Elliptic PDE's, Stochastic Processes & Dynamic Programming Equations

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☞ $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$ denotes a bounded open domain.

The **Laplacian** of $u \in C^2(\Omega)$:

$$\Delta u = \mathrm{Tr}\{D^2 u\} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

A function $u \in C^2(\Omega)$ is **harmonic** in Ω if $\Delta u = 0$.

u is harmonic in Ω $\iff$ u satisfies the **mean value property** in Ω:

$$u(x) = \fint_{B_r(x)} u(z)\, dz \qquad \text{for every } B_r(x) \Subset \Omega.$$

– The asymptotic mean value property

Let $u \in C^2(\Omega)$ and fix $x \in \Omega$. Average with respect to $z \in B_1$ the Taylor's expansion of u:

$$u(x + \varepsilon z) = u(x) + \varepsilon \langle \nabla u(x), z \rangle + \frac{\varepsilon^2}{2} \operatorname{Tr}\{D^2 u(x) \cdot z \otimes z\} + o(\varepsilon^2)$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\fint_{B_\varepsilon(x)} u(y)\, dy = u(x) + \quad 0 \quad + \quad \frac{\varepsilon^2}{2} \cdot \frac{\Delta u(x)}{n+2} \quad + o(\varepsilon^2)$$

Blaschke, Privaloff, Zaremba (1900's)

Let $u \in C(\Omega)$. Then

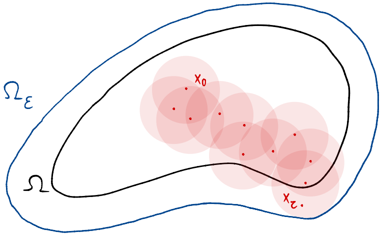
$$u \text{ is harmonic in } \Omega \quad \Longleftrightarrow \quad \fint_{B_\varepsilon(x)} u(y)\, dy = u(x) + o(\varepsilon^2), \quad x \in \Omega.$$

The **Random Walk** is a stochastic process describing the location of a particle jumping from one position to another inside Ω.

– $\{x_0, x_1, x_2, \dots\} \subset \Omega$ describes the location of a particle at each time $j \geqslant 0$.

– $x_{j+1} \in \Omega$ is randomly chosen in $B_\varepsilon(x_j)$ for each $j \geqslant 0$.

– The process stops the first time the particle exits Ω at some $x_\tau \notin \Omega$.



– Then the amount $F(x_\tau)$ is collected, where F is a pay-off function defined outside Ω;

$$F \in C(\Omega_\varepsilon \setminus \Omega),$$

where Ω_ε is the ε-extension of Ω,

$$\Omega_\varepsilon := \{x \in \mathbb{R}^n \ : \ \mathrm{dist}(x, \Omega) < \varepsilon\}.$$

Let $u_\varepsilon(x_0)$ be the expected pay-off of a random walk starting from $x_0 \in \Omega$, i.e.

$$u_\varepsilon(x_0) := \mathbb{E}\big[F(x_\tau)\,|\,x_0\big].$$

Then, $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ and, by conditional probability, u_ε satisfies

$$\boxed{u_\varepsilon(x) = \begin{cases} \displaystyle\fint_{B_\varepsilon(x)} u_\varepsilon(y)\, dy & \text{if } x \in \Omega, \\ F(x) & \text{if } x \in \Omega_\varepsilon \setminus \Omega. \end{cases}}$$

☞ u_ε satisfies the mean value property for each ball $B_\varepsilon(x) \subset \Omega_\varepsilon$.

☞ This is the so-called **Dynamic Programming Equation (DPE)**.

