



# The dual processes hypothesis in mathematics performance: Beliefs, cognitive reflection, working memory and reasoning

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## ABSTRACT

In this paper, using data provided by an empirical study of students in a high school science course, we discuss key variables in the interaction between System 1 (S1) (intuitive and unconscious processes) and System 2 (S2) (analytical and conscious processes) in mathematical reasoning. These variables are: beliefs about oneself and about mathematics; cognitive reflection understood as a self-regulatory skill; working memory; and the evaluation of the deductive and probabilistic reasoning of students. The results confirm the interaction between these variables and their predictive power on performance in mathematics. The study also adds novel considerations related to the function and interaction of cognitive and metacognitive components involved in mathematical performance.

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## 1. Introduction

In the past decades, several scholars in Mathematical Studies (De Corte, 2004; Schoenfeld, 1992, 2005) have shown that mathematical competence depends on factors such as: (1) mathematical knowledge, (2) heuristic methods, (3) meta-knowledge, (4) self-regulatory skills, (5) positive beliefs about oneself in relation to mathematical learning and problem solving, and (6) beliefs about mathematics and mathematical learning.

These authors point out that much of the complexity involved in learning and teaching mathematics is due to the necessary interconnections the student must make between their knowledge and pre-existing skills and attitudes. In this article we will focus on some of the more relevant factors, such as those related to metacognitive self-regulation and control, as well as student beliefs about mathematics and its learning as predictors of mathematical achievement. Metacognitive refers to knowledge regarding the cognitive process itself, as well as the active monitoring and consequent regulation and orchestration of the decisions and processes involved in problem solving. Schoenfeld (1987) described it as follows:

1. Your knowledge about your own thought processes, the description of your own thinking.

2. Self-awareness or self-regulation. How well do you keep track of what you're doing when (for example) you're solving problems, and how well (if at all) do you use the input from such observations to guide your problem solving actions?
3. Beliefs and intuitions. What ideas about mathematics do you bring to your work in mathematics, and how does that shape the way you do mathematics? (p. 190).

This definition illustrates the relevance beliefs and intuition can have. What somebody believes about this discipline determines the way s/he selects a particular direction or method to solving a problem. We agree with Schoenfeld (1985) who perceives mathematical beliefs as "the set of understandings about mathematics that establish the psychological context within which individuals do mathematics and within which work resources, heuristics and control strategies".

This approach to metacognition adds to its new contents of high functionality, as has been shown in previous studies. The results of our own previous work on these variables make it clear that the scores of subjects in cognitive reflection (i.e., a metacognitive measure) as well as the scores regarding subjects' beliefs about mathematics, and their beliefs regarding their own selves, all correlated positively and significantly with mathematical achievement. It was also found that working memory and reasoning, along with mathematical beliefs, were variables involved in this performance (Gómez-Chacón, García-Madruga, Vila, Elosúa, & Rodríguez, in press).

Another reason to consider these variables is the distinction between two types of cognitive processes in reasoning and judgement: those that run quickly and without conscious deliberation, and the

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slower and more reflective kind. A good part of research in mathematics education has been aimed (explicitly or implicitly) at the relationship between these modes of intuitive and analytical thinking (e.g., Fischbein, 1987; Stavy & Tirosh, 2000). Various studies have explained the conceptual errors in mathematics as consisting of a gap between student intuition and the requirements of the formal system of mathematics. Additionally, the distinction between modes of intuitive and analytical thinking has been comprehensively treated in the theory of dual processes in cognitive psychology (e.g., Evans, 2007; Evans & Over, 1996; García-Madruga, Gutiérrez, Carriedo, Vila, & Luzón, 2007). Authors such as Stanovich and West (2000) refer to these dual processes as “System 1” and “System 2” respectively.

System 1 has been characterized as unconscious, associative, fast, and not linked to individual resources of working memory and fluid intelligence. This system, which humans share to a large extent with other animals, allows individuals to quickly access answers that are often valid, but also lead them to commit mistakes. System 2 is considered conscious, slow, controlled and linked to the individual resources of working memory and fluid intelligence. The performance of System 2 involves the overriding of System 1 and depends on intellectual capability, as well as the disposition and mental styles of individuals.

The role of this dual process theory of reasoning applied to mathematics education is remarkable, and this is a growing field of study (see e.g. Inglis, Mejia-Ramos, & Simpson, 2007; Vamvakoussia, Van Dooren, & Verschaffel, 2012). There are several themes that have been productive: a) comparative studies with different groups (mathematicians, the general population, college students) used to analyse the influence the so-called belief bias has on responses in conditional inference tasks; b) studies on mathematical processes – such as demonstration – that attempt to identify different types of informal arguments, and the possible differences between expert and novice; and c) studies establishing the intuitive nature of fallacious reasoning in the domain of rational numbers, as well as problems of proportion and persistence in adulthood.

In this paper, we seek to deepen our knowledge about the relationship these belief systems might have when applied to mathematics education, a topic scarcely addressed in the study of dual process theories of reasoning. We would like to identify and understand the positive as well as the negative influences of beliefs that can serve to foster as well as present barriers during cognitive reflection (a metacognitive measure) and reasoning.

