y$. It follows from the definition of pullbacks in Top that

$$\overline{\pi}^{-1}(f^{-1}(U)) = \{(y, \overline{x}) \mid y \in f^{-1}(U) \text{ and } f(y) = \pi(\overline{s})\}$$

Clearly, $\overline{x} \in \pi^{-1}(U)$. Let $V_i = \{(y, \overline{x}) \mid \overline{x} \in U_i\}$, we are claiming that

$$\overline{\pi}^{-1}(f^{-1}(U)) = \bigsqcup_{i\in I} V_i \text{ with } \overline{\pi}|_{V_i} \colon V_i \longrightarrow f^{-1}(U) \text{ an homeomorphism}$$

The map clearly is surjective because given $y \in f^{-1}(U)$ we can take $\overline{x}_i \in \pi^{-1}(f(y))$ and injectivity is guaranteed by the fact that $\overline{X}$ is a covering space. It is trivial to check that we have an homeomorphism. □

Corollary 5.2.10. The inverse image of local system is a local system.

Proof. This is inmediate since the inverse image is constructed using the pullback of the étalé space of $\mathcal{L}$. □

Corollary 5.2.11. Given a map $f\colon Y \longrightarrow X$ with Y simply connected, then $f^*\mathcal{L}$ is a constant sheaf on Y.

Proof. Let $f^*\mathcal{L}$ and consider its étalé space. Clearly by Proposition 5.2.5 it is a covering space but the covering spaces of Y are trivial so $f^*\mathcal{L}$ is constant sheaf by Proposition 5.2.4. □

$$\begin{array}{ccc} \Lambda\gamma^*\mathcal{L} & \xrightarrow{\overline{\gamma}} & \Lambda\mathcal{L} \\ \big\downarrow{\scriptstyle\overline{\pi}} & & \big\downarrow{\scriptstyle\pi} \\ I & \xrightarrow{\gamma} & X \end{array}$$

Diagram 5.2.1

$$\begin{array}{ccccc} \mathcal{L}_{H(0,0)} & \xrightarrow{\sim} & \mathcal{L}(I\times I) & \xrightarrow{\sim} & \mathcal{L}_{H(0,1)} \\ \| & & \| & & \| \\ \mathcal{L}_{H(1,0)} & \xrightarrow{\sim} & \mathcal{L}(I\times I) & \xrightarrow{\sim} & \mathcal{L}_{H(1,1)} \end{array}$$

Diagram 5.2.2

Definition 5.2.12 (Monodromy). Given a loop $\gamma\colon I \longrightarrow X$ the composition of isomorphisms:

$$\gamma^*\mathcal{L}_{\gamma(0)} \cong \gamma^*\mathcal{L}([0,1]) \cong \gamma^*\mathcal{L}_{\gamma(1)}$$

is called the monodromy of $\mathcal{L}$ along γ. The isomorphism will be denoted by $\rho(\gamma)$.

Remark 5.2.13. We will drop the notation of γ^* when it is clear from the context and we will denote the stalk of the inverse image simply by $\mathcal{L}_{\gamma(t)}$.

Remark 5.2.14. In order to make Definition 5.2.12 effective we need to see how this isomorphism works. Following Diagram 5.2.1 let $\mathrm{germ}_{\gamma(0)}\, s \in \mathcal{L}_{\gamma(0)}$ this section gives rise to a section

$(0, \mathrm{germ}_{\gamma(0)}\, s) \in \Lambda\gamma^*\mathcal{L}$. This section can be extended to a section of $s \in \gamma^*\mathcal{L}([0,1])$ with the property

$$\pi \circ \overline{\gamma} \circ s = \gamma$$

So $\overline{\gamma} \circ s$ is the unique lifting of γ. Finally we can see that $\overline{\gamma} \circ s(1) \in \pi^{-1}(\gamma(0))$. We have shown that the monodromy sends each element of $\mathcal{L}_{\gamma(0)}$ to the element of that stalk determined by the lifting of γ.

Proposition 5.2.15. The monodromy of $\mathcal{L}$ only depends on the homotopy class of γ.

Proof. Let γ_0, γ_1 two paths and an homotopy between them given by $I \times I \xrightarrow{H} X$ such that $H(i,t) = \gamma_i,\ \ i = 0, 1$. Following the same argument as in Corollary 5.2.11 we can see that $H^*\mathcal{L}$ is a constant sheaf on $I \times I$. Commutativity of Diagram 5.2.2 is guaranteed by Remark 5.2.3 and the composition of the top and bottom rows are respectively the monodromy of γ_0 and the monodromy of γ_1. □

Remark 5.2.16. From now on we will be dealing with local systems with stalks a $\mathbb{K}$-vector space L. In this setting $\rho_{\mathcal{L}}(\gamma) \in (\mathrm{Hom}_{\mathsf{Vec}_{\mathbb{K}}}(L,L))^* = \mathrm{Aut}_{\mathbb{K}}(L)$, the the vector space of linear automorphisms of L.

The aim of this section is to show that there is an equivalence of categories between the category of local systems on X and the category of representations of $\pi_1(X,p)$ on L. We will introduce the very first definitions of representation theory.

Definition 5.2.17. Given a group G an its associated group category $\mathscr{G}$ (see Example 2.1.8). The category of representations of G is simply $[\mathscr{G}, \mathsf{Vec}_{\mathbb{K}}]$.

Proposition 5.2.18. Given a local system $\mathcal{L}$ and let $p \in X$. Then the mapping

$$\begin{aligned} \pi_1(X,p) &\longrightarrow \mathrm{Aut}_{\mathbb{K}}(L) \\ \gamma &\longmapsto \rho_{\mathcal{L}}(\gamma) \end{aligned}$$

is a representation of the fundamental group on L.

Proof. Since ρ only depends of the homotopy class of each loop the map is well defined. Let $p\colon I \longrightarrow X$ be the constant map in X. Clearly the constant loop in X lifts to a constant loop in $\Lambda\mathcal{L}$ proving that the induced automorphism is the identity. Now we only need to check that composition of loops yields composition of automorphisms of L.

Let $\gamma_1 * \gamma_2 \in \pi_1(X,p)$[3], and liftings $\overline{\gamma}_1, \overline{\gamma_2}$, clearly composition of the liftings is a lifting of $\gamma_1 \circ \gamma_2$ this is a translation of what we were looking for in the language of monodromy. □

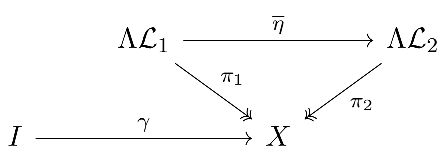

Diagram 5.3.1

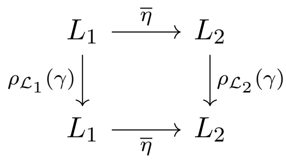

Diagram 5.3.2

Proposition 5.2.19. The construction of Proposition 5.2.18 is functorial.

[3]"$*$" denotes the composition of paths in the fundamental group.

Proof. Given two local systems $\mathcal{L}_1, \mathcal{L}_2$ with stalks L_1, L_2. Let $\mathcal{L}_1 \overset{\eta}{\Longrightarrow} \mathcal{L}_2$ yielding a continuous map between the corresponding étalé spaces, denoted by $\overline{\eta}$. Then given a loop γ, the lifting $I \xrightarrow{\overline{\gamma}^2} \Lambda\mathcal{L}_2$ is given by the composition $\overline{\eta} \circ \overline{\gamma}^1$ by commutativity of Diagram 5.3.1. Let $\text{germ}_p\, s \in \Lambda\mathcal{L}_1$ this germ is mapped via monodromy to some other $\text{germ}_p\, t$. By the above remarks $\overline{\eta}(\text{germ}_p\, s)$ is mapped via monodromy to $\overline{\eta}(\text{germ}_q\, t)$ making Diagram 5.3.2 commute. □

Proposition 5.2.20. Consider a representation ρ of $\pi_1(X, p)$ on L then we can associate to ρ a local system $\mathcal{L}_\rho$.

Proof. For every $x \in X$ let's fix a path α_x joining p and x. Then we can give the following definition

$$\mathcal{L}_\rho(U) = \{f\colon U \longrightarrow L \mid \text{ for every path } \gamma,\ \ f(\gamma(1)) = \rho(\alpha_\gamma) \cdot f(\gamma(0))\}$$

$$\alpha_\gamma = \alpha^{-1}_{\gamma(1)} * \gamma * \alpha_{\gamma(0)}$$

Lemma 5.2.21. *$\mathcal{L}_\rho$ is a sheaf.*

Proof. The first axiom is trivial, both functions must agree if they agree when restricted to an open cover. The only thing to see is that the glued function is equivariant with respect to the representation. Let $U \subset X$ together with an open cover $\{U_i\}_{i\in I}$ and compatible sections $f_i \in \mathcal{L}_\rho(U_i)$. Forgetting about the equivariant condition we might glue the sections to form $U \xrightarrow{f} L$. Considering a path γ, since $\gamma(I)$ is compact in X we can substract a finite subcovering of $\{U_i\}_{i\in I}$ and find a partition of I into $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ such that $\gamma([a_i, a_{i+1}]) \subset U_i$. Let's denote by $\gamma_i = \gamma|_{[a_i, a_{i+1}]}$. Now $f(\gamma(1)) = \rho(\alpha_{\gamma_{n-1}}) \cdot f(\gamma(a_{n-1}))$. Applying induction to this process yields the desired result. □

Lemma 5.2.22. *$\mathcal{L}_\rho$ is locally constant. $\mathcal{L}_\rho|_U \cong \mathscr{L}|_U$. $\mathscr{L}$ is the constant sheaf with stalk L.*

Proof. Let $U \subset X$ such that every loop in U is nullhomotopic in X. Fix $x \in U$ and set

$$\begin{aligned} \mathcal{L}_\rho(U) &\longrightarrow \mathscr{L}(U) \\ f &\longmapsto f(x) \end{aligned}$$

Every $f \in \mathcal{L}_\rho(U)$ is completely defined by its image in x. Indeed, given another $y \in U$ we have $f(y) = \rho(\alpha_\gamma) \cdot f(x)$ with γ a path joining x and y. This is well defined since given another path γ' the composition $\gamma^{-1} * \gamma'$ is nullhomotopic in X. This map is clearly injective, suppose $f(x) = g(x)$ then

$$f(y) = \rho(\alpha_\gamma) \cdot f(x) = \rho(\alpha_\gamma) \cdot g(x) = g(y) \implies f = g$$

Moreover, defining for every $l \in L$ the function

$$\begin{aligned} f_l\colon U &\longrightarrow L \\ y &\longmapsto \rho(\alpha_y^{-1} * \gamma * \alpha_x) \cdot l \end{aligned}$$

with γ any path joining x and y we can see that this map is surjective and that $f_l(x) = l$. □

This finishes the proof of Proposition 5.2.20. □

Proposition 5.2.23. The construction of Proposition 5.2.20 is functorial.