If $u \in C^2(\Omega)$ and $\nabla u \neq 0$, then $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

$$\boxed{\mathcal{T}_\varepsilon u(x) := \frac{p-2}{n+p} \cdot \frac{1}{2}\left\{ \sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \right\} + \frac{n+2}{n+p} \fint_{B_\varepsilon(x)} u(y)\, dy}$$

Manfredi-Parviainen-Rossi, 2010

Let $u \in C^2(\Omega)$ and $x \in \Omega$ such that $\nabla u(x) \neq 0$. Then $\Delta_p u(x) = 0$ if and only if

$$\mathcal{T}_\varepsilon u(x) = u(x) + o(\varepsilon^2).$$

There is a stochastic interpretation of the p-Laplacian in terms of **Tug-of-war games with noise**, whose corresponding value functions are the solutions of the DPE

$$u_\varepsilon = \begin{cases} \mathcal{T}_\varepsilon u_\varepsilon & \text{in } \Omega, \\ F & \text{in } \Omega_\varepsilon \setminus \Omega. \end{cases}$$

Given $0 < \lambda \leqslant \Lambda < \infty$, let us consider the PDE in non-divergence form

$$Lu(x) = \mathrm{Tr}\{A(x) \cdot D^2 u(x)\} = 0$$

where the coefficients $A(\cdot) = \{a_{ij}\}_{ij}$ are (measurable, bounded) symmetric and uniformly elliptic, i.e.

$$\lambda|\xi|^2 \leqslant \langle A(x) \cdot \xi, \xi \rangle \leqslant \Lambda|\xi|^2 \qquad (\xi \in \mathbb{R}^n,\ x \in \Omega)$$

Krylov-Safonov (1979)

Solutions of $Lu = 0$ are $C^{0,\alpha}$ for some $\alpha \in (0, 1]$.

☞ Is there a DPE related to this PDE?

In what follows we consider a more general class of Dynamic Programming Equations that take the form

$$\int_{B_1} u(x+\varepsilon z)\, d\mu_x(z) = u(x), \qquad x \in \Omega.$$

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Let μ be a Radon measure supported in B_1 satisfying:

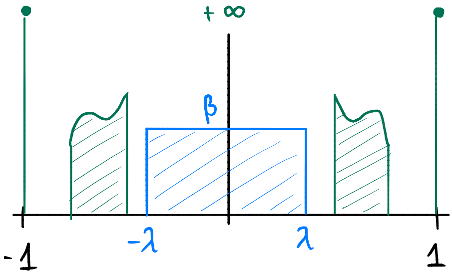
1. *Unit measure*: $\mu(B_1) = 1$;
2. *Symmetry*: $\mu(E) = \mu(-E)$ for every $E \subset \mathbb{R}^n$ measurable;
3. *Ellipticity*: there exist $\beta, \lambda \in (0,1]$ and ν a unit Radon measure supported in B_1 such that

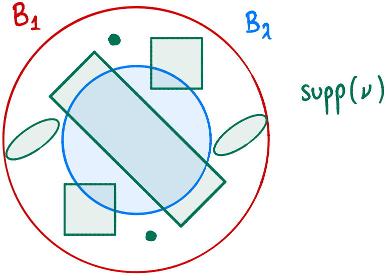
$$\mu(E) = \beta \frac{|E \cap B_\lambda|}{|B_\lambda|} + (1-\beta)\nu(E) \qquad \forall\, E \subset \mathbb{R}^n \text{ meas.}$$

☞ We denote by $\mathcal{M}_{\beta,\lambda}$ the set of all measures μ satisfying these conditions.

– Uniformly elliptic, bounded and measurable increments

$$\mu(E) = \beta \frac{|E \cap B_\lambda|}{|B_\lambda|} + (1-\beta)\nu(E)$$





$$\boxed{\int_{B_1} u(x+\varepsilon z)\, d\mu_x(z) = u(x), \qquad x \in \Omega}$$

☞ In addition, we assume that the choice of measures $x \longmapsto \mu_x$ is:

1. *Uniformly elliptic*: $\mu_x \in \mathcal{M}_{\beta,\lambda}$ for each $x \in \Omega$;
2. *Bounded and measurable*: the map

$$x \longmapsto \int_{B_1} w\, d\mu_x$$

defines a Borel measurable bounded function $\Omega \to \mathbb{R}$ for every $w : B_1 \to \mathbb{R}$ Borel measurable.