The proposal of Evans, like that of Stanovich and West, provides a convergent line of research worth exploring. Evans (see Evans, 2009) proposes a third metacognitive system, System 3, responsible for the activation of working memory, as well as resolving possible conflicts occurring between Systems 1 and 2. Similarly, Stanovich and West (2000); (e.g., Stanovich et al., 2011) propose a tripartite structure. In addition to System 1 and System 2, the so-called reflexive mind exists, and this is responsible for the overall control of the individual's behaviour depending on one's general purpose and goals.

In this article we reflect on a few key variables in the interaction between System 1 (S1) and System 2 (S2) dual processes in mathematics: 1) at the metacognitive level, beliefs provide immediate psychological context and directly affect the performance of subjects in mathematical tasks; 2) cognitive reflection is a measure of metacognitive executive control (regulation) that the subject applies to the resolution of tasks and allows for the inhibition of S1 automatic responses; 3) at the cognitive level, working memory is a fundamental cognitive structure that makes reference to processing and information storage limitations while performing cognitive tasks; and, finally, 3) at the performance level, reasoning abilities are made up of three basic components: deductive inferences (propositional and syllogistic), meta-deductive knowledge, and probabilistic reasoning. Likewise, mathematical achievement scores are at this performance level. Before describing the study and presenting

the results, we will review a few theoretical considerations from a historical point of view underlying the variables used in the study.

## 2. Theoretical aspects

### 2.1. Belief systems regarding mathematics and learning

Many recent studies have been completed on the essential role of beliefs in learning and teaching mathematics (e.g., Leder, Pehkonen, & Törner, 2002; Maass & Schloeglmann, 2009; Roesken & Casper, 2011). As a unifying framework for the study of belief systems, we refer to the proposal of Op't Eynde, De Corte, and Verschaffel (2002). This proposal allows for a better understanding of the interactions between different types of beliefs, such as reflected in the Mathematics-Related Beliefs Questionnaire (MRBQ) of Op'Eynde and De Corte (2003). These authors refer to the following building blocks for the analysis of the nature and structure of belief systems: the social context, the self, and the object. In previous studies (Gómez-Chacón, Op't Eynde, & De Corte, 2006a, 2006b), we found the need to operationalize the MRBQ questionnaire for the Spanish population. Also, in another project, researchers who designed the CreeMat questionnaire used in the current study also sought to understand the relationship between working memory, reading comprehension, cognitive thinking, deductive reasoning and mathematical belief systems (Gómez-Chacón, García-Madruga, Rodríguez, Vila, & Elosúa, 2011).

Our previous work has led to prioritizing four dimensions in the development of our questionnaire of beliefs: 1) student beliefs about mathematics (MathBe), 2) beliefs about learning and solving math problems (ProsoolvBe), 3) student beliefs about themselves (beliefs about the meaning of personal competence in mathematics, that is, the confidence and perception of a student's own ability) (ConfBe), and 4) an affective and behavioural dimension regarding a student's engagement in individual mathematical learning (EngBehav). The first three have been discussed in the studies mentioned above, while the fourth dimension does not appear to be integrated in questionnaires such as the MRBQ. In relation to this latter dimension, we would like to point out that we have considered two aspects of engagement in mathematical learning: affective and behavioural engagement. In this sense, experts such as Fredricks, Blumenfeld, and Paris (2004) provide a more comprehensive understanding regarding engagement at school. In our own context regarding learning the discipline of mathematics, we refer only to the school's engagement in the cognitive domain of mathematics. It is within this area that we decided to examine how students feel about the discipline itself (i.e., the affective dimension of commitment) and how they behave when learning the subject (i.e., the engagement expressed in their conduct).

### 2.2. Cognitive reflection

The proper resolution of arithmetic problems often requires a deep understanding of the problem. The cognitive reflection test (CRT) used in this study is adapted from the evidence presented by Frederick (2005) which, in addition to the three problems used in the initial test, also includes two additional issues proposed and used by the author. The test attempts to evaluate the depth of reasoning of a participant through a simple mathematical reasoning task, similar to the following (problem 1):

*A bat and a ball cost \$1.10. The bat costs \$1.00 more than the ball. How much does the ball cost? \_\_\_ cents.*

Faced with this kind of mathematical problem, subjects tend to give an impulsive response that comes readily to mind: “10 cents”. However, this answer is wrong, as a little reflection will make clear: the difference between \$1.00 and 10 cents is 90 cents and not \$1.00, as the problem

states. The correct answer “5 cents” now comes readily to mind and, therefore, requires that individuals act in a reflective way.

The problems of Frederick (2005) provide a simple measure of a very important metacognitive skill: cognitive reflection in problem solving. For this reason, the test measures the ability of individuals to control their behaviour in a thoughtful way and the ability to inhibit the first answer that comes “to mind.” The author investigates the relationship of CRT to various aspects of decision-making, and emphasizes that people with lower cognitive skills of reflection tends to make a wrong first choice, the “intuitive” one.

To explain the actions of individuals in the cognitive reflection test, Frederick (2005) applies the dual process theory of reasoning and decision-making. According to the dual process hypothesis, spontaneous responses produced by System 1 are fast and involve a superficial understanding of the problem. Conversely, the correct answers produced by System 2 involve cognitive and motivational effort as well as concentration, and for this reason they require more time to solve.

