

SUBVARIETIES OF GRASSMANNIANS

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These notes are an expanded version of a seminar I gave at the University of Trento in the spring of 1996. I would like to thank Edoardo Ballico for his kind invitation and also for encouraging me to write these notes, which helped me to clear my mind up. I hope they will also be useful for the reader interested in having a few background on subvarieties in Grassmannians. The idea is to start from the very beginning, giving all definitions and examples for then immediately increase the difficulty in an exponential way without even warning, so that we can reach some of the last open problems on the topic. In this way, the beginner student could get a basic background and realize the kind of techniques he or she will need in order to be able to make some contribution (of course the standard techniques in algebraic geometry are those in [H] and/or [Sh]) and the expert reader can go directly to the main points. I preferred not to give in detail proofs that can be found elsewhere, but just the main ideas. I would appreciate any kind of comments to these notes, even just to tell me that I did not get my goals.

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0. Preliminaries on Grassmannians

General notation: Given a vector space V (over a field that we will assume only for simplicity to be the field of complex numbers) of dimension $n + 1$, we will denote by $\mathbf{P}(V) = \mathbf{P}^n$ to the projective space of all hyperplanes of V . This description, rather than the one of lines in V is the most usual one after Grothendieck. The “old-fashioned” reader can view \mathbf{P}^n as the set of lines in the dual V^* (I will use a star for the dual of a vector space, and a check for any other duals: of projective spaces, sheaves,...), as in fact we will do many times. We will also write $G(k, n) = G(k, \mathbf{P}^n)$ for the Grassmann variety of k -dimensional linear subspaces of \mathbf{P}^n . This variety is naturally identified with the set of $(k + 1)$ -dimensional linear subspaces of the dual V^* , or with the set of $(n - k)$ -dimensional quotients of V (probably the latter is more in fashion nowadays). I would like to remark that, according to any of these descriptions, this variety is also usually denoted by $G(k + 1, n + 1)$ or $G(n + 1, n - k)$ (among many other notations in different texts). Hence the reader must be aware that my notation is not the standard one (if there is any). By abuse of notation, very often I will identify (and denote with the same letter, usually Λ) a k -plane of \mathbf{P}^n with the corresponding $(k + 1)$ -dimensional linear subspace of V^* . I should mention here that there is also a definition of Grassmannians in the language of categories (see Remark 2.2).

Note that the projective space \mathbf{P}^n appears as the particular case $k = 0$ and its dual \mathbf{P}^n appears for $k = n - 1$. In a similar way, there is a duality among Grassmannians. Indeed the Grassmann variety of k -planes in \mathbf{P}^n is naturally identified with the Grassmann variety of $(n - k - 1)$ -planes in $\check{\mathbf{P}}^n$. Geometrically, if you fix a k -plane in \mathbf{P}^n , the space of hyperplanes containing it form an $(n - k - 1)$ -plane in $\check{\mathbf{P}}^n$. Algebraically, an inclusion $\Lambda \subset V^*$ is equivalent to an inclusion $(V^*/\Lambda)^* \subset V$.

Let us give now $G(k, n)$ the structure of a variety (analytic, algebraic, or whatever category you prefer). For this purpose we will cover it by affine charts and determine the patchings in the intersections. So let us fix a system of coordinates x_0, \dots, x_n for \mathbf{P}^n or equivalently a basis $\{w_0, \dots, w_n\}$ of V^* . Then we represent an element $\Lambda \in G(k, n)$ by a $(k + 1) \times (n + 1)$ matrix (we will call it *Plücker matrix*)

$$\begin{pmatrix} a_{00} & \dots & a_{0n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix} \tag{0.1}$$

where the rows are the coordinates of a basis of Λ . Of course this representation is not unique. If we change the basis of Λ , the matrix (0.1) changes by multiplying on the left by the non-degenerate square matrix of order $k + 1$ corresponding to the change of basis in Λ . Assume for instance that the minor corresponding to the first $k + 1$ columns is not zero.

This means that, after multiplying by a suitable matrix one can represent Λ in a unique way by a matrix

$$\begin{pmatrix} 1 & \dots & 0 & b_{0k+1} & \dots & b_{0n} \\ & \ddots & & \vdots & & \vdots \\ 0 & \dots & 1 & b_{kk+1} & \dots & b_{kn} \end{pmatrix} \quad (0.2)$$

Hence $G(k, n)$ contains an open affine subset of dimension $(k+1)(n-k)$ (of coordinates b_{0k+1}, \dots, b_{kn}); this subset can be described as the set of k -planes that do not meet the $(n-k-1)$ -plane of equations $x_0 = \dots = x_k = 0$. Since at least one of the minors of order $k+1$ of the Plücker matrix (0.1) is not zero, $G(k, n)$ can be covered by $\binom{n+1}{k+1}$ affine pieces.

Notation: We will denote by U_{i_0, \dots, i_k} the open affine subset of $G(k, n)$ corresponding to subspaces that do not meet the $(n-k-1)$ -plane of equations $x_{i_0} = \dots = x_{i_k} = 0$, or equivalently those subspaces such that the maximal minor of the Plücker matrix (0.1) obtained when considering the columns i_0, \dots, i_k is not zero.

It is very easy (but tedious to write, so we will just do an example below) to describe the change of coordinates from one piece to another. In conclusion we have that $G(k, n)$ can be viewed as an abstract manifold of dimension $(k+1)(n-k)$.

Example 0.1. Consider $G(1, 3)$ and its two affine pieces U_{01} and U_{02} corresponding to matrix representations of the form:

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a' & 0 & b' \\ 0 & c' & 1 & d' \end{pmatrix}$$

Then they have intersection when $c \neq 0$ in U_{01} and $c' \neq 0$ in U_{02} . It is clear then that the first matrix represents the same line as

$$\begin{pmatrix} 1 & \frac{-a}{c} & 0 & b - \frac{ad}{c} \\ 0 & \frac{1}{c} & 1 & \frac{d}{c} \end{pmatrix}$$

This means that the change of coordinates from U_{01} to U_{02} is given by the equation

$$\phi(a, b, c, d) = (a', b', c', d') = \left(\frac{-a}{c}, b - \frac{ad}{c}, \frac{1}{c}, \frac{d}{c} \right),$$

which is clearly an isomorphism (in whatever category you decided to work in) from $\{c \neq 0\}$ to $\{c' \neq 0\}$. More precisely, its Jacobian matrix (which we will need later) is

$$J(\phi) = \begin{pmatrix} \frac{-1}{c} & 0 & \frac{a}{c^2} & 0 \\ \frac{-d}{c} & 1 & \frac{ad}{c^2} & \frac{-a}{c} \\ 0 & 0 & \frac{-1}{c^2} & 0 \\ 0 & 0 & \frac{-d}{c^2} & \frac{1}{c} \end{pmatrix}$$

The Plücker embedding.

In order to view $G(k, n)$ as a projective variety, one needs to consider the so-called *Plücker embedding*

$$\begin{aligned} \varphi_{k,n} : G(k, \mathbf{P}(V)) &\rightarrow \mathbf{P}\left(\bigwedge^{k+1} V\right) \\ L[v_0, \dots, v_k] &\mapsto [v_0 \wedge \dots \wedge v_k] \end{aligned}$$

Here $L[v_0, \dots, v_k]$ represents the linear span in $\mathbf{P}(V)$ of the points represented by the independent vectors $v_0, \dots, v_k \in V^*$ and $[v_0 \wedge \dots \wedge v_k]$ means the point of $\mathbf{P}(\bigwedge^{k+1} V)$ represented by $v_0 \wedge \dots \wedge v_k$. If you want to see this map in coordinates, fix a basis for V^* and the induced one for $\bigwedge^{k+1} V^*$. Then $\varphi_{k,n}$ associates to the space generated by the rows v_0, \dots, v_k of the Plücker matrix (0.1) the point in $\mathbf{P}(\bigwedge^{k+1} V)$ whose coordinates are just the maximal minors of the matrix.

It is an easy exercise to check that $\varphi_{k,n}$ is well defined: If v'_0, \dots, v'_k is another basis of the same space Λ and A is the matrix that changes from one basis into another, then $v'_0 \wedge \dots \wedge v'_k = (\det A)(v_0 \wedge \dots \wedge v_k)$, so that we obtain the same point in $\mathbf{P}(\bigwedge^{k+1} V)$.

Definition: The homogeneous coordinates in $\mathbf{P}(\bigwedge^{k+1} V)$ induced by a choice of coordinates in $\mathbf{P}(V)$ are called *Plücker coordinates* and they are denoted by p_{i_0, \dots, i_k} .

It is not very hard to see that $\varphi_{k,n}$ provides indeed an embedding of $G = G(k, n)$ in $\mathbf{P}(\bigwedge^{k+1} V)$ as an algebraic subvariety. For each affine open set $V_{i_0, \dots, i_k} = \{p_{i_0, \dots, i_k} \neq 0\}$ of $\mathbf{P}(\bigwedge^{k+1} V)$ we observe that $\varphi_{k,n}(G) \cap V_{i_0, \dots, i_k} = \varphi_{k,n}(U_{i_0, \dots, i_k})$, so it is enough to prove that $\varphi_{k,n}$ restricted to each U_{i_0, \dots, i_k} is an algebraic embedding in V_{i_0, \dots, i_k} . Let us work (for simplicity of notation) in $U_{0, \dots, k}$. Then we can use (0.2) as a matrix representation. For this matrix we have $p_{0, \dots, k} = 1$, so that we can consider the rest of the Plücker coordinates as the coordinates for $V_{0, \dots, k}$. But also observe that, up to probably a sign, all the coordinates of $U_{0, \dots, k}$ (the entries b_{ij} of the matrix (0.2)) appear as coordinates of the map $\varphi_{k,n}$. More precisely we have that each b_{ij} appears up to a sign as the minor of the Plücker matrix (0.2) in which we take the columns $0, \dots, i-1, i+1, \dots, k, j$. This proves that $\varphi_{k,n}$ is an embedding. It is also algebraic because the rest of the Plücker coordinates are just polynomials in the previous coordinates.

A natural question at this point is to ask for a set of polynomials defining $G(k, n)$ in $\mathbf{P}(\bigwedge^{k+1} V)$. We will just mention that one can prove that the homogeneous ideal is generated by quadrics and that these quadrics are essentially obtained by developing by blocks of size $k+1$ the determinant of a matrix with two identical rows, as we will illustrate now with an example. For a complete study of the Plücker equations, see for example [ACGH].

Example 0.2: Let us consider $G = G(1, 3)$ and choose a system of coordinates for \mathbf{P}^3 .

Let us represent a line in \mathbf{P}^3 by a matrix like (0.1), i.e. of the form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \end{pmatrix}$$

Then it has six Plücker coordinates

$$\begin{aligned} p_{01} &= \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix} & p_{02} &= \begin{vmatrix} a_{00} & a_{02} \\ a_{10} & a_{12} \end{vmatrix} & p_{03} &= \begin{vmatrix} a_{00} & a_{03} \\ a_{10} & a_{13} \end{vmatrix} \\ p_{12} &= \begin{vmatrix} a_{01} & a_{02} \\ a_{11} & a_{12} \end{vmatrix} & p_{13} &= \begin{vmatrix} a_{01} & a_{03} \\ a_{11} & a_{13} \end{vmatrix} & p_{23} &= \begin{vmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{vmatrix} \end{aligned}$$

Just developing by the two first rows the (trivially zero) determinant of the matrix

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \end{pmatrix}$$

it follows that the image of G in \mathbf{P}^5 satisfies the following quadratic equation:

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0 \tag{0.3}$$

By just counting dimensions it is easy to conclude that this is the only equation of G , since it defines an irreducible variety of dimension four containing G , which also has dimension four. But it is more enlightening for understanding the general case to do it by hand. So take a point $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbf{P}^5$ verifying (0.3). One of its coordinates must be non-zero. Assume for instance that $p_{01} \neq 0$ and normalize so that $p_{01} = 1$. Then it is immediate to check that $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$ is the image of the point Λ of G represented by the Plücker matrix

$$\begin{pmatrix} 1 & 0 & -p_{12} & -p_{13} \\ 0 & 1 & p_{02} & p_{03} \end{pmatrix}$$

Indeed the only non-trivial thing to check is that the last Plücker coordinate of Λ is p_{23} . This is so because, when $p_{01} = 1$, equation (0.3) just reads

$$p_{23} = \begin{vmatrix} -p_{12} & -p_{13} \\ p_{02} & p_{03} \end{vmatrix}.$$

The beginner reader is invited to check in the same way that the Plücker equations for $G(1,4)$ in \mathbf{P}^9 are

$$\begin{aligned} p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} &= 0 \\ p_{01}p_{24} - p_{02}p_{14} + p_{04}p_{12} &= 0 \\ p_{01}p_{34} - p_{03}p_{14} + p_{04}p_{13} &= 0 \\ p_{02}p_{34} - p_{03}p_{24} + p_{04}p_{23} &= 0 \\ p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} &= 0 \end{aligned} \tag{0.4}$$

and try to figure out the form of the equations for a general Grassmannian of lines.

The universal bundles.

Fix again $G = G(k, n)$. As remarked above, we can consider an element $\Lambda \in G$ as a k -dimensional linear subspace of a projective space $\mathbf{P}(V)$ or as a $(k + 1)$ -dimensional subspace of the vector space V^* . Depending on which point of view we take there are two incidence diagrams in which the maps are the natural projections:

$$\begin{array}{ccc} & I & = \{(p, \Lambda) \in \mathbf{P}(V) \times G \mid p \in \Lambda\} \\ \mathbf{P}(V) & \swarrow \scriptstyle p & \searrow \scriptstyle q \\ & G & \end{array} \quad (0.5)$$

$$\begin{array}{ccc} & \check{Q} & = \{(v, \Lambda) \in V^* \times G \mid v \in \Lambda\} \\ V^* & \swarrow \scriptstyle \bar{p} & \searrow \scriptstyle \bar{q} \\ & G & \end{array} \quad (0.6)$$

In (0.6), notice that the map \bar{q} provides \check{Q} a natural structure of a vector bundle over G . And more precisely, it is a vector subbundle of the trivial bundle $V^* \times G$. Hence we can also consider the quotient vector bundle, which we will call S . We can associate to these two bundles their corresponding locally free sheaves on G , that we will denote with the corresponding script letters $\check{\mathcal{Q}}$ and \mathcal{S} .