Indeed, we consider non-homogeneous dynamic programming equations:

$$\boxed{\int_{B_1} u(x+\varepsilon z)\, d\mu_x(z) = u(x) + \varepsilon^2 f(x), \qquad x \in \Omega}$$

where f is a Borel measurable bounded function.

For convenience, we rewrite the DPE as

$$\boxed{\mathcal{L}_\varepsilon u(x) := \int_{B_1} \frac{u(x+\varepsilon z) - u(x)}{\varepsilon^2}\, d\mu_x(z) = f(x)}$$

$$\Updownarrow$$

$$\boxed{\mathcal{L}_\varepsilon u(x) := \int_{B_1} \frac{u(x+\varepsilon z) + u(x-\varepsilon z) - 2u(x)}{2\varepsilon^2}\, d\mu_x(z) = f(x)}$$

Let $u \in C^2(\Omega)$. Then:

$$u(x+\varepsilon z) = u(x) + \varepsilon \langle \nabla u(x), z \rangle + \frac{\varepsilon^2}{2} \operatorname{Tr}\{D^2 u(x) \cdot z \otimes z\} + o(\varepsilon^2)$$

Integrating with respect μ_x we get

$$\begin{aligned}
\mathcal{L}_\varepsilon u(x) &= \int_{B_1} \frac{1}{2} \operatorname{Tr}\{D^2 u(x) \cdot z \otimes z\}\, d\mu_x(z) + o(\varepsilon^0) \\
&= \operatorname{Tr}\left\{ D^2 u(x) \cdot \underbrace{\left(\frac{1}{2} \int_{B_1} z \otimes z \, d\mu_x(z) \right)}_{=A(x)} \right\} + o(\varepsilon^0) \\
&= Lu(x) + o(\varepsilon^0)
\end{aligned}$$

Let $u \in C^2(\Omega)$. Then:

$$Lu(x) = 0 \qquad \Longleftrightarrow \qquad \mathcal{L}_\varepsilon u(x) = u(x) + o(\varepsilon^2)$$

☞ For $A(x) = \dfrac{1}{2}\displaystyle\int_{B_1} z \otimes z \; d\mu_x(z)$, then

$$\langle A(x) \cdot \xi, \xi \rangle = \frac{|\xi|^2}{2} \int_{B_1} \langle \frac{\xi}{|\xi|}, z \rangle^2 \; d\mu_x(z).$$

It is easy to estimate

$$\frac{\beta\lambda^2}{2(n+2)} |\xi|^2 \leqslant \langle A(x) \cdot \xi, \xi \rangle \leqslant \frac{1}{2} |\xi|^2 ,$$

so $A(\cdot)$ is uniformly elliptic in Ω.

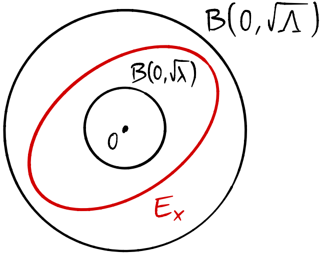
☞ Each positive definite matrix $A(x)$ defines an ellipsoid.

Assuming that $\det\{A(x)\} = 1$ for each $x \in \Omega$ we can define

$$E_x = \sqrt{A(x)}\, B_1 = \{y \in \mathbb{R}^n \,:\, \langle A(x)^{-1} \cdot y, y\rangle < 1\}$$

The uniform ellipticity of $A(\cdot)$ yields that

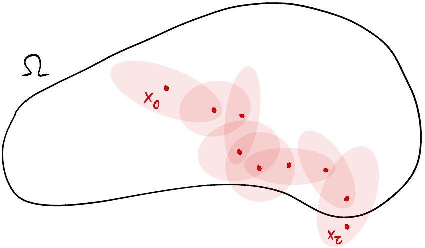
$$B_{\sqrt{\lambda}} \subset E_x \subset B_{\sqrt{\Lambda}}, \qquad x \in \Omega.$$



If $u \in C^2(\Omega)$ then

$$\mathrm{Tr}\{D^2u(x) \cdot A(x)\} = 0 \ \text{ in } \Omega \quad \Longleftrightarrow \quad \fint_{x+\varepsilon E_x} u(z)\, dz = u(x) + o(\varepsilon^2).$$

The **Ellipsoid Process** is a generalization of the random walk in which the next step in the process is taken inside a space-dependent ellipsoid.



