Presentations of the unit group of an order in a non-split quaternion algebra ☆

Capi Corrales,[a,*] Eric Jespers,[b] Guilherme Leal,[c] and Ángel del Río[d]

[a] *Facultad de Matemáticas, Universidad Complutense de Madrid, Madrid, Spain*
[b] *Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussels, Belgium*
[c] *Instituto de Matemática, Universidade Federal do Rio de Janeiro, 21910 Rio de Janeiro, Brazil*
[d] *Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain*

Abstract

We give an algorithm to determine a finite set of generators of the unit group of an order in a non-split classical quaternion algebra $\mathbb{H}(K)$ over an imaginary quadratic extension K of the rationals. We then apply this method to obtain a presentation for the unit group of $\mathbb{H}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$. As a consequence a presentation is discovered for the orthogonal group $SO_3(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$. These results provide the first examples of a characterization of the unit group of some group rings that have an epimorphic image that is an order in a non-commutative division algebra that is not a totally definite quaternion algebra.
© 2003 Elsevier Inc. All rights reserved.

MSC: primary 16U60; secondary 11R27; 16A26

1. Introduction

The unit group of an order in a finite dimensional semisimple algebra A over the rationals is an important example of an arithmetic group. Hence it forms a

fundamental topic of interest. Recall that a subring Γ of A is said to be an order if Γ is a finitely generated $\mathbb{Z}$-module that contains a $\mathbb{Q}$-basis of A. Prominent examples of orders are group rings RG of finite groups G over the ring of integers R of an algebraic number field. The unit group RG^* of RG has received a lot of attention and most of it has been given to the case $R = \mathbb{Z}$; for surveys we refer to [7,9,12]. It is well known that the unit group Γ^* of an order Γ is a finitely presented group. However, only for very few finite non-abelian groups G the unit group $\mathbb{Z}G^*$ has been described, and even for fewer groups G a presentation of $\mathbb{Z}G^*$ has been obtained. Nevertheless, for many finite groups G a specific finite set B of generators of a subgroup of finite index in $\mathbb{Z}G^*$ has been given. The only groups G excluded in this result are those for which the rational group algebra $\mathbb{Q}G$ has a simple component that is either a non-commutative division algebra different from a totally definite quaternion algebra or a 2×2 matrix ring $M_2(F)$, where F is either $\mathbb{Q}$, a quadratic imaginary extension of $\mathbb{Q}$ or a non-commutative division algebra. One of the important tools used to prove that the group generated by B (which we will denote by $\langle B \rangle$) is of finite index, is to show that if $M_n(D)$ is a simple component of $\mathbb{Q}G$ and Γ is an order in the division algebra D then $\langle B \rangle$ contains a subgroup of finite index in $\mathrm{SL}_n(\Gamma)$, the group of matrices in $M_n(\Gamma)$ of reduced norm one. As mentioned above, the case $n = 1$ and D a non-commutative division algebra different from a totally definite quaternion algebra is excluded. If Γ is an order in D then the unit group Γ^* is a $\mathbb{Q}$-group that is anisotropic. Hence even describing some generic classes of units in this group is hard. For the known results on unit groups in division algebras we refer to [8]; the author mainly concentrates on the case that the non-commutative division algebra D splits over $\mathbb{R}$.

The aim of this paper is to present a finite algorithm to compute a finite set of generators of the unit group of an order in a non-split classical quaternion algebra $\mathbb{H}(K)$ over an imaginary quadratic extension K of the rationals. We then apply this method in case $K = \mathbb{Q}(\sqrt{-7})$ and obtain a presentation of $\mathbb{H}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])^*$. Since the latter group is closely related to $\mathrm{SO}_3(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$ we also obtain a presentation of this group. The difficulty with this is that the bilinear form associated to the group $\mathrm{SO}_3(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$ is non-singular because 2 is not invertible in $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$. Hence we will have to obtain a new exact sequence from the well-known Cartan–Diedonné sequence [6, 7.2.20] relating the two mentioned groups.

Let Q_8 denote the quaternion group of order 8. Clearly $\mathbb{Q}(\sqrt{-7})Q_8 = 4\mathbb{Q}(\sqrt{-7}) \oplus \mathbb{H}(\mathbb{Q}(\sqrt{-7}))$ and thus $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]Q_8 \subseteq 4\mathbb{Z}[\frac{1+\sqrt{-7}}{2}] \oplus \mathbb{H}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$. As $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]^* = \{1, -1\}$ it follows that $(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]Q_8)^* \subseteq \{1, -1\}^4 \times \mathbb{H}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])^*$. Hence the above gives us a description of the unit group of the group ring $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]Q_8$. This is the "simplest example" of a group ring for which no finite set of generators of a subgroup of finite index in its unit group is known (see [7,12]). For integral group rings, the group of least order for which this problem is still open is $Q_8 \times C_7$, the direct product of Q_8 with the cyclic group C_7 of order 7. In this situation, $\mathbb{Z}[Q_8 \times C_7]^* \subseteq \{1, -1\}^4 \times \pm Q_8 \times (\mathbb{Z}[\xi_7]^*)^4 \times (\mathbb{Z}[\xi_7]^*)^4 \mathbb{H}\mathbb{Q}(\mathbb{Z}[\xi_7])^*$, where ξ_7 is a primitive

7th root of unity. Hence the study of the unit group is reduced to that of $\mathbb{H}(\mathbb{Z}[\xi_7])$. Now $\mathbb{H}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]) \subseteq \mathbb{H}(\mathbb{Z}[\xi_7])$ and there are 6 complex embeddings of $\mathbb{Q}(\xi_7)$. Hence the calculations become much more involved and it remains a challenge to obtain a presentation for $\mathbb{H}(\mathbb{Z}[\xi_7])^*$.

2. Preliminaries

In this section we introduce some notation and recall a fundamental result that is essential for our investigations.

Let K be an algebraic number field and R its ring of integers. For nonzero $a, b \in K$ we denote by $\mathbb{H}(K) = (\frac{a,b}{K})$ the quaternion K-algebra induced by a, b, that is, $\mathbb{H}(K)$ is the K-algebra given by

$$\mathbb{H}(K) = K[i,j|i^2 = a, j^2 = b, ji = -ij].$$

As usual we write $k = ij$, so that $\{1, i, j, k\}$ is a K-basis of $\mathbb{H}(K)$. If $a, b \in R$ then let

$$\mathbb{H}(R) = \left(\frac{a,b}{R}\right) = R[i,j],$$

that is, $\mathbb{H}(R)$ is the R-algebra consisting of the R-linear sums of $\{1, i, j, k\}$. Let $n : \mathbb{H}(K) \to K$ denote the norm map, that is,

$$n(x) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 \tag{1}$$

for $x = x_0 + x_1i + x_2j + x_3k \in \mathbb{H}(K)$. The group of units $\mathbb{H}(R)^*$ of $\mathbb{H}(R)$ is commensurable with (i.e., it has a common subgroup of finite index with) $R^* \times SL_1(\mathbb{H}(R))$ where

$$SL_1(\mathbb{H}(R)) = \{x \in \mathbb{H}(R) \mid n(x) = 1\}.$$

Therefore the study of the structure of $\mathbb{H}(R)^*$ is reduced to that of R^* and $SL_1(\mathbb{H}(R))$. Since the Dirichlet Unit Theorem deals with the structure of R^* we will investigate $SL_1(\mathbb{H}(R))$. In case $R^* = \{\pm 1\}$ then of course $SL_1(\mathbb{H}(R))$ is of index at most 2 in $\mathbb{H}(R)^*$.

