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Department of Mathematics
University of Nebraska at Omaha
Omaha, NE 68182-0243
USA
johnkon@unomaha.edu

Another Elementary Proof of the Nullstellensatz

Enrique Arrondo

In [**1**], May reproduced an elegant and elementary proof of the Nullstellensatz provided by Munshi in [**2**]. We offer here an alternative elementary proof, in which we avoid some of the algebraic technicalities needed in [**1**] and [**2**]. As a counterpart, we need a simple version of the Noether normalization lemma (Lemma 1). On the other hand, our proof requires the resultant of two polynomials, but in such a simple way that only one property of it is needed (and we include it within the proof).

We restrict ourselves to the weak form of the Nullstellensatz (Theorem 2), since the strong form is easily derived from the weak one with the aid of the Rabinowitsch trick. For this and other historical comments we refer to [**1**], from which we have tried to preserve the notation. We also assume the elementary background given there.

The geometric idea behind the proof is very simple. We prove that the zero locus of an ideal is not empty by doing induction on the dimension of the ambient affine space. To do this we project to a smaller affine space (this is why we need to use a resultant). But a projection can miss some points, so we need to put the zero locus in good position before projecting (this is why we need Lemma 1). Surprisingly, by combining these two things in a suitable way we obtain a complete proof.

We proceed with the proof, starting with this simple (and standard) version of Noether's normalization lemma.

Lemma 1. *If F is an infinite field and f is a nonconstant polynomial in $F[x_1, \dots, x_n]$ with $n \geq 2$, then it is possible to find $\lambda_1, \dots, \lambda_{n-1}$ in F such that in*

$$f(x_1 + \lambda_1 x_n, \dots, x_{n-1} + \lambda_{n-1} x_n, x_n)$$

the coefficient of x_n^d (where d is the total degree of f) is nonzero.

Proof. If f_d is the homogeneous component of f of degree d, then the coefficient of x_n^d in $f(x_1 + \lambda_1 x_n, \dots, x_{n-1} + \lambda_{n-1} x_n, x_n)$ is $f_d(\lambda_1, \dots, \lambda_{n-1}, 1)$. Since

$f_d(x_1, \ldots, x_{n-1}, 1)$ is a nonzero polynomial in $F[x_1, \ldots, x_{n-1}]$ and F is infinite, there is some point $(\lambda_1, \ldots, \lambda_{n-1})$ of F^{n-1} at which it does not vanish (this can be established in a straightforward way by induction on the number of variables). This proves the lemma. ■

Theorem 2. *Let I be a proper ideal of $F[x_1, \ldots, x_n]$. If F is algebraically closed, then there exists $(a_1, \ldots, a_n)$ in F^n such that $f(a_1, \ldots, a_n) = 0$ for all f in I.*

Proof. Let us assume $I \neq 0$, since otherwise the result is trivial. We prove the theorem by induction on n. The case $n = 1$ is immediate, because any nonzero proper ideal I of $F[x]$ is generated by a nonconstant polynomial. A generator of I necessarily has some root a in F, for F is algebraically closed. Therefore, $f(a) = 0$ for all f in I.

We assume now that $n > 1$. Lemma 1 allows us, modulo a change of coordinates and scaling, to suppose that I contains a polynomial g monic in the variable x_n. Fixing such a polynomial g, we consider the ideal I' of $F[x_1, \ldots, x_{n-1}]$ consisting of those polynomials in I that do not contain the undeterminate x_n. Since 1 is not in I, it follows that I' is a proper ideal. Therefore, by the induction hypothesis there is a point $(a_1, \ldots, a_{n-1})$ at which all the polynomials of I' vanish. We now assert the following:

Claim. The set $J = \{f(a_1, \ldots, a_{n-1}, x_n) : f \in I\}$ is a proper ideal of $F[x_n]$.

Suppose to the contrary that there exists f in I such that $f(a_1, \ldots, a_{n-1}, x_n) = 1$. Thus we can write $f = f_0 + f_1 x_n + \cdots + f_d x_n^d$, with all the f_i in $F[x_1, \ldots, x_{n-1}]$, $f_1(a_1, \ldots, a_{n-1}) = \cdots = f_d(a_1, \ldots, a_{n-1}) = 0$, and $f_0(a_1, \ldots, a_{n-1}) = 1$. On the other hand, we can express the monic polynomial g in the form $g = g_0 + g_1 x_n + \cdots + g_{e-1} x_n^{e-1} + x_n^e$ with g_j in $F[x_1, \ldots, x_{n-1}]$ for $j = 0, \ldots, e - 1$.