The present work focuses on the study of this metacognitive dimension called cognitive reflection. It takes not only the aspects of readiness, motivation and affective engagement in individual mathematical learning into account, but also an individual's belief system. In the field of mathematics, decisions are affected by the beliefs, goals and an individual's prior knowledge (Schoenfeld, 2005). That is, an individual's ideas about mathematics shape the way he or she does mathematics.

### 2.3. Reasoning is limited by working memory capacity

The study of deductive reasoning, as related to mathematics and education in general, has revealed that in addition to building the necessary processes and conscious manipulation of semantic representations, reasoning is often affected by other unconscious processes that lead to certain errors. Thus, some have argued that reasoning operates by way of differing dual processes (Evans, 2007), each described somewhat differently by various authors: semantic/surface (García-Madruga, 1983), analytical/heuristic (Evans, 1984a, 1984b), and explicit/tacit (Evans & Over, 1996). In this way, the deductive capabilities of individuals would be based on the development of semantic processes, explicit and analytical, just as the increased ability to inhibit responses would be based on surface, or heuristic and tacit processes.

As we have been arguing, the theoretical idea underlying the current work is that a student's working memory capacity, as well as their beliefs and metacognitive skills, may restrict or foster their reasoning ability. This also implies, in turn, that these variables will necessarily affect a student's learning processes in school, in particular in mathematics tasks (Schlögglmann, 2007). According to Johnson-Laird's Mental Models Theory (MMT) (Johnson-Laird & Byrne, 1991), working memory plays a key role in explaining reasoning and its development. Researchers in the current study propose that the key to explaining reasoning lies not so much on working memory storage, be it verbal or visual, but rather on WM's executive processes (García-Madruga et al., 2007; García-Madruga, Gutiérrez, Carriedo, Luzón, & Vila, 2005). Solving deductive reasoning problems not only requires a capacity to store information. It makes special use of efficient cognitive resources in a series of tasks that require the control and regulation of the problem-solving process. Thus, deductive tasks require attentional focus for an in-depth understanding of the statements in the premises, the activation of knowledge and representations in long-term memory, and a shift of attention from the task of comprehension to the task of integrating the meanings of various premises in search of an appropriate solution. In other words, it requires the execution of the principal functions of the central executive. Furthermore, as highlighted by dual process theories, individuals must also be able to, whenever necessary, inhibit the automatic responses inherent to System 1.

### 3. The current study

This paper examines students' beliefs about mathematics, their cognitive reflection skills, their working memory capacity, and their reasoning abilities, as well as the relationships these variables have with mathematical achievement. Reasoning and mathematical achievement are typical cognitive tasks in which Systems 1 and 2 are directly involved. As diverse authors have claimed (Evans, 2009; Thompson, 2010), the metacognitive processes are in charge of the activation of working memory, as well as the resolution of possible conflicts between the outputs of Systems 1 and 2. In the same vein, Stanovich, West, and Toplak (2011, pp. 374–376) maintain that in order to override System 1, System 2 must be able to perform two related abilities: to interrupt System 1 functioning and inhibit their response tendencies, and to provide alternative responses by means of hypothetical analytical reasoning and cognitive simulation. Overriding System 1, however, requires executive control, something that cannot be directly carried out by System 2. It requires a metacognitive or reflective kind of processing that allows an individual to control how his or her general dispositions, beliefs and aims influence his or her behaviour. Fig. 1 represents the proposed relationships between the variables examined in this work, in particular the relationships between the metacognitive variables (beliefs and cognitive reflection) with basic cognitive capacity (working memory) and the two performance variables of reasoning and mathematical achievement. This study does not examine the various other contextual and academic influences involved in mathematical achievement, such as learning and teaching methods and practices, and academic motivation.

The following hypotheses guided the work: 1) there will be positive correlations between positive beliefs about mathematics and learning and cognitive reflection; 2) those participants with more positive beliefs, better cognitive reflection abilities or wider working memory span, are those who will perform better in reasoning; and 3) all these variables: beliefs, cognitive reflection, working memory, and reasoning, will be positively related to academic achievement in mathematics.

#### 3.1. Method

##### 3.1.1. Participants

Our research is primarily correlational and our sample was composed of 56 high school science students (mean age = 16.58, SD = 0.64). All participants had received their academic qualifications and performed every one of the tests. We measured mathematical achievement by the mathematics scores each student earned (1 to 10) at the end of the course, when the data was collected. The students have four evaluations per year and a final evaluation at the end of the course. In these assessments, students were evaluated on their understanding and their problem solving ability with regard to the various kinds of content. The conceptual categories that make up secondary school mathematics standards are the following: Number and Algebra, Geometry, Analysis, Statistics and Probability.

##### 3.1.2. Description of the testing instruments used

**3.1.2.1. The beliefs questionnaire (CreeMat).** The beliefs questionnaire (CreeMat) is a Likert-like scale on which 1 represents “completely disagree” and 5 “completely agree.” The purpose of this test is to evaluate the systems of beliefs about mathematics in high school students.