The expected pay-off function u_ε satisfies the dynamic programming principle

$$u_\varepsilon(x) = \begin{cases} \fint_{x+\varepsilon E_x} u_\varepsilon(z)\, dz & \text{if } x \in \Omega, \\ F(x) & \text{if } x \in \Omega_\varepsilon \setminus \Omega. \end{cases}$$

A.-Blanc-Parviainen (2020 & 2021)

Let $f : \Omega \to \mathbb{R}$ and $g : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ be measurable bounded functions. There exists a unique solution u to the DPE

$$\begin{cases} \mathcal{L}_\varepsilon u = f & \text{in } \Omega, \\ u = g & \text{in } \Omega_\varepsilon \setminus \Omega. \end{cases}$$

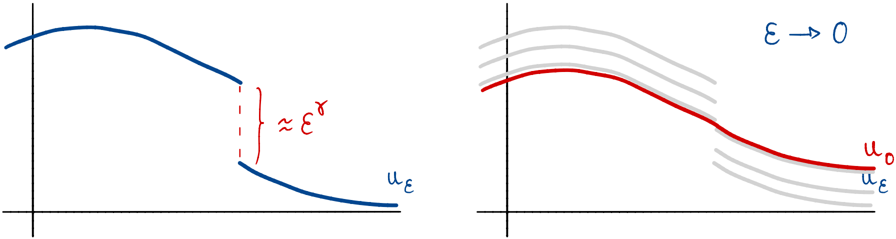
What is the regularity of the solutions of $\mathcal{L}_\varepsilon u = f$?

☞ The main problem is that the solutions of $\mathcal{L}_\varepsilon u = f$ might present discontinuities. This is due to the fact that the averages are take in balls that might exit de domain.

☞ $\{u_\varepsilon \ : \ \varepsilon > 0\}$ is **asymptotically Hölder/Lipschitz continuous** if

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leqslant C\big(|x-y|^\gamma + \varepsilon^\gamma\big),$$

for some $0 < \gamma \leqslant 1$.



$$\Downarrow$$

$$u_0 := \lim_{\varepsilon \to 0} u_\varepsilon \in C^{0,\gamma}$$

The **coupling method** for asymptotic Hölder regularity uses the stochastic interpretation of the DPE for the estimates using probabilistic techniques.

Luiro-Parviainen (2018): Several variants of the tug-of-war game in connection to the p-Laplacian.

Parviainen-Ruosteenoja (2016): Parabolic version.

A.-Heino-Parviainen (2017): Space-dependent version in connection to the **normalized $p(x)$-Laplacian**.

Han (2018): Space-dependent parabolic version.

A.-Luiro-Parviainen-Ruosteenoja (2019): Asymptotic Lipschitz estimate for the space-dependent version with $p(\cdot)$ Hölder continuous.

A.-Parviainen (2020): The coupling method can be adapted for the Ellipsoid Process in the following cases:

1. The assignment of coefficients $x \mapsto A(x)$ is continuous.
2. The assignment of coefficients $x \mapsto A(x)$ is measurable and the *maximum distortion* Λ/λ of the ellipsoids satisfies

$$1 \leqslant \frac{\Lambda}{\lambda} < \frac{n+1}{n-1}.$$

☞ The DPE's described with a choice of measures $x \longmapsto \mu_x$ can be extended to a more general class of equations.

The **maximal Pucci-type operators** is

$$\mathcal{L}_\varepsilon^+ u(x) := \sup_{\mu \in \mathcal{M}_{\beta,\lambda}} \int_{B_1} \frac{u(x+\varepsilon z) - u(x)}{\varepsilon^2} \, d\mu(z),$$

while the **minimal Pucci-type operator** $\mathcal{L}_\varepsilon^- u(x)$ is defined analogously taking the infimum.