Proof. Given two representations ρ, θ of $\pi_1(X, p)$ and $\rho \stackrel{\eta}{\Longrightarrow} \theta$. Let's denote the unique component of this natural transformation by T. Then we can define the following natural transformation componentwise

$$\begin{aligned} \mathcal{L}_\rho(U) &\longrightarrow \mathcal{L}_\theta(U) \\ f &\longmapsto T \circ f \end{aligned}$$

The rest of the comprobations are trivial. □

Theorem 5.2.24. *There is an equivalence of categories between the local systems on X with stalk $\mathbb{K}$-vector spaces and representations of $\pi_1(X, p)$ on $\mathbb{K}$-vector spaces.*

Proof. The results of Proposition 5.2.18 and Proposition 5.2.20 show that we have functors in both directions. We will show that the compositions are naturally isomorphic to the corresponding identity functors. Consider the sheaf given by equivariant functions with respect to the representation induced by the monodromy. We are going to identify $\mathcal{L}$ with the sheaf of sections of its étalé bundle $\Lambda\mathcal{L}$. Let $f \in \mathcal{L}_\rho(U)$ with U simply connected, we want to send f to a section of $\Lambda\mathcal{L}$. First we note that every stalk of $\mathcal{L}$ is isomorphic to some L the codomain of f. After identifying $L \cong \mathcal{L}_p$ we can set

$$\begin{aligned} s_f \colon U &\longrightarrow \Lambda\mathcal{L} \\ x &\longmapsto \rho(\alpha_x) \cdot f(x) \end{aligned}$$

Let s be a section of $\Lambda\mathcal{L}$ such that $s(x) = s_f(x)$ we can assume that s is defined in U without losing generality. We need to check that $s(y) = s_f(y),\ \ \forall y \in U$. We can see that $s(y) = \rho(\gamma) \cdot s(x)$ because the section s is a proper lift of a any path in U, also any two paths in U are homotopic by assumption. Finally, given γ joining x and y,

$$s_f(y) = \rho(\alpha_y)\cdot f(y) = \rho(\alpha_y)\rho(\alpha_\gamma)\cdot f(x) = \rho(\alpha_y * \alpha_y^{-1} * \gamma * \alpha_x)\cdot f(x) = \rho(\gamma)\rho(\alpha_x)\cdot f(x) = \rho(\gamma)\cdot s(x) = s(y)$$

To find the inverse to this natural transformation let $s \in \mathcal{L}(U)$ and consider

$$\begin{aligned} f_s \colon U &\longrightarrow \mathcal{L}_p \\ x &\longmapsto \rho(\alpha_x^{-1}) \cdot s(x) \end{aligned}$$

We need to check that f_s is equivariant, let γ be a path joining x and y.

$$f_s(y) = \rho(\alpha_y^{-1}) \cdot s(y) = \rho(\alpha_y^{-1})\rho(\gamma) \cdot s(x) = \rho(\alpha_y^{-1} * \gamma * \alpha_x)\rho(\alpha_x^{-1}) \cdot s(x) = \rho(\alpha_y^{-1} * \gamma * \alpha_x) \cdot f_s(x)$$

The reader might think that this is not a valid natural transformation since is only defined on some special open sets. However, our hypothesis guarantee that these sets form a basis of the topology of X so after a gluing process similar to the one exposed in Theorem 3.1.12 we will find a natural isomorphism.

To finish the proof we need to see that starting with a representation ρ of $\pi_1(X, p)$ on L the representation defined by the monodromy of the sheaf of equivariant functions satisfies $\rho_{\mathcal{L}} \cong \rho$. Consider $\operatorname{germ}_x f \in \Lambda\mathcal{L}$ such that $f \in \mathcal{L}(U)$ a simply connected set. We remark the fact that the stalks of $\mathcal{L}$ are isomorphic to L and that $\operatorname{germ}_x f$ can be identified with $f(x) \in L$. Given $I \xrightarrow{\gamma_0} U$ between x and y, it is clear that $f(y) = \rho_{\mathcal{L}}(\gamma_0) \cdot f(x)$, moreover

$$f(y) = \rho(\alpha_y^{-1} * \gamma_0 * \alpha_x) \cdot f(x) \text{ since } f \text{ is equivariant } \implies \rho_{\mathcal{L}}(\gamma_0) = \rho(\alpha_y^{-1} * \gamma_0 * \alpha_x)$$

Let $I \xrightarrow{\gamma} X$ be an arbitrary loop. Select a cover by simply connected open sets of $\gamma(I)$, namely

$\{U_i\}_{i\in I}$ and find a partition of I as in the proof of Proposition 5.2.20. Using the same notation we can see

$$\rho_{\mathcal{L}}(\gamma) = \rho(\alpha^{-1}_{\gamma(0)} * \gamma_0 * \alpha_{\gamma(a_1)}) \cdot \rho(\alpha^{-1}_{\gamma(a_1)} * \gamma_1 * \alpha_{\gamma(a_2)}) \cdots \rho(\alpha^{-1}_{\gamma(a_{n-1})} * \gamma_{n-1} * \alpha_{\gamma(a_n)}) = \rho(\gamma)$$

We have used the fact that α_p is the constant loop. This completes the proof. □

6 Appendix I: Internal structure of $\mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ and sheaf categories

Definition 6.0.1 (Category of elements). Consider a functor $F\colon \mathscr{A} \longrightarrow \mathsf{Set}$ from a category $\mathscr{A}$ to the category of sets and mappings. The category $\mathsf{Elts}(F)$ is defined in the following way:

- The objects of $\mathsf{Elts}(F)$ are the pairs (A, a) where $A \in |\mathscr{A}|$ and $a \in FA$.
- A morphism $f\colon (A, a) \longrightarrow (B, b)$ is an arrow $f\colon A \longrightarrow B$ in $\mathscr{A}$ such that $Ff(a) = b \in FB$.
- The composition of $\mathsf{Elts}(F)$ is that induced by the composition of $\mathscr{A}$.

Remark 6.0.2. In the case of a contravariant functor $F : \mathscr{A}^{\mathbf{op}} \to \mathsf{Set}$ we will obviously require that for $f\colon (A, a) \longrightarrow (B, b)$ we get $Ff(b) = a$.

Definition 6.0.3. Let $F\colon \mathscr{A} \longrightarrow \mathsf{Set}$ and the category of elements $\mathsf{Elts}(F)$ then we can define the projection functor

$$\begin{aligned} \pi_F\colon \mathsf{Elts}(F) &\longrightarrow \mathscr{A} \\ (A, a) &\longmapsto A \end{aligned}$$

with the obvious action on morphisms.

$$\begin{array}{ccc} P(C_1) & \xrightarrow{\eta_{C_1}} & \mathrm{Hom}_{\mathscr{E}}(A(C_1), E) \\ \uparrow{\scriptstyle P(f)} & & \uparrow{\scriptstyle -\circ A(f)} \\ P(C_2) & \xrightarrow{\eta_{C_2}} & \mathrm{Hom}_{\mathscr{E}}(A(C_2), E) \end{array}$$

Diagram 6.1.1

Theorem 6.0.4. *Given a functor $A\colon \mathscr{C} \longrightarrow \mathscr{E}$ from a small category to a cocomplete category we can define a functor $R\colon \mathscr{E} \longrightarrow \mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ given by*

$$\begin{aligned} R(E)\colon \mathscr{C}^{\mathbf{op}} &\longrightarrow \mathsf{Set} \\ C &\longmapsto \mathrm{Hom}_{\mathscr{E}}(A(C), E) \end{aligned}$$

This functor has a left adjoint L:

$$\begin{aligned} L\colon \mathsf{Set}^{\mathscr{C}^{\mathbf{op}}} &\longrightarrow \mathscr{E} \\ P &\longmapsto \operatorname*{colim}_{\mathsf{Elts}(P)}(A \circ \pi_P) \end{aligned}$$

Proof. Let $\eta \in \mathrm{Hom}_{\mathsf{Psh}(C)}(P, R(E))$. For every $(C, p) \in \mathsf{Elts}(P)$ consider

$$\eta_C(p) \in R(E)C = \mathrm{Hom}_{\mathscr{E}}(A(C), E).$$

We need to show that these maps form a cocone, consider a morphism $f\colon (C_1, p_1) \longrightarrow (C_2, p_2)$ of $\mathsf{Elts}(P)$, then commutativity of Diagram 6.1.1 shows that we have just defined a cocone on $A \circ \pi_P$ getting a unique factorization map $f_\eta \in \mathrm{Hom}_{\mathscr{E}}(L(P), E)$. In a similar way given $f \in \mathrm{Hom}_{\mathscr{E}}(L(P), E)$ we can form a family of maps namely $\{f \circ s_{(C_i,p)}\}_{i\in I}$ that can be clearly seen as a natural transformation $P \xrightarrow{\eta_f} R(E)$. The universal property of the colimit shows that both mappings are inverse to each other. Naturality follows from the construction. □

Corollary 6.0.5. Every presheaf $\mathscr{F}$ in $\mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ with $\mathscr{C}$ a small category, is the colimit of representable presheaves.