Let R be the resultant of f and g with respect to the variable x_n. In other words, R is the polynomial in $F[x_1, \ldots, x_{n-1}]$ given by the determinant

$$R = \begin{vmatrix} f_0 & f_1 & \ldots & f_d & 0 & 0 & \ldots & 0 \\ 0 & f_0 & \ldots & f_{d-1} & f_d & 0 & \ldots & 0 \\ & & \ddots & & & & & \\ 0 & \ldots & 0 & f_0 & f_1 & \ldots & f_{d-1} & f_d \\ g_0 & g_1 & \ldots & g_{e-1} & 1 & 0 & \ldots & 0 \\ 0 & g_0 & \ldots & g_{e-2} & g_{e-1} & 1 & 0\ldots & 0 \\ & & \ddots & & & & \ddots & \\ 0 & \ldots & 0 & g_0 & g_1 & \ldots & g_{e-1} & 1 \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} e \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} d \text{ rows} \end{matrix}$$

It is then well known that R belongs to I. (In the determinant defining R, add to the first column the second one multiplied by x_n, then the third column multiplied by x_n^2, and so on until one adds the last column multiplied by x_n^{d+e-1}. Expanding the resulting determinant along the first column reveals that R is a linear combination of f and g.) Therefore R is a member of I'. But direct inspection of the determinant defining the resultant shows that, when evaluated at $(a_1, \ldots, a_{n-1})$, it reduces to the determinant of a lower-triangular matrix whose entries on its main diagonal are all 1s. Hence $R(a_1, \ldots, a_{n-1}) = 1$, which contradicts the fact that R is in I'. This proves the claim.

Therefore J is a proper ideal of $F[x_n]$, hence is generated either by a polynomial $h(x_n)$ of positive degree or by $h = 0$. Since F is algebraically closed, in the former case h has at least one root a_n in F. In either case this means that $f(a_1, \ldots, a_{n-1}, a_n) = 0$ for all f in I, which completes the proof. ■

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Departamento de Álgebra, Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain
Enrique_Arrondo@mat.ucm.es

The Sphere Is Not Flat

P. L. Robinson

The circumstance announced in our title explains why it is impossible to construct (plane) maps of the (ideal, spherical) Earth in which distances are faithfully represented. More technically, each sphere $\mathbf{S}$ carries a metric under which the distance between points P and Q of $\mathbf{S}$ is the length of the (shorter) great circular arc PQ. The theorem of our title asserts that there is no isometric (that is, distance-preserving) function from $\mathbf{S}$ (or indeed from any of its nonempty open subsets) to the Euclidean plane; more generally, there is no isometry to any Euclidean space. This theorem may be traced back to Euler, in his *De repraesentatione superficiei sphaericae super plano* of 1778. A relatively sophisticated proof of this result involves regarding $\mathbf{S}$ as a Riemannian manifold and verifying that its curvature does not vanish; a more elementary proof compares areas of plane triangles and spherical triangles. Here we offer a proof that is more fundamental (in referring only to distances) and is still elementary (in using only trigonometry and calculus).

We begin our proof by considering the problem of isometrically embedding *finite* metric spaces in Euclidean spaces. It is immediately evident that any two-point metric space embeds isometrically in the real line with its standard Euclidean metric. It is slightly less evident (but becomes clear upon inspecting the triangle inequality) that any three-point metric space embeds isometrically in the Euclidean plane. In view of this geometrically rather seductive progression, it is perhaps surprising that there exist four-point metric spaces that admit no isometric embeddings into Euclidean $\mathbf{R}^3$ or indeed into Euclidean space of any dimension.

To construct a simple example of such a space, we start from a three-point space $Y = \{n, p, q\}$ with metric given by $d(n, p) = d(n, q) = d(p, q) = 2L > 0$; the image of Y under any isometric embedding in a Euclidean space comprises the vertices of an equilateral triangle with $2L$ as side. Now add to Y a point t so as to obtain $X = \{n, p, q, t\}$ and extend the metric d to X by declaring that $d(p, t) = d(q, t) = L$ but leaving $d(n, t)$ temporarily unspecified. Any isometric embedding f from X into a Euclidean space necessarily maps t to the midpoint T of the line joining $P = f(p)$ and $Q = f(q)$. As the Euclidean distance between $N = f(n)$ and T is $\sqrt{3}L$, it follows that the isometric nature of f forces $d(n, t) = \sqrt{3}L$ as well. Accordingly, in order to ensure that X admits no isometric embedding into any Euclidean space we need arrange only that $d(n, t)$ have a different value, for example, as follows:

Theorem 1. *The four-point metric space $X = \{n, p, q, t\}$ with*

$$d(n, t) = d(n, p) = d(n, q) = d(p, q) = 2L,\ d(p, t) = d(q, t) = L$$

admits no isometric embedding into any Euclidean space.