By dualizing we get the so-called *universal exact sequence* on G :

$$0 \rightarrow \check{\mathcal{S}} \rightarrow V \otimes \mathcal{O}_G \rightarrow \mathcal{Q} \rightarrow 0 \quad (0.7)$$

Definition: The sheaves $\check{\mathcal{S}}$ and \mathcal{Q} appearing in the sequence are called respectively the *universal subbundle* and the *universal quotient bundle*. Note that they have respective ranks $n - k$ and $k + 1$.

Notation: We will try always to make the distinction between vector bundles and their associated locally free sheaves, using script letters for the second ones. However we cannot avoid the use of the standard expression “universal bundles” to indicate in fact their corresponding locally free sheaves. Also I would like to remark that it is not standard to which of the universal bundles one should call quotient or subbundle: it depends on the point of view from you are considering the Grassmannian. My notation is consistent when you view the Grassmannian as the space of quotients of V . However my notation with duals may not be quite standard, but I prefer S and Q to have the same behavior when dualizing (see remark below).

Remark 0.3: From this construction it becomes clear, using the identification $G(k, \mathbf{P}^n) \cong G(n - k - 1, \check{\mathbf{P}}^n)$ that the universal exact sequence for the second Grassmannian is just the dual of (0.7), and hence the bundles S and Q are interchanged.

An alternative way of constructing the universal bundles is to consider in $\mathbf{P}^n = \mathbf{P}(V)$ the Euler sequence

$$0 \rightarrow \Omega_{\mathbf{P}^n}(1) \rightarrow V \otimes \mathcal{O}_{\mathbf{P}^n} \rightarrow \mathcal{O}_{\mathbf{P}^n}(1) \rightarrow 0$$

and pull it back to I via p and then push it forward to G (see diagram (0.5)). Then you get nothing the universal exact sequence. In particular

$$\begin{aligned} \check{\mathcal{S}} &= q_* p^* (\Omega_{\mathbf{P}^n}(1)) \\ \check{\mathcal{Q}} &= q_* p^* (\mathcal{O}_{\mathbf{P}^n}(1)) \end{aligned} \tag{0.8}$$

Remark 0.4. If we consider the trivial Grassmannian $G(0, n) = \mathbf{P}^n$, then the universal exact sequence is just the Euler sequence and hence the universal bundles are $\check{\mathcal{Q}} = \mathcal{O}_{\mathbf{P}^n}(1)$ and $\check{\mathcal{S}} = \mathcal{T}_{\mathbf{P}^n}(-1)$. And if we take the other trivial Grassmannian $G(n-1, n) = \check{\mathbf{P}}^n$, then as observed in Remark 0.3, we have $\check{\mathcal{Q}} = \mathcal{T}_{\check{\mathbf{P}}^n}(-1)$ and $\check{\mathcal{S}} = \mathcal{O}_{\check{\mathbf{P}}^n}(1)$

From the identities (0.8) it follows immediately (just need to use Leray theorem) that there is a natural identification $H^0(G, \check{\mathcal{Q}}) = V$. Considering dual Grassmannians, it also follows that $H^0(G, \check{\mathcal{S}}) = V^*$. In particular, giving a non-zero section of $\check{\mathcal{Q}}$ is the same, up to a constant, as giving a hyperplane of \mathbf{P}^n , and a non-zero section of $\check{\mathcal{S}}$ is equivalent to giving a point in \mathbf{P}^n . Let us look at this more closely. More precisely, take s_0, \dots, s_r $r+1$ to be independent sections of $\check{\mathcal{S}}$. They span a linear subspace $W \subset V^*$ of dimension $r+1$ that defines an r -plane $\Omega \subset \mathbf{P}^n$. Consider the following commutative diagram of exact sequences of vector bundles:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & W \times G & & & \\ & & & \downarrow & & \searrow^s & \\ 0 & \rightarrow & \check{\mathcal{Q}} & \rightarrow & V^* \times G & \rightarrow & S \rightarrow 0 \\ & & & & \downarrow & & \\ & & & & V^*/W \times G & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

where the map s is defined by the given sections s_0, \dots, s_r of $\check{\mathcal{S}}$. We want to study the dependency locus of s_0, \dots, s_r , i.e. the locus of elements of G for which the map $s : W \times G \rightarrow S$ (or the associated map of sheaves $W \times \mathcal{O}_G \rightarrow \mathcal{S}$) is not injective. From the diagram it follows that an element $(w, \Lambda) \in W \times G$ is mapped to zero by s if and only if its image in $V^* \times G$ is in $\check{\mathcal{Q}}$, i.e. if and only if $w \in \Lambda$. This means that the map s is not injective in the fiber of Λ if and only if Λ has non-zero intersection with W . In other words, the degeneracy locus of s_0, \dots, s_r is exactly the set of k -planes $\Lambda \subset \mathbf{P}^n$ such that Λ meets Ω .

A similar result holds for sections of $\check{\mathcal{Q}}$ by using duality, as explained in Remark 0.3. For future use, we summarize these results in the following:

Proposition 0.5. *We have natural identifications $H^0(G, \mathcal{Q}) = V$ and $H^0(G, \mathcal{S}) = V^*$. Under this identification s independent sections of \mathcal{Q} correspond to a linear subspace $A \subset \mathbf{P}^n$ of codimension s . If $s \leq k + 1$, the dependency locus of the sections is just the set of k -planes meeting A in dimension at least $k - s + 1$ (i.e. one more than expected). Analogously, $r + 1$ independent sections of \mathcal{S} correspond to a linear subspace $B \subset \mathbf{P}^n$ of dimension r . If $r + 1 \leq n - k$, the dependency locus of this sections is the set of k -planes meeting B .*

If we take the particular cases $s = k + 1$ and $r + 1 = n - k$ in Proposition 0.5, then we get the same dependency locus: the set of k -planes meeting a fixed $(n - k - 1)$ -plane. This is just because we have an identification of invertible sheaves $\bigwedge^{k+1} \mathcal{Q} \cong \bigwedge^{n-k} \mathcal{S}$, as can be seen for example from the universal exact sequence (0.7).

Notation: Since this invertible sheaf is the one giving the Plücker embedding, we will denote it by $\mathcal{O}_G(1)$.

The tangent bundle.

Once we have $G(k, n)$ covered by affine pieces, it is immediate to describe the tangent space at each point Λ ; it is just the underlying vector space of any affine piece in which Λ lies in. But we want to do it in a more intrinsic way so that it is coordinate-free and we can recognize the tangent bundle of $G(k, n)$. So start with a choice of a basis $B = \{w_0, \dots, w_n\}$ of V^* . As we have already seen, this provides the decomposition of $G(k, n)$ in (non-disjoint) affine pieces. Look for simplicity at $U_{0, \dots, k}$, the affine piece represented by the variable entries of the matrix

$$\begin{pmatrix} 1 & \dots & 0 & b_{0k+1} & \dots & b_{0n} \\ & \ddots & & \vdots & & \vdots \\ 0 & \dots & 1 & b_{kk+1} & \dots & b_{kn} \end{pmatrix}.$$

Now fix an element $\Lambda \in G(k, n)$ which is in this affine piece. This is the same as fixing such a matrix, or equivalently, a basis $B' = \{v_0, \dots, v_k\}$ of the vector subspace that produces Λ , and whose coordinates with respect to the basis B are the rows of the given matrix. Since we are in an affine space, giving a tangent vector at the point Λ is the same as giving $(k + 1)(n - k)$ numbers $\beta_{0k+1}, \dots, \beta_{kn}$. Since the zero vector must correspond to Λ , the right way to interpret these numbers is as a matrix

$$\begin{pmatrix} 1 & \dots & 0 & b_{0k+1} + \beta_{0k+1} & \dots & b_{0n} + \beta_{0n} \\ & \ddots & & \vdots & & \vdots \\ 0 & \dots & 1 & b_{kk+1} + \beta_{kk+1} & \dots & b_{kn} + \beta_{kn} \end{pmatrix}.$$

It is very natural to interpret such a matrix as the matrix of a linear map $\Lambda \rightarrow V^*$ with respect to the basis B' and B . Observe first that this homomorphism maps any vector

v_i to itself plus a linear combination (depending only on the β_{ij} 's) of the last vectors w_{k+1}, \dots, w_n of the basis B . And also note that the space of linear maps from Λ to V^* has “too big” dimension for encoding the numbers $\beta_{0k+1}, \dots, \beta_{kn}$. It is therefore natural to compose our map with the projection $V^* \rightarrow V^*/\Lambda$, for finally getting a linear map $\Lambda \rightarrow V^*/\Lambda$ whose matrix is precisely

$$\begin{pmatrix} \beta_{0k+1} & \cdots & \beta_{0n} \\ \vdots & & \vdots \\ \beta_{kk+1} & \cdots & \beta_{kn} \end{pmatrix}.$$

when taking B' as basis for Λ , and for V^*/Λ the basis given by the classes of the vectors w_{k+1}, \dots, w_n .

This same process can be done for any of the affine charts U_{i_0, \dots, i_k} we have given (just changing the appropriate columns in the Plücker matrix), and the important fact (which we will check only in one example) is that this map $\Lambda \rightarrow V^*/\Lambda$ is *independent on the affine chart*. As a consequence, we canonically identify the tangent space of $G(k, n)$ at a point Λ to be $\text{Hom}(\Lambda, V^*/\Lambda)$. Recalling our description of the universal bundles we conclude the following:

Theorem 0.6. *The tangent sheaf of $G(k, n)$ is naturally isomorphic to $\mathcal{H}om(\check{\mathcal{Q}}, \mathcal{S}) \cong \mathcal{Q} \otimes \mathcal{S}$.*

By recalling that $\bigwedge^{k+1} \mathcal{Q} = \bigwedge^{n-k} \mathcal{S} = \mathcal{O}_G(1)$, we have

Corollary 0.7. *The canonical sheaf of $G = G(k, n)$ is $\omega_G \cong \mathcal{O}_G(-n-1)$.*

Observe that when we identify \mathbf{P}^n with $G(0, n)$ we have that the universal sequence is just the Euler sequence, so that the above results are obvious in this case.

Example 0.8. Just to check how the above construction is independent of the affine chart, let us go back to the situation of example 0.1. So let us take an element $\Lambda \in G(1, 3)$ that is in $U_{01} \cap U_{02}$. Hence Λ can be represented by a Plücker matrix

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

(with $c \neq 0$) and also by the matrix

$$\begin{pmatrix} 1 & \frac{-a}{c} & 0 & b - \frac{ad}{c} \\ 0 & \frac{1}{c} & 1 & \frac{d}{c} \end{pmatrix}$$

Now take a tangent vector, represented in the first chart U_{01} by four numbers $\alpha, \beta, \gamma, \delta$. If we denote by $\{w_0, w_1, w_2, w_3\}$ the basis we have chosen for V^* , then we have that Λ is

generated by $v_0 = w_0 + aw_2 + bw_3$ and $v_1 = w_1 + cw_2 + dw_3$. Hence we have that $(\alpha, \beta, \gamma, \delta)$ defines the linear map $\Lambda \rightarrow V^*/\Lambda$ determined by

$$\begin{array}{lll} w_0 & +aw_2 + bw_3 & \mapsto [\alpha w_2 + \beta w_3] \\ w_1 & +cw_2 + dw_3 & \mapsto [\gamma w_2 + \delta w_3] \end{array}$$

(where the square brackets represent classes modulo Λ). If we work on the second chart U_{02} , then we would need to take $v'_0 = w_0 - \frac{a}{c}w_1 + (b - \frac{ad}{c})w_3$ and $v'_1 = \frac{1}{c}w_1 + w_2 + \frac{d}{c}w_3$ as a new basis for Λ . Then the same linear map is determined now by

$$\begin{array}{lll} w_0 & -\frac{a}{c}w_1 & + (b - \frac{ad}{c})w_3 & \mapsto [(\alpha - \frac{a}{c}\gamma)w_2 + (\beta - \frac{a}{c}\delta)w_3] \\ \frac{1}{c}w_1 & +w_2 & +\frac{d}{c}w_3 & \mapsto [\frac{1}{c}\gamma w_2 + \frac{1}{c}\delta w_3] \end{array}$$

Taking into account that $[w_2] = [-\frac{1}{c}w_1 - \frac{d}{c}w_3]$ (just using the expression of v'_1) one gets that this maps can be defined as

$$\begin{array}{lll} w_0 & -\frac{a}{c}w_1 & + (b - \frac{ad}{c})w_3 & \mapsto [(-\frac{1}{c}\alpha + \frac{a}{c^2}\gamma)w_1 + (-\frac{d}{c}\alpha + \beta + \frac{ad}{c^2}\gamma - \frac{a}{c}\delta)w_3] \\ \frac{1}{c}w_1 & +w_2 & +\frac{d}{c}w_3 & \mapsto [-\frac{1}{c^2}\gamma w_1 + (-\frac{d}{c^2}\gamma + \frac{1}{c}\delta)w_3] \end{array}$$

This means, that in the second chart U_{02} the tangent vector corresponding to this map would have coordinates $(-\frac{1}{c}\alpha + \frac{a}{c^2}\gamma, -\frac{d}{c}\alpha + \beta + \frac{ad}{c^2}\gamma - \frac{a}{c}\delta, -\frac{1}{c^2}\gamma, -\frac{d}{c^2}\gamma + \frac{1}{c}\delta)$. But this is nothing but the image of $(\alpha, \beta, \gamma, \delta)$ under the differential of the map ϕ (of change of coordinates from U_{01} to U_{02}) at the point (a, b, c, d) , as we computed in example 0.1. This proves that this linear map does not depend on the change ϕ of coordinates.