The field $F = K[i]$ is a maximal subfield of $\mathbb{H}(K)$, and $\mathbb{H}(K) = F \oplus Fj$ is a crossed product over F. The Galois group of the extension F/K is a cyclic group of order two generated by the restriction σ of the inner automorphism of $\mathbb{H}(K)$ induced by j, that is $\sigma(x) = jxj^{-1}$, for every $x \in F$. Then $\mathbb{H}(K)$ can be embedded in $M_2(\mathbb{C})$ by the map

$$x + yj \mapsto \begin{pmatrix} x & y \\ b\sigma(y) & \sigma(x) \end{pmatrix}. \tag{2}$$

Moreover, this embedding maps the elements of norm 1 into $SL_2(\mathbb{C})$. Therefore we may identify $SL_1(\mathbb{H}(K))$ and $SL_1(\mathbb{H}(R))$ with subgroups of $SL_2(\mathbb{C})$.

Since $PSL_2(\mathbb{C})$ is the group of orientation preserving isometries of the three-dimensional hyperbolic space H^3, the group $SL_1(\mathbb{H}(R))$ acts on H^3 and we can use this action to study $\mathbb{H}(R)^*$. The best situation is when this action is discontinuous or equivalently when $SL_1(\mathbb{H}(R))$ is discrete. In case $\mathbb{H}(K)$ splits, then, using that the rank of a free abelian discrete subgroup of $SL_2(\mathbb{C})$ is at most 2 [3, Theorem 1.1.8] and at most 1 if it is embedded in $PSL_2(\mathbb{R})$ (see Theorems 2.2.5 and 2.2.7 in [5]), it is easy to see that the action of $\mathbb{H}(R)$ on H^3 is discontinuous if and only if $K = \mathbb{Q}$ or an imaginary quadratic field. In this situation $SL_1(\mathbb{H}(R)) = SL_2(R)$ and the investigations reduce to the study of the corresponding Bianchi group $PSL_2(R)$. These groups have been widely studied [2,3,5,11]. In this paper we are interested in the non-splitting case and mainly in the case that K is imaginary quadratic. The following theorem shows that then the action of $SL_1(\mathbb{H}(R))$ on H^3 is discontinuous.

Theorem 2.1 (Elstrodt et al. [3, Theorem 10.1.2]). *Let* $\mathbb{H}(K) = K[i,j|i^2 = a, j^2 = b, ji = -ij]$ *be a quaternion algebra over a number field K and assume that the following conditions hold*:

- K *has exactly one pair of complex embeddings* (*also known as complex archemedean places*);
- $\mathbb{H}(K)$ *is ramified at all the real places, that is,* $\mathbb{H}(\sigma(K)) \otimes_K \mathbb{R}$ *is a division ring* (*necessarily the ring of Hamiltonian quaternions* $\mathbb{H}(\mathbb{R})$), *for every real embedding σ of K.*

Then for every order Γ in $\mathbb{H}(K)$:

1. $SL_1(\Gamma)$ *is discrete.*
2. $SL_1(\Gamma)$ *has finite covolume* (*i.e. the fundamental domains have finite volume*) *and hence* [3, *Theorem* 2.27] *it is geometrically finite* (*i.e., all Dirichlet or Poincaré normal polyhedra have finitely many sides*).
3. $SL_1(\Gamma)$ *is cocompact* (*i.e. it has a compact fundamental domain*) *if and only if* $\mathbb{H}(K)$ *is a division ring.*

Assume that the conditions of Theorem 2.1 hold. Then both the Poincaré [10] and the Swan methods [15] give presentations of $SL_1(\mathbb{H}(R))$. For more details on these methods and on groups acting discontinuously on H^3 we refer the reader to [1,3,5]. Note that to use the Poincaré method it is necessary to compute a convex locally finite fundamental polyhedron for $SL_1(\mathbb{H}(R))$. For the Swan method, it is enough [3, Theorem 2.7.1] to find a connected bounded open subset X of H^3 that contains a fundamental domain (indeed, since the fundamental domain is compact the set $\{g \in SL_1(\mathbb{H}(R)): X \cap g(X) \neq \emptyset\}$ is finite).

As in [3], we identify H^3 with the following subset of the Hamiltonian quaternions $\mathbb{H}(\mathbb{R})$:

$$H^3 = \{z + rj | z \in \mathbb{C}, r \in \mathbb{R}^+\},$$

where $\mathbb{C}$ is considered inside $\mathbb{H}(\mathbb{R})$ in the obvious way. The hyperbolic distance ρ in H^3 is determined by the formula

$$\cosh \rho(P, P') = \delta(P, P') = 1 + \frac{d(P, P')}{2rr'},$$

where d is the Euclidean distance and $P = z + rj$ and $P' = z' + r'j$ are two elements of H^3. The open and closed ball of radius $r > 0$ centred at $P \in H^3$ are denoted respectively by

$$B(P, r) = \{x \in H^3 | \rho(x, P) < r\},$$

$$\bar{B}(P, r) = \{x \in H^3 | \rho(x, P) \leqslant r\}.$$

The norm in $M_2(\mathbb{C})$ is denoted by $|| \ ||$, that is, if $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{C})$ then

$$||g||^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2.$$

If c and d are two different elements of H^3 then the set of points equidistant (in the metric ρ) to both c and d is a geodesic plane called the bisector between c and d and it intersects orthogonally the hyperbolic segment joining c and d. This bisector is the border of two open convex subsets of H^3 (called half spaces) one containing c and the other containing d. If $g \in \mathrm{SL}_2(\mathbb{C})$ and $c \in H^3$ is not a fixed point of g, then put

$$D_g(c) = \{x \in H^3 | \rho(x, c) \leqslant \rho(x, g(c))\},$$

the half space containing c. If G is a discrete group contained in $\mathrm{SL}_2(\mathbb{C})$ and $c \in H^3$ is not fixed by any non-trivial element of G then

$$D_G(c) = \bigcap_{1 \neq g \in G} D_g(c)$$

is known as a Dirichlet or Poincaré fundamental polyhedron of G with centre c and it is well known that $D_G(c)$ is a convex locally finite fundamental polyhedron of G.

It is easy to see that every Dirichlet fundamental domain of a cocompact discrete group is compact.

3. A finite algorithm to compute a fundamental domain of cocompact discrete groups

The difficulty with computing $D_G(c)$ is that, theoretically, it is necessary to compute the intersection of as many sets as the cardinality of G. We now describe a finite algorithm to compute a Dirichlet fundamental polyhedron for cocompact discrete groups. For a positive real number r let

$$D_r(c) = \bigcap \{D_g(c) | 1 \neq g \in G, g(c) \in \bar{B}(c, r)\}.$$

If $D_G(c)$ is compact, then there is a positive real number r such that $D_G(c) = D_r(c)$.