The questionnaire is composed of 13 items (Table 1) and covers 4 subscales or dimensions: affective and behavioural engagement in mathematical learning (EngBehav: items 1, 2 and 3); confidence and beliefs regarding one's personal competence in mathematics (ConfBe: items 4, 6, 8 and 12); mathematical beliefs (MathBe: items 5, 9 and 13); and beliefs about mathematical problem solving (ProsolBe: items 7, 10 and 11) (See Table 1). The Cronbach  $\alpha$  value for the test is 0.645.

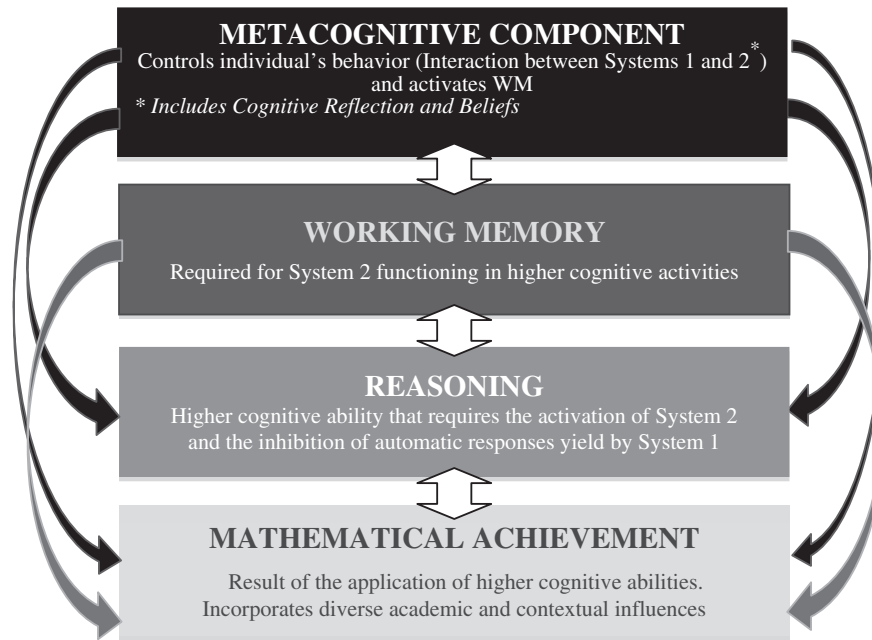


Fig. 1. Theoretical model that contains all essential variables and their mutual relationships.

**3.1.2.2. Cognitive reflection test.** We used an adapted version of Frederick's (2005) cognitive reflection test with 5 items. The task is presented as a "paper and pencil" test consisting of 5 arithmetic problems, such as the problem described in Section 2.2.

Each problem has three possible answers: the correct one, a superficial alternative (which is incorrect), and the response "impossible to determine". The test is evaluated by scoring one point for the correct response and a zero for the two incorrect response alternatives. In this way, we obtain the scores for each type of response. It is worth noting that, contrary to that done by Frederick in his test, and in order to reduce the test's difficulty, we have chosen to make our version a multiple-choice evaluation task instead of a generation task.

**3.1.2.3. Test of working memory.** The Semantic Anaphora Working Memory Test (Carriedo, Elosúa, & García-Madruga, 2011) was used in the current study. In it, a participant must read a sentence aloud that has a pronominal anaphora. For example, "Eladio really encouraged her to interpret such a demanding role". Then, we present two words, for example, "career" and "actress", and the subject has to choose the correct

answer between the two and remember the word selected to resolve the anaphora. The anaphora problems were really easy and the difficulty of each problem was similar, (97% of correct responses; Elosúa, Carriedo, & García-Madruga, 2009). The task consisted of 42 anaphora problems that shifted through three series of different levels of 2, 3, 4 and 5 problems each. Participants have to solve anaphora and at the end of each series to remember the 2, 3, 4 or 5 word-solution of each group of anaphora.

The task ends when the participant fails at least 2 of the 3 series comprising the same level. For a series to be deemed successful, the participant must remember all the words in the series, even if repeated in a different order. A series is deemed incorrect if a word is omitted from the series or is replaced by another one. Only the correct series were scored. For the score, 1 point is awarded to the word recalled in the same order and 0.5 points to the word remembered in a different order. The approximate duration of the task is 20 min.

**3.1.2.4. The deductive and probabilistic reasoning questionnaire.** The purpose of this test is to obtain a measure of competence in reasoning (deductive and probabilistic) of subjects. The reasoning task is important not only to measure one of the most relevant higher-level cognitive skills, but also to serve as a good predictor of cognitive development, learning and academic performance. Specifically, this reasoning test consists of understanding and solving various problems by selecting some of the alternatives posed to solve three kinds of problems: deductive inference problems (propositional and syllogistic), meta-deductive knowledge problems, and probabilistic reasoning problems (García-Madruga et al., 2009). The test of deductive inferences includes three types of deductive problems: four propositional inferences, two conditionals (denial of antecedent: DA; and modus tollens: MT) and two inclusive disjunctions (affirmative and negative); and three syllogistic problems of varying difficulty: easy (one model), difficult (two models) and very difficult (three models). The meta-deductive part of the reasoning task consists of three truth table problems, which involve a consistent interpretation of the three true models of the conditional ( $p \rightarrow q$ ,  $\neg p \rightarrow \neg q$ ,  $\neg p \rightarrow q$ ); and three necessity-possibility problems requiring a correct interpretation of syllogistic premises. The probabilistic reasoning part seeks to evaluate a few relevant aspects of this kind of