Proof. Set $\mathscr{E} = \mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ and $A = Y$ the Yoneda embedding functor. It is immediate to see that

$$R(\mathscr{F})C = \mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(Y(C), \mathscr{F}) \cong \mathscr{F}(C)$$

so R is isomorphic to the identity functor. Recalling Corollary 2.3.4 we can see that L must be isomorphic to the identity. □

Before showing that the category of presheaves on a small category is cartesian closed we will need a previous result about limits in presheaf categories.

Proposition 6.0.6. Limits in $\mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ are constructed pointwise. In other words, given a diagram $F\colon \mathscr{I} \longrightarrow \mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ with the notation $FI = F_I$

$$\lim_{I\in\mathscr{I}} F_I \colon \mathscr{C}^{\mathbf{op}} \longrightarrow \mathsf{Set}$$
$$C \longmapsto \lim_{I\in\mathscr{I}} F_I(C)$$

Proof. It is necessary to construct the natural transformations $\lim_{I\in\mathscr{I}} F_I \xrightarrow{p_I} F_I$ and verify the universal property of the limit. First note that every for every $C \in |\mathscr{C}|$ we have a map

$$\lim_{I\in\mathscr{I}} F_I C \xrightarrow{p_I^C} F_I C.$$

Given $C_1 \xrightarrow{f} C_2$ by setting $\{F_I(f) \circ p_I^{C_2}\}_{I\in|I|}$ we can form a cone on $F_I(C_1)$ inducing a factorization map:

$$\lim_{I\in\mathscr{I}} F_I C_2 \longrightarrow \lim_{I\in\mathscr{I}} F_I C_1$$

It is immediate to see that this construction shows that every p_I is a natural transformation. For the universal property suppose a presheaf P with natural transformations $P \xrightarrow{\overline{p}_I} F_I$. Then evaluating the diagram of P in each $C \in |\mathscr{C}|$ we can find factorization maps in Set that form a natural transformation. We omit this last comprobation since is analogous to the previous one. □

Remark 6.0.7. This construction can be dualized to show that colimits in Set are constructed pointwise. Since the category of sets and mappings is complete and cocomplete we get as a corollary that $\mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ is also complete and cocomplete.

Corollary 6.0.8. Finite products commute with colimits in $\mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$

Proof. Let F_I denote an $\mathscr{I}$-indexed diagram in $\mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ with $\lim_{I\in|\mathscr{I}|} F_I = F$ then our corollary asserts that

$$\lim_{I\in|\mathscr{I}|} (F_I \times P) \cong F \times P$$

Since limits are constructed pointwise $\forall C \in |\mathscr{C}|$:

$$\lim_{I \in |\mathscr{I}|} (F_I \times P)(C) = \lim_{I \in |\mathscr{I}|} (F_I C \times PC) \cong \lim_{I \in |\mathscr{I}|} F_I C \times PC = FC \times PC$$

Where we have used the fact that Set is cartesian closed in the first isomorphism to see that since the functor $- \times PC$ is a left adjoint preserves colimits. ☐

Theorem 6.0.9. *Given a small category $\mathscr{C}$ then the presheaf category $\mathsf{Set}^{\mathscr{C}^{\mathrm{op}}}$ is cartesian closed*

Proof. Suppose that the exponential of P exists, so we can use a Yoneda argument as follows

$$\mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(Y(C) \times P, Q) \cong \mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(Y(C), Q^P) \cong Q^P(C)$$

Now we can drop the assumption of the existence of $-^P$ and define the exponential with the formula above. It is left to prove that we still have an adjunction even in the non-representable case. Let $f \in \mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(F \times P, Q)$ and use Corollary 6.0.5 to find a diagram $\{F_I\}_{I \in \mathscr{I}}$ with colimit F and each $F_I = Y(C_I)$ representable. Then $F \times P \cong \lim\limits_{I \in \mathscr{I}} F_I \times P$ so f can be identified with a family of maps

$$f_I \in \mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(F_I \times P, Q) \cong \mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(Y(C_I), Q^P) \quad I \in |\mathscr{I}|$$

Naturality of the isomorphism shows that $\{f_I\}_{I \in |\mathscr{I}|}$ is a cocone on Q^P so the universal property induces a unique map

$$\mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(F \times P, Q) \xrightarrow{\theta} \mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(F, Q^P)$$

To find an inverse for θ we only need to do this construction backwards. It is important to note that every step used in this proof is natural so there is nothing left to prove. ☐

Remark 6.0.10. If $\mathscr{C} = \mathcal{O}(X)$ then the internal $\underline{\mathrm{Hom}}$ for presheaves is given by

$$\underline{\mathrm{Hom}}_{\mathsf{Psh}(X)}(\mathscr{F}, \mathscr{G})(U) = \{\eta \colon \mathscr{F}|_U \Longrightarrow \mathscr{G}|_U\}$$

It is convenient now to reduce the level of generality in our disgression. Although every result that we will prove generalizes in the obvious way to sheaves on "topological categories" (sites) we find this point of view out of the scope of this memoir. An interested reader might want to look at [4]. We will focus on sheaves on topological spaces with the aim of showing that this category inherits the internal logic of a cartesian closed category. Before doing this some results about reflective categories will be given to describe limits in sheaf categories. We recall that a reflective subcategory is a full subcategory replete in the sense of [2] with a left adjoint to the inclusion functor.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{A}}(GFA, GFA) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{B}}(FA, FGFA) \\
{\scriptstyle \alpha_A \circ -}\uparrow & & \uparrow{\scriptstyle \eta_{FA} \circ -} \\
\mathrm{Hom}_{\mathscr{A}}(GFA, A) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{B}}(FA, FA)
\end{array}$$

Diagram 6.2.1

Lemma 6.0.11. *Consider a functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$ with G left adjoint to F. Denote by (η, ε) the unit and the counit respectively then if F is full and faithful, ε is an isomorphism.*

Proof. Let $FA \xrightarrow{\eta_{FA}} FGFA$ since F is a full functor there exists some $A \xrightarrow{\alpha_A} GFA$ with $F(\alpha_A) = \eta_{FA}$. We claim that α_A is the inverse to ε_A.

$$F(\varepsilon_A \circ \alpha_A) = F(\varepsilon_A) \circ F(\alpha_A) = F(\varepsilon_A) \circ \eta_{FA} = \mathbb{1}_{FA} \implies (\varepsilon_A \circ \alpha_A) = \mathbb{1}_A \text{ since } F \text{ is faithful.}$$

Now observing Diagram 6.2.1 it easy to see that

$$\theta(\alpha_A \circ \varepsilon_A) = \eta_{FA} = \theta(\mathbb{1}_{GFA}) \implies \alpha_A \circ \varepsilon_A = \mathbb{1}_{GFA}$$ □

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{B}}(L, irL) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{A}}(rL, rL) \\ \downarrow{\scriptstyle iq_D \circ -} & & \downarrow{\scriptstyle q_D \circ -} \\ \mathrm{Hom}_{\mathscr{B}}(L, iFD) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{A}}(rL, FD) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathscr{B}}(L, irL) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{A}}(rL, rL) \\ \downarrow{\scriptstyle i(\nu_L) \circ -} & & \downarrow{\scriptstyle \nu_L} \\ \mathrm{Hom}_{\mathscr{B}}(L, irL) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{A}}(rL, rL) \end{array}$$

Diagram 6.3.1 Diagram 6.3.2

Proposition 6.0.12. Given a full reflective subcategory of a complete category,

$$\mathscr{A} \underset{r}{\overset{i}{\rightleftarrows}} \mathscr{B}$$

then $\mathscr{A}$ is also complete.

Proof. Consider a diagram in $\mathscr{A}$ given by $F\colon \mathscr{D} \longrightarrow \mathscr{A}$ and let L denote the limit of $i \circ F$ with cannonical maps $L \xrightarrow{p_D} i \circ FI$. Let $q_D = \theta(p_D) \in \mathrm{Hom}_{\mathscr{A}}(rL, FD)$ then commutativity of Diagram 6.3.1 shows that $iq_D \circ \eta_L = p_D$. Given $d\colon D \longrightarrow D'$, the relation

$$i(Fd \circ q_D) \circ \eta_L = iFd \circ iq_D \circ \eta_L = iFd \circ p_D = p_{D'} = iq_{D'} \circ \eta_L$$

implies that $Fd \circ q_D = q'_D$ by naturality of the bijection of the adjunction. So the morphisms $(q_D)_{D \in |\mathscr{D}|}$ constitute a cone on F and thus $(iq_D)_{D \in |\mathscr{D}|}$ constitutes a cone on iF. This yields an unique factorization map $\mu_L\colon irL \longrightarrow L$ such that $p_D \circ \mu_L = iq_D$. Then

$$p_D \circ \mu_L \circ \eta_L = iq_D \circ \eta_L = p_D \implies \mu_L \circ \eta_L = \mathbb{1}_L \text{ by the universal property}$$

Finally $\eta_L \circ \mu_L\colon irL \longrightarrow irL$ can be written as $i(\nu_L)$ since i is full and faithfull. Finally

$$i(\nu_L) \circ \eta_L = \eta_L \implies \nu_L = \mathbb{1}_{rL} \text{ by } \mathit{Diagram}\ 6.3.2$$

We have shown that $irL \cong L$. Since $\mathscr{A}$ is replete this means that L belongs already to $\mathscr{A}$. □

Proposition 6.0.13. Given a full reflective subcategory of a cocomplete category,

$$\mathscr{A} \underset{r}{\overset{i}{\rightleftarrows}} \mathscr{B}$$

then $\mathscr{A}$ is also cocomplete.