1. Schubert calculus.

Schubert calculus is a tool for studying intersections in Grassmannians. We will just give here the results that are useful for our purposes. A complete survey can be found in [KL] (see also [F] and [GH] as other good references). The general idea comes from the analogous situation in projective spaces: What are the numerical invariants for a projective variety that can be obtained by performing some intersections? Clearly it is the degree the only invariant we have. More precisely, if $X \subset \mathbf{P}^n$ has codimension s , then its degree is the number of points obtained when intersecting X with a sufficiently general s -plane of \mathbf{P}^n . On the other hand, we observe that, fixing a complete flag of linear subspaces $\emptyset \subsetneq L_0 \subsetneq \dots \subsetneq L_n = \mathbf{P}^n$ (here each L_i has dimension i), we have a cell decomposition of the projective space

$$\mathbf{P}^n = (L_n \setminus L_{n-1}) \sqcup (L_{n-1} \setminus L_{n-2}) \sqcup \dots \sqcup (L_1 \setminus L_0) \sqcup L_0$$

Hence what happens in the projective space is: take a cell decomposition of \mathbf{P}^n ; take the closure of its cells (in our case we get a linear subspace of each dimension); then the degree of a subvariety of codimension s is just the intersection number of this subvariety with the closure of the cell of complementary dimension (maybe you need to change your cell by a projectivity in order to get a finite number of points of intersection). When you change \mathbf{P}^n with a Grassmannian, the method will be again the same: find a cell decomposition and take the closure of the cells. But in this case you can have several cells of the same dimension, so that you obtain more than one degree.

Before stating any concrete result, let us work out completely the case of $G(1, 3)$. So we start by taking an affine piece of $G(1, 3)$, more precisely we consider U_{01} , which is a four-dimensional cell corresponding to points uniquely represented by a Plücker matrix of the form:

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \tag{1.1}$$

(where a $*$ means a free entry). One point of $G(1, 3)$ is not in that cell if the corresponding line meets the line $L : x_0 = x_1 = 0$. So now we just need to cover by cells the points of $G(1, 3)$ of this last form. One affine cell of dimension three contained there is the set of points represented (uniquely) by matrices of the form

$$\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \tag{1.2}$$

One line in this cell is a line meeting L in a point different from $P = (0 : 0 : 0 : 1)$ and not entirely contained in the plane $\Pi : x_0 = 0$. So now the missing points (i.e. those not covered by any of the two disjoint cells above) are those corresponding to lines

either passing through the point P or contained in the plane Π . These two sets contain respectively the (disjoint) affine cells represented by matrices of the form

$$\begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1.3)$$

(representing the lines passing through P but not contained in Π) and

$$\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \quad (1.4)$$

(representing the lines contained in Π but not passing through P). Now finally we are only missing the set of points corresponding to lines contained in Π and passing through P . This can be covered easily (for example observing that this is nothing but a projective line) by two cells represented by matrices

$$\begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1.5)$$

(representing the lines contained in Π , passing through P and different from L) and

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (1.6)$$

(representing the line L). So we get exactly six cells corresponding to the closures of the sets defined by matrices of the forms (1.1)-(1.6). More precisely, looking at dimensions we get the following closures of cells:

Dim. 4: The whole $G(1, 3)$.

Dim. 3: The set of lines meeting the line L .

Dim. 2: The set of lines passing through the point P , and
the set of lines contained in the plane Π .

Dim. 1: The set of lines contained in Π and passing through P .

Dim. 0: The set consisting of the line L .

The beginner reader is highly encouraged to perform the same calculations for $G(1, 4)$.

Now one of the main questions in Schubert calculus is: How to find such a cell decomposition for a general Grassmannian? One standard method is to use a torus action and compute the orbits. This is the sometimes called *Bialynicki-Birula method*. We are not going to describe it here (an expository paper in which the case of $G(1, 3)$ is also studied in detail, is [V]). We will just give here the final result. So fix a Grassmannian $G(k, n)$ of k -planes in the projective space of dimension n . We need first some definitions.

Definitions: Fix a flag of (non-empty) $k + 1$ linear subspaces of \mathbf{P}^n : $\Lambda_0 \subsetneq \Lambda_1 \subsetneq \dots \subsetneq \Lambda_k$. We define the *Schubert variety* associated to this flag to be the set

$$\Omega(\Lambda_0, \dots, \Lambda_k) = \{\Lambda \in G(k, n) \mid \dim(\Lambda \cap \Lambda_i) \geq i \text{ for } i = 0, \dots, k\}.$$

Since the projectivities of \mathbf{P}^n act transitively on the set of flags with the same dimensions, hence it also does on the corresponding set of Schubert varieties. A class of equivalence under projectivities is called a *Schubert cycle* and is denoted by $\Omega(l_0, \dots, l_k)$ (where the l_i 's are the dimensions of the Λ_i 's of any representative of the class).

A *special Schubert cycle* is $\sigma_a = \Omega(n - k - a, n - k + 1, \dots, n)$, i.e. the cycle of k -planes meeting a linear space of dimension $n - k - a$. Of course this makes sense only for $0 \leq a \leq n - k$.

Example 1.1: The reader can immediately verify that the six sets found for $G(1, 3)$ are respectively $\Omega(2, 3)$ (dim. 4), $\sigma_1 = \Omega(1, 3)$ (dim. 3), $\sigma_2 = \Omega(0, 3)$ and $\Omega(1, 2)$ (dim. 2), $\Omega(0, 2)$ (dim. 1) and $\Omega(0, 1)$ (dim. 0).

In the above example one can see that the dimension of the Schubert cycles decreases with the dimension of the varieties in the flag. In fact one easily verifies that

$$\dim \Omega(l_0, \dots, l_k) = \sum_{i=0}^k (l_i - i).$$

And now we come to the intersection of Schubert cycles. First of all, we need to define a space where intersections will take place. The natural place to work in is the so-called *Chow ring* of $G = G(k, n)$, that we will denote $A(G) = \bigoplus A^i(G)$. If you are not familiar with the Chow ring, just think of it as a certain quotient of the free abelian group generated by all irreducible subvarieties of $G(k, n)$; this is graded by the codimension and we say that two pieces of the same dimension are equivalent if it is possible to deform one into another in an “algebraic way”. The multiplicative structure is given by the intersection (for this you need a *Chow’s moving lemma* to guarantee that you can move the representatives of the classes you want to intersect so that they meet transversally). Also if you want you can quotient out by this intersection product, just saying that two classes are the same if they have the same intersections with all other classes. Of course I am cheating here a lot, and gave (intuitive) definitions that are different in general. But they all agree for example for a variety admitting a cell decomposition, as it is the case for Grassmannians. Also this special property of Grassmannians allows to regard them just as topological spaces (more precisely, what is called CW-complexes) and have natural isomorphisms

$$A^i(G) \cong H^{2i}(G, \mathbf{Z}) \cong H_{2(k+1)(n-k)-2i}(G, \mathbf{Z})$$

(the reason of multiplying by two is that the underlying topological spaces have double dimension). Now you can think of the intersection product as just the usual topological product.

So in fact what we called a Schubert cycle is essentially nothing but the corresponding class of a Schubert variety in the Chow ring, in the sense that two Schubert varieties define the same point in the Chow ring (but a variety having the same class as a Schubert variety is not necessarily a Schubert cycle; see examples 1.11 and 1.12 below). In this language, the precise result is:

Theorem 1.2. *For any $i = 0, \dots, (k+1)(n-k)$, the group $A^i(G(k, n))$ is freely generated by all Schubert cycles $\Omega(l_0, \dots, l_k)$ such that $\sum_{j=0}^k (l_j - j) = (k+1)(n-k) - i$.*

Note that this result is obvious when regarding $G(k, n)$ as a CW-complex and having in mind the above identifications. Of course the interesting part is to know the multiplication structure. This is exactly the Schubert calculus. We will just state here a few results.

Theorem 1.3. *(Pieri's formula) The intersection product of a Schubert cycle and a special Schubert cycle is given by the following formula:*

$$\Omega(l_0, \dots, l_k) \cdot \sigma_a = \sum \Omega(m_0, \dots, m_k)$$

where the sum is taken over all m_i 's verifying that $l_{i-1} \leq m_i \leq l_i$ and $\sum m_i = \sum l_i - a$

Example 1.4: If we consider $G(1, 3)$, then the special Schubert cycles are $\sigma_1 = \Omega(1, 3)$ and $\sigma_2 = \Omega(0, 3)$. We have for example:

$$\sigma_1^2 = \Omega(1, 3) \cdot \sigma_1 = \Omega(0, 3) + \Omega(1, 2)$$

Of course this (and any) product can also be obtained geometrically: σ_1 represents the set $\Omega(L, \mathbf{P}^3)$ of lines meeting the line L ; so if we take the intersection of $\Omega(L_1, \mathbf{P}^3)$ and $\Omega(L_2, \mathbf{P}^3)$ where we choose L_1 and L_2 to meet in one point P (and hence to lie in the same plane Π) we have that the intersection is, as sets,

$$\Omega(L_1, \mathbf{P}^3) \cap \Omega(L_2, \mathbf{P}^3) = \Omega(P, \mathbf{P}^3) \cup \Omega(L_1, \Pi)$$

i.e. the union of the set of lines passing through P and the set of lines contained in Π .

Theorem 1.3 above result implies in fact that we can know the whole multiplicative structure of the Chow ring. This is implied by the following theorem.

Theorem 1.5. *The Chow ring of a Grassmann variety is generated by the special Schubert cycles.*

Remark 1.6: If you are familiar with the theory of Chern classes, you will recognize immediately (by looking at Proposition 0.5) that $\sigma_a = c_a(\mathcal{S})$ (in particular σ_1 is the class of the hyperplane section of (k, n) under the Plücker embedding). Hence the above result is saying that *the Chow ring of a Grassmann variety is generated by the Chern classes of its universal subbundle*. In particular *the Picard group of $G(k, n)$ is generated by the class of the hyperplane section after the Plücker embedding*. Because of the duality between Grassmannians, it also holds that *the Chow ring of $G(k, n)$ is generated by the Chern classes of its universal quotient bundle* (the most natural choice is to take in any case the universal bundle of smallest rank). In fact the philosophy to have the whole multiplicative structure for a Grassmann variety (also valid for Grassmann bundles) is the following: The Chow ring is generated by the Chern classes of say the universal subbundle \mathcal{S} , and the relations are obtained by using the universal exact sequence to compute the Chern classes of \mathcal{Q} and impose that $c_i(\mathcal{Q}) = 0$ for $i = k + 2, \dots, (k + 1)(n - k)$. Let us see this in the following example.

Example 1.7: We will apply the above technique for getting once more the Chow ring of $G = G(1, 3)$. It must be generated by $\sigma_1 = c_1(\mathcal{S})$ and $\sigma_2 = c_2(\mathcal{S})$ (and if you want to be rigorous you would have to include $\sigma_0 = \Omega(2, 3)$), which we know that are respectively $\Omega(1, 3)$ and $\Omega(0, 3)$. From the universal exact sequence (0.7) we get that $c_3(\mathcal{Q}) = \sigma_1^3 - 2\sigma_1\sigma_2$ and $c_4(\mathcal{Q}) = \sigma_1^4 - 3\sigma_1^2\sigma_2 + \sigma_2^2$. Using the relations coming from making zero these last two classes we get the following bases for each degree:

$$\begin{aligned} A^0(G) &= \mathbf{Z}\sigma_0 \\ A^1(G) &= \mathbf{Z}\sigma_1 \\ A^2(G) &= \mathbf{Z}\sigma_2 \oplus \mathbf{Z}\sigma_1^2 \\ A^3(G) &= \mathbf{Z}(\sigma_1\sigma_2) \\ A^4(G) &= \mathbf{Z}\sigma_2^2 \end{aligned}$$

Using Pieri's formula we check immediately that these generators agree with the ones we already computed, except in degree two. In this degree what we just got as a basis is $\Omega(0, 3)$ and $\Omega(0, 3) + \Omega(1, 2)$ (see Example 1.4), which is clearly equivalent to the usual basis.

And finally we state a result that will be enough for the purposes we want.

Theorem 1.8. *If two Schubert cycles have complementary dimension, then its intersection $\Omega(l_0, \dots, l_k) \cdot \Omega(m_0, \dots, m_k)$ is zero unless $l_i + m_{k-i} = n$ for all $i = 0, \dots, k$. In this last case $\Omega(l_0, \dots, l_k) \cdot \Omega(m_0, \dots, m_k) = \Omega(n - k, \dots, n)$ (i.e., it is the class of a point).*

Notation: Since all the Grassmannians G we are working on are connected, it follows that $A^{(k+1)(n-k)}(G) = H_0(G, \mathbf{Z})$ is generated by the class $\Omega(n-k, \dots, n)$ of a point. By abuse of notation we will identify $A^{(k+1)(n-k)}(G)$ with \mathbf{Z} and usually write its elements just as integers, just meaning the corresponding multiple of the class of a point.

To understand the importance of Theorem 1.8, let us go back to the case of varieties in a projective space. So fix $X \subset \mathbf{P}^n$ of codimension s . How can we interpret its degree? First we study the Chow ring of \mathbf{P}^n . It is isomorphic to the graded ring $\mathbf{Z}[H]/(H^{n+1})$, where H is the class of a hyperplane (if you want, you can even apply the Schubert calculus to this trivial Grassmannian and then $H = c_1(\mathcal{Q}) = c_1(\mathcal{O}_{\mathbf{P}^n}(1))$). Hence the class of X in the Chow ring is an element of $A^s(\mathbf{P}^n) = \mathbf{Z}H^s$, so a multiple of H^s ; this multiple is precisely the degree d of X . The way of checking this is by verifying that both classes (the class of X and dH^s) have the same intersection products with all elements of complementary dimension; since $A^{n-s}(\mathbf{P}^n) = \mathbf{Z}H^{n-s}$ we just need to check that both classes have the same intersection product with H^{n-s} . But both products are clearly equal to dH^n , i.e. the class of d points, (the first one comes from the very definition of degree).

With this point of view, the above theorem is just telling us that *the bases we have found for two complementary dimensions are orthogonal with respect to the intersection product*. So we can easily extend the notion of degree to any Grassmannian. The precise way is the following:

Definition: Let X be a subvariety of codimension s in $G = G(k, n)$. We call the *multidegree* of X to be the set of coefficients of the class of X in $A^s(G)$ with respect to the basis given by the Schubert cycles of codimension s . Equivalently, the multidegree of X will be the set of the degrees of the intersection numbers of X with all Schubert cycles of dimension s .