**Table 1**  
Descriptive data of the items in the CreeMat questionnaire.

Items	Mean	SD
Item 1: I work hard in mathematics	3.13	1.16
Item 2: If I make mistakes, I work until I correct them	3.61	1.28
Item 3: I like to invent new problems	1.70	1.01
Item 4: I learn mathematics quickly	3.30	0.87
Item 5: Mathematics allows us to better understand the world we live in	3.41	0.95
Item 6: When I am asked to solve math problems, I get nervous	3.32	1.24
Item 7: When I cannot solve a math problem quickly, I keep on trying	3.77	1.08
Item 8: I feel confident when I study or work on mathematics	3.14	0.96
Item 9: Anybody can learn mathematics	4.18	1.05
Item 10: I prefer challenging tasks in order to learn new things	3.68	0.90
Item 11: Mathematics classes should not place much importance on problem solving	3.52	1.04
Item 12: I feel happy when I solve math problems	3.93	0.99
Item 13: Mathematics is about concepts and procedures that we have to memorize	3.21	1.14
Mean	3.37	0.46
Total	43.75	6.09

reasoning: 1) the variability of small samples, 2) the actual frequency of an event and the gambler's fallacy, and 3) the distinction between direct probability P(A/B) and its inverse P(B/A).

3.2. Results

Below are the results structured according to the specified variables and their interactions.

3.2.1. Belief systems about mathematics and learning (CreeMat)

The data show that the group tested has low individual affective and behavioural commitment to learning in mathematics (mean 2.80). However, the group does illustrate an acceptable perspective regarding their beliefs about mathematics and mathematical problem solving (mean 3.6) and beliefs about the meaning of personal competence in mathematics. That is, the confidence and perception of a student's own ability (mean 3.4). We note the apparent discrepancy in their answers when asked about their beliefs about themselves. This indicates that the trend is to memorize mathematical concepts and procedures, even though their responses are often influenced by social desirability, which makes them respond more in terms of problem solving.

3.2.2. Cognitive reflection

The results achieved on the cognitive reflection test (RC) can be seen in Table 2. The data are presented in percentages. You can also view the mean percentage of the difficulty level participants estimated having with each problem. In this test, the total percentage of correct answers was slightly higher than the total percentage of superficial answers. There were also a significant percentage of answers with "no conclusion", although the variability between different items in this type of response was high. Participants underestimated the difficulty of the problems: the total percentage estimate of correct answers was 69%, well above the total percentage of correct answers (42%). The correlation between the difficulty estimate and the percentage of superficial responses was 0.14 ( $p = .15$ ). Overall, the performance of participants was relatively high compared with the results Frederick (2005) reported with college students. That is, our students gave more correct, less superficial answers. We believe that amending the version of the test, making it a multiple-choice recognition task rather than a recall task, may explain these results.

3.2.3. Reasoning

The results of the study illustrate that high school science students have difficulties in conditional inferences and categorical syllogisms. Table 3 shows the percentage of correct responses for each item and the type of reasoning and inference.

More detailed analysis of various aspects of deductive reasoning show that inclusive disjunctions and especially conditionals, continue to challenge students. For example, only 27% of participants chose the correct conclusion (no conclusion) in denying the antecedent inference. Also, difficulties were confirmed in conditional inferences in its truth table interpretation: only 23% of participants felt that the model "not-p q" is true. Moreover, the correlation of correct responses on both

Table 2

Response rates for the cognitive reflection test, and the difficulty estimation for each of the five problems (correlations between superficial responses and estimation of difficulty appear in parentheses, \*  $p < .05$ ).

Item	Correct responses	Superficial responses	Other incorrect responses	Difficulty estimation
1	36	61	3	88 (.12)
2	57	43	0	72 (.33*)
3	21	25	54	53 (.13)
4	50	36	14	65 (.28*)
5	45	23	32	65 (.08)
Mean	42	38	21	69 (.14)

Table 3

Percentage of correct answers and standard deviations on the diverse reasoning problems.

	Type of inference		Mean	SD
Propositional reasoning	Conditional	Deny antecedent	27	.45
		Modus tollens	70	.46
	Disjunctive	Positive	64	.48
		Negative	80	.40
	"Consistency"		96	.19
			89	.31
		23	.43	
Syllogistic reasoning	Syllogisms	AI1-I	70	.46
		IA-No conclusion	48	.50
		EA-O	9	.29
	"Necessity/possibility"		89	.31
			84	.37
			66	.48
Probabilistic reasoning	Variability of small samples		29	.46
		Gambler's fallacy	66	.48
		Distinguish between P(A/B) and P(B/A)	68	.47
Overall reasoning			65.48	11.74

items was high and significant ( $r = .53, p < .01$ ). As we can see, most high school freshmen continue making a bi-conditional interpretation, and merely a quarter of the participants reach fully conditional interpretations and inferences. As shown in Table 3, the resolution of the three categorical syllogisms confirms the predictions regarding difficulty: more than half of the students made mistakes on the difficult syllogisms. In addition, the results show that teens have difficulty with the concept of possibility in relation to necessity and impossibility.