Proposition 3.1. *Let G be a cocompact discrete group and let $c \in H^3$ not fixed by any non-trivial element of G. Then the following algorithm computes the Dirichlet fundamental polyhedron $D = D_G(c)$ in a finite number of steps.*

Input: *G.*

Step 1: *Select a strictly increasing unbounded sequence (k_n) of positive numbers.*

Step 2: *Compute $D_{k_1}(c), D_{k_2}(c), \ldots$ until $D_{k_n}(c)$ is compact (or, equivalently, bounded).*

Step 3: *Compute $r = \max(\{\frac{k_n}{2}\} \cup \{\rho(c, x): x \in D_{k_n}(c)\})$.*

Step 4: *Compute $D = D_{2r}(c)$.*

Output: *D.*

Proof. The fact that Step 2 stops after finitely many computations is a consequence of the fact that $D_G(c)$ is compact and has finitely many sides. To prove that $D = D_G(c)$, note that

$$D_G(c) = D_{2r}(c) \cap \bigcap_{\rho(c,g(c)) > 2r} D_g(c).$$

Thus it is enough to show that for every $g \in G$ such that $\rho(c, g(c)) > 2r$, $D \subseteq D_{k_n}(c) \subseteq \bar{B}(c, r) \subseteq D_g(c)$. The first equality is obvious and the second is a consequence of the selection of r in Step 3. Finally, if $x \in \bar{B}(c, r) \backslash D_g(c)$, then $d(x, g(c)) < d(x, c) \leqslant r$. Thus $2r < d(c, g(c)) \leqslant d(c, x) + d(x, g(c)) < 2r$, a contradiction. $\quad\square$

Note also that for the Swan method it is enough to stop the algorithm of Proposition 3.1 when Step 2 has finished because then $D_{k_n}(c)$ is contained in a bounded connected open subset O of H^3 and therefore $O \cap g(O) = \emptyset$ for all but finitely many $g \in G$. However even for the Swan method it is convenient to perform Steps 3 and 4 in order to make the open and connected set O smaller. Indeed, the open set can be taken inside the open ball $B(c, r + \varepsilon)$ for small ε. Then the group elements $g \in G$ that satisfy $O \cap g(O) \neq \emptyset$ also satisfy $d(c, g(c)) \leqslant 2(r + \varepsilon)$.

Let G be a cocompact discrete group. In order to make the algorithm of Proposition 3.1 effective we need effective algorithms for the following tasks.

Task 1: Compute all the elements $g \in G$, so that $g(c) \in \bar{B}(c, r)$, for a given $r > 0$.

Task 2: Compute $\max\{\rho(c, x): x \in P\}$, for a convex and compact polyhedron P in H^3 containing c and with finitely many sides.

Task 2 is easy to achieve because the maximum distance to a point in a convex polyhedron is realized at one of its vertices and if the polyhedron has finitely many sides this can be calculated with finitely many computations.

For Task 1 remember that for every $g \in SL_2(\mathbb{C})$ [1, Theorem 4.2.1]

$$\|g\|^2 = 2 \cosh \rho(j, g(j)). \tag{3}$$

Therefore if $c = j$, Task 1 reduces to

Task 1$'$: For a given positive number r, compute all elements of G of norm at most r.

For most of the interesting groups, j is fixed by some non-trivial element. Theoretically, this problem can be solved by replacing G by $g^{-1}Gg$ where $g \in SL_2(\mathbb{C})$ is such that $g(j) = c$, and hence we may assume that $c = j$. Nevertheless, this usually increases the difficulty of Task $1'$ and consequently decreases the effectiveness of the possible algorithms to accomplish it. In order to avoid this conjugation process we propose the following alternative method to compute a fundamental domain of G. Let $G_j = \{g \in G : g(j) = j\}$ the stabilizer of G, which is finite because G is discrete. Then it is usually easy to compute a convex fundamental polyhedron F of G_j. For every $g \in G \backslash G_j$ let $F_g = F \cap D_g(j)$ and for every positive real number r let F_r be the intersection of all the sets F_g with $g \in G$ so that $0 < \rho(c, g(c)) \leqslant r$.

Proposition 3.2. *Let G be a cocompact discrete group, $G_j = \{g \in G: g(j) = j\}$ the stabilizer of j under the action of G on H^3 and F a convex fundamental polyhedron of G_j. Then the following algorithm computes a convex fundamental polyhedron of G in a finite number of steps.*
   Input: G.
   Step 1: Compute a fundamental domain of the stabilizer G_j of j.
   Step 2: Select a strictly increasing unbounded sequence of positive numbers k_n.
   Step 3: Compute $F_{k_1}, F_{k_2}, \ldots$ until F_{k_n} is compact (or, equivalently, bounded).
   Step 4: Compute $r = \max(\{\frac{k_n}{2}\} \cup \{\rho(j, x): x \in F_{k_n}\})$.
   Step 5: Compute $D = F_{2r}$.
   Output: D.

Proof. After proving that $D_1 = \bigcap_{g \in G \backslash G_j} F_j$ is a convex and compact fundamental domain of G one can proceed as in the proof of Proposition 3.1. So let us prove that D_1 has these properties.

First, D_1 is convex as a consequence of the fact that it is the intersection of convex sets. To prove that D_1 is a fundamental domain it is enough to use the ideas of the proof of [1, Theorem 9.6.1]. We include a self-contained proof for completeness. The boundary of D_1 is embedded in the union of countably many geodesic planes of H^3 and hence it has Lebesgue measure zero. So it only remains to prove that if $g(x) = y$ with x and y interior points of D_1 and $g \in G$ then $x = y$ and every orbit contains one element in the closure of D_1. To prove the former, let g, x and y be as above and assume that $x \neq y$. Then $g \neq 1$ and x and y belong to the interior of F. Since F is a fundamental domain of G_j we have that $g \notin G_j$. Furthermore, x and y belong to the interior of $D_g(j)$ and hence

$$\rho(j, y) < \rho(g(j), y) = \rho(j, x) < \rho(j, g(x)) = \rho(j, y),$$

a contradiction. Now let O be an orbit of the action of G on H^3; we have to prove that it contains one element in the closure of D_1. Since the intersection of O with every compact set of H^3 is finite, there is an $u \in O \cap F$ so that $\rho(u, j) \leqslant \rho(v, j)$ for every $v \in O$. We claim that $u \in D_1$. Suppose the contrary, then there is $g \in G$ so that

$\rho(u,j) > \rho(u,g(j))$. Let $h \in G_j$ be so that $hg^{-1}(u) \in F$. It follows that $\rho(u,j) > \rho(u,g(j)) = \rho(hg^{-1}(u),h(j)) = \rho(hg^{-1}(u),j)$, in contradiction with the choice of u.

Finally, we prove that D_1 is compact. Notice that since G is cocompact it has a compact fundamental domain, X say, containing j. Let $X \subseteq \bar{B}(j,r)$ for $r > 0$. We claim that $D_1 \subseteq \bar{B}(j,r)$ (and thus the compactness of D_1 follows). Indeed, if $z \notin \bar{B}(j,r)$ then $z \notin X$ and, hence, there is $g \in G$ such that $g(z) \in X$. Then $d(g^{-1}(j),z) = d(j,g(z)) \leqslant r$ and, consequently, $d(z,j) > d(g^{-1}(j),z)$; it follows that $g^{-1} \notin G_j$ and $z \notin F_{g^{-1}}(j)$. Thus $z \notin D_1$. $\square$

4. An example

In this section we show how to apply the algorithm of Proposition 3.2 to the group $SL_1(\mathbb{H}(R))$ where $R = \mathbb{Z}[\omega]$ is the ring of integers of $K = \mathbb{Q}(\sqrt{-7})$, $\omega = \frac{1+\sqrt{-7}}{2}$ and $\mathbb{H}(K) = \left(\frac{-1,-1}{K}\right)$, the classical quaternion algebra over K.

The first two steps can be done in the more general context of quadratic imaginary extensions $K = \mathbb{Q}(\sqrt{-d})$, where d is a square-free positive integer congruent with 7 modulo 8. It then follows that the ring of integers is $R = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{-d}}{2}$ and that the prime integer 2 splits in R, that is, $2R = pq$ where p and q are different prime ideals of R. As a consequence of the behaviour of the prime 2 one can deduce using local class field theory [13] that $\mathbb{H}(K)$ is a division algebra. An alternative approach is noticing that $\mathbb{H}(K)$ splits if and only if -1 is a sum of squares in K. Using this approach the fact that $\mathbb{H}(K)$ is a division ring for d prime is a consequence of the results of [4].