Example 1.9: Let S be a surface in $G = G(1, 3)$ (this is what classically was called a *congruence of lines*). Then it has codimension two in G , which means that its class in $A^2(G) = \mathbf{Z}\Omega(0, 3) \oplus \mathbf{Z}\Omega(1, 2)$ is an integral linear combination $[S] = a\Omega(0, 3) + b\Omega(1, 2)$. A representative of $\Omega(0, 3)$ is classically called α -*plane*, while one of $\Omega(1, 2)$ is called β -*plane*. The pair (a, b) is hence the bidegree of S in G . To compute it we have to multiply by the Schubert cycles of complementary dimension (in this case again of dimension two). We have that

$$\begin{aligned} a &= (a\Omega(0, 3) + b\Omega(1, 2)) \cdot \Omega(0, 3) = [S] \cdot \Omega(0, 3) \\ b &= (a\Omega(0, 3) + b\Omega(1, 2)) \cdot \Omega(1, 2) = [S] \cdot \Omega(1, 2) \end{aligned}$$

In other words, a is the number of lines passing through a general fixed point (classically called the *order* of S), and b is the number of lines contained in a general fixed plane (classically called the *class* of S). Since $\sigma_1^2 = \Omega(0, 3) + \Omega(1, 2)$ (see Example 1.4), one has that the degree of S in \mathbf{P}^5 after the Plücker embedding is $d = [S] \cdot \sigma_1^2 = a + b$.

Example 1.10: To see a more concrete example, take S to be the union of all bisecant lines to a twisted cubic C . These form a surface in G . Its class b is easily seen to be three, since a general plane Π meets C in three points, and the lines of S contained in Π are the three ones joining any two of these three points. To compute the order a of S we need to count how many bisecants to C pass through a general point $p \in \mathbf{P}^3$. This is equivalent to count the number of singular points of the plane projection of C from p . But a plane rational cubic has one singular point, hence $a = 1$. So we have seen that the bidegree of S is $(a, b) = (1, 3)$.

Example 1.11: If we apply Theorem 1.8 to a Grassmannian of lines $G(1, n)$ we see that $\Omega(0, 2) \cdot \Omega(n-2, n) = 1$ (we could have also applied Pieri's formula since $\Omega(n-2, n) = \sigma_1$). This means in particular that the pencil of lines contained in a fixed plane and passing through a fixed point of the plane forms a line in $\mathbf{P}^{\binom{n+1}{2}-1}$ after the Plücker embedding. In fact all lines contained in $G(1, n)$ are of this form. Indeed let Ω be a line contained in $G(1, n)$. Let $\Pi \subset \mathbf{P}^n$ be the ruled surface obtained as the union of all lines in Ω . Since $[\Omega] \cdot \Omega(n-2, n) = 1$, this means that a general $\mathbf{P}^{n-2} \subset \mathbf{P}^n$ meets Ω in only one point (since through any point of Π there passes at least one line of Ω). Therefore Π is a plane and we have that through a general point of it there passes only one line of Ω . In other words, Ω is a line in the dual plane $\check{\Pi}$. Hence Ω is a pencil $\Omega(p, \Pi)$ for some point $p \in \Pi$. Note that in particular this proves that any subvariety in $G(1, n)$ whose class in the Chow ring is $\Omega(1, 2)$ is in fact a Schubert variety.

Example 1.12: Let us first verify again by using Schubert calculus that the degree of $G(1, 3)$ in \mathbf{P}^5 after the Plücker embedding is two (we obtained explicitly the equation (0.3) of the quadric in Example 0.2). The class of the hyperplane section is σ_1 , so we just need to compute σ_1^4 . We have already seen in Example 1.4 that $\sigma_1^2 = \Omega(0, 3) + \Omega(1, 2)$. It is immediate to check that its square is two. Let us now study the Schubert cycle $\Omega(1, 3)$:

Let us consider a representative $\Omega(L, \mathbf{P}^3)$ of σ_1 (i.e. the set of lines meeting the line L). It has also degree two. If we fix some coordinates x_0, x_1, x_2, x_3 in \mathbf{P}^3 and take for example L to have equations $x_0 = x_1 = 0$ it is immediate to see that $\Omega(L, \mathbf{P}^3)$ is the intersection of $G(1, 3)$ with the hyperplane $p_{01} = 0$. So looking again at the equation (0.3) of $G(1, 3)$ in \mathbf{P}^5 , we see that the equations of $\Omega(L, \mathbf{P}^3)$ are

$$\begin{aligned} p_{01} &= 0 \\ p_{02}p_{13} &= p_{03}p_{12} \end{aligned}$$

This is a cone over a quadric surface whose vertex is the point of Plücker coordinates $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) = (0 : 0 : 0 : 0 : 0 : 1)$, which is precisely the point corresponding to L . This is easy to see geometrically. In fact for any $L' \in \Omega(L, \mathbf{P}^3)$, we consider the point $p = L \cap L'$ and the plane Π spanned by L and L' . Then the pencil

$\Omega(p, \Pi)$ is contained in $\Omega(L, \mathbf{P}^3)$. But this pencil is nothing but the line of \mathbf{P}^5 joining L and L' (see Example 1.11). This proves that $\Omega(L, \mathbf{P}^3)$ is a cone with vertex L . Let us see that the base of the cone is a product of two lines, hence a quadric surface. This is because any pencil $\Omega(p, \Pi)$ as above is determined by the choice of the point p in the line L and a plane Π containing L , and the set of such planes is also isomorphic to \mathbf{P}^1 .

This example shows that the general hyperplane section of $G(1, 3)$ is not of the form $\Omega(L, \mathbf{P}^3)$, since by Bertini's theorem it has to be smooth (however its class in the Chow ring is of course $\Omega(1, 3)$). In fact with the above coordinates one can see that the hyperplane $p_{01} = 0$ is precisely the embedded tangent hyperplane of $G(1, 3)$ at L . It is not difficult to see that, up to projectivity, there are only two types of hyperplane sections, depending on whether the hyperplane is or not tangent to $G(1, 3)$. The classical terminology is the following:

Definition: A hyperplane section of $G(1, 3) \subset \mathbf{P}^5$ is called a *linear complex*. If it is smooth it is called a *general linear complex*, and if it is of the form $\Omega(L, \mathbf{P}^3)$ is called a *special linear complex*.

Example 1.13: Let us repeat for $G(1, 4)$ without details (left to the reader) the steps in Example 1.12. It turns out that the degree of its Plücker embedding in \mathbf{P}^9 is $\sigma_1^6 = 5$. Also a set $\Omega(L, \mathbf{P}^4)$ is a cone with vertex at the point $L \in G(1, 4) \subset \mathbf{P}^4$ and of degree $\Omega(1, 4) \cdot \sigma^4 = 3$. Again if we take L to be the line $x_0 = x_1 = x_2 = 0$ in a suitable system of coordinates, the equations of this cone in $G(1, 4)$ are $p_{01} = p_{02} = p_{12} = 0$. Using the equations (0.4) of $G(1, 4)$ in \mathbf{P}^9 we get as equations for $\Omega(L, \mathbf{P}^4)$ in \mathbf{P}^9 :

$$\begin{aligned} p_{01} = p_{02} = p_{12} &= 0 \\ p_{03}p_{14} &= p_{04}p_{13} \\ p_{03}p_{24} &= p_{04}p_{23} \\ p_{13}p_{24} &= p_{14}p_{23} \end{aligned} \tag{1.7}$$

The base of the cone is the set of points in $p_{01} = p_{02} = p_{12} = p_{34} = 0$ satisfying the last three equations of (1.7), which are equivalent to the rank of the matrix

$$\begin{pmatrix} p_{03} & p_{13} & p_{23} \\ p_{04} & p_{14} & p_{24} \end{pmatrix}$$

to be one. This is nothing but the equations of the Segre embedding of $\mathbf{P}^2 \times \mathbf{P}^1$ in \mathbf{P}^5 , so it has indeed degree three. As in Example 1.12, this product can be seen geometrically as the choice of a pencil $\Omega(p, \Pi)$: again p varies in a line L , but now Π is a plane containing a line in \mathbf{P}^4 , and these planes form a \mathbf{P}^2 .

2. Construction of subvarieties of Grassmannians

It is well-known that giving a non-degenerate map (i.e. such that the image is not contained in a hyperplane) from a smooth algebraic variety into a projective space is equivalent to giving an invertible sheaf \mathcal{L} on X and a linear space of global sections of \mathcal{L} that generate it. A criterion for the map to be an embedding is also known in terms of subschemes of length two. We will give here a general result for maps to Grassmannians that yields as a particular case that for projective morphisms.

Proposition 2.1. *Let X be a smooth algebraic variety. Then giving a map $\varphi : X \rightarrow G(k, n)$ is equivalent to giving a locally free sheaf \mathcal{F} on X of rank $k+1$ and an epimorphism $\phi : V \otimes \mathcal{O}_X \rightarrow \mathcal{F}$, where V is an $(n+1)$ -dimensional vector space. Moreover, in this situation, ϕ is the pull-back by φ of the universal epimorphism appearing in (0.7).*

Idea of the proof: Assume we have such an epimorphism $\phi : V \otimes \mathcal{O}_X \rightarrow \mathcal{F}$. Then, for each point $p \in X$ we have an epimorphism $\phi_p : V \rightarrow \mathcal{F}|_p$. This defines a point $\varphi(p) \in G(1, \mathbf{P}(V))$ (if you see it better this other way, observe that this last epimorphism is equivalent to V^* to contain the $(k+1)$ -dimensional subspace $(\mathcal{F}|_p)^*$). This defines the map φ . The geometric idea is that we want to define φ in such a way that V is naturally the set of hyperplanes of \mathbf{P}^n and $V(-p) = \ker \phi_p$ is the set of hyperplanes that contain the k -dimensional space $\varphi(p)$. Hence what we need in order to have the map is the codimension of $V(-p)$ to be $k+1$.

For the second part, think in terms of vector bundles rather than locally free sheaves. The inclusion $\check{F} \rightarrow V^* \times X$ (obtained by dualizing ϕ) corresponds in each fiber at a point p to the inclusion $(F|_p)^* \subset V^*$. By just recalling the definition of \check{Q} and that $(F|_p)^*$ is the $(k+1)$ -dimensional linear space of V^* defining $\phi(p)$ we deduce that $\check{F} \rightarrow V^* \times X$ is the pull-back by φ of the universal inclusion $\check{Q} \rightarrow V^* \times G$. Dualizing back we get the result.

Remark 2.2: The above result can be interpreted in the language of categories. Specifically, fix a vector space V and an integer k , consider the functor from the category of schemes over the ground field to the category of sets that associates to each scheme X the set of epimorphisms from $V \otimes \mathcal{O}_X$ to a (non fixed) locally free sheaf of rank $k+1$. Then Proposition 2.1 says that this functor is represented by $G(k, \mathbf{P}(V))$. In fact this is the nowadays standard definition of Grassmannians (and also of Grassmann bundles if you change V by a locally free sheaf over a given scheme).

Remark 2.3: The epimorphism $\phi : V \otimes \mathcal{O}_X \rightarrow \mathcal{F}$ is saying that \mathcal{F} is spanned (i.e. generated by its global sections). More precisely, if we consider the associated map of sections $\bar{\phi} : V \rightarrow H^0(X, \mathcal{F})$ and call $W = \text{im}(\bar{\phi})$, we see that \mathcal{F} is spanned by the sections in W . We can have the following situations:

(i) The map $\bar{\phi} : V \rightarrow W$ is not injective. In this case let L be the kernel of $\bar{\phi}$. Then for all $p \in X$ we have a factorization $\phi_p : V \rightarrow V/L \rightarrow F_p$ (or dually $F_p^* \rightarrow (V/L)^* \rightarrow V^*$). This indicates that in fact the image of ϕ is contained in $\Omega(k, \mathbf{P}(V/L))$, which is a smaller Grassmannian of k -planes. In this case we will say that ϕ is a *degenerate map*. On the contrary, if $\bar{\phi}$ is injective (hence an isomorphism onto W), since the sections of W generate \mathcal{F} , none of its section vanishes at all points of X . This means that the image of ϕ is not contained in a smaller Grassmannian $G(k, n-1)$. We will say that ϕ is *non-degenerate*.

(ii) The map $\bar{\phi} : V \rightarrow H^0(X, \mathcal{F})$ is not surjective. Then we can take the map $\phi' : X \rightarrow G(k, \mathbf{P}(H^0(X, \mathcal{F})))$ defined by the evaluation map $H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \rightarrow \mathcal{F}$. It is clear that $\phi(p)$ is the k -plane in $\mathbf{P}(V)$ obtained by projecting the k -plane $\phi'(p)$ in $\mathbf{P}(H^0(X, \mathcal{F}))$. This projection is obtained by first projecting $\mathbf{P}(H^0(X, \mathcal{F}))$ onto $\mathbf{P}(W)$ and then considering the natural inclusion $\mathbf{P}(W) \subset \mathbf{P}(V)$. We will study this kind of phenomenon in the last section of these notes.

Notation: In the above situation, for a subscheme $Z \subset X$ of finite length, we will denote by $V(-Z)$ to the subspace of V consisting of those vectors $v \in V$ such that the section $\phi(v) \in H^0(X, \mathcal{F})$ is zero at Z . With this notation we see that the fact that ϕ is an epimorphism (i.e. we have a morphism ϕ , as we have seen in Proposition 2.1) is equivalent to the fact that for all $p \in X$, $\dim V(-p) = \dim V - k - 1$. Let us prove an analogous criterion for ϕ to be an embedding:

Proposition 2.4. *The map ϕ defined by an epimorphism $\phi : V \otimes \mathcal{O}_X \rightarrow \mathcal{F}$ is an embedding if and only if for any subscheme $Z \subset X$ of length 2 we have that $V(-Z)$ has dimension at most $n - k - 1$.*

Idea of the proof: If we want ϕ to be an embedding, we need first to separate points. So let p, q be two different points of X . Then the k -spaces $\phi(p)$ and $\phi(q)$ will be different if and only if the set of hyperplanes containing both of them is strictly smaller than the set of hyperplanes containing one of them. This condition is easily seen to be equivalent to the condition that the codimension of $V(-p - q)$ in V is at most $k + 2$. To separate tangent directions we just need essentially to repeat the same argument by replacing the scheme $Z = \{p, q\}$ by a scheme Z of length two supported at a single point p .