Proof. Consider a diagram in $\mathscr{A}$ of a colimit $F\colon \mathscr{D} \longrightarrow \mathscr{A}$ and let L denote the colimit of $i \circ F$. Since r is a left adjoint then rL is the colimit of $r \circ i \circ F \cong F$ by Lemma 6.0.11. □

Corollary 6.0.14. The category of sheaves on a topological space, $\mathsf{Sh}(X)$ is complete and cocomplete.

Theorem 6.0.15. *The category of sheaves on a topological space is cartesian closed.*

Proof. We showed previously that the category of presheaves over X has an internal Hom. We only need to check that if $\mathscr{F},\mathscr{G}$ are sheaves, then $\underline{\mathrm{Hom}}_{\mathsf{Sh}(X)}(\mathscr{F},\mathscr{G})$ is also a sheaf.

Let be $U \subset X$ an open set together with a cover $\{U_i\}$ and suppose that we have compatible natural transformations $\eta^i \in \underline{\mathrm{Hom}}_{\mathsf{Sh}(X)}(\mathscr{F},\mathscr{G})(U_i)$. We want to be able to define a natural transformation:

$$\eta\colon \mathscr{F}|_U \Longrightarrow \mathscr{G}|_U \text{ with the property } \eta|_{U_i} = \eta^i$$

We start by taking $V \subset U$ and using the open cover $\{U_i\}$ to form an open cover $\{V_i\}$ of V. Now given $t \in \mathscr{F}(V)$ we use the restriction maps to get $t_i \in \mathscr{F}(V_i)$ and since each $V_i \subset U_i$ we can map the elements via $\eta^i|_{V_i} = \bar{\eta}^i$. We are going to denote $\bar{\eta}^i|_{V_i\cap V_j} = \bar{\eta}^{ij}$. The condition of compatibility translates to $\bar{\eta}^{ij} = \bar{\eta}^{ji}$. In a similar way we denote $t_i|_{V_i\cap V_j} = t_{ij}$. Moreover, if $V_i \cap V_j \neq \emptyset$ since the natural transformations match in the intersection and every natural transformation commute with the restriction maps we can see that:

$$\bar{\eta}^i(t_i)|_{V_i\cap V_j} = \bar{\eta}^{ij}(t_{ij}) = \bar{\eta}^{ji}(t_{ji}) = \bar{\eta}^j(t_j)|_{V_i\cap V_j}$$

Observing the fact that $\bar{\eta}^i(t_i)$ is a family of compatible sections of $\mathscr{G}(V)$ we can glue them to form $\eta(t)$. It's clear that we have found a way of sending elements of $\mathscr{F}(V)$ to $\mathscr{G}(U)$, the last thing to check is that this map constitutes a natural transformation but we omit that last comprobation since all the maps in our construction are natural so they won't change much when restricted. □

Remark 6.0.16. Component-wise addition of natural transformations shows that $\mathsf{Sh}^{\mathsf{Ab}}(X)$ has an internal $\underline{\mathrm{Hom}}_{\mathsf{Sh}^{\mathsf{Ab}}(X)}(-,-)$. However, this category is not cartesian closed since it has a zero object.

Remark 6.0.17. Sheafification of the tensor product of presheaves of $\mathcal{O}_X$-Modules yields the right definition of tensor product of sheaves. As expected, a familiar argument to the one used in $\mathscr{R}$-Mod shows that the tensor product of sheaves is adjoint to $\underline{\mathrm{Hom}}$.

6.1 Subobject classifiers

Definition 6.1.1 (Subobject). Let $\mathscr{C}$ be a category and $C \in |\mathscr{C}|$. We say that two monomorphisms $S \overset{f}{\rightarrowtail} C,\ R \overset{g}{\rightarrowtail} C$ are equivalent if there exists an isomorphism $\tau\colon S \longrightarrow R$ such that $g \circ \tau = f$. An equivalence class of monomorphisms with codomain C is called a subobject of C. Dually we have the notion of quotient of C. We usually refer to this class (which is not always a set!) as $\mathsf{Sub}(C)$.

Definition 6.1.2. A category is said to be well-powered when the subobjects of every object constitute a set.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle m} \\ D & \xrightarrow{f} & C \end{array} \qquad \begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \text{true}} \\ C & \xrightarrow{\chi_S} & \Omega \end{array} \qquad \begin{array}{ccccc} X & \longrightarrow & S & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{true}} \\ D & \xrightarrow{f} & C & \xrightarrow{\chi_S} & \Omega \end{array}$$

Diagram 6.4.1 Diagram 6.4.2 Diagram 6.4.3

Proposition 6.1.3. Given a well-powered category $\mathscr{C}$ there is a subobject functor

$$\mathsf{Sub}\colon \mathscr{C} \longrightarrow \mathsf{Set}$$

Proof. The action on objects is clear. Let $C \longrightarrow D$ be an arrow in $\mathscr{C}$ and $X \overset{m}{\rightarrowtail} C$ and form the pullback square of Diagram 6.4.1. Then given $u, v\colon M \rightrightarrows S$ such that $n \circ u = n \circ v$ we can see that:

$$m \circ g \circ u = f \circ n \circ u = f \circ n \circ v = m \circ g \circ v \implies g \circ u = g \circ v.$$

Meaning that u, v are possible factorizations for the maps $n \circ u, g \circ u$ then the universal property shows that $u = v$ so n is a monomorphism and S is a subobject of D. □

Definition 6.1.4 (Subobject classifier). In a finitely complete category $\mathscr{C}$, a pair (true, Ω), $\Omega \in |\mathscr{C}|$ and $\text{true}\colon 1 \longrightarrow \Omega$ is called a subobject classifier if for every object $C \in |\mathscr{C}|$ and every subobject $S \rightarrowtail C$ there exists a map $C \xrightarrow{\chi_S} \Omega$ such that Diagram 6.4.2 is a pullback square.

$$\begin{array}{ccc} \mathsf{Sub}(\Omega) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{C}}(\Omega, \Omega) \\ \downarrow{\scriptstyle \mathsf{Sub}\chi_S} & & \downarrow{\scriptstyle -\circ\chi_S} \\ \mathsf{Sub}(C) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathscr{C}}(C, \Omega) \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\chi} & \Omega_0 \\ \| & & \downarrow{\scriptstyle t_0} \\ C & \xrightarrow{t_0\circ\chi} & \Omega \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\chi'} & \Omega_0 \\ \| & & \downarrow{\scriptstyle t_0} \\ C & \xrightarrow{t_0\circ\chi'} & \Omega \end{array}$$

Diagram 6.5.1 Diagram 6.5.2 Diagram 6.5.3

Proposition 6.1.5. A category $\mathscr{C}$ has a subobject classifier Ω is and only if

$$\mathsf{Sub} \cong \mathrm{Hom}_{\mathscr{C}}(-, \Omega)$$

Proof. $(\Rightarrow)$ Let $C \in |\mathscr{C}|$ and a subobject $S \rightarrowtail C$ then define

$$\begin{aligned} \varepsilon_C\colon \mathsf{Sub}(C) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(C, \Omega) \\ S &\longmapsto \chi_S \end{aligned}$$

this map is well defined since the arrow χ_S is unique and does not depend on the representative S. Conversely to define ε_C^{-1} we set for every $f \in \mathrm{Hom}_{\mathscr{C}}(C, \Omega)$ the pullback as in Diagram 6.4.2 obtaining a subobject of C. It is convinient to recall that any map from the terminal object to any other object in the category is a monomorphism. To finish this proof we need to check that this family of maps indeed defines a natural transformation. Given $D \xrightarrow{f} C$ looking at Diagram 6.4.3 we can see that the outer rectangle is a pullback since both inner squares are by the same argument as Proposition 3.4.6 that shows that $f \circ \chi_C = \chi_D$.

$(\Leftarrow)$ Suppose that Sub is representable with representing object Ω. Then there exists some subobject $\Omega_0 \overset{t_0}{\rightarrowtail} \Omega$ mapped to $\mathbb{1}_\Omega \in \mathrm{Hom}_{\mathscr{C}}(\Omega, \Omega)$. Similarly for every subobject $S \rightarrowtail C$ there is some $\chi_S \in \mathrm{Hom}_{\mathscr{C}}(C, \Omega)$. Then since naturality guarantees that Diagram 6.5.1 commutes it is clear that $\mathsf{Sub}\chi_S(\Omega_0) = S$ and the subobject can be obtained by a pullback square along a unique function. It is left to see that $\Omega_0 \cong 1$. Suppose $\chi, \chi'\colon C \longrightarrow \Omega_0$ and consider C as a subobject of itself via $\mathbb{1}_C$. Then both Diagram 6.5.2 and Diagram 6.5.2 are pullbacks squares since t_0 is a monomorphism. Now uniqueness of the pullback square shows that $t_0 \circ \chi = t_0 \circ \chi' \implies \chi = \chi'$, so $\Omega_0 \cong 1$. □

Remark 6.1.6 (Subobject of a functor). Let $\mathsf{Set}^{\mathscr{C}^{\mathrm{op}}}$ with $\mathscr{C}$ small. We are interested in finding a good description of subobjects in this category. Starting from the definition a subfunctor is a equivalence class of monomorphisms $Q \Longrightarrow P$, monomorphisms in this category are constructed pointwise since they are a special case of a pullback. Then it is possible to define a subfuctor (more formally is it possible to pick a representative of the class) as a functor satisfying

- For every $C \in |\mathscr{C}|$, $Q(C) \subset P(C)$
- For every $D \xrightarrow{f} C$, $Qf = Pf|_{QC}$

Remark 6.1.7. In the case of representable presheaves a subfunctor $F \Longrightarrow YC$ sends every $X \in |\mathscr{C}|$ to a subset of $\mathrm{Hom}_{\mathscr{C}}(X, C)$ with the property that for all arrows $f \in FX$ and $R \xrightarrow{h} X$ the composite $f \circ h \in FX$.