Clearly, the maximal subfield of $\mathbb{H}(K)$ mentioned in the introduction is $F = K[i] = \mathbb{Q}[\sqrt{-d},i]$. The Galois group of $F/\mathbb{Q}$ has four elements $\{1,\sigma,\tau,\sigma\tau\}$ and these act as follows:

$$\sigma(\omega) = \omega, \quad \sigma(i) = -i,$$
$$\tau(\omega) = \bar{\omega}, \quad \tau(i) = i.$$

Let $S = R[i] = \mathbb{Z}[\omega,i]$ and $B = \{1,i,\omega,\omega i\}$. Note that the automorphism σ of F is induced by conjugating by j. Hence, we may identify $SL_1(\mathbb{H}(R))$ with the subgroup G of $SL_2(\mathbb{C})$ consisting of all matrices of the form

$$g = \begin{pmatrix} g_1 & g_2 \\ -\sigma(g_2) & \sigma(g_1) \end{pmatrix} \tag{4}$$

with

$$g_1 = a_0 + b_0\omega + a_1 i + b_1\omega i \in S \quad \text{and} \quad g_2 = a_2 + b_2\omega + a_3 i + b_3\omega i \in S,$$

where all a_i's and b_i's are rational integers. (Note that the discreteness of G, ensured by Theorem 2.1, is now elementary because it is well known from algebraic number theory that the map $(1, \sigma): S \to \mathbb{C}^2$ maps S onto a lattice of $\mathbb{C}^2 = \mathbb{R}^4$.)

In order to go through the several steps, we first deal with Task 1'. Thus, let $g \in G$ be as in (4) and set

$$\alpha = a_0^2 + a_1^2 + a_2^2 + a_3^2,$$

$$\beta = b_0^2 + b_1^2 + b_2^2 + b_3^2,$$

$$\gamma = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Note that

$$|g_1|^2 + |\sigma(g_1)|^2 = 2(a_0^2 + a_1^2) + \frac{d+1}{2}(b_0^2 + b_1^2) + 2(a_0 b_0 + a_1 b_1)$$

and similarly

$$|g_2|^2 + |\sigma(g_2)|^2 = 2(a_2^2 + a_3^2) + \frac{d+1}{2}(b_2^2 + b_3^2) + 2(a_2 b_2 + a_3 b_3).$$

Therefore

$$||g||^2 = 2\alpha + \frac{d+1}{2}\beta + 2\gamma. \tag{5}$$

Moreover, $\det(g) = \alpha - \frac{d+1}{4}\beta + (\beta + 2\gamma)\omega$, so that

$$\det(g) = 1 \quad \text{if and only if} \quad \gamma - \frac{d+1}{4}\beta = 1 \quad \text{and} \quad \beta + 2\gamma = 0. \tag{6}$$

Lemma 4.1. *Let $g \in G$ be as in (4), with $\det(g) = 1$ and $l = ||g||^2$. Then*

$$\alpha = 1 + \frac{(d+1)(l-2)}{4d}, \quad \beta = \frac{l-2}{d}, \quad and \quad \gamma = \frac{2-l}{2d}.$$

In particular $l \equiv 2 \bmod 2d$.

Proof. Because of (5) and (6), α, β and γ are integral solutions of the following linear system of equations:

$$2\alpha + \frac{d+1}{2}\beta + 2\gamma = l,$$

$$\alpha - \frac{d+1}{4}\beta = 1,$$

$$\beta + 2\gamma = 0.$$

Hence the result follows. $\square$

Hence, in order to deal with Task $1'$, we need an algorithm to compute the set

$$G_n = \{g \in G : ||g||^2 = 2 + 2dn\}$$

for every non-negative integer n. Lemma 4.1 reduces this problem to compute all the matrices

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix} \in M_{2,4}(\mathbb{Z})$$

satisfying the following equations:

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 + \frac{d+1}{2}n,$$

$$b_0^2 + b_1^2 + b_2^2 + b_3^2 = 2n,$$

$$a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3 = -n.$$

We identify G_n with the set of these matrices.

Let C_2 denote the cyclic group of order 2, put $\mathscr{C} = \prod_{i=1}^4 \langle s_i \rangle \simeq C_2^4$ and let S_4 denote the symmetric group on four letters. Obviously S_4 acts on $\mathscr{C}$ by permutations of the generators, that is, for $\gamma \in S_4$, $\gamma(s_i) = s_{\gamma(i)}$. Let $X = \mathscr{C} \rtimes S_4$ denote the induced semidirect product. Then the following is a right action of X on $M_{2,4}(\mathbb{Z})$:

$$(c_1, c_2, c_3, c_4) \cdot s_1 = (-c_1, c_2, c_3, c_4),$$

$$(c_1, c_2, c_3, c_4) \cdot s_2 = (c_1, -c_2, c_3, c_4),$$

$$(c_1, c_2, c_3, c_4) \cdot s_3 = (c_1, c_2, -c_3, c_4),$$

$$(c_1, c_2, c_3, c_4) \cdot s_4 = (c_1, c_2, c_3, -c_4),$$

$$(c_1, c_2, c_3, c_4) \cdot \gamma = (c_{\gamma(1)}, c_{\gamma(2)}, c_{\gamma(3)}, c_{\gamma(4)}),$$

where $c_1, \ldots, c_4$ are the columns of an element in $M_{2,4}(\mathbb{Z})$ and $\gamma \in S_4$. Clearly X induces an action on G_n, and hence to know G_n it is enough to produce a set of representatives R_n of each X-orbit inside G_n.

For example, G_0 has only one orbit represented by

$$r_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and for $d = 7$, G_1 has two orbits represented by

$$r_{1,1} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad r_{1,2} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

Since the stabilizers of r_0, $r_{1,1}$ and $r_{1,2}$ are $\langle s_2, s_3, s_4, (2\,3\,4), (2\,3) \rangle$, $\langle s_3, s_4, (3\,4) \rangle$ and $\langle s_4 \rangle$ respectively, the respective orbits have 8, 48 and 192 elements. Hence $|G_0| = 8$ and $|G_1| = 240$.

Thus, to compute G_n we first compute a set R_n of representatives of the X-orbits of G_n and, next, for each $x \in R_n$ we first compute a right transversal S_x of X/X_x, where X_x is the stabilizer of x in X, and then we compute xg for every $g \in S_x$.

We now go through the several steps of the algorithm of Proposition 3.2. We complete Steps 1 and 2 for the general case, and the other steps for $d = 7$.

Step 1: The stabilizer G_j of j is precisely G_0 which has one X-orbit represented by the matrix r_0 and hence G_j has eight elements and

$$G_j = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle,$$

which is isomorphic to the quaternion group of order 8. Further

$$F = \{ P = x + yi + rj \in H^3 : |P| \leqslant 1, y \geqslant 0 \}$$

is a fundamental domain of G_j.

Step 2: Because of Lemma 4.1 and (3), a good selection for the sequence (k_n) is given by the formula $2 \cosh k_n = 2 + 2dn$.