Remark 2.5: The conditions in the above Propositions 2.1 and 2.4 are more easily stated if we take V to be the space $V = H^0(X, \mathcal{F})$ of all sections of \mathcal{F} and ϕ to be the evaluation map. In this case, for any finite subscheme Z of X we have an exact sequence

$$0 \rightarrow \mathcal{F} \otimes \mathcal{J}_{Z,X} \rightarrow \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{O}_Z \rightarrow 0$$

Taking global sections we get another exact sequence

$$0 \rightarrow H^0(X, \mathcal{F} \otimes \mathcal{J}_{Z,X}) \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(Z, \mathcal{F} \otimes \mathcal{O}_Z) \quad (2.1)$$

Note that the dimension of the last vector space is $(k + 1) \cdot \text{length}Z$, and that $V(-Z) = H^0(X, \mathcal{F} \otimes \mathcal{J}_{Z,X})$. Now the hypotheses in Propositions 2.1 and 2.4 can be read as the last morphism to be surjective (for schemes of length one and two). Hence a sufficient condition for \mathcal{F} to give a map (which is also necessary if $H^1(X, \mathcal{F}) = 0$) is that $h^1(X, \mathcal{F} \otimes \mathcal{J}_{P,X}) = 0$ for all points of X . And for \mathcal{F} to give also an embedding that $h^1(X, \mathcal{F} \otimes \mathcal{J}_{Z,X}) \leq k$ for any subscheme of length two.

Remark 2.6: Propositions 2.1 and 2.4 are closely related to the corresponding ruled variety obtained by taking the union of all k -planes defined by the points of X . Indeed consider $Y = \mathbf{P}(\mathcal{F})$ to be the projective bundle associated to \mathcal{F} and let $\pi : Y \rightarrow X$ be the natural projection. Recall that Y has a tautological invertible sheaf $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ (with the property that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)) = \mathcal{F}$) and there is an epimorphism

$$\pi^* \mathcal{F} \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$$

that induces an isomorphism $H^0(X, \mathcal{F}) \cong H^0(Y, \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1))$. Hence an epimorphism $\phi : V \otimes \mathcal{O}_X \rightarrow \mathcal{F}$ induces an epimorphism $V \otimes \mathcal{O}_Y \rightarrow \pi^* \mathcal{F} \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$. And conversely, an epimorphism $V \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ induces an epimorphism $V \otimes \mathcal{O}_X \rightarrow \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)) = \mathcal{F}$. Hence Proposition 2.1 is just saying that a map $\varphi : X \rightarrow G(k, n)$ is equivalent to a map $\bar{\varphi} : Y \rightarrow \mathbf{P}^n$ of the corresponding ruled variety. However the hypothesis of Proposition 2.4 is much weaker than saying that $\bar{\varphi}$ is an embedding. Indeed for a subscheme Z of length two, from the pull-back via $\bar{\phi}$ of the exact sequence (2.1) we get

$$0 \rightarrow V(-Z) \rightarrow V \rightarrow H^0(Z, \mathcal{F} \otimes \mathcal{O}_Z)$$

This shows that the dimension of $V(-Z)$ (if Z has length two) is always at least $(n + 1) - 2(k + 1)$. Equality means geometrically that any two k -linear subspaces (even those “infinitely close”) do not meet. In other words, this would be equivalent to $\bar{\varphi}$ to be an embedding. If this is so (i.e. if $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ is very ample), the sheaf \mathcal{F} is said to be *very ample*. But this is a much stronger condition. In general *the ruled variety defined by an embedding in a Grassmannian does not need to be smooth* (see for instance Example 2.9 below).

Notation: When we take the epimorphism ϕ to be the evaluation map $H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \rightarrow \mathcal{F}$ of a spanned sheaf \mathcal{F} , we will write $\varphi_{\mathcal{F}}$ for the corresponding map from X to $G(k, n)$.

Example 2.7: Take $X = \mathbf{P}^1$ and $\mathcal{F} = \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. Then it is easy to check that \mathcal{F} gives an embedding in $G(1, 3)$ (in fact \mathcal{F} is very ample since it is the direct sum of two very ample invertible sheaves). Each of the summands of \mathcal{F} yields an embedding of \mathbf{P}^1 as a line in \mathbf{P}^3 . So we have two lines L_1 and L_2 in \mathbf{P}^3 . For each point $p \in \mathbf{P}^1$, we get then two points $p_1 \in L_1$ and $p_2 \in L_2$. The corresponding line $\varphi_{\mathcal{F}}(p)$ is hence the line joining p_1

and p_2 . Summing up, we have two skew lines L_1 and L_2 in \mathbf{P}^3 and an isomorphism among them. Then the image of \mathbf{P}^1 in $G(1, 3)$ under the map $\varphi_{\mathcal{F}}$ defined by \mathcal{F} is the set of lines joining two corresponding points of L_1 and L_2 . In other words, it is the set of lines of one of the rulings of a smooth quadric in \mathbf{P}^3 .

Example 2.8: More generally, if we take again $X = \mathbf{P}^1$ and now $\mathcal{F} = \mathcal{O}_{\mathbf{P}^1}(a_0) \oplus \dots \oplus \mathcal{O}_{\mathbf{P}^1}(a_k)$ (with all $a_i > 0$) we get an embedding of \mathbf{P}^1 in $G(k, N)$ (where $N = a_0 + \dots + a_k + k$). The k -planes in \mathbf{P}^N are precisely those contained in the scroll of k -planes of type (a_0, \dots, a_k) .

Example 2.9: If \mathcal{L} is an invertible sheaf, then $\mathcal{F} = \mathcal{O}_X \oplus \mathcal{L}$ is easily seen to verify the conditions for giving an embedding in a Grassmannian of lines if and only if \mathcal{L} is very ample. In fact the corresponding set of lines are those lines in the cone with vertex one point over the image of X by \mathcal{L} . In particular in Example 2.8 above we can allow some of the a_i 's to be zero. The reader can easily figure out what happens if \mathcal{L} is in general a locally free sheaf instead of an invertible sheaf.

Ruled surfaces.

Let us concentrate for a while in the case of curves in Grassmannians (and more specifically Grassmannians of lines). So take C to be a curve and \mathcal{F} a locally free sheaf over C of rank $k + 1$. In this case, it is very easy to state when \mathcal{F} defines an embedding. In fact one has that \mathcal{F} is generated by its global sections if and only if $h^0(C, \mathcal{F}(-p)) = h^0(C, \mathcal{F}) - k - 1$ for any point $p \in C$, and it produces an embedding if and only if it also holds that $h^0(C, \mathcal{F}(-p - q)) \leq h^0(C, \mathcal{F}) - k - 2$ for any two points $p, q \in C$ (not necessarily different).

Remark 2.10: If \mathcal{F} is an invertible sheaf it is known that one can be sure that the above cohomological conditions hold as soon as \mathcal{F} has degree at least $2g + 1$, since in this case $h^1(C, \mathcal{F}(-p - q)) = 0$ (see Remark 2.5). However in higher rank there is nothing similar since such a vanishing depends strongly on the stability of \mathcal{F} .

Note also that since $C \subset G(k, n)$ has dimension one, it has only one degree in $G(k, n)$, namely the intersection with σ_1 . If we call Σ the ruled variety obtained as the union of all the k -planes determined by Σ , then it has dimension $k + 1$. If $[C] \cdot \sigma_1 = d$ this means that a general $(n - k - 1)$ -plane meets d k -planes of C . This is the same as saying that Σ has degree d .

As it happens for projective curves, where we have Castelnuovo's theorem (see for instance [GH]), the genus of a curve in a Grassmannian is bounded by its degree. The precise result (see [GGS] for $k = 1$ and [G] for general k) is:

Theorem 2.11. *If $C \subset G(k, n)$ has genus g and degree d and is not contained in an $\Omega(n - k - 1, \dots, n - 1)$ nor in an $\Omega(0, n - k + 1, \dots, n)$ then $g \leq G(d, n)$, where $G(d, n)$ is Castelnuovo's bound for the genus of nondegenerate curves in \mathbf{P}^n of degree d . Moreover this bound is sharp.*

The hypotheses of the theorem just say that the curve is not degenerate neither in $G(k, n)$ nor in the dual $G(n - k - 1, n)$. The idea of the proof is to show that, under these hypotheses, one has $h^0(C, \mathcal{O}_G(1)|_C) \geq n + 1$, so that one can apply Castelnuovo's theorem to the image of C under the Plücker embedding. As far as I know, the sharpness of the bound has been shown only in case $k = 1$.

In the case $k = 1$, the union of all lines form what is called a ruled surface or scroll in lines. This is a surface whose irregularity $q(S) = h^1(S, \mathcal{O}_S)$ is precisely the genus g of C . Sometimes the word "scroll" is reserved for ruled surfaces that are not what is called developable.

Definition: A ruled surface is called *developable* if for each point of the corresponding $C \subset G(1, n)$, its embedded tangent line in the Plücker embedding is contained in $G(1, n)$.

Developable ruled surfaces are characterized by the following:

Proposition 2.12. *A ruled surface is developable if and only if it is a cone or it is the tangent developable to a curve of \mathbf{P}^n .*

Proof: Take a point $p \in C$. Since its embedded tangent line is contained in $G(1, n)$, this must be (see Example 1.11) a pencil of the form $\Omega(q_p, \Pi_p)$. We consider now the map

$$\begin{aligned} C &\rightarrow \mathbf{P}^n \\ p &\mapsto q_p \end{aligned}$$

If the map is constant, then all lines of the ruled surface pass through the same point $q \in \mathbf{P}^n$, so our ruled surface S is a cone. If the map is not constant, its image will be a curve D . In this case, we want to show that S is precisely the set of tangent lines to D . So fix a point $p \in C$, and choose coordinates in \mathbf{P}^n and the corresponding Plücker coordinates such that:

- (i) The line of \mathbf{P}^n corresponding to p has equations $x_2 = \dots = x_n = 0$.
- (ii) The point q_p is the point $(1 : 0 : \dots : 0)$.
- (iii) The plane Π_p has equations $x_3 = \dots = x_n = 0$.

Because of (i), the point p is the one with all Plücker coordinates but p_{01} equal to zero. From (ii) and (iii) we see that the pencil $\Omega(q_p, \Pi_p)$ is the line defined by making zero

all Plücker coordinates but p_{01}, p_{02} . We choose now a local parameter t at p , in such a way that C has a parametrization around p of the form

$$\begin{aligned} p_{01} &= 1 \\ p_{ij} &= f_{ij}(t) \text{ for other } i, j \end{aligned}$$

The functions f_{ij} are linked by the Plücker equations, they all vanish at $t = 0$ and the order of vanishing of f_{02} at $t = 0$ must be strictly smaller than for any other f_{ij} (since we are assuming $\Omega(q_p, \Pi_p)$ to be the tangent line of C at p). Since we are working in the affine chart $\{p_{01} = 1\}$ we can represent all elements in this chart by matrices

$$\begin{pmatrix} 1 & 0 & -p_{12} & \cdots & -p_{1n} \\ 0 & 1 & p_{02} & \cdots & p_{0n} \end{pmatrix}$$

Hence the only Plücker relations to verify are

$$p_{ij} = \det \begin{pmatrix} -p_{1i} & -p_{1j} \\ p_{0i} & p_{0j} \end{pmatrix} \text{ for } 2 \leq i < j \leq n.$$

The embedded tangent line to a point of C corresponding to a parameter value t can be parametrized by

$$p_{ij} = f_{ij}(t) + \lambda f'_{ij}(t)$$

(where of course f'_{ij} means the derivative of f_{ij} with respect to the parameter t). Hence the condition for this line to be contained in $G(1, n)$ turns out to be (using the above Plücker relations and checking that it is only necessary to work with the coefficients of λ^2):

$$\det \begin{pmatrix} -f'_{1i} & -f'_{1j} \\ f'_{0i} & f'_{0j} \end{pmatrix} \text{ for } 2 \leq i < j \leq n.$$

This means that $(f'_{12}, \dots, f'_{1n})$ is a multiple of $(f'_{02}, \dots, f'_{0n})$. Recalling that the order of vanishing of f_{02} at $t = 0$ was strictly smaller than the rest, this means that there is a function g vanishing at $t = 0$ such that $f'_{1i} = g f'_{0i}$ for all $i = 2, \dots, n$. Therefore all points in such tangent line can be represented by a Plücker matrix

$$\begin{pmatrix} 1 & 0 & -f_{12} - \lambda g f'_{02} & \cdots & -f_{1n} - \lambda g f'_{0n} \\ 0 & 1 & f_{02} + \lambda f'_{02} & \cdots & f_{0n} + \lambda f'_{0n} \end{pmatrix}.$$

Multiplying the second row by g and adding it to the first row, we see that the point $(1, g, g f_{02} - f_{12}, \dots, g f_{0n} - f_{1n})$ is in all the lines of the pencil corresponding to a parameter value t . Hence this is a parametrization of the curve D . It is now a straightforward calculation to check with this parametrization that the tangent line to D at any point t is precisely the line whose point in $G(1, n)$ corresponds to t .

Double-point formulas.

It is well-known that a smooth projective subvariety whose dimension is at least its codimension cannot have arbitrary invariants, since there are numerical relations among them. In the limit case, if we have $X \subset \mathbf{P}^{2r}$ of dimension r , the relation comes from the so-called *double point formula*, which simply states that $c_r(\mathcal{N}) = [X]^2$ (here \mathcal{N} means the normal sheaf of X in \mathbf{P}^{2r}). Since we are in a projective space, we know that $[X] = dH^r$, where H is the class of the hyperplane section and d is the degree of X . Hence the second expression in the formula is just d^2 . The name of the formula comes from the fact that, if you have a smooth $X' \subset \mathbf{P}^{2r+1}$ of dimension r , then the number of double points that you get for its general linear projection $X \subset \mathbf{P}^{2r}$ is $\frac{1}{2}([X]^2 - c_r(\mathcal{N}))$ (in particular this number is non-negative, so that you always have some inequality relating the invariants of any projective variety).