Definition 6.1.8 (Sieve). Given an $C \in |\mathscr{C}|$ a sieve on C is a set S of arrows with codomain C such that for every arrow h composable with $f \in S$ we have $f \circ h \in S$. It is trivial to check that sieves on C are exactly the subfunctors of YC. Going further since given a map $B \xrightarrow{g} C$ there is a natural transformation $YB \to YC$ it is natural to consider the corresponding case in the language of sieves. This is given by

$$S \cdot g = \{h \mid g \circ h \in S\}$$

$$\begin{array}{ccc} Q & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ P & \xrightarrow{\chi} & \Omega \end{array} \qquad \begin{array}{ccc} Q(C) & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ P(C) & \xrightarrow{\chi_C} & \Omega(C) \end{array} \qquad \begin{array}{ccc} P(A) & \xrightarrow{\theta_A} & \Omega(A) \\ \downarrow Pf & & \downarrow \Omega(f) \\ P(C) & \xrightarrow{\theta_C} & \Omega(C) \end{array}$$

Diagram 6.6.1 Diagram 6.6.2 Diagram 6.6.3

Proposition 6.1.9. The category $\mathsf{Set}^{\mathscr{C}^{\mathbf{op}}}$ has a subobject classifier.

Proof. Following the same “Yoneda” point of view suppose that Ω exists and set

$$\mathrm{Hom}_{\mathsf{Psh}(\mathscr{C})}(Y(C), \Omega) \cong \mathsf{Sub}(YC) \cong \Omega(C).$$

Now dropping the assumption we set $\Omega(C) = \{S|S \text{ is a sieve on } C\}$ and the action on a morphism $B \xrightarrow{g} C$ is defined to be:

$$\Omega(g) \colon \Omega(C) \longrightarrow \Omega(B)$$
$$S \longrightarrow S \cdot g$$

For every $C \in |\mathscr{C}|$ the maximal sieve $t(C)$ is defined to be the set of all arrows with codomain C. It is readily clear that $t(C) \cdot g = t(B)$ so there is a natural transformation $true \colon 1 \Longrightarrow \Omega$ with $\mathrm{true}_C(*) = t(C)$.

Let Q be a subfunctor of P. For every $x \in PC$ set

$$\chi_C(x) = \{f \mid Pf(x) \in Q(\mathrm{dom}(f))\}.$$

This definition considers the set of maps $f \colon A \longrightarrow C$ that sends an element $x \in PC$ to $Pfx \in QA \subset PA$. Given $g \colon B \longrightarrow A$ we can see that $P(f \circ g)(x) = Pg \circ Pf(x)$ and since $Pf(x) \in Q(A)$ then clearly $Pg(Pf(x)) \in Q(B)$ by definition of subfunctor implying that $\chi_C(x)$ is a sieve on C. We can observe that

$$\chi_A(Pf(x)) = \{h \mid Ph(Pf(x)) = P(f \circ h)(x) \in Q(\mathrm{dom}\, h)\} = \chi_C(x) \cdot f$$

proving naturality of the natural transformation $P \overset{\chi}{\Longrightarrow} \Omega$. Moreover, if $x \in Q(C)$ we have that $\chi_C(x) = t(C)$ showing commutativity of Diagram 6.6.1 and that Diagram 6.6.2 is a pullback square. We need to show that χ is the unique natural transformation satisfying the pullback condition, to see this suppose $P \overset{\theta}{\Longrightarrow} \Omega$ such that Q is the pullback of true and θ. Given $x \in P(C)$ and $A \xrightarrow{f} C$, the pullback condition means that $Pf(x) \in Q(A)$ if and only if $\theta_A(Pf(x)) = \text{true}_A$; by naturality of Diagram 6.6.3 this equals to $\theta_C(x) \cdot f = \text{true}_A$, showing that $f \in \theta_C(x)$. The elements of $\theta_C(x)$ then are those with $Pf(x) \in Q(A)$ so the pullback condition forces the definition of χ uniquely. □

7 Appendix II: Functorial algebraic geometry

In this section we are going to consider functors on non-small categories giving rise to some set theoretical issues. We refer to [3] for a discussion of these problems and its solution via universes.

Definition 7.0.1 (Affine schemes). Let $\mathsf{Set}^{\mathsf{Rng}}$ every representable functor $\mathscr{F}$ with representing object A is called the affine scheme of the ring A.

Definition 7.0.2 (Affine line). The functor $\mathscr{L}$ that sends every $A \in |\mathsf{Rng}|$ to its underlying set is called the affine line.

Remark 7.0.3. One can easily verify that $\mathscr{L} \cong \text{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathbb{Z}[T], -)$.

Remark 7.0.4. The set $\text{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathscr{F}, \mathscr{L})$ has a natural ring structure. Given two natural transformations we can easily perfom a pointwise sum in each component. This ring is called the ring of functions on $\mathscr{F}$ and will be denoted by $\mathcal{O}(\mathscr{F})$.

In the process of showing that this approach is essentially equivalent to the geometric vision of locally ringed spaces we need to stablish some category theoretical results. In this section locally ringed spaces will be called geometric spaces and its category will be denoted by GSp.

Definition 7.0.5. We have a contravariant functor Γ from the category of geometric spaces with values in the ring category defined as

$$\begin{aligned} \Gamma\colon \mathsf{GSp} &\longrightarrow \mathsf{Rng}^{\mathbf{op}} \\ (X, \mathcal{O}_X) &\longmapsto \mathcal{O}_X(X) \end{aligned}$$

Theorem 7.0.6. *The functor* Γ *is left adjoint to* Spec*:*

$$\mathsf{GSp}^{\mathbf{op}} \underset{\text{Spec}}{\overset{\Gamma}{\rightleftarrows}} \mathsf{Rng}^{\mathbf{op}}$$

Proof. We will show the natural isomorphism $\text{Hom}_{\mathsf{GSp}}(X, \text{Spec}\, A) \cong \text{Hom}_{\mathsf{Rng}}(A, \Gamma X)$, note the subscript of the second Hom-set where we can see that we will be working with Spec as a contravariant functor instead of working in the opposite category.

Given a morphism of geometrical spaces $(f, f^\sharp)\colon X \longrightarrow \text{Spec}\, A$ it is clear that we can send $(f, f^\sharp)$ to the global component of the natural transformation. Conversely let $\varphi \in \text{Hom}_{\mathsf{Rng}}(A, \Gamma X)$ we can define the pair $(f_\varphi, f_\varphi^\sharp)$ as follows,

$$\begin{aligned} f_\varphi\colon X &\longrightarrow \text{Spec}\, A \\ x &\longmapsto \varphi^{-1} \circ p_x^{-1}(\mathfrak{m}_x) \end{aligned}$$

where p_x denotes the canonical morphism $\mathcal{O}_X(X) \xrightarrow{p_x} \mathcal{O}_{X,x}$. We need to check that

$$f_\varphi^{-1}(D(g)) = \{x \in X \mid f_\varphi(x) \in D(g)\} = \{x \in X \mid \varphi(g) \text{ is invertible in } \mathcal{O}_{X,x}\}$$

is open. To see this consider $x \in f_\varphi^{-1}(D(g))$ and $s_\varphi, s_{\varphi^{-1}}$ two sections on an open set U representing $\operatorname{germ}_x \varphi(g), \operatorname{germ}_x \varphi(g)^{-1}$. First we note that there is some neighborhood $V_1 \subset U$ such that $s_\varphi, s_{\varphi^{-1}}$ are mutual inverses also it is easy to see that exists some $V_2 \subset U$ such that $\varphi(g)|_{V_2} = s_\varphi|_{V_2}$, then it follows that we can find and open set $V \subset f_\varphi^{-1}(D(g))$ such that $x \in V$. Now we need to construct the natural transformation $\mathcal{O}_A \xRightarrow{f_\varphi^\sharp} f_{\varphi,*}\mathcal{O}_X$. Consider a basic open set $D(g) \subset \operatorname{Spec} A$ it is an easy exercise to show that $\varphi(g)|_{f_\varphi^{-1}(D(g))}$ is invertible in $f_{\varphi,*}\mathcal{O}_X(D(g))$ since this section is invertible at each stalk. Now the universal property of the localization induces a map,

$$f_\varphi^\sharp|_{D(g)} \colon A_g \longrightarrow f_{\varphi,*}\mathcal{O}_X(D(g)).$$

which is clearly natural due to the universal property of the localization.

To finish the proof we need to see that both assignations are inverse to each other. Let $\varphi \in \operatorname{Hom}_{\mathsf{Rng}}(A, \Gamma X)$ is clear that the global component of $f_\varphi^\sharp$ equals φ. Finally let $(f, f^\sharp) \in \operatorname{Hom}_{\mathsf{GSp}}(X, \operatorname{Spec} A)$, and denote by $\varphi = f^\sharp(\operatorname{Spec} A)$. We will finish this proof in two lemmas. □

$$\begin{array}{ccc} A_{f(x)} & \xrightarrow{f^\sharp_{f(x)}} & \mathcal{O}_{X,x} \\ {\scriptstyle p_{f(x)}}\uparrow & & \uparrow{\scriptstyle p_x} \\ A & \xrightarrow{\varphi} & \mathcal{O}_X(X) \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ A_g & \xrightarrow{f^\sharp|_{D(g)}} & f_*\mathcal{O}_X(D(g)) \end{array}$$

Diagram 7.1.1 Diagram 7.1.2

Lemma 7.0.7. *In the conditions of Theorem 7.0.6 we have* $(f, f^\sharp) = (f_\varphi, f_\varphi^\sharp)$.