Step 3: To proceed with the computations, we need to calculate $G_n j$ for $n = 1, 2, \ldots$. We have seen how to compute G_n and thus this task can be executed for specific d. Furthermore as G_n is a union of G_0-cosets, we can simplify these calculations by working with a set T_n of representatives of the left G_0-cosets of the elements of G_n and note that $G_n j = T_n j$. Consequently, we only need to compute a left transversal of G_n/G_0. Now note that the right action of G_0 on G coincides with the action of the subgroup $Y = \langle s_2 s_3 (1\,2)(3\,4), s_3 s_4 (1\,3)(2\,4) \rangle$ of X, because for every $g \in G$ one has

$$g \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = g \cdot s_2 s_3 (1\,2)(3\,4) \quad \text{and} \quad g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = g \cdot s_3 s_4 (1\,3)(2\,4).$$

Therefore, to compute T_n it is enough to compute a left transversal T_x of $\langle Y, X_x \rangle$ in X for every $x \in X_n$, and then $T_n = \cup_{x \in R_n} x T_x$.

In order to do the actual calculations we need to specify a concrete value for p. *From now on we assume $d = 7$.* As will be explained later we need to compute X_n and T_n for $n \leqslant 10$. Table 1 shows the results obtained using Mathematica. In the table, $S_3 = \langle (1\,2), (1\,2\,3) \rangle$ and $A_4 = \langle (1\,2)(3\,4), (1\,2\,3) \rangle$.

A little word of explanation on how, for example, the second row (with $n = 1$) is obtained. We have put $R_1 = \{r_{1,1}, r_{1,2}\}$, $X_{r_{1,2}} = \langle s_4 \rangle$ and $X_{r_{1,1}} = \langle s_3, s_4, (3\,4) \rangle$. Then a right transversal of $\langle Y, X_{r_{1,1}} \rangle$ in X is $T_{r_{1,1}} = S_3$ and a right transversal of $\langle Y, X_{r_{1,2}} \rangle$ in X is $T_{r_{1,2}} = \langle s_1, s_2 \rangle \times S_3$ (note that the latter only is a subset and not a subgroup of X; so one should not confuse it with $\langle s_1, s_2, S_3 \rangle = X$).

Next, we first compute T_1 by applying the six elements of $T_{r_{1,1}}$ to $r_{1,1}$ and the twenty four elements of $T_{r_{1,2}}$ to $r_{1,2}$, and then we compute gj and the bisectors between j and gj for all the $g \in T_1$. The results appear in the following table, where C_g and R_g denote the centre and the radius, respectively, of the bisector between j and gj.

We now can compute F_{k_1}, the intersection of F with the 30 $D_g(j)$'s for which $||g||^2 = 16$, that is the 30 half spaces containing j and limited by the 30 spheres of Table 2. To visualize the result we represent in Fig. 1 the base of these spheres, that is, the intersection of these spheres with the plane $\mathbb{C}$, border of H^3. In Fig. 1, we represent also the base of the fundamental domain F of G_j computed above:

It is clear that many of the spheres displayed in Fig. 1 do not contribute to the intersection. The biggest sphere (the one with radius $\sqrt{8 + 3\sqrt{7}}$) contains F, while the eight spheres with next biggest radius ($\frac{2}{3}\sqrt{\frac{11}{2} + 2\sqrt{7}}$) do not intersect F, so they can be dropped from the list of spheres. We can also ignore the seven spheres completely embedded in the half space not containing j limited by the plane $Y = 0$. Of the remaining 14 spheres, the two of biggest radius and the two centred below the line $y = 0$ are also unnecessary. The remaining spheres are displayed in Fig. 2. It is now clear that the remaining ten spheres cover the base F and, hence, it follows that F_{k_1} is compact. A three dimensional representation of the ten spheres and the boundary of F is given in Fig. 3. For further use in the paper we introduce names and notation for the bisectors. As the picture reminds us of the set up of an orchestra, we have chosen the names accordingly. In Fig. 4 we illustrate the projection on $\mathbb{C}$ of the 10 contributing spheres. In Table 3 we list the bisectors (via their center and radius) and the group element associated to each of them.

So following the notation of Proposition 3.2 F_{k_1} is compact and hence we already have completed Step 3.

Step 4: We now need to compute $r = \max(\{k_1/2\} \cup \{\rho(j, x) : x \in F_{k_1}(j)\})$. Since F_{k_1} is convex, the maximum of the distances $\rho(j, x)$ for $x \in F_{k_1}$ is realized at one of the vertices of F_{k_1}. These vertices are the points of intersection of three of the 12 sides of F_{k_1}. There are 20 such vertices (we present them below as the intersection of three bisectors):

$$C \cap P \cap S_l; \ C \cap P \cap S_r; \ C \cap S_l \cap V_l; \ C \cap S_r \cap V_r; \ C \cap V_l \cap V_r; \ S_l \cap D_1 \cap P; \ S_r \cap D_r \cap P;$$

Table 1
R_n for $n \leqslant 10$

n	R_n	X_x	T_x	$|T_x|$
0	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\langle s_2, s_3, s_4, (2\,3\,4), (2\,3) \rangle$	1	1
1	$\begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$	$\langle s_3, s_4, (3, 4) \rangle$	S_3	6
	$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$	$\langle s_4 \rangle$	$\langle s_1, s_2 \rangle \times S_3$	24
2	$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}$	$\langle s_4, (1, 2) \rangle$	$\langle s_1, s_2 \rangle \times \langle (1\,2\,3) \rangle$	12
3	$\begin{pmatrix} 2 & 2 & 2 & 1 \\ -2 & 0 & 1 & -1 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 3 & 2 & 0 & 0 \\ -1 & 0 & -2 & -1 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
4	$\begin{pmatrix} 3 & 2 & 2 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
5	$\begin{pmatrix} 3 & 2 & 2 & 2 \\ -1 & -2 & -1 & 2 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 4 & 2 & 1 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 4 & 2 & 1 & 0 \\ -2 & 2 & -1 & -1 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 4 & 2 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix}$	$\langle s_4 \rangle$	$\langle s_1, s_2 \rangle \times S_3$	24
6	$\begin{pmatrix} 4 & 2 & 2 & 1 \\ -2 & 0 & 2 & -2 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 4 & 2 & 2 & 1 \\ 0 & -2 & -2 & 2 \end{pmatrix}$	$\langle (2, 3) \rangle$	$\langle s_1 \rangle \times A_4$	24
	$\begin{pmatrix} 4 & 3 & 0 & 0 \\ 0 & -2 & -2 & -2 \end{pmatrix}$	$\langle (3, 4) \rangle$	$\langle s_1 \rangle \times A_4$	24
7	$\begin{pmatrix} 4 & 3 & 2 & 0 \\ 0 & -3 & 1 & -2 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 4 & 3 & 2 & 0 \\ 0 & -1 & -2 & -3 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
8	$\begin{pmatrix} 4 & 3 & 2 & 2 \\ 0 & 0 & -4 & 0 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 5 & 2 & 2 & 0 \\ 0 & -4 & 0 & 0 \end{pmatrix}$	$\langle s_4 \rangle$	$\langle s_1, s_2 \rangle \times S_3$	24
9	$\begin{pmatrix} 4 & 4 & 2 & 1 \\ -3 & 0 & 0 & 3 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 4 & 4 & 2 & 1 \\ -3 & 2 & -2 & -1 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
		$\langle (1, 2) \rangle$	$\langle s_1 \rangle \times A_4$	24