For dimension $r = 1$, the double-point formula is nothing but the well-known formula for the genus of a plane curve: $g = \frac{(d-1)(d-2)}{2}$. For dimension $r = 2$, i.e. for a smooth surface $S \subset \mathbf{P}^4$, the double-point formula reads (see [H] p.434):

$$d^2 = 5d + 5(2\pi - 2) + 2K_S^2 - 12\chi(\mathcal{O}_S)$$

Here d is the degree of S , π is the *sectional genus* (i.e. the genus of the curve obtained by taking a general hyperplane section), K_S^2 is the self-intersection of the canonical divisor K_S and $\chi(\mathcal{O}_S)$ is the Euler characteristic of the structure sheaf \mathcal{O}_S of S .

We are interested in double-point formulas for subvarieties of Grassmannians (in fact the above double-point formula is valid for any smooth r -dimensional subvariety X of a smooth variety Y of dimension $2r$). For dimension $r = 1$ there is nothing new, since the only Grassmann varieties of dimension two are \mathbf{P}^2 and its dual. For dimension $r = 2$, the only four-dimensional Grassmann variety that is not a projective space is $G(1, 3)$. So if we have a smooth surface $S \subset G(1, 3)$ the double-point formula takes the form (see for example [AS]):

$$a^2 + b^2 = 3d + 4(2\pi - 2) + 2K_S^2 - 12\chi(\mathcal{O}_S) \tag{2.2}$$

Here a and b are the order and the class of the congruence (see Example 1.9) and $d = a + b$ and the rest of invariants are as for \mathbf{P}^4 (after the Plücker embedding we can see now S as a surface in \mathbf{P}^5).

We come finally to the case of dimension $r = 3$. The Grassmann varieties of dimension six that are not projective spaces are $G(1, 4)$ and $G(2, 4)$. Since they are naturally isomorphic by duality, we will just study the case of smooth threefolds $X \subset G(1, 4)$. The main reason to do so is to see how complicated the double-point formula could become and

the kind of invariants appearing in it. First of all, we observe that $A^3(G(1, 4))$ is generated by $\Omega(0, 4)$ and $\Omega(1, 3)$ so X will have a bidegree (a, b) , where a is the number of lines passing through a general fixed point of \mathbf{P}^4 and b is the number of lines contained in a given hyperplane and meeting a given line of it. One can check by using Schubert calculus that $\sigma_1^3 = \Omega(0, 4) + 2\Omega(1, 3)$ (this relation will come also from the computations below), so that the degree of X as a threefold in \mathbf{P}^9 (after the Pücker embedding) is $d = a + 2b$.

The interesting fact in this case, is that there are more genera appart from the sectional genus π (defined as usual as the genus of the curve obtained by intersecting X with two hyperplanes of \mathbf{P}^9). Indeed one can observe that $A^2(G(1, 4))$ is generated by $\Omega(1, 4)$ and $\Omega(2, 3)$, so that we can consider the intersection of X with a representative of any of these classes (lines meeting a given line in the first case, lines contained in a hyperplane in the second). In the first case, we obtain a curve of degree $[X] \cdot \Omega(1, 4) \cdot \sigma_1 = (a\Omega(0, 4) + b\Omega(1, 3)) \cdot (\Omega(0, 4) + \Omega(1, 3)) = a + b$. In the second case, the curve will have degree $[X] \cdot \Omega(2, 3) \cdot \sigma_1 = (a\Omega(0, 4) + b\Omega(1, 3)) \cdot \Omega(1, 3) = b$. We will denote these curves respectively C_{a+b} and C_b (of course we have used again Schubert calculus in both cases). Then we can consider as invariants the genera of these curves, which we will denote by g_{a+b} and g_b .

These genera are related to the sectional genus g . Indeed take two special hyperplane sections of the form $\Omega(\Pi, \mathbf{P}^4)$ and $\Omega(\Pi', \mathbf{P}^4)$, where Π and Π' is a pair of general planes meeting along a line L and hence contained in a hyperplane $H \subset \mathbf{P}^4$. The intersection of X with these two hyperplanes is therefore (as Schubert calculus also predicted) the intersection of two curves C_{a+b} and C_b , obtained by intersecting X respectively with $\Omega(L, \mathbf{P}^4)$ and $\Omega(\Pi, H)$. These two curves meets in the intersection of X with $\Omega(L, \mathbf{P}^4) \cap \Omega(\Pi, H) = \Omega(L, H)$, so exactly in b points (by definition of b). So the situation is that a sectional curve of genus π can be obtained as the union of two curves of genera g_{a+b} and g_b meeting in b points. In this situation it holds that $\pi = g_{a+b} + g_b + b - 1$.

With all this notation we can give now the double-point formula for small threefolds in $G(1, 4)$. This formula can be found in [A], although we have changed it a little bit so that the invariants can be readily recognized. It is as follows:

$$a^2 + b^2 = 3a + 5b + 6(2\pi - 2) + (2g_b - 2) - 60\chi(\mathcal{O}_S) + 10K_S^2 + K_X^3 + 48\chi(\mathcal{O}_X) - \chi(X)$$

Here S is the surface obtained by intersecting S with a general hyperplane, and $\chi(X)$ is the topological Euler-Poincaré characteristic of X .

Construction of subvarieties in Grassmannians by means of sheaves.

We have already seen how some particular subvarieties of $G(k, n)$, for instance the special Schubert varieties, can be obtained from the universal bundles on $G = G(k, n)$ (see Proposition 0.5). What we want to do now is to illustrate this with more general examples

(which will be also used later). The general idea is to take a locally free sheaf \mathcal{F} of certain rank R that is generated by its global sections. Then the dependency locus X of m general sections of \mathcal{F} (with $m \leq R$) has the expected codimension, namely $R - m + 1$, and also the singular locus of X has the expected codimension (in G), namely $2(R - m + 2)$. If you are familiar with Porteous formula (see for example [ACGH] or [F]), this dimension counting comes from the fact that the dependency locus of m sections of \mathcal{F} is the locus X where the morphism $\mathcal{O}_G^m \rightarrow \mathcal{F}$ defined by the sections has rank at most $m - 1$, and the singular locus of X is (for general sections) the locus in which that morphism has rank at most $m - 2$. Also Porteous formula tells us that the class of X in G is $c_{R-m+1}(\mathcal{F})$ (it can also predict the class of the singular locus of X but it seldom occurs that it is zero). So for example if $2(R - m + 2) > (k + 1)(n - k)$ you obtain in this way a smooth subvariety X of codimension $R - m + 1$. Note that this method allows to construct only smooth subvarieties of dimension smaller than approximately half the dimension of the ambient space.

Example 2.13: Consider $G = G(1, 3)$. Certainly the universal bundle \mathcal{S} is generated by its global sections, so that three general sections of $\mathcal{S} \oplus \mathcal{S}$ will produce a smooth congruence S of bidegree $(3, 1)$, since one can check that $c_2(\mathcal{S} \oplus \mathcal{S}) = 3\Omega(1, 2) + \Omega(1, 2)$ (we will not do the details; they can be found in [AS] p.47 and/or can be completed by the patient reader). From this description one finds a Koszul exact sequence

$$0 \rightarrow \mathcal{O}_G^3 \rightarrow \mathcal{S} \oplus \mathcal{S} \rightarrow \mathcal{O}_G(2) \rightarrow \mathcal{O}_S(2) \rightarrow 0$$

From this exact sequence one can get all the information needed to identify the congruence S . For example one has $H^0(G, \mathcal{O}_G(1)) \cong H^0(S, \mathcal{O}_S(1))$, so that S is a non-degenerate surface of degree four in \mathbf{P}^5 . Hence it is a surface of minimal degree, i.e. a Veronese surface or a rational scroll. By looking more carefully at the invariants produced by the construction (one needs to compute $K_S^2 = 9$) one checks that S is the Veronese surface, hence isomorphic to \mathbf{P}^2 .

Also by tensoring the Koszul sequence with $\mathcal{Q}(-2)$ one can derive, after several cohomological computations, that $\mathcal{Q}|_S$ has a six-dimensional space of sections and that has no intermediate cohomology (i.e. that $h^1(\mathbf{P}^2, \mathcal{Q}|_S \otimes \mathcal{O}_{\mathbf{P}^2}(i)) = 0$ for all $i \in \mathbf{Z}$). A theorem of Horrocks implies then that $\mathcal{Q}|_S \cong \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(1)$. Hence the geometrical description of the congruence is, as in Example 2.7, the set of lines joining corresponding points of two planes $\Pi_1, \Pi_2 \subset \mathbf{P}^3$ for which there is a given isomorphism $\Pi_1 \rightarrow \Pi_2$. Note that this surface is the projection of a surface in $G(1, 5)$. Using this geometric description, one can check (see again [AS]) that the corresponding congruence in $G(1, \check{\mathbf{P}}^3)$ is the set of bisecants to a twisted cubic in $\check{\mathbf{P}}^3$ (we have already considered this congruence in Example 1.10).

Example 2.14: It is clear from Example 2.13 that with a lot of work one can get all the information about a subvariety from such a description as a dependency locus. Sometimes

this information can be obtained in a more direct way. For example, take S to be the dependency locus of two general sections of $\mathcal{S} \oplus \mathcal{O}_G(1)$ (this example is described in [AS] p.46). As before you can look at the second Chern class to conclude that S has bidegree $(2, 1)$ and also can find a Koszul exact sequence

$$0 \rightarrow \mathcal{O}_G^2 \rightarrow \mathcal{S} \oplus \mathcal{O}_G(1) \rightarrow \mathcal{O}_G(2) \rightarrow \mathcal{O}_S(2) \rightarrow 0$$

From this one can deduce this time that S is contained in a hyperplane of \mathbf{P}^5 (as it should be, because it has degree three), and hence is a surface of minimal degree in \mathbf{P}^4 , hence a rational scroll. But this last information was clear *a priori*:

Indeed a point of G is in the dependency locus of two sections s_0, s_1 of a locally free sheaf $(\mathcal{S} \oplus \mathcal{O}_G(1))$ in this case) if and only if it is in the zero locus of a linear combination of s_0 and s_1 . But a section of $\mathcal{S} \oplus \mathcal{O}_G(1)$ vanishes in the intersection of a section of \mathcal{S} (an α -plane) and a section of $\mathcal{O}_G(1)$ (a hyperplane section); this intersection is clearly a pencil of lines (hence a line contained in G , according to Example 1.11). Therefore the situation is that we have a natural morphism from S to the pencil of sections determined by s_0 and s_1 whose fibers are, as we just saw, lines contained in G . This provides S a natural structure of scroll over \mathbf{P}^1 .

One last remark about this example. We have seen that S is degenerate as a surface in \mathbf{P}^5 , since it is contained in a hyperplane. However S is not degenerate as a surface in $G(1, 3)$ On the contrary one can see from the Koszul sequence that $h^0(S, \mathcal{Q}_{|S}) = 5$, which implies that S is projected from a surface in $G(1, 4)$.

Of course a natural question is to know whether any smooth subvariety of a Grassmannian (of suitable small dimension, as we have seen) can be obtained as the dependency locus of some sections of an appropriate sheaf. The answer is positive for codimension two (the so-called *Serre's construction*) but very little is known for bigger codimension. And from what we have seen before, codimension two would imply the dimension of G to be very small (at most five) for the construction to produce smooth subvarieties.

Let us say just few words about Serre's construction in $G = G(k, n)$. First assume that you have $X \subset G$ of codimension two obtained as the dependency locus of $R - 1$ sections of a locally free sheaf \mathcal{F} of rank R . Then there is a Koszul exact sequence

$$0 \rightarrow \mathcal{O}_G^{R-1} \rightarrow \mathcal{F} \rightarrow \mathcal{J}_{X,G}(c_1) \rightarrow 0 \tag{2.3}$$

where $\bigwedge^{R-1} \mathcal{F} = \mathcal{O}_G(c_1)$. Dualizing this exact sequence we get an epimorphism

$$\mathcal{O}_G^{R-1} \rightarrow \mathcal{E}xt^1(\mathcal{J}_{X,G}(c_1), \mathcal{O}_G) \cong \mathcal{E}xt^2(\mathcal{O}_X, \omega_G) \otimes \mathcal{O}_G(n+1-c_1) \cong \omega_X(n+1-c_1)$$

where we used the isomorphism $\omega_G \cong \mathcal{O}_G(-n-1)$ of Corollary 0.7. Then Serre's construction is the reciprocal construction. It can be proved that, if for some integer c_1 the twist $\omega_X(n+1-c_1)$ of the canonical sheaf of X is generated by $R-1$ sections, then there is a locally free sheaf \mathcal{F} and an exact sequence like (2.3).

Example 2.15: Let us consider \dot{X} to be the disjoint of an α -plane and a β -plane, then \dot{X} is a smooth congruence of bidegree $(1,1)$ and its canonical sheaf is $\omega_X \cong \mathcal{O}_{\dot{X}}(-3)$ (since the same property holds for any of its components, which are planes). Then Serre's construction guarantees that \dot{X} is the zero section of a locally free sheaf of rank two, usually denoted with a twist $\dot{\mathcal{E}}(1)$ (so that it has normalized first Chern class). In fact this sheaf is not unique and there is a whole module space of dimension six for these sheaves (see [AS] for instance). The Chern classes of $\dot{\mathcal{E}}$ are $c_1(\dot{\mathcal{E}}) = -\Omega(1,3)$, $c_2(\dot{\mathcal{E}}) = \Omega(0,3) + \Omega(1,2)$. There is an exact sequence

$$0 \rightarrow \mathcal{O}_G \rightarrow \dot{\mathcal{E}}(1) \rightarrow \mathcal{O}_G(1) \rightarrow \mathcal{O}_{\dot{X}}(1) \rightarrow 0$$

from where it is possible to get all necessary information about $\dot{\mathcal{E}}$.

Example 2.16: It can be shown that $\dot{\mathcal{E}}(2)$ is generated by its global sections, so that a general section defines a smooth congruence S . The corresponding Koszul exact sequence is

$$0 \rightarrow \mathcal{O}_G \rightarrow \dot{\mathcal{E}}(2) \rightarrow \mathcal{O}_G(3) \rightarrow \mathcal{O}_S(3) \rightarrow 0$$

One can see that S has bidegree $(3,3)$ (hence degree six), that $h^0(\mathcal{O}_S(1)) = 7$ (hence S is projection from a surface in \mathbf{P}^6) and that $\omega_S = \mathcal{O}_S(-1)$. From this we see that S is a Del Pezzo surface, i.e. the blow-up of \mathbf{P}^2 in three points p_0, p_1, p_2 , and embedded in the projective space by the system of cubics passing through p_0, p_1, p_2 . It should be also remarked that $h^0(S, \mathcal{Q}_{|S}) = 5$, and hence S is besides projected from a surface in $G(1,4)$.