Proof. Considering Diagram 7.1.1 we can see that $\varphi^{-1} \circ p_x^{-1}(\mathfrak{m}_x) = f_\varphi(x) = p_{f(x)}^{-1} \circ f_{f(x)}^{\sharp,-1}(\mathfrak{m}_x)$. Then we can note that since $f_{f(x)}^\sharp(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$ implies that $f_{f(x)}^{\sharp,-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ but the inverse image of $\mathfrak{m}_{f(x)}$ under $p_{f(x)}$ equals $f(x)$. We can finish the proof of this lemma together with the proof of Theorem 7.0.6 by noting that Diagram 7.1.2 commutes and applying the universal property of the localization. We left naturality comprobations to the reader. □

Corollary 7.0.8. Denote by AffSch the category of geometric spaces isomorphic to the spectrum of a given ring then the previous adjuction restricts to an isomorphism of categories

$$\mathsf{AffSch} \cong \mathsf{Rng}^{\mathbf{op}}$$

Proof. Obvious. □

Definition 7.0.9. For every $X \in |\mathsf{GSp}|$ we have its associated functor given by

$$\begin{aligned} \mathscr{H} \colon \mathsf{GSp} &\longrightarrow \mathsf{Set}^{\mathsf{Rng}} \\ X &\longmapsto \mathscr{H}X = \operatorname{Hom}_{\mathsf{GSp}}(\operatorname{Spec} -, X) \end{aligned}$$

Remark 7.0.10. If $\operatorname{Spec} A \in |\mathsf{AffSch}|$ then clearly $\mathscr{H} \operatorname{Spec} A \cong \operatorname{Hom}_{\mathsf{Rng}}(A, -)$.

Theorem 7.0.11. *There is a functor* $|-| \colon \mathsf{Set}^{\mathsf{Rng}} \longrightarrow \mathsf{GSp}$ *which is left adjoint to* $\mathscr{H}$

Proof. This is a consequence of Theorem 6.0.4. A thoughtful reader might ask why we did not show that GSp is cocomplete, this can be seen in [3]. We skip this proof since it is a technicality with no interest to this section. We remind the reader that this proof depends on the two universes setting defined in [3] since Rng is not small. □

Proposition 7.0.12. For every affine scheme $\mathscr{F} = \mathrm{Hom}_{\mathsf{Rng}}(A, -)$ we have $|\mathscr{F}| \cong \mathrm{Spec}\, A$.

Proof. Again consider the category $\mathsf{Elts}(\mathscr{F})$ and the diagram of the functor $\mathrm{Spec} \circ \pi$ we can see that every $\mathrm{Spec}\, R_i$ in the diagram comes from some (R_i, f_i) such that $f_i \in \mathrm{Hom}_{\mathsf{Rng}}(A, R_i)$. Then we can set a family of maps given by $\{\mathrm{Spec}\, f_i\}_{i \in I}$ (modulo set theory) such that

$$\mathrm{Spec}\, f_i \colon R_i \longrightarrow A$$

Given $(R_i, f_i) \xrightarrow{\varphi} (R_j, f_j)$ it clear that since this is a morphism in $\mathsf{Elts}(\mathscr{F})$ we have $\varphi \circ f_i = f_j$ which amounts to the fact that the family of maps $\{\mathrm{Spec}\, f_i\}_{i \in I}$ forms a cocone. We have now a unique factorization map $|\mathscr{F}| \xrightarrow{\alpha} \mathrm{Spec}\, A$. Let $(|\mathscr{F}|, s_i)$ denote the colimit together with its canonical cocone then we have a map $\mathrm{Spec}\, A \xrightarrow{s_A} |\mathscr{F}|$. It is now inmediate to check that we have just defined an isomorphism. □

In our task of finding a functorial foundation of algebraic geometry we can see that the idea of representable functor captures correctly the notion of affine scheme. We seek to find a functorial description of general schemes avoiding locally ringed spaces this amounts to finding a subcategory of $\mathsf{Set}^{\mathsf{Rng}}$ such that $|-|$ restricts to an equivalence of categories with the subcategory Sch of schemes.

7.1 Some reflections about the notion of a point

The revolutionary ideas introduced by Alexander Grothendieck in the field of algebraic geometry had a tremendous impact on the way we look at very familiar notions in geometry such as points. For example, considering a parabola X, we may ask what are the points of this geometric object. At first sight these points are solutions of a polynomial equation, namely, $y = x^2$. However, translating this idea to affine schemes we encounter a problem: Which of the following schemes truly represent the parabola?

$$X_1 = \mathrm{Spec}\, \mathbb{Z}[t_1, t_2]/(t_2 - t_1^2) \quad X_2 = \mathrm{Spec}\, \mathbb{R}[t_1, t_2]/(t_2 - t_1^2) \quad X_3 = \mathrm{Spec}\, \mathbb{C}[t_1, t_2]/(t_2 - t_1^2)$$

A first answer would be to say that it depends on which kind solutions to the equation you are interested in, but geometrically this is not very satisfactory. We should have one parabola and different kinds of points in it. A categorical solution to this problem is to understand points through morphisms.

Example 7.1.1. Consider a point $(a_1, a_2) \in \mathbb{R}$ in the parabola. Then we can construct the morphism,

$$\begin{aligned} \varphi \colon \mathbb{Z}[t_1, t_2] &\longrightarrow \mathbb{R} \\ t_i &\longmapsto a_i \end{aligned}$$

then clearly this maps factorizes to

$$\overline{\varphi} \in \mathrm{Hom}_{\mathsf{Rng}}(\mathbb{Z}[t_1, t_2]/(t_2 - t_1^2), \mathbb{R}) \cong \mathrm{Hom}_{\mathsf{GSp}}(\mathrm{Spec}\, \mathbb{R}, \mathrm{Spec}\, \mathbb{Z}[t_1, t_2]/(t_2 - t_1^2)).$$

Definition 7.1.2 (k-points). Let X be a locally ringe space. A k-point is an element of $(x_k, x_k^\sharp) \in \mathrm{Hom}_{\mathsf{GSp}}(\operatorname{Spec} k, X)$.

Remark 7.1.3. Clearly as a set $\operatorname{Spec} k$ is just one point, accordingly the topological map is determined by $x_k(*) = x \in X$. The map on the structure sheaves should be a local ring map in the stalks, $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{k,*}$, so $\mathfrak{m}_x$ must be mapped to zero getting a factorization $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \rightarrowtail k$. The reader might check that the data of a k-point is equivalent to the data of a point $x \in X$ together with a field extension of the residue field $k(x) \rightarrowtail k$.

Definition 7.1.4. Let $\mathcal{F} \in \mathsf{Set}^{\mathsf{Rng}}$, we shall define the underlying set of $\mathcal{F}$ as

$$|\mathcal{F}|^s = \operatorname*{colim}_{\mathsf{Field}} \mathcal{F}$$

Proposition 7.1.5. The underlying set of $\mathrm{Hom}_{\mathsf{Rng}}(A, -)$ coincides with the set of prime ideals of A. This bijection is natural in A.

Proof. We start by giving a specific construction for this underlying set,

$$\operatorname*{colim}_{\mathsf{Field}} \mathrm{Hom}_{\mathsf{Rng}}(A, -) \cong \coprod_{K \in \mathsf{Field}} \mathrm{Hom}_{\mathsf{Rng}}(A, K)/\sim$$

where "$\sim$" is the smallest equivalence relation making the disjoint union inclusions the maps of a cocone. Then, every $x \in |\mathrm{Hom}_{\mathsf{Rng}}(A, -)|^s$ can be represented by a function $f\colon A \longrightarrow K_1$ for some field whose kernel is an element $\mathfrak{p}_x \in \operatorname{Spec} A$. To see that this map is well defined, we need to check that given $u\colon K_1 \longrightarrow K_2$, we have $\operatorname{Ker}(u \circ f) = \operatorname{Ker}(f)$. This follows inmediatly from the fact that every field morphism is injective. Conversely given a prime ideal of A we can send $\mathfrak{p}$ to the cannonical map of A into its residue field at $\mathfrak{p}$, i.e. the fraction field of $A/\mathfrak{p}$. It is easy to check that we have defined a natural bijection. $\square$

Theorem 7.1.6. *The underlying set of the functor* $\mathcal{F} : \mathsf{Rng} \longrightarrow \mathsf{Set}$ *is naturally isomorphic to the underlying set of the topological space* $|\mathcal{F}|$.

Proof. Define the bifunctor,

$$\begin{aligned} \mathsf{Elts}(\mathcal{F}) \times \mathsf{Field} &\longrightarrow \mathsf{Set} \\ (A, a) \times K &\longmapsto \mathrm{Hom}_{\mathsf{Rng}}(A, K) \end{aligned}$$

and note that $\operatorname*{colim}_{\mathsf{Elts}(\mathcal{F})} \mathrm{Hom}_{\mathsf{Rng}(A,-)} \cong \mathcal{F}$. Therefore, the underlying set functor can be interpreted as a double colimit of a bifunctor. Since Set is cocomplete the two colimits involved commute with each other. Finally,

$$|\mathcal{F}|^s = \operatorname*{colim}_{\mathsf{Elts}(\mathcal{F})} \operatorname*{colim}_{\mathsf{Field}} \mathrm{Hom}_{\mathsf{Rng}}(A, K) \cong \operatorname*{colim}_{\mathsf{Elts}(\mathcal{F})} \operatorname{Spec} A \cong |\mathcal{F}|. \qquad \square$$

$$\begin{array}{ccc} \operatorname{Spec} k(x) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k(f(x)) & \longrightarrow & Y \end{array}$$

Diagram 7.2.1

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Rng}}(A, -)_{D(I)} & \Longrightarrow & \mathrm{Hom}_{\mathsf{Rng}}(A, -) \\ \Downarrow & & \Downarrow \eta \\ \mathcal{U} & \overset{i}{\Longrightarrow} & \mathcal{F} \end{array}$$

Diagram 7.2.2

Proposition 7.1.7. Let F be the functor that sends every geometric space to its underlying set. Then $F \cong \operatorname*{colim}_{K \in \mathsf{Field}} \mathscr{H}-$.