With respect to probabilistic reasoning, this group of students has difficulty when considering the variability of small samples (29%). 66% of the students are able to distinguish the "gambler's fallacy" (the fact of expecting the actual frequency of an event to manifest itself in very few trials is a common mistake). 68% of the students distinguish between P (A/B) and P (B/A).

3.2.4. Interrelationships between variables

In order to evaluate the interactions between belief systems, cognitive reflection working memory, reasoning, and mathematical achievement—we performed a correlation analysis. In addition, we conducted two multiple linear regression analyses, one to assess the predictive beliefs, cognitive reflection and working memory span on reasoning, and another one to assess the predictive ability of all these variables on mathematical achievement.

We describe these interrelationships in response to the hypotheses. There is a significant positive correlation between beliefs and cognitive reflection, confirming our first hypothesis (Table 4).

Relative to our second hypothesis, the data shows a pattern of positive and mostly significant correlations between the variables: beliefs, cognitive reflection, working memory and reasoning (Table 4).

Table 4

Correlations between belief Systems, cognitive reflection, working memory, reasoning and achievement in mathematics.

	1	2	3	4	5
1. Beliefs	1	.341**	.284*	.338**	.471**
2. Cognitive reflection		1	.173#	.515**	.246*
3. Working memory			1	.446**	.164
4. Overall reasoning				1	.323**
5. Math achievement					1

One tailed.

\*\*  $p < .01$ .

\*  $p < .05$ .

#  $p < .10$ .

However, we expected to find a clearer relationship between tests of cognitive reflection and working memory: this correlation is the only one that is positive, even if marginally significant. A multiple linear regression analysis was used to test the predictive capacity of the variables. Beliefs (total), cognitive reflection and working memory explain 37% of the variance:  $F(3,52) = 11.73$   $p < .0001$  of overall reasoning. The significant variables were cognitive reflection ( $B = 3.17$ ,  $Beta = .42$ ,  $p < .001$ ) and working memory ( $B = .17$ ,  $Beta = .35$ ,  $p < .003$ ). As we can see, results confirm on the whole our second hypothesis.

Lastly, Table 4 also shows evidence of positive correlations between beliefs, cognitive reflection, working memory and reasoning, with mathematic achievement. Working memory is again the only variable whose correlation is not significant. The multiple linear regression analysis used to test the predictive capacity of all variables on mathematical achievement explains 20% of the variance:  $F(4,51) = 4.32$   $p < .004$ , the significant variable being beliefs ( $B = .186$ ,  $Beta = .412$ ,  $p < .003$ ). The results also confirm our third hypothesis.

#### 4. Discussion and conclusion

The results of this study were derived from the incorporation of a set of new variables for predicting performance in mathematics, and an analysis of the interaction of S1 and S2 in the context of mathematical reasoning and achievement. Regarding these relationships between the metacognitive variables (beliefs and cognitive reflection) with working memory (basic cognitive capacity) and the two performance variables of reasoning and mathematical achievement, this study confirmed that belief systems and cognitive reflection may explain some of the relationships between the two systems and the gap between the intuitive and analytical system (Stanovich et al., 2011). In particular, this study confirms the trend noted in previous research regarding the relationship between the beliefs of individuals about the relevance of solving problems, and the confidence they have in their own personal competence. In addition, it provides two novel findings, one regarding the positive interaction of cognitive reflection with respect to reasoning, and another one relative to the relationship between belief systems and working memory.

We found that the belief systems of students correlate with working memory. Likewise, we found a more positive trend between tests of cognitive reflection and working memory than in our previous study (Gómez-Chacón et al., in press). These interrelations may provide a key interpretation of the possible conflicts between System 1 and System 2 outputs, as well as a better understanding of metacognitive processes. Student errors may be caused by misconceptions or they may also be due to what are known as “slips”, which are linked to the concept of working memory. In the context of working memory, these open representations are in direct relationship with the central executive, and involve the planning and execution of a cognitive process. In addition, working memory correlated significantly with reasoning, something we already mentioned and emphasised from mental model theory (García-Madruga et al., 2007). We can conclude that subjects with appropriate beliefs, high cognitive reflection, and high working memory show better reasoning and academic achievement in mathematics. We think that these results contribute to a better understanding of the important role that metacognitive processes have in mathematical reasoning (Evans, 2009; Santamaría, Tse, Moreno-Ríos, & García-Madruga, 2013; Schoenfeld, 2005; Thompson, 2010).

The data confirm the difficulties students have with deductive and probabilistic reasoning problems. They highlight the existence of various sources of error, both in terms of the complexity of the inferences as well as with respect to the influence of previous ideas and beliefs. As noted in the introduction, these results allow us to state that, in addition to processes of construction and the conscious manipulation of semantic representations, other unconscious processes often affect reasoning and lead to certain errors.