Example 2.17: Another way of producing interesting examples of subvarieties or sheaves in a Grassmannian is to consider a finite map to the projective space of the same dimension and pull back via this map all interesting examples there. For instance, if we view $G(1,3)$ as a quadric hypersurface in \mathbf{P}^5 , we can project it from an exterior point to \mathbf{P}^4 , getting a double covering. Then we can pull back all exotic things in \mathbf{P}^4 , such as the Horrocks-Mumford bundle or its associated abelian variety. One checks immediately that the pull-back of a surface of degree d becomes a congruence of bidegree (d,d) , and in general all the new invariants can be computed. In particular, the pull-back of the Veronese surface in \mathbf{P}^4 is a congruence of bidegree $(4,4)$ and it can be checked that it is isomorphic to the blow-up of \mathbf{P}^2 in seven points, embedded in the projective space by the system of plane sextics passing doubly through these seven points. From its construction (or from this last description) we see that it is a projection from a surface in \mathbf{P}^6 . However, it is not a projection from any surface of $G(1,4)$.

3. Some questions on subvarieties of Grassmannians

Of course there are a lot of subjects one can discuss about subvarieties in Grassmannians and there is also a lot of work done and still to do: classification of subvarieties satisfying certain numerical or geometrical conditions, analogous results to those obtained for projective varieties,... I will just pick some concrete topics. I am not sure they will be completely representative, but at least are my favorite ones.

Let us start as usual recalling some facts happening in projective space. The first question is which is the “natural” projective space in which to embed a variety X of dimension r . The first thing one can do is to use a very ample invertible sheaf on X to embed X in a probably very large projective space. Then certainly one can start performing linear projections to smaller projective spaces. But how far can we continue projecting if we want the image of X to be smooth? It is clear that a necessary and sufficient condition for having a smooth projection from a point p is that no bisecant to X passes through p (here the word “bisecant” has a wide meaning: it is any line meeting X in at least two points, maybe infinitely close points; in particular a tangent line is a bisecant). So in order to be able to find a center of projection we need that the variety $Bis(X)$ consisting of the union of all bisecants does not fill the whole projective space. The expected dimension of $Bis(X)$ is $2r + 1$, since we have r parameters to choose each of the two points spanning the bisecant plus one more parameter to choose a point in the bisecant line. In any case, $Bis(X)$ has dimension at most $2r + 1$. Hence any smooth projective variety of dimension r can be smoothly projected to \mathbf{P}^{2r+1} , and probably some of them can be projected to a smaller projective space. This agrees with the double-point formula, which predicts a number of double points when projecting X to \mathbf{P}^{2r} . There are easy results for $r = 1, 2$. First let us recall some definitions valid also for any Grassmannian.

Definitions: A subvariety $X \subset G(k, n)$ is said to be *non-degenerate* if the inclusion map $X \hookrightarrow G(k, n)$ is non-degenerate, i.e. if X is not contained in any $G(k, n - 1)$. As we have seen in Remark 2.3(i) this is equivalent to the map $H^0(G, \mathcal{Q}) \rightarrow H^0(X, \mathcal{Q}|_X)$ to be injective. On the other hand, X is said to be *linearly normal* if it is not projected from any non-degenerate subvariety of $G(k, n + 1)$. In this case we know from Remark 2.3(ii) that this is equivalent to the map $H^0(G, \mathcal{Q}) \rightarrow H^0(X, \mathcal{Q}|_X)$ to be surjective. Recall that for the projective space \mathbf{P}^n it is $\mathcal{Q} = \mathcal{O}_{\mathbf{P}^n}(1)$.

Proposition 3.1. *No smooth non-degenerate curve in \mathbf{P}^3 can be smoothly projected to \mathbf{P}^2 .*

Proof: It is enough to prove that any smooth curve $C \subset \mathbf{P}^2$ is not projected from a non-degenerate curve in \mathbf{P}^3 . To see this, it is enough to show that the map $H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)) \rightarrow$

$H^0(C, \mathcal{O}_{\mathbf{P}^2}(1)|_C)$ is an epimorphism. This map comes from taking cohomology in the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^2}(1-d) \rightarrow \mathcal{O}_{\mathbf{P}^2}(1) \rightarrow \mathcal{O}_{\mathbf{P}^2}(1)|_C \rightarrow 0$$

(here d is the degree of C). Then we get our result because $H^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1-d)) = 0$.

Let us give a more geometric proof in order to illustrate the general phenomenon. So we go back to our original situation and take a smooth curve $C \subset \mathbf{P}^3$. Also assume that $Bis(C)$ is not the whole \mathbf{P}^3 . Since this is a closed subvariety it is easily seen to have dimension two (unless C is a line). But it is not difficult to prove that an irreducible surface with a two-dimensional family of lines is a plane (the idea is that through any point of the surface there is a one-dimensional family of lines; hence any two points of the surface can be joined by a line). Hence we get that C is already contained in a plane.

Theorem 3.2. (*Severi*) *The only non-degenerate surface that can be projected from \mathbf{P}^5 to \mathbf{P}^4 is the Veronese surface, i.e. the image of \mathbf{P}^2 in \mathbf{P}^5 by the invertible sheaf $\mathcal{O}_{\mathbf{P}^2}(2)$.*

The proof of this theorem (my favorite one) can be found in [S] (I think that it is worth to learn Italian just to read it) or, in a more detailed and general way (allowing the surfaces to have some singularities), in [M]. The general idea is more or less to make a dimension counting to conclude that your surface contains a two-dimensional family of conics and deduce from this that it should be the Veronese surface.

More generally, what is known is which r -dimensional varieties can be smoothly projected from $\mathbf{P}^{\frac{3r+4}{2}}$ to $\mathbf{P}^{\frac{3r+2}{2}}$. These are called *Severi varieties*, they are of four different types, their dimensions being $r = 2, 4, 8, 16$ (see [Z] for such a result and general results about secant varieties). There is a uniform construction for these four varieties (due to Banchoff, Roberts and Zak) using the four real division fields, i.e. the reals, the complex numbers, the quaternions and the octonions. Unfortunately, to the best of my knowledge there is no reference for it.

Linear normality in Grassmannians.

Coming now to the case of Grassmannians, let us proceed in the same way. Of course in this context it is still possible to speak about linear projections, since a linear projection from one projective space to another induces a map from the corresponding Grassmannians of k -spaces (since a k -space not meeting the center of projection is projected to another k -space). Of course it should be said first that it is also possible to prove that any algebraic variety can be embedded in a large Grassmann variety (see for example [G] p. 207). The only known results about projections are so far those concerning Grassmannians of lines. They all have even dimension, more precisely $G(1, n)$ has dimension $2n - 2$. Hence the

idea is that a variety of dimension r would not fit well in $G(1, r + 1)$, since its dimension is too small. Then in general such a variety could not be projected smoothly from $G(1, r + 2)$ to $G(1, r + 1)$.

We should remark before going on that it is not true in general however that a smooth subvariety $X \subset G(1, N)$ of dimension r can be smoothly projected to $G(1, r + 2)$ or even to a bigger Grassmannian. For example a cone over a smooth surface $S \subset \mathbf{P}^5$ produces a smooth surface in $G(1, 6)$ (see Example 2.9) that cannot be projected to $G(1, 5)$ unless S is the Veronese surface (from Theorem 3.2).

We come now to the case of the projection of curves This is easy, and we will give two proofs as in Proposition 3.1:

Proposition 3.3. *The only non-degenerate curve in $G(1, 3)$ that can be smoothly projected to $G(1, 2)$ is the one of Example 2.7, corresponding to the lines of a ruling of a smooth quadric.*

Proof: For a first (algebraic) proof, we start from a curve $C \subset G(1, 2)$ and want to know whether it is projected from $G(1, 3)$. Hence we just have to check the surjectivity of the map $H^0(G(1, 2), \mathcal{Q}) \rightarrow H^0(C, \mathcal{Q}|_C)$. But this Grassmannian $G(1, 2)$ is nothing but $\check{\mathbf{P}}^2$, and its universal bundle (see Remark 0.4) is $\mathcal{T}_{\check{\mathbf{P}}^2}(-1)$. Now we use the exact sequence

$$0 \rightarrow \mathcal{T}_{\check{\mathbf{P}}^2}(-1 - d) \rightarrow \mathcal{T}_{\check{\mathbf{P}}^2}(-1) \rightarrow \mathcal{T}_{\check{\mathbf{P}}^2}(-1)|_C \rightarrow 0$$

(where again d is the degree of C). Using the dual of the Euler sequence or the isomorphism $\mathcal{T}_{\check{\mathbf{P}}^2} \cong \Omega_{\check{\mathbf{P}}^2}(3)$ coming from the fact that $\mathcal{T}_{\check{\mathbf{P}}^2}$ has rank two and its determinant is $\mathcal{O}_{\check{\mathbf{P}}^2}(3)$, it follows that $H^1(\mathcal{T}_{\check{\mathbf{P}}^2}(l)) \neq 0$ if and only if $l = -3$. This proves that that $d = 2$ if C is projected from $G(1, 3)$. Hence $C \cong \mathbf{P}^1$ (since it is a plane conic) and since $d = 2$ it must be embedded in $G(1, 3)$ by either $\mathcal{O}_{\check{\mathbf{P}}^2} \oplus \mathcal{O}_{\check{\mathbf{P}}^2}(2)$ or $\mathcal{O}_{\check{\mathbf{P}}^2}(1) \oplus \mathcal{O}_{\check{\mathbf{P}}^2}(1)$. In the first case we get the set of lines of a quadric cone and in the second case the lines of one of the rulings of a smooth quadric in \mathbf{P}^3 . In the first case any projection to $G(1, 2)$ is not smooth (in fact it is always two to one), so it is always the second case.

And now let us prove the same result geometrically. Now we consider $C \subset G(1, 3)$ (or equivalently a ruled surface in \mathbf{P}^3) and assume it can be projected to $G(1, 2)$ by projecting from a point of \mathbf{P}^3 . Let us analyze first what is the locus of points from which we cannot project. We will obtain a singular point in $G(1, 2)$ as soon as two lines (maybe infinitely close) of the ruled surface are projected to the same line. This happens when the two lines are coplanar and the plane containing them (we will call such a plane a bad plane) passes through the center of projection. So if C can be projected it means that the union of all bad planes is not the whole \mathbf{P}^3 . This is the same as saying that there is only a finite number of such planes.

But now, if we call d the degree of C (i.e. the degree of the ruled surface), we know by definition that the intersection number of C with the special Schubert cycle $\sigma_1 = \Omega(1, 3)$ is d . If we take now any line L of the ruled surface (corresponding to a point $p \in C$) to represent this Schubert cycle, we know (see Example 1.12) that in \mathbf{P}^5 this Schubert variety $\Omega(L, \mathbf{P}^3)$ is a quadric cone with vertex p . Hence for a general p the intersection multiplicity at p of C and the Schubert variety is two. This means that there are other $d - 2$ lines of the ruled surface (counted with possible multiplicities) meeting the given line. This proves that if $d \geq 3$ a general line of the ruled surface meets other lines; in particular there are infinitely many bad planes. So this proves again that it must be $d = 2$.

In case of dimension $r = 2$ it has been proved (see [AS]) that there are five families of smooth surfaces $S \subset G(1, 4)$ that can be smoothly projected to $G(1, 3)$. The idea is essentially as the geometric proof we just gave for Proposition 3.3. If you take a special Schubert variety in the class $\sigma_2 = \Omega(1, 4)$, you have now that it is a cone of degree three. Hence, if $a = [S] \cdot \Omega(1, 4) \neq 3$ you prove as in Proposition 3.3 that each line is contained in at least one bad plane (bad plane meaning again a plane containing at least two lines of S). Since we assumed that S can be projected, the union of all bad planes cannot fill the whole \mathbf{P}^4 , so it is a threefold. Hence there cannot be a two-dimensional family of bad planes (since a threefold with a two-dimensional family of planes is a \mathbf{P}^3 ; this is proved again as in Proposition 3.1, showing that any two points can be joined by a line contained in the threefold). Therefore there is a one-dimensional family of planes each of one containing infinitely many lines of S .

You can finish immediately by looking at the classification of smooth surfaces in $G(1, 3)$ with order $a = 3$ (see [G2]) and with such a family of planes (see [AG]) and checking which ones are projected from $G(1, 4)$. If you do not want to use such strong classification results, you can look at the original proof in [AS], which was done before [AG] was finished. Also you can avoid the use of [G2] by studying the possible cubic hypersurfaces in \mathbf{P}^4 with a two-dimensional family of lines that can be projected to a smooth family of lines in \mathbf{P}^3 .

We have seen in Examples 2.13, 2.14 and 2.16 three of the five families of surfaces in $G(1, 3)$ that are projected from higher Grassmannians. It is worth to recall that the Veronese surface of 2.13 is in fact projected from $G(1, 5)$. This can be put in a more general context:

Proposition 3.4. *For any positive integer r , the image of \mathbf{P}^r in $G(1, 2r + 1)$ by the sheaf $\mathcal{O}_{\mathbf{P}^r}(1) \oplus \mathcal{O}_{\mathbf{P}^r}(1)$ can be smoothly projected to $G(1, r + 1)$.*

Proof: We will just use coordinates. The embedding of \mathbf{P}^r in $G(1, 2r + 1)$ can be given

by associating to each $(t_0 : \dots : t_r) \in \mathbf{P}^r$ the Plücker matrix

$$\begin{pmatrix} t_0 & \dots & t_r & 0 & \dots & 0 \\ 0 & \dots & 0 & t_0 & \dots & t_r \end{pmatrix}$$

We take the projection from \mathbf{P}^{2r+1} to \mathbf{P}^{r+1} defined by

$$(x_0 : \dots : x_{2r+1}) \mapsto (x_0 : x_1 + x_{r+1} : \dots : x_r + x_{2r} : x_{2r+1})$$

Then the map from \mathbf{P}^r to $G(1, r+1)$ is now given by associating to the point $(t_0 : \dots : t_r)$ the Plücker matrix

$$\begin{pmatrix} t_0 & t_1 & \dots & t_r & 0 \\ 0 & t_0 & \dots & t_{r-1} & t_r \end{pmatrix}$$

This is still an embedding, since it is an embedding in $\mathbf{P}^{\binom{r+2}{2}-1}$. Indeed just observe that the minors of the last matrix form a basis of the space of homogeneous polynomials of degree two (to see this, check by induction on i that all monomials of the form $t_i t_j$ are generated by those minors). Hence the composed map $\mathbf{P}^r \rightarrow G(1, r+1) \hookrightarrow \mathbf{P}^{\binom{r+2}{2}-1}$ is the double Veronese embedding of \mathbf{P}^r . This proves that the first map is also an embedding, as wanted.