Proof. Let $x \in X$, we can define a morphism of locally ringed space by setting

$$\begin{aligned} \varphi_x \colon \operatorname{Spec} k(x) &\longrightarrow X \\ * &\longmapsto x \end{aligned}$$

and the identity map on the stalk, $\mathcal{O}_{X,x}$. Conversly for every element in the colimit represented by a morphism $\varphi\colon \operatorname{Spec} k \longrightarrow X$ we can associate the point $\varphi(*) \in X$. This map is clearly well defined and it can be shown that both maps are mutual inverses. Given a morphism of locally ringed spaces $X \xrightarrow{f} Y$, a simple verification shows that Diagram 7.2.1 commutes, proving the naturality of the bijection. $\square$

7.2 Open subfunctors

$$\begin{array}{ccc} |\mathscr{G}| & \xrightarrow{\quad i \quad} & |\mathscr{F}| \\ \big\uparrow & & \big\uparrow{\scriptstyle s_R} \\ s_R^{-1}(|\mathscr{G}|) & \longrightarrow & \operatorname{Spec} R \end{array}$$

Diagram 7.3.1

Definition 7.2.1. Let $\mathscr{F} \in |\mathsf{Set}^{\mathsf{Rng}}|$, and $U \subset |\mathscr{F}|$, then we shall define the following subfunctor,

$$\mathscr{F}_U(R) = \{p \in \mathscr{F}(R) \mid \forall \varphi\colon R \longrightarrow K \text{ with } K \text{ a field }, [\mathscr{F}\varphi(x)] \in U\}.$$

Here we make the following abuse of notation, we identify

$$[\mathscr{F}\varphi(x)] \in \operatorname*{colim}_{K \in \mathsf{Field}} \mathscr{F}$$

with its image under the canonical bijection of Theorem 7.1.6.

Remark 7.2.2. It is a consequence of the previous definition that

$$\operatorname*{colim}_{K \in \mathsf{Field}} \mathscr{F}_U \cong U$$

Proposition 7.2.3. Let $\mathscr{G} \overset{\varepsilon}{\Longrightarrow} \mathscr{F}$, then for every $U \in |\mathscr{F}|$ the pullback of ε and the canonical inclusion of $\mathscr{F}_U$ is isomorphic to $\mathscr{G}_Q$ where $Q = |\varepsilon|^{-1}(U)$.

Proof. Let P denote the pullback because limits of presheaves are compute pointwise we have,

$$P(R) = \{x \in \mathscr{G}(R) \mid \varepsilon_R(x) \in \mathscr{F}_U(R)\}.$$

Now we might use the naturality of ε to see that for every morphism φ of R into a field K, $\mathscr{F}\varphi(\varepsilon_R(x)) = \varepsilon_K(\mathscr{G}\varphi(x))$. The right hand side of this equality just states that $x \in P(R)$, if and only if, $|\varepsilon|(G\varphi(x)) \in U$, for every $\varphi\colon R \longrightarrow K$. $\square$

Definition 7.2.4 (Open subfunctor). A subfunctor $\mathscr{U} \overset{i}{\Longrightarrow} \mathscr{F}$ is said to be open if for every representable functor and every natural transformation $\eta\colon \operatorname{Hom}_{\mathsf{Rng}}(A, -) \Longrightarrow \mathscr{F}$, the pullback of this two morphisms is given by Diagram 7.2.2.

Proposition 7.2.5. If U is open in $|\mathscr{F}|$ then $\mathscr{F}_U$ is an open subfunctor of $\mathscr{F}$.

Proof. This is an inmediate consequence of Proposition 7.2.3. □

Proposition 7.2.6. If $\mathscr{U}$ is an open subfunctor of $\mathscr{F}$ then the geometric realization of $\mathscr{U}$ is open in $|\mathscr{F}|$. Moreover, $\mathscr{U} \cong \mathscr{F}_{|\mathscr{U}|}$.

Proof. From the construction of $|\mathscr{F}|$ it is obvious that a set U, is open in the geometric realization if and only if, the inverse image of U via the canonical maps of the colimit are open in each $\operatorname{Spec} R$. Let (R, p) with $p \in \mathscr{F}(R)$. We construct the natural transformation η_p given by the contravariant Yoneda lemma, namely

$$\begin{aligned} \operatorname{Hom}_{\mathsf{Rng}}(R, A) &\xrightarrow{\eta_{p,A}} \mathscr{F}(A) \\ f &\longmapsto \mathscr{F} f(p) \end{aligned}.$$

From this point is just a matter of colimit-"diagram chasing" to see that the diagram of the pullback of η, i maps under the geometric realization functor to Diagram 7.3.1. Therefore, applying Proposition 7.2.3 we find that $s_R^{-1}(|\mathscr{G}|) = D(I)$. In a similar way we might check that $\mathscr{G} \cong \mathscr{F}_{|\mathscr{G}|}$. □

Corollary 7.2.7. There is a bijection between the open subfunctors of $\mathscr{F}$ and the open subsets of $|\mathscr{F}|$.

Proposition 7.2.8. Given an open subfunctor $\mathscr{F}_U$, it can be expressed as a canonical colimit,

$$\mathscr{F}_U \cong \operatorname*{colim}_{(R,p)\in|\mathsf{Elts}(\mathscr{F}_U)|} \operatorname{Hom}_{\mathsf{Rng}}(R, -)_{U_R}.$$

Here U_R denotes the inverse image of U under the canonical colimit map of the geometric realization functor associated to (R, p).

Proof. This is a restatement of Corollary 6.0.5. What we should note is that if we compute the canonical colimit of $\mathscr{F}_U$ as usual we will encounter the natural transformations

$$\begin{aligned} \eta^p \colon \operatorname{Hom}_{\mathsf{Rng}}(R, -) &\Longrightarrow \mathscr{F}_U \\ \mathbb{1}_R &\longmapsto p \end{aligned}.$$

These morphisms can be used together with Proposition 7.2.3 to show that $\operatorname{Hom}_{\mathsf{Rng}}(R, -) \cong \operatorname{Hom}_{\mathsf{Rng}}(R, -)_{U_R}$. □

Definition 7.2.9 (Covering of a functor). A family $\{\mathscr{F}_i\}_{i\in I}$ of subfunctors of $\mathscr{F}$ is said to cover $\mathscr{F}$ if the induced map

$$\coprod_{i\in I} \operatorname*{colim}_{K\in\mathsf{Field}} \mathscr{F}_i \twoheadrightarrow \operatorname*{colim}_{K\in\mathsf{Field}} \mathscr{F},$$

is an epimorphism. Similarly a cover by open subfunctors of $\mathscr{F}$ is called an open covering.

Example 7.2.10. It is clear that if $\{U_i\}_{i\in I}$ is a covering of a locally ringed space then $\{\mathscr{H}U_i\}_{i\in I}$ is a open covering of the functor $\mathscr{H}X$. The scenario when X is a scheme is more interesting. In this case X is covered by opens isomorphic to $\operatorname{Spec} A$ for some ring. This amounts to the fact that $\mathscr{H}X$ has an open covering by representable functors.

7.3 Local functors

Given an affine scheme $\operatorname{Spec} A$ we can find a basis for the topology of the form $D(f_i)$ with $f_i \in A$. Clearly this induces an open inmersion of $SpecA_{f_i} \rightarrowtail \operatorname{Spec} A$. Moreover, the data of a morphism $\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$ is equivalent to the data of a family of morphisms

$$\operatorname{Spec} B \xrightarrow{f_i} \operatorname{Spec} A_i$$

which agree on the intersections $\operatorname{Spec} A_{f_i f_j}$. Translating this behaviour into the functorial language we have the notion of a local functor.

Definition 7.3.1 (Ring covering). Let $A \in \mathsf{Rng}$, and a family of elements $f_i \in A$ such that the set $\{f_i\}_{i \in I}$ generates the unit ideal. For every $i \in I$ we consider the canonical morphism

$$\varphi_i \colon A \longrightarrow A_i \quad A_i = A_{f_i}.$$

Then a covering of A is a family of morphisms of the form $\{\varphi_i\}_{i \in I}$.

Definition 7.3.2 (Local functor). A functor $\mathscr{F} : \mathsf{Rng} \longrightarrow \mathsf{Set}$ is said to be local if for every ring A and every covering $\{\varphi_i\}_{i \in I}$ of A the following diagram is an equalizer,

$$\mathscr{F}(A) \xrightarrow{u} \prod_i \mathscr{F}(A_i) \underset{s}{\overset{t}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(A_{ij}).$$

Here u denote the map induced by $\mathscr{F}\varphi_i$, t denotes the map induced by $\mathscr{F}\varphi_{ij}$ and s denotes the map induced by $\mathscr{F}\varphi_{ji}$.

Remark 7.3.3. The reader will realize that this definition is that of a covariant sheaf.

Proposition 7.3.4. For every locally ringed space X, the functor $\mathscr{H}X$ is local.

Proof. Topologically we can glue the maps because they agree on the intersections. In addition, we can use the fact that $f^{-1}(U)$ can be covered by $f_i^{-1}(U)$ to send sections of $\mathcal{O}_X(U)$ to $\mathcal{O}_A(f^{-1}U)$. We leave the comprobations to the reader. □

7.4 Structure sheaf of the geometric realization of a functor

Proposition 7.4.1. Given $\mathscr{F} \in |\mathsf{Set}^{\mathsf{Rng}}|$ the following conditions are equivalent:

1. $\mathscr{F}$ is local.
2. For each ring A, the presheaf $U \longmapsto \operatorname{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\operatorname{Hom}_{\mathsf{Rng}}(A,-)_U, \mathscr{F})$ is a sheaf of sets over $\operatorname{Spec} A$.
3. For each functor $\mathscr{G}$, the presheaf $U \longmapsto \operatorname{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathscr{G}_U, \mathscr{F})$ is a sheaf of sets over $|\mathscr{G}|$.