Table 1 (*continued*)

n	R_n	X_x	T_x	$|T_x|$
	$\begin{pmatrix} 4 & 4 & 2 & 1 \\ -2 & -2 & 3 & 1 \end{pmatrix}$			
	$\begin{pmatrix} 4 & 4 & 2 & 1 \\ 0 & 0 & -3 & -3 \end{pmatrix}$	$\langle (1,2) \rangle$	$\langle s_1 \rangle \times A_4$	24
	$\begin{pmatrix} 4 & 4 & 2 & 1 \\ 2 & -3 & -2 & -1 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 5 & 2 & 2 & 2 \\ -3 & -1 & 2 & 2 \end{pmatrix}$	$\langle (3,4) \rangle$	$\langle s_1 \rangle \times A_4$	24
	$\begin{pmatrix} 5 & 2 & 2 & 2 \\ -3 & 0 & 0 & 3 \end{pmatrix}$	$\langle (2,3) \rangle$	$\langle s_1 \rangle \times A_4$	24
	$\begin{pmatrix} 5 & 2 & 2 & 2 \\ 1 & -3 & -2 & -2 \end{pmatrix}$	$\langle (3,4) \rangle$	$\langle s_1 \rangle \times A_4$	24
	$\begin{pmatrix} 6 & 1 & 0 & 0 \\ -2 & 3 & -2 & -1 \end{pmatrix}$	1	$\langle s_1, s_2, s_3 \rangle \times S_3$	48
	$\begin{pmatrix} 6 & 1 & 0 & 0 \\ -1 & -3 & -2 & -2 \end{pmatrix}$	$\langle (3,4) \rangle$	$\langle s_1 \rangle \times A_4$	24
10	$\begin{pmatrix} 4 & 4 & 3 & 0 \\ -4 & 0 & 2 & 0 \end{pmatrix}$	$\langle s_4 \rangle$	$\langle s_1, s_2 \rangle \times S_3$	24
	$\begin{pmatrix} 5 & 4 & 0 & 0 \\ -2 & 0 & -4 & 0 \end{pmatrix}$	$\langle s_4 \rangle$	$\langle s_1, s_2 \rangle \times S_3$	24
	$\begin{pmatrix} 6 & 2 & 1 & 0 \\ 0 & -4 & -2 & 0 \end{pmatrix}$	$\langle s_4 \rangle$	$\langle s_1, s_2 \rangle \times S_3$	24

$S_l \cap D_l \cap W_l;\ S_r \cap D_r \cap W_r;\ S_l \cap V_l \cap W_l;\ S_r \cap V_r \cap W_r;\ Ch \cap V_r \cap W_r;\ Ch \cap V_l \cap W_l;$

$Ch \cap V_l \cap V_r;\ D_r \cap R \cap P;\ D_l \cap R \cap P;\ D_l \cap W_l \cap R;\ D_r \cap W_r \cap R;\ Ch \cap W_l \cap R;$

$Ch \cap W_r \cap R$

Computing $\rho(j, v)$ for all the vertices v, one obtains that the maximum is $\cosh^{-1} 6$. Moreover, $2\cosh k_1 = 16$ and therefore $k_1 = \cosh^{-1} 8$. Thus $r = \max(\frac{\cosh^{-1} 8}{2}, \cosh^{-1} 6) = \cosh^{-1} 6$.

Step 5: Next we calculate $D = F_{2r}$, which is the fundamental domain of our group. Using $\cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{1 + \cosh(2x)}{2}$ we obtain that $\cosh 2r = 2\cosh^2 r - 1 = 71$. Because of (3) the elements of the orbit of j in $\bar{B}(j, 2r)$ coincide with the elements of the form gj with $g \in G$ satisfying $||g||^2 \leqslant 142$. These are the group elements associated to the matrices displayed in the second column of Table 1. Hence we can compute the set O of elements in the orbit of j at distance less than or equal to $2r$. Next, for any $x \in O$ we have to compute the bisector between j and x and we have to check if