It should be noted that the dual-process framework offers more methodological tools than we have taken advantage of in this kind of study (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009a). However, we highlight some limitations of this study both in terms of its conceptualization and in terms of certain methodological aspects that might lead to new perspectives for future study. In this study we have limited ourselves to the exploration of metacognitive ability by way of cognitive reflection and the influence of beliefs. It would be interesting to extend, on a micro and macro scale, how decision-making affects the resolution of tasks and how it depends on the knowledge, goals, beliefs and implications of the self (Gómez-Chacón, 2000, 2008; Malara & Zan, 2000; Schoenfeld, 2005). We think that individual beliefs, as part of the metacognitive system, serve to model one's goals in interaction with the context. Another way to extend this work is to use other methodological tools in order to deepen the assessment of an individual's conscious experience of the kinds of conflicts involved in solving mathematical reasoning problems. Qualitative data based on protocols of timed sequences of thinking aloud might be useful to establish the more precise nature of certain heuristics (Babai, Brecher, Stavy, & Tirosh, 2006; Gillard, Van Dooren, Schaeken, & Verschaffel, 2009b). Likewise, methodologies coming from neuroscientific research could be valuable. For instance, it might be useful in cases of unconscious conflict detection (De Neys, Moyens, & Vansteenwegen, 2010). This methodology focuses on the autonomic nervous system modulation during biased reasoning and can provide new evidence about the possibility and kind of reasoning conflicts that come from the intuitive beliefs. The resulting conflict should elicit autonomic arousal, which should be reflected in an increased skin conductance response (SCR) in the face of the conflict. In such a case, it may be possible to establish a link between autonomic modulation and conflict detection that comes from beliefs and the lack of cognitive reflection. These results might help to provide more solid conceptual ground for the idea of dual processes in mathematics performance.

Finally, we would like to stress the educational significance of this study. Considering the perspective of dual processes in mathematical reasoning, we believe that our approach has direct application to schooling, given that a teacher directs attention towards two basic educational objectives: firstly, the promotion of in-depth understanding of mathematical concepts, and secondly, the inhibition of superficial processes and strategies that otherwise lead to error.

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#### References

- Babai, R., Brecher, T., Stavy, R., & Tirosh, D. (2006). Intuitive interference in probabilistic reasoning. *International Journal of Science and Mathematics Education*, 4, 627–639.
- Carriedo, N., Elosúa, M. R., & García-Madruga, J. A. (2011). Working memory, text comprehension, and propositional reasoning: A new semantic anaphora WM Test. *Spanish Journal of Psychology*, 14(1), 37–49.
- De Corte, E. (2004). Mainstreams and perspectives in researcher on learning (mathematics) from instruction. *Applied Psychology: An International Review*, 53(2), 279–310.
- De Neys, W., Moyens, E., & Vansteenwegen, D. (2010). Feeling were biased: Autonomic arousal and reasoning conflict. *Cognitive, Affective, & Behavioral Neuroscience*, 10, 208–216.
- Elosúa, M. R., Carriedo, N., & García-Madruga, J. A. (2009). Dos nuevas pruebas de memoria operativa de anáforas. *Infancia y Aprendizaje, Journal for the Study of Education and Development*, 32(1), 97–118.
- Evans, J. S. B. T. (1984a). Heuristic and analytic processes in reasoning. *British Journal of Psychology*, 75, 451–468.
- Evans, J. S. B. T. (1984b). The heuristic-analytic theory of reasoning. Extension and evaluation. *Psychonomic Bulletin and Review*, 13, 321–329.