Proof. (1) ⇒ (2) For simplicity, we show that we have a sheaf on a basis of $\operatorname{Spec} A$. Nevertheless, this is no loss of generality because we know how to glue the information on a basis. Consider an open set of the form $D(f)$ covered by $D(f_i)$. The only thing that we will need to show is that $\operatorname{Hom}_{\mathsf{Rng}}(A,-)_{D(g)} = \operatorname{Hom}_{\mathsf{Rng}}(A_g,-)$ but this is fairly trivial. After this we can apply the Yoneda lemma to see that $\operatorname{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\operatorname{Hom}_{\mathsf{Rng}}(A_g,-), \mathscr{F}) \cong \mathscr{F}(A_g)$ and conclude.

(2) ⇒ (3) We remind that sheaf categories are complete and cocomplete therefore, we have the

following isomorphisms,

$$\mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathscr{G}_U, \mathscr{F}) \cong \mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\operatorname{colim} \mathrm{Hom}_{\mathsf{Rng}}(R, -), \mathscr{F}) \cong \lim \mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathrm{Hom}_{\mathsf{Rng}}(R, -), \mathscr{F}),$$

where we have introduced the following abuse of notation,

$$\operatorname{colim} \mathrm{Hom}_{\mathsf{Rng}}(R, -) := \mathscr{G}_U \cong \operatorname*{colim}_{(R,p)\in|\mathsf{Elts}(\mathscr{G}_U)|} \mathrm{Hom}_{\mathsf{Rng}}(R, -)_{U_R}$$

(3) ⇒ (1) The last implication can be proved using the Yoneda lemma to identify

$$\mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathrm{Hom}_{\mathsf{Rng}}(A_i, -), \mathscr{F}) \cong \mathscr{F}(A_i). \qquad \square$$

Remark 7.4.2. We have the following isomorphisms,

$$\mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathscr{F}, \mathrm{Hom}_{\mathsf{Rng}}(A, -) \cong \mathrm{Hom}_{\mathsf{GSp}}(|\mathscr{F}|, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathsf{Rng}}(A, \Gamma\mathcal{O}_{|\mathscr{F}|}).$$

Proposition 7.4.3. The structure sheaf of $\operatorname{Spec} A$ is naturally isomorphic to,

$$\mathcal{O}_{|A|} : U \longmapsto \mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathrm{Hom}_{\mathsf{Rng}}(A, -)_U, \mathscr{H} \operatorname{Spec} \mathbb{Z}[T])$$

Proof. As usual, let's choose a distinguished open set $D(f)$ which is mapped to,

$$\mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathrm{Hom}_{\mathsf{Rng}}(A_f, -), \mathscr{H} \operatorname{Spec} \mathbb{Z}[T]) \cong \mathrm{Hom}_{\mathsf{Rng}}(\mathbb{Z}[T], \Gamma\mathcal{O}_{A_f}) \cong A_f$$

showing that both sheaves coincide on a basis implying that they must be isomorphic. $\square$

Theorem 7.4.4. *Given a functor* $\mathscr{F} \in \mathsf{Set}^{\mathsf{Rng}}$ *the structure sheaf of* $|\mathscr{F}|$ *is isomorphic to*

$$\mathcal{O}_{\mathscr{F}} : U \longmapsto \mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathscr{F}_U, \mathscr{H} \operatorname{Spec} \mathbb{Z}[T])$$

Proof. We have already shown that this is true in the affine case. For the general situation consider

$$\mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathscr{F}_U, \mathscr{H} \operatorname{Spec} \mathbb{Z}[T]) \cong \lim_{\mathsf{Elts}(\mathscr{F}_U)} \mathrm{Hom}_{\mathsf{Set}^{\mathsf{Rng}}}(\mathrm{Hom}_{\mathsf{Rng}}(R, -)_{U_R}, \mathscr{H} \operatorname{Spec} \mathbb{Z}[T]) \cong \lim \mathcal{O}_R(s_R^{-1}(U)).$$

The reader should recall that given a diagram $(X_i, \mathcal{O}_{X_i})$ of locally ringed spaces the structure sheaf of the colimit is just the limit of the family of sheaves $(s_{R,*}\mathcal{O}_{X_i})$. This finishes the proof. $\square$

7.5 The Comparison Theorem

$$\begin{array}{ccccc} U & \xrightarrow{i_U} & |\mathscr{H}X| & \xrightarrow{\Psi} & X \\ \uparrow & & \uparrow s_A & & \uparrow c_A \\ D(I) & \xrightarrow{i_A} & \operatorname{Spec} A & = & \operatorname{Spec} A \end{array}$$

Diagram 7.4.1

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Rng}}(A, -) & \xRightarrow{i} & \mathscr{H}|\mathscr{F}| \\ \Uparrow & & \Uparrow \Phi \\ \mathscr{F}_{\operatorname{Spec} A} & \Longrightarrow & \mathscr{F} \end{array}$$

Diagram 7.4.2

Definition 7.5.1 ($\mathbb{Z}$-scheme). A functor $\mathscr{F} \in |\mathsf{Set}^{\mathsf{Rng}}|$ is called a $\mathbb{Z}$-scheme if it is local and has an open covering of representable functors. The category of $\mathbb{Z}$-schemes will be denoted by $\mathbb{Z}$-Sch.

Theorem 7.5.2 (Comparison Theorem). *There is an equivalence of categories between the category of schemes and the category of* $\mathbb{Z}$*-schemes. More precisely, the functors* $\mathscr{H}, |-|$ *restrict to an equivalence of categories.*

$$\mathsf{Sch} \underset{|-|}{\overset{\mathscr{H}}{\rightleftarrows}} \mathbb{Z} - \mathsf{Sch}$$

Proof. We split the proof in some lemmas. □

Lemma 7.5.3 (Topological lemma). *Let* X *be an scheme, then* X *and* $|\mathscr{H}X|$ *are homeomorphic.*

Proof. Let X be a scheme, and consider $|\mathscr{H}X|$. For every morphism $\operatorname{Spec} A \xrightarrow{f} X$ we have an element $(A, f) \in |\mathsf{Elts}(\mathscr{H}X)|$. This induces a morphism $\Psi\colon |\mathscr{H}X| \longrightarrow X$. We will show that Ψ induces an homeomorphism on the underlying topological spaces. It is already known that Ψ is a bijection (Proposition 7.1.7), so we need to show that Ψ is open. Consider the diagram Diagram 7.4.1, where the left hand side square is a pullback. Due to the fact that X is an scheme we can check that $\Psi(U)$ is open in X locally on the affine base. This amounts to show that the big square in the diagram is a pullback. We leave to the reader this routinary comprobation, as a hint we should use that Ψ is a injective. □

Lemma 7.5.4. *An open subfunctor* $\mathscr{F}_U$ *of* $\mathscr{F}$ *gives rise to an open subspace in the geometric realization.*

Proof. This is clear once we invoke Corollary 7.2.7, to see that U is open in $|\mathscr{F}|$ then the description of the structure sheaf given in Theorem 7.4.4 shows that the structure sheaf of U coincides with the restriction of $\mathcal{O}_{|\mathscr{F}|}$ to U. □

Corollary 7.5.5. If X is an scheme then $|\mathscr{H}X| \cong X$.

Proof. Obvious once we see that the structure sheaf of both locally ringed spaces coincides on a open covering. □

It is clear that given an scheme X the associated functor is a $\mathbb{Z}$-scheme, the covering of $\mathscr{H}X$ being induced by the affine covering of X. Moreover, we already saw that the associated functor of an scheme is local. Conversely given a $\mathbb{Z}$-scheme, the covering by representable functors induces an affine covering on the geometric realization by affine open subspaces as we saw previously.

Lemma 7.5.6. *Given a local functor* $\mathscr{F}$*, we have the following isomorphism* $\mathscr{F} \cong \mathscr{H}|\mathscr{F}|$.

Proof. We are going to use that $\mathscr{F}$ is local and Proposition 7.4.1 to glue natural isomorphisms. Firt of all, there is a canonical morphism,

$$\begin{aligned} \mathscr{F}(A) &\xrightarrow{\Phi_A} \mathscr{H}|\mathscr{F}|(A) \\ a &\longmapsto |\eta^a| \end{aligned}$$

where η^a denotes the natural transformation,

$$\begin{aligned} \operatorname{Hom}_{\mathsf{Rng}}(A, -) &\overset{\eta^a}{\Longrightarrow} \mathscr{F} \\ \mathbb{1}_A &\longmapsto \eta^a(\mathbb{1}_A) = a \in \mathscr{F}(A) \end{aligned}.$$

The inverse of this morphism is constructed locally. Consider $\operatorname{Hom}_{\mathsf{Rng}}(A, -) \overset{i}{\Longrightarrow} \mathscr{H}|\mathscr{F}|$ a subfunctor associated to the affine open covering of $\mathscr{H}|\mathscr{F}|$. It is an easy comprobation to see that $|\Phi| = \Psi^{-1}$. Therefore, we can see in Diagram 7.4.2 that the pullback is isomorphic to $\mathrm{Hom}_{\mathsf{Rng}}(A, -)$.

We already knew that the covering of $\mathscr{F}$ is also a covering of $\mathscr{H}|\mathscr{F}|$ the crucial point is that Φ restricted to this covering is an isomorphism. Thus, we can construct a compatible family of local inverses $\{\Phi^{-1}|_{\mathrm{Spec}\,A}\}$ and use Proposition 7.4.1 to glue them and form Φ^{-1}. This finishes the proof of the Comparison Theorem. □

The Comparison Theorem states that the study of schemes can be perfomed in a pure functorial way. This has some clear advantages, for example the construction of limits is well known in presheaf categories and carries an easier description than in the category of locally ringed spaces. Another example is the definition of an algebraic group which is more natural in the functorial language as a $\mathbb{Z}$-scheme that factorizes through the category of groups.

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