Table 2
The bisectors for the elements of T_1

$g \in T_1$	gj	C_g	R_g
$1 + 2i + w - iw$	$(8 + 3i - 6iw)j$	$(0, 0)$	$\sqrt{8 + 3\sqrt{7}}$
$2 + i - w + iw$	$(8 - 3i + 6iw)j$	$(0, 0)$	$\sqrt{8 - 3\sqrt{7}}$
$i + iw + (2 - w)j$	$\dfrac{3i - 6iw + j}{8}$	$\left(\dfrac{3}{\sqrt{7}}, 0\right)$	
$1 + w + (2 - w)j$	$\dfrac{3 - 6w + j}{8}$	$\left(0, -\dfrac{3}{\sqrt{7}}\right)$	$\sqrt{\dfrac{2}{7}}$
$2i - iw + (1 + w)j$	$\dfrac{-3i + 6iw + j}{8}$	$\left(-\dfrac{3}{\sqrt{7}}, 0\right)$	
$(2 - w) + (1 + w)j$	$\dfrac{-3 + 6w + j}{8}$	$\left(0, \dfrac{3}{\sqrt{7}}\right)$	
$-2 - iw + (1 - w)j$	$-\dfrac{1}{18} - i + \dfrac{8w}{9} + \dfrac{4iw}{9} + \left(\dfrac{2}{9} - \dfrac{i}{18} + \dfrac{iw}{9}\right)j$	$\left(-\dfrac{1}{3}(-2 + \sqrt{7}), \dfrac{2}{3}(-2 + \sqrt{7})\right)$	
$-2 + i - iw - wj$	$-\dfrac{5}{6} - \dfrac{5i}{9} + \dfrac{8w}{9} - \dfrac{4iw}{9} + \left(\dfrac{2}{9} - \dfrac{i}{18} + \dfrac{iw}{9}\right)j$	$\left(\dfrac{1}{3}(-2 + \sqrt{7}), \dfrac{2}{3}(-2 + \sqrt{7})\right)$	
$-1 - 2i + w - wj$	$\dfrac{5}{9} - \dfrac{5i}{6} + \dfrac{4w}{9} + \dfrac{8iw}{9} + \left(\dfrac{2}{9} - \dfrac{i}{18} + \dfrac{iw}{9}\right)j$	$\left(-\dfrac{2}{3}(-2 + \sqrt{7}), \dfrac{1}{3}(-2 + \sqrt{7})\right)$	
$-1 + 2i + w - wj$	$\dfrac{5}{9} + \dfrac{5i}{6} + \dfrac{4w}{9} - \dfrac{8iw}{9} + \left(\dfrac{2}{9} + \dfrac{i}{18} - \dfrac{iw}{9}\right)j$	$\left(-\dfrac{2}{3}(-2 + \sqrt{7}), -\dfrac{1}{3}(-2 + \sqrt{7})\right)$	$\dfrac{2}{3}\sqrt{\dfrac{11}{2} - 2\sqrt{7}}$
$2i - w + (1 - w)j$	$-1 + \dfrac{i}{18} + \dfrac{4w}{9} - \dfrac{8iw}{9} + \left(\dfrac{2}{9} - \dfrac{i}{18} + \dfrac{iw}{9}\right)j$	$\left(\dfrac{2}{3}(-2 + \sqrt{7}), -\dfrac{1}{3}(-2 + \sqrt{7})\right)$	
$1 - 2i - w - wj$	$-\dfrac{5}{9} - \dfrac{5i}{6} - \dfrac{4w}{9} + \dfrac{8iw}{9} + \left(\dfrac{2}{9} + \dfrac{i}{18} - \dfrac{iw}{9}\right)j$	$\left(-\dfrac{1}{3}(-2 + \sqrt{7}), -\dfrac{2}{3}(-2 + \sqrt{7})\right)$	
$1 - w + iw + 2j$	$-\dfrac{10}{19} + \dfrac{2i}{57} + \dfrac{32w}{57} - \dfrac{32iw}{57} + \left(\dfrac{8}{57} - \dfrac{i}{57} + \dfrac{2iw}{57}\right)j$	$\left(\dfrac{1}{3}(-2 + \sqrt{7}), -\dfrac{2}{3}(-2 + \sqrt{7})\right)$	
$2 - iw + (1 - w)j$	$\dfrac{1}{18} - i - \dfrac{8w}{9} + \dfrac{4iw}{9} + \left(\dfrac{2}{9} + \dfrac{i}{18} - \dfrac{iw}{9}\right)j$	$\left(\dfrac{2}{3}(-2 + \sqrt{7}), \dfrac{1}{3}(-2 + \sqrt{7})\right)$	
$-2 - i + iw - wj$	$-\dfrac{5}{6} + \dfrac{5i}{9} + \dfrac{8w}{9} + \dfrac{4iw}{9} + \left(\dfrac{2}{9} + \dfrac{i}{18} - \dfrac{iw}{9}\right)j$	$\left(\dfrac{1}{3}(-2 - \sqrt{7}), \dfrac{2}{3}(2 + \sqrt{7})\right)$	
$-2 + iw + (1 - w)j$	$-\dfrac{1}{18} + i + \dfrac{8w}{9} - \dfrac{4iw}{9} + \left(\dfrac{2}{9} + \dfrac{i}{18} - \dfrac{iw}{9}\right)j$	$\left(\dfrac{1}{3}(2 + \sqrt{7}), \dfrac{2}{3}(2 + \sqrt{7})\right)$	
$-2i + w + (1 - w)j$	$1 - \dfrac{i}{18} - \dfrac{4w}{9} + \dfrac{8iw}{9} + \left(\dfrac{2}{9} - \dfrac{i}{18} + \dfrac{iw}{9}\right)j$	$\left(\dfrac{2}{3}(2 + \sqrt{7}), \dfrac{1}{3}(2 + \sqrt{7})\right)$	
$2i + w + (1 - w)j$	$1 + \dfrac{i}{18} - \dfrac{4w}{9} - \dfrac{8iw}{9} + \left(\dfrac{2}{9} + \dfrac{i}{18} - \dfrac{iw}{9}\right)j$	$\left(\dfrac{2}{3}(2 + \sqrt{7}), -\dfrac{1}{3}(+2 + \sqrt{7})\right)$	$\dfrac{2}{3}\sqrt{\dfrac{11}{2} + 2\sqrt{7}}$
$-2i - w + (1 - w)j$	$-1 - \dfrac{i}{18} + \dfrac{4w}{9} + \dfrac{8iw}{9} + \left(\dfrac{2}{9} + \dfrac{i}{18} - \dfrac{iw}{9}\right)j$	$\left(-\dfrac{2}{3}(2 + \sqrt{7}), \dfrac{1}{3}(2 + \sqrt{7})\right)$	
$1 - w - iw + 2j$	$-\dfrac{10}{19} - \dfrac{2i}{57} + \dfrac{32w}{57} + \dfrac{32iw}{57} + \left(\dfrac{8}{57} + \dfrac{i}{57} - \dfrac{2iw}{57}\right)j$	$\left(-\dfrac{1}{3}(2 + \sqrt{7}), -\dfrac{2}{3}(2 + \sqrt{7})\right)$	
$1 + 2i - w - wj$	$\dfrac{5}{9} + \dfrac{5i}{6} - \dfrac{4w}{9} - \dfrac{8iw}{9} + \left(\dfrac{2}{9} - \dfrac{i}{18} + \dfrac{iw}{9}\right)j$	$\left(\dfrac{1}{3}(2 + \sqrt{7}), -\dfrac{2}{3}(2 + \sqrt{7})\right)$	
$2 + i - iw - wj$	$\dfrac{5}{6} - \dfrac{5i}{9} - \dfrac{8w}{9} - \dfrac{4iw}{9} + \left(\dfrac{2}{9} + \dfrac{i}{18} - \dfrac{iw}{9}\right)j$	$\left(-\dfrac{2}{3}(2 + \sqrt{7}), -\dfrac{1}{3}(2 + \sqrt{7})\right)$	
$-1 + w - iw + 2j$	$\dfrac{10}{19} - \dfrac{2i}{57} - \dfrac{32w}{57} + \dfrac{32iw}{57} + \left(\dfrac{8}{57} - \dfrac{i}{57} + \dfrac{2iw}{57}\right)j$	$\left(-\dfrac{1}{3}(-1 + \sqrt{7}), -\dfrac{1}{3}(-1 + \sqrt{7})\right)$	
$-i + w + iw + 2j$	$\dfrac{2}{57} + \dfrac{10i}{19} - \dfrac{32w}{57} - \dfrac{32iw}{57} + \left(\dfrac{8}{57} - \dfrac{i}{57} + \dfrac{2iw}{57}\right)j$	$\left(\dfrac{1}{3}(-1 + \sqrt{7}), -\dfrac{1}{3}(-1 + \sqrt{7})\right)$	$\sqrt{\dfrac{4 - \sqrt{7}}{3}}$

Table 2 (*continued*)

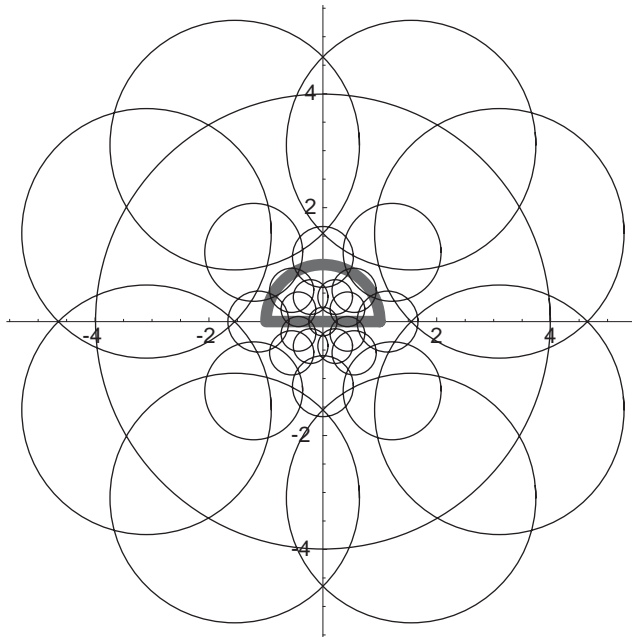| $g \in T_1$	gj	C_g	R_g
$i - w - iw + 2j$	$-\dfrac{2}{57} - \dfrac{10i}{19} + \dfrac{32w}{57} + \dfrac{32iw}{57} + \left(\dfrac{8}{57} - \dfrac{i}{57} + \dfrac{2iw}{57}\right)j$	$\left(-\dfrac{1}{3}(-1+\sqrt{7}), \dfrac{1}{3}(-1+\sqrt{7})\right)$	
$2 - i + iw - wj$	$\dfrac{5}{6} + \dfrac{5i}{9} - \dfrac{8w}{9} + \dfrac{4iw}{9} + \left(\dfrac{2}{9} - \dfrac{i}{18} + \dfrac{iw}{9}\right)j$	$\left(\dfrac{1}{3}(-1+\sqrt{7}), \dfrac{1}{3}(-1+\sqrt{7})\right)$	
$-1 + w + iw + 2j$	$\dfrac{10}{19} + \dfrac{2i}{57} - \dfrac{32w}{57} - \dfrac{32iw}{57} + \left(\dfrac{8}{57} + \dfrac{i}{57} - \dfrac{2iw}{57}\right)j$	$\left(\dfrac{1}{3}(1+\sqrt{7}), -\dfrac{1}{3}(1+\sqrt{7})\right)$	
$i + w - iw + 2j$	$\dfrac{2}{57} - \dfrac{10i}{19} - \dfrac{32w}{57} + \dfrac{32iw}{57} + \left(\dfrac{8}{57} + \dfrac{i}{57} - \dfrac{2iw}{57}\right)j$	$\left(-\dfrac{1}{3}(1+\sqrt{7}), -\dfrac{1}{3}(1+\sqrt{7})\right)$	$\sqrt{\dfrac{4+\sqrt{7}}{3}}$
$-i - w + iw + 2j$	$-\dfrac{2}{57} + \dfrac{10i}{19} + \dfrac{32w}{57} - \dfrac{32iw}{57} + \left(\dfrac{8}{57} + \dfrac{i}{57} - \dfrac{2iw}{57}\right)j$	$\left(\dfrac{1}{3}(1+\sqrt{7}), \dfrac{1}{3}(1+\sqrt{7})\right)$	
$2 + iw + (1-w)j$	$\dfrac{1}{18} + i - \dfrac{8w}{9} - \dfrac{4iw}{9} + \left(\dfrac{2}{9} - \dfrac{i}{18} + \dfrac{iw}{9}\right)j$	$\left(-\dfrac{1}{3}(1+\sqrt{7}), \dfrac{1}{3}(1+\sqrt{7})\right)$	