Question 3.5: Are these all possible subvarieties of $G(1, r+1)$ that can be projected from $G(1, 2r+1)$?

As we have seen, the answer for $r = 1, 2$ is yes and I conjecture it is still so for any r .

Projective linear normality in Grassmannians.

There is also another question about projection you can think of (an in fact related to the previous one, as we will see in the next subsection). Take a smooth $X \subset G(k, n)$ of small codimension (for example not bigger than its dimension). You can also see X as a subvariety of \mathbf{P}^N (where $N = \binom{n+1}{k+1} - 1$) by using the Plücker embedding. So the question is to know when X can be projected from a higher projective space. In the case of curves, the natural question would be to see which smooth curves $C \subset G(1, 2) \cong \check{\mathbf{P}}^2$ are projected from a higher projective space. We have already checked before that none of them. For surfaces, one has the following

Problem 3.6: Classify all smooth surfaces in $G(1, 3) \subset \mathbf{P}^5$ that are projected from at least a \mathbf{P}^6 .

Three examples are known: they have bidegrees $(3, 3)$ (Example 2.16), $(4, 4)$ (Example 2.17) and $(5, 5)$. While the two first are relatively easy to construct, the last one requires an elaborated technique (see [G3]). It is also worth to remark that in the list of possible smooth congruences of degree twelve there is another candidate of bidegree $(6, 6)$. Since

$G(1, 3)$ can be viewed as a smooth quadric in \mathbf{P}^5 , we can think of the analogous problem for curves, which would be to find all smooth curves on a smooth quadric of \mathbf{P}^3 (which we will view as the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$) that are not linearly normal. So let $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ be a smooth curve of bidegree (a, b) (of course here the word “bidegree” has nothing to see with Grassmannians: a and b are the intersection numbers of C with the two rulings of the quadric). Consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1 - a, 1 - b) \rightarrow \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 1) \rightarrow \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 1)|_C \rightarrow 0.$$

and its associated map $H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1 - a, 1 - b)) \rightarrow H^0(C, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1, 1)|_C)$. We just need to know when this map is not surjective. By taking cohomology in the exact sequence, this is equivalent to know when $H^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1 - a, 1 - b))$ is not zero. We see that this is not zero if and only if either $a = 1$ and $b \geq 3$ or $b = 1$ and $a \geq 1$ (we are assuming $a, b > 0$). Hence we have that there are infinitely many such curves. This can give some evidence that also the answer to Problem 3.6 could consist of infinitely families of surfaces.

The natural framework to study Problem 3.6 is just to take a smooth surface $S \subset \mathbf{P}^6$ that is contained in a hyperquadric with one singular point in such a way that no bisecant of S passes through the singular point. Unfortunately, this just provides a numerical condition (which at the end is just the double-point formula) instead of a dimensional condition. This is what makes the problem difficult.

One can also change the statement of the problem. As stated, we are just looking for smooth surfaces in \mathbf{P}^6 that have a (probably special) projection to \mathbf{P}^5 such that the image lies on a smooth quadric hypersurface. Instead of this, one can be interested on smooth surfaces in \mathbf{P}^6 such that their general projection to \mathbf{P}^5 lies on a smooth quadric hypersurface. Then the answer is given by the following:

Proposition 3.7. *If the general projection of a smooth non-degenerate surface $S \in \mathbf{P}^6$ lies in a smooth quadric, then S is the Del Pezzo surface of degree six (Example 2.16).*

Proof: From our hypothesis, a general point of \mathbf{P}^6 is the vertex of a singular quadric containing S . Hence there is a 6-dimensional family of singular quadrics containing S . Since the general quadric containing S must be smooth outside S (by Bertini’s theorem), it follows that the linear system of quadrics containing S has in fact (projective) dimension at least 7. In cohomological terms, $h^0(\mathbf{P}^6, \mathcal{I}_{S, \mathbf{P}^6}(2)) \geq 8$. We will just use now the standard Castelnuovo’s technique (the one used to prove the genus bound for curves) to conclude. So consider two consecutive hyperplane sections: the intersection C of S with a general \mathbf{P}^5 and the intersection Z of C with a general \mathbf{P}^4 . By using that S and C are non-degenerate

and taking cohomology in the exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{J}_{S, \mathbf{P}^6}(1) \rightarrow \mathcal{J}_{S, \mathbf{P}^6}(2) \rightarrow \mathcal{J}_{C, \mathbf{P}^5}(2) \rightarrow 0 \\ 0 &\rightarrow \mathcal{J}_{C, \mathbf{P}^5}(1) \rightarrow \mathcal{J}_{C, \mathbf{P}^5}(2) \rightarrow \mathcal{J}_{Z, \mathbf{P}^4}(2) \rightarrow 0 \end{aligned}$$

we still get that $h^0(\mathbf{P}^4, \mathcal{J}_{Z, \mathbf{P}^4}(2)) \geq 8$. Assume $d \geq 9$ and take $Z' = \{p_1, \dots, p_9\}$ to be a subset of nine points of Z . Then the map

$$f : H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2)) \rightarrow H^0(Z', \mathcal{O}_{Z'}) = k(p_1) \oplus \dots \oplus k(p_9)$$

is surjective. Indeed since the points are in general position (we took general hyperplane sections) we see that for each point p_i we can take two hyperplanes whose union contains the other eight points but not p_i (just take them so that each of them contains four of the remaining points). Then the image by f of the product of the equations of these hyperplanes has all coordinates in $k(p_1) \oplus \dots \oplus k(p_9)$ equal to zero, except the one corresponding to $k(p_i)$. This proves the surjectivity of f , and implies that the map $H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2)) \rightarrow H^0(Z, \mathcal{O}_Z)$ has rank at least nine. So its kernel $H^0(\mathbf{P}^4, \mathcal{J}_{Z, \mathbf{P}^4}(2))$ has dimension at most $\binom{6}{2} - 9 = 6$, which is a contradiction. Hence it follows that $d \leq 8$, and looking at the list of smooth congruences of degree at most 8 (see for example [AS]) we conclude our result.

Remark 3.8: If $S \subset \mathbf{P}^6$ is contained in a 7-dimensional family of quadrics, then through a general point $p \in \mathbf{P}^6$ there passes a singular quadric of the family with singularity at p . But in principle this quadric could have more singularities and in this case the general projection into \mathbf{P}^5 does not necessarily lie in a smooth quadric. For instance, blow up \mathbf{P}^2 in six general points and embed it in \mathbf{P}^6 by the system of quartics passing doubly through one of the points and simply through the rest. This is a surface of degree $d = 7$ and invariants $\pi = 2$, $K_S^2 = 3$ and $\chi(\mathcal{O}_S) = 1$. It is also easy to check that there is a 7-dimensional system of quadrics of \mathbf{P}^6 containing the surface, but its projection into \mathbf{P}^5 is never contained in a smooth quadric (for example, its invariants do not satisfy the double point formula (2.2) for any possible bidegree). In fact, it can be shown (using Macaulay program for example) that the quadric containing the general projection of S into \mathbf{P}^5 is singular along a line. This proves that in the 7-dimensional family of quadrics containing the surface S , the one having a singularity at a general point of \mathbf{P}^6 is singular along a whole plane. I thank Ciro Ciliberto for pointing this example to me.

Relation between projective and Grassmannian linear normality.

As we have seen in Examples 2.13, 2.14, 2.16 and 2.17, there is in principle no relation between a surface in $G(k, n) \subset \mathbf{P}^{\binom{n+1}{k+1}-1}$ to be projected from $G(k, n+1)$ and to be projected from $\mathbf{P}^{\binom{n+1}{k+1}}$. However both things are not completely independent. Indeed take

a point $p \in \mathbf{P}^{n+1}$ and project from it $G(k, n+1)$ to $G(k, n)$. Then the set of k -planes containing p is naturally isomorphic to $G(k-1, n)$ (just take a hyperplane H not containing p , and map each k -plane Λ containing p to $\Lambda \cap H \in G(k-1, H)$). This $G(k-1, n)$ spans an $\binom{n+1}{k} - 1$ -dimensional linear space M inside the $\binom{n+2}{k+1} - 1$ -dimensional space in which $G(k, n+1)$ lies after the Plücker embedding. Then the projection from $G(k, n+1)$ to $G(k, n)$ from the point p is the restriction of the projection from $\mathbf{P}^{\binom{n+2}{k+1}-1}$ to $\mathbf{P}^{\binom{n+1}{k+1}-1}$ from the subspace M . Of course what can happen is for instance that a non-degenerate variety in $G(k, n+1)$ is already very degenerate upstairs as a projective variety.

To see the above description in coordinates, choose them in such a way that p is the point of coordinates $(0 : \dots : 0 : 1)$. Then H is the linear space defined by all Plücker coordinates p_{i_0, \dots, i_k} with $0 \leq i_0 < \dots < i_k < n+1$ equal to zero. Hence the projection from H just consists of forgetting all Plücker coordinates of the form $p_{i_0, \dots, i_{k-1}, n+1}$.

In case $k = 1$ the situation is more clear, since the set of all lines passing through a point forms already an n -dimensional linear space contained in $G(1, n+1)$. In fact in this case we have the following relation among both kind of projections:

Proposition 3.9. *Let X be a subvariety of $G = G(1, n) \subset \mathbf{P}^{\binom{n+1}{2}-1}$ that is a projection from a subvariety in $G(1, n+1)$. Then either X is projected from $\mathbf{P}^{\binom{n+1}{2}}$ or it is contained in the zero locus of a section of $\mathcal{S}(1)$.*

Proof: By using the exact sequences

$$0 \rightarrow \mathcal{J}_{X,G}(1) \rightarrow \mathcal{O}_G(1) \rightarrow \mathcal{O}_X(1) \rightarrow 0$$

$$0 \rightarrow \mathcal{Q} \otimes \mathcal{J}_{X,G} \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}|_X \rightarrow 0$$

and the fact that $h^1(G, \mathcal{O}_G(1)) = h^1(G, \mathcal{Q}) = 0$, we see that X being projected from $\mathbf{P}^{\binom{n+1}{2}}$ is equivalent to $h^1(G, \mathcal{J}_{X,G}(1)) \neq 0$, and that X being projected from $G(1, n+1)$ is equivalent to $h^1(G, \mathcal{Q} \otimes \mathcal{J}_{X,G}) \neq 0$. We now dualize and tensor with $\mathcal{J}_{X,G}(1)$ the universal sequence (0.7) and get an exact sequence

$$0 \rightarrow \check{\mathcal{Q}} \otimes \mathcal{J}_{X,G}(1) \rightarrow V^* \otimes \mathcal{J}_{X,G}(1) \rightarrow \mathcal{S} \otimes \mathcal{J}_{X,G}(1) \rightarrow 0$$

Since \mathcal{Q} has rank two and determinant $\mathcal{O}_G(1)$, it follows that $\check{\mathcal{Q}}(1) \cong \mathcal{Q}$. So now taking cohomology in the above exact sequence, we get that $h^1(G, \mathcal{Q} \otimes \mathcal{J}) \neq 0$ implies that either $h^1(G, \mathcal{J}_{X,G}(1)) \neq 0$ or $h^0(\mathcal{S} \otimes \mathcal{J}_{X,G}(1)) \neq 0$. This proves the result.

Remark 3.10: In particular Proposition 3.9 says that in the case of $G(1, 3)$, an answer to Problem 3.6 would give an immediate classification of smooth congruences projected from $G(1, 4)$. Indeed the proposition says that those not projected from \mathbf{P}^6 are contained in the zero locus of a section of $\mathcal{S}(1)$. If this zero locus has the expected dimension two,

then it is a congruence of bidegree $(3, 2)$ (by looking at the second Chern class of $\mathcal{S}(1)$). If it has dimension one, it can only be a hyperplane, but smooth congruences contained in a \mathbf{P}^4 are very well-known and classified.

Distribution of bidegrees for congruences.

Let S be a smooth surface in $G(1, 3)$ of bidegree (a, b) . By duality, the roles of a and b are similar, so we will discuss for instance a bound of a in terms of b . Looking at examples, one realizes that the distance between a and b is not quite big. In fact, the biggest known difference is achieved by taking the degeneracy locus of n general sections of the symmetric power $\text{Sym}^n \mathcal{S}$ of the universal subbundle. These have bidegree

$$\left(\frac{1}{24}(n-1)n(n+1)(3n+2) + \frac{1}{6}n(n+1)(n+2), \frac{1}{24}(n-1)n(n+1)(3n+2) \right).$$

Note that if n is big, the ratio a/b tends to 1. However, the best known bound is of the type $a \leq O(b^{4/3})$ (see [G1]). Note that in our example it holds $a = b + O(b^{3/4})$. On the other hand we have the following:

Conjecture 3.11. (*Dolgachev-Reider*) *If S is not contained in a special linear complex then the restriction of the universal bundle \mathcal{Q} to S is semistable.*

The importance of this conjecture is that, by a theorem of Bogomolov, semistability implies $c_1^2 \leq 4c_2$. Applied to $\mathcal{Q}|_S$ and assuming it is semistable, this reads $a + b \leq 4b$, which yields $a \leq 3b$. It should be remarked that a smooth surface contained in a linear complex is known to verify $|a - b| \leq 1$. The interested reader can find any known result about this problem in [G1].

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