This can be *explicitly* computed as

$$\begin{aligned}\mathcal{L}_\varepsilon^+ u(x) = \beta \fint_{B_\lambda} \frac{u(x+\varepsilon z) - u(x)}{\varepsilon^2} \, dz \\ + (1-\beta) \sup_{z \in B_1} \left\{ \frac{u(x+\varepsilon z) + u(x-\varepsilon z) - 2u(x)}{2\varepsilon^2} \right\}.\end{aligned}$$

The extremal operators allow to cover a more general range of DPE's.

☞ For example, if $2 \leqslant p < +\infty$, for $\beta = \frac{n+2}{n+p}$ and $\lambda = 1$ it holds that

$$\mathcal{L}_\varepsilon^- u(x) \leqslant \frac{\mathcal{T}_\varepsilon u(x) - u(x)}{\varepsilon^2} \leqslant \mathcal{L}_\varepsilon^+ u(x),$$

where $\mathcal{T}_\varepsilon u = u$ is the DPE for the p-Laplacian.

Thus, if u is a solution to the DPE

$$\mathcal{T}_\varepsilon u(x) = u(x) + \varepsilon^2 f(x),$$

then

$$\mathcal{L}_\varepsilon^- u \leqslant f \leqslant \mathcal{L}_\varepsilon^+ u.$$

A.-Blanc-Parviainen (2020 & 2021)

There exists $\varepsilon_0 > 0$ such that if u satisfies

$$\mathcal{L}_\varepsilon^+ u \geqslant -|f| \qquad \text{and} \qquad \mathcal{L}_\varepsilon^- u \leqslant |f| \qquad \text{in } B_2$$

with $\varepsilon \in (0, \varepsilon_0)$, then

$$|u(x) - u(y)| \leqslant C\left(\|u\|_\infty + \|f\|_\infty\right)\left(|x-y|^\gamma + \varepsilon^\gamma\right), \qquad x, y \in B_1,$$

for some $C > 0$ and $\gamma \in (0, 1]$ independent of ε.

If in addition $u \geqslant 0$, the following **asymptotic Harnack inequality** holds,

$$\sup_{B_1} u \leqslant C\left(\inf_{B_1} u + \|f\|_\infty + \varepsilon^\sigma \|u\|_\infty\right).$$

☞ There are two different proofs of the asymptotic Hölder estimate:

1. A stochastic proof using probabilistic techniques.
2. A entirely analytic proof.

Both of them are based in two main ingredients:

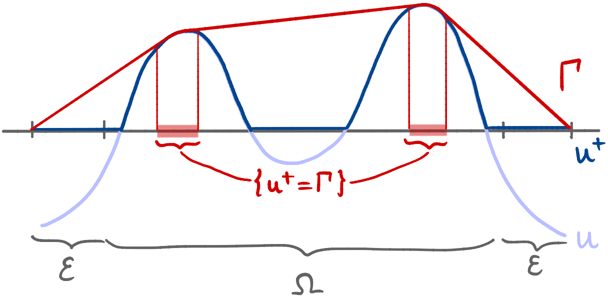
1. A discrete version of the Aleksandrov-Bakelman-Pucci (ABP) estimate.
2. A truncated Calderón-Zygmund dyadic cube decomposition.

Aleksandrov-Bakelman-Pucci (ABP) estimate: if $f \in C_b(\Omega)$ and $u \in C(\overline{\Omega})$ satisfies