- Evans, J. S. B. T. (2007). *Hypothetical thinking. Dual processes in reasoning and judgement*. Hove: Psychology Press.
- Evans, J. S. B. T. (2009). How many dual-process theories do we need? One, two, or many? In J. S. B. T. Evans, & K. Frankish (Eds.), *In two minds: Dual processes and beyond* (pp. 33–54). Oxford: Oxford University Press.
- Evans, J. S. B. T., & Over, D. E. (1996). *Rationality and reasoning*. Hove: Psychology Press.
- Fischbein, E. (1987). *Intuition in science and mathematics. An educational approach*. Dordrecht: Kluwer Academic Publishers.
- Frederick, S. (2005). Cognitive reflection and decision making. *Journal of Economic Perspectives*, 19(4), 25–42.
- Fredricks, J., Blumenfeld, P., & Paris, A. (2004). School engagement: potential of the concept, state of the evidence. *Review of Educational Research*, 74(1), 59–109.
- García-Madruga, J. (1983). Un modelo general sobre el razonamiento silogístico: Doble procesamiento y fase de comprobación con verificación. *Revista de Psicología General y Aplicada*, 3, 439–466.
- García-Madruga, J. A., Gutiérrez, F., Carriedo, N., Luzón, J. M., & Vila, J. O. (2005). Working memory and propositional reasoning: Searching for new working memory tests. In V. Girotto, & P. N. Johnson-Laird (Eds.), *The shape of reason. Essays in honour of Paolo Legrenzi* (pp. 69–90). London: Psychology Press.
- García-Madruga, J. A., Gutiérrez, F., Carriedo, N., Vila, J. O., & Luzón, J. M. (2007). Mental models in propositional reasoning and working memory's central executive. *Thinking and Reasoning*, 13(4), 370–393.
- García-Madruga, J. A., Santamaría, C., Moreno Ríos, S., Vila, J. O., Gómez-Chacón, I. M.<sup>a</sup>, Orenés, I., et al. (2009). Prueba de razonamiento para secundaria. *I+D+i project: Cognition and education: working memory, reading comprehension and reasoning*. Spanish Ministry of Science and Innovation.
- Gillard, E., Van Dooren, W., Schaeken, W., & Verschaffel, L. (2009a). Dual processes in the psychology of mathematics education and cognitive psychology. *Human Development*, 52, 95–108.
- Gillard, E., Van Dooren, W., Schaeken, W., & Verschaffel, L. (2009b). Proportional reasoning as a heuristic-based process: Time pressure and dual-task considerations. *Experimental Psychology*, 56, 92–99.
- Gómez-Chacón, I. M.<sup>a</sup>. (2000). Affective influences in the knowledge of mathematics. *Educational Studies in Mathematics*, 43, 149–168.
- Gómez-Chacón, I. M.<sup>a</sup>. (2008). Suggesting practical advances in the research on affect in mathematical learning. *ICME-11, 11th International Congress on Mathematical Education* (<http://tsg.icme11.org/tsg/show/31>)
- Gómez-Chacón, I. M.<sup>a</sup>, García-Madruga, J. A., Rodríguez, R., Vila, J. O., & Elosúa, M. R. (2011). Mathematical beliefs and cognitive reflection: Do they predict academic achievement? In B. Roesken, & M. Casper (Eds.), *Current state of research on mathematical beliefs. XVII Proceedings of the MAVI-17 Conference* (pp. 64–73). Bochum, Germany: Professional School of Education, Ruhr- Universität Bochum.
- Gómez-Chacón, I. M.<sup>a</sup>, García-Madruga, J. A., Vila, J. O., Elosúa, M. R., & Rodríguez, R. (2013). *La relación entre las creencias, las habilidades cognitivas y metacognitivas, y el rendimiento en Matemáticas*. (in press).
- Gómez-Chacón, I. M.<sup>a</sup>, Op't Eynde, P., & De Corte, E. (2006a). Creencias de los estudiantes de matemáticas, la influencia del contexto de clase. *Enseñanza de las Ciencias*, 24(3), 309–324.
- Gómez-Chacón, I. M.<sup>a</sup>, Op't Eynde, P., & De Corte, E. (2006b). Flemish and Spanish high school student's Mathematics-related beliefs systems: A comparative study. *Proceedings of the 30th PME, 16–21 July*. Prague, Czech Republic: PME.
- Inglis, M., Mejía-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, 66, 3–21.
- Johnson-Laird, P. N., & Byrne, R. M. J. (1991). *Deduction*. Hove: Lawrence Erlbaum Associates.
- Leder, G., Pehkonen, E., & Törner, G. (Eds.). (2002). *Beliefs: A hidden variable in mathematics education?*. Dordrecht: Kluwer Academic Publishers.
- Maass, J., & Schloeglmann, W. (Eds.). (2009). *Beliefs and attitudes in mathematics education: New research results*. Rotterdam: Sense Publishers.
- Malara, N. A., & Zan, R. (2000). The problematic relationship between theory and practice. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 553–580). Mahwah, NJ: Lawrence Erlbaum Associates.
- Op'Eynde, P., & De Corte, E. (2003). Junior High students' mathematics-related beliefs systems: An empirical analysis of their internal and external structure. *10th Biennial Conference of the European Association for Research on Learning and Instruction, Italy: Padua*.
- Op't Eynde, P., De Corte, E., & Verschaffel, L. (2002). Framing students' mathematics-related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 13–38). Dordrecht: Kluwer Academic Publishers.
- Current state of research on mathematical beliefs. Roesken, B., & Casper, M. (Eds.). (2011). *XVII Proceedings of the MAVI-17 Conference Sept. 17–20*. Bochum, Germany: Professional School of Education, Ruhr- Universität Bochum.
- Santamaría, C., Tse, P. P., Moreno-Ríos, S., & García-Madruga, J. A. (2013). Deductive reasoning and metalogical knowledge in preadolescence: A mental model appraisal. *Journal of Cognitive Psychology*, 25(2), 192–200.
- Schlöglmann, W. (2007). Student errors in task-solving processes. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the fifth congress of the European Society for Research in Mathematics education* (pp. 359–368). University of Cyprus.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic Press.
- Schoenfeld, A. H. (1987). What is all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Mahwah, NJ: Lawrence Erlbaum Associates.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.
- Schoenfeld, A. H. (2005). Problem Solving from Cradle to Grave. *Annales de Didactique et de Sciences Cognitives*, 11, 41–73.
- Stanovich, K. E., & West, R. F. (2000). Individual differences in reasoning. Implications for the rational debate? *Behavioral and Brain Sciences*, 23, 645–726.
- Stanovich, K. E., West, R. F., & Toplak, M. E. (2011). Individual differences as essential components of heuristic and biases research. In K. Manktelow, D. Over, & S. Elqayam (Eds.), *The science of reason. A festschrift for Jonathan St. B. T. Evans* (pp. 355–396). Hove: Psychology Press.
- Stavy, R., & Tirosh, D. (2000). *How students (mis-)understand science and mathematics: Intuitive rules*. New York: Teachers College Press.
- Thompson, V. A. (2010). Towards a metacognitive dual process theory of conditional reasoning. In M. Oaksford, & N. Chater (Eds.), *Cognition and conditionals: Probability and logic in human thinking* (pp. 335–354). Oxford: Oxford University Press.
- Vamvakoussia, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior*, 31(3), 344–355.