$$\mathrm{Tr}\{A(x) \cdot D^2 u(x)\} + f(x) \geqslant 0, \qquad x \in \Omega,$$

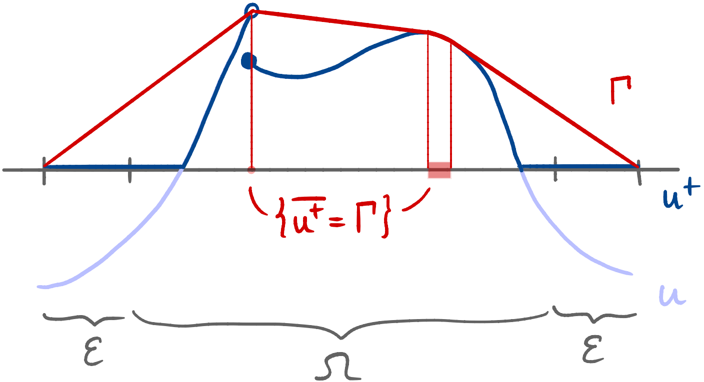
then there exists $C = C(n, \operatorname{diam}\Omega, \lambda, \Lambda)$ such that

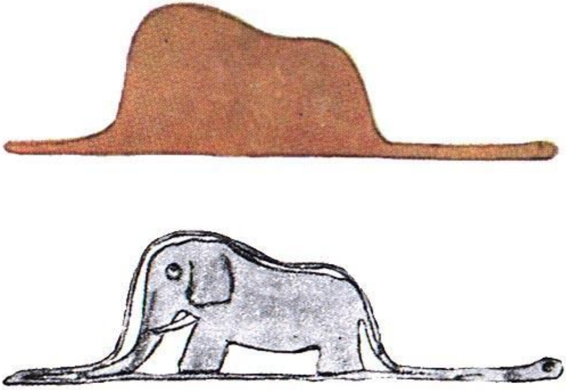
$$\sup_{\Omega} u \leqslant \sup_{\partial\Omega} u + C \operatorname{diam}\Omega \left(\int_{\{u^+ = \Gamma\}} |f(x)|^n \, dx \right)^{1/n}.$$



☞ **Caffarelli-Silvestre (2009):** There is a ABP type-estimate for integro-differential equations.

☞ However, in the case of the DPE, the non-local nature of the setting forces to consider non-continuous subsolutions, so Γ might not be $C^{1,1}$...





"Mon dessin ne représentait pas un chapeau. It représentait un serpent boa qui digérait un éléphant." Le Petit Prince.

The ε-ABP estimate (A.-Blanc-Parviainen, 2020 & 2021)

Let $f \in C(\overline{\Omega})$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded measurable function satisfying

$$\mathcal{L}_\varepsilon^+ u + f \geqslant 0 \qquad \text{in } \Omega.$$

Then

$$\sup_\Omega u \leqslant \sup_{\mathbb{R}^n \setminus \Omega} u + C(\operatorname{diam}\Omega + \varepsilon)\bigg(\sum_{Q \in \mathcal{Q}_\varepsilon} |Q| \sup_{x \in Q} |f(x)|^n\bigg)^{1/n}.$$

where $\mathcal{Q}_\varepsilon$ is certain pairwise disjoint family of open cubes Q satisfying:

1. $|Q| \simeq \varepsilon$ for all $Q \in \mathcal{Q}_\varepsilon$,
2. $\overline{Q} \cap \{\overline{u^+} = \Gamma\}$ for all $Q \in \mathcal{Q}_\varepsilon$,
3. $\displaystyle\bigcup_{Q \in \mathcal{Q}_\varepsilon} \overline{Q} \supseteq \{\overline{u^+} = \Gamma\}$.

☞ Is it possible to obtain an asymptotic Hölder regularity estimate for solutions of the mean value property

$$\int_{B_1} \frac{u(x+\rho(x)y)-u(x)}{\rho(x)^2}\, d\mu_x(y) = f(x)$$

where ρ is an admissible radius function in Ω such that

$$\rho(x) \simeq \varepsilon \operatorname{dist}(x, \partial\Omega)?$$

☞ Non symmetric probability measures?

☞ Further regularity? Conditions for asymptotic Lipschitz estimates? And "asymptotic $C^{1,\gamma}$ estimates"?

- ☞ Is it possible to weaken the uniform ellipticity condition of the measures?
- ☞ What happens if we replace μ_x by $\mu_{x,\varepsilon,u}$?
- ☞ Parabolic version? Lower order terms?

Thanks for your attention!