Fig. 1.

this bisector intersects $D = F_{k_1}$. If it does then we reduce D by taking off the half space limited by this bisector containing gj and then we go on to the next g. Using Mathematica to execute these calculations one discovers that $D = F_{k_1}$ and the algorithm finishes giving as an output the region represented by Fig. 3. Hence, we also obtain generators for the group under consideration.
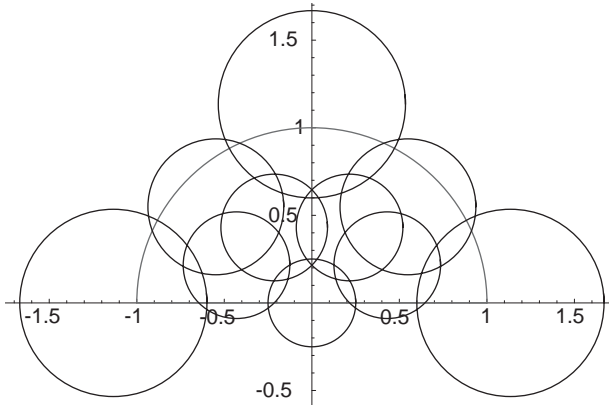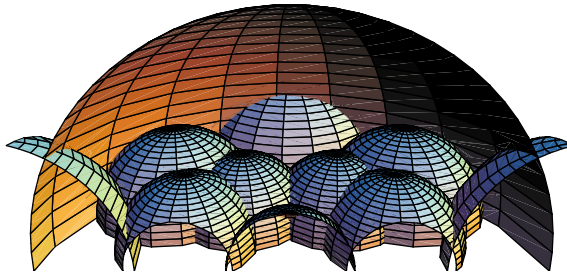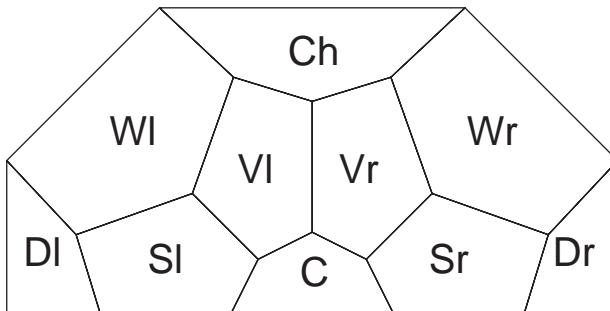
Fig. 2.

Fig. 3.

Fig. 4.

Theorem 4.2. *Let* $R = \mathbb{Z}\left[\dfrac{1 + \sqrt{-7}}{2}\right]$. *The set* $D = F_{k_1}$ *is a convex fundamental domain for* $\mathrm{SL}_1(\mathbb{H}(R))$ *and the group* $\mathrm{SL}_1(\mathbb{H}(R))$ *is generated by the following* 12 *elements*:

$$i, j, 2 + i - w + iw, -1 - 2i + w - wj, -2 - iw + (1 - w)j,$$

Table 3
The boundary of F_{k_1}

Notation	Name	Bisector	g
P	Public	$Y = 0$	i
R	Roof	$S((0,0),1)$	j
C	Conductor	$S\left((0,0), \sqrt{8-3\sqrt{7}}\right)$	$2+i-w+iw$
S_l	Soloists	$S\left(-\left(\frac{2}{3}(-2+\sqrt{7}), \frac{1}{3}(-2+\sqrt{7})\right), \frac{2}{3}\sqrt{\frac{11}{2}-2\sqrt{7}}\right)$	$-1-2i+w-wj$
S_r		$S\left(\left(\frac{2}{3}(-2+\sqrt{7}), \frac{1}{3}(-2+\sqrt{7})\right), \frac{2}{3}\sqrt{\frac{11}{2}-2\sqrt{7}}\right)$	$2-iw+(1-w)j$
V_l	Violins	$S\left(-\left(\frac{1}{3}(-2+\sqrt{7}), \frac{2}{3}(-2+\sqrt{7})\right), \frac{2}{3}\sqrt{\frac{11}{2}-2\sqrt{7}}\right)$	$-2-iw+(1-w)j$
V_r		$S\left(\left(\frac{1}{3}(-2+\sqrt{7}), \frac{2}{3}(-2+\sqrt{7})\right), \frac{2}{3}\sqrt{\frac{11}{2}-2\sqrt{7}}\right)$	$-2+i-iw-wj$
W_l	Winds	$S\left(-\left(\frac{-1+\sqrt{7}}{3}, \frac{-1+\sqrt{7}}{3}\right), \frac{\sqrt{4-\sqrt{7}}}{3}\right)$	$i-w-iw+2j$
W_r		$S\left(\left(\frac{-1+\sqrt{7}}{3}, \frac{-1+\sqrt{7}}{3}\right), \frac{\sqrt{4-\sqrt{7}}}{3}\right)$	$2-i+iw-wj$
Ch	Chorus	$S\left(\left(0, \frac{3}{\sqrt{7}}\right), \sqrt{\frac{2}{7}}\right)$	$(2-w)+(1+w)j$
D_l	Drums	$S\left(-\left(\frac{3}{\sqrt{7}}, 0\right), \sqrt{\frac{2}{7}}\right)$	$2i-iw+(1+w)j$
D_r		$S\left(\left(\frac{3}{\sqrt{7}}, 0\right), \sqrt{\frac{2}{7}}\right)$	$i+iw+(2-w)j$

$$-2+i-iw-wj, 2-iw+(1-w)j,$$

$$i-w-iw+2j, 2-i+iw-wj, (2-w)+(1+w)j,$$

$$2i-iw+(1+w)j, i+iw+(2-w)j.$$

Proof. This is a direct consequence of the description of the fundamental domain and the fact that the group generated by the listed elements has the same fundamental domain as $\mathrm{SL}_1(\mathbb{H}(R))$ and contains -1 (the latter is needed to lift the information from $\mathrm{PSL}_1(\mathbb{H}(R))$ to $\mathrm{SL}_1(\mathbb{H}(R))$. $\quad\square$

The knowledge of the fundamental domain gave at once generators for the considered group G. However, to obtain a presentation for the group G we now have to apply the method of Poincaré. Of course this method will first rediscover a set of generators and then it will also provide us with a complete set of relations. The generators are the group elements g so that $D \cap g(D)$ has dimension 2. The latter are called sides and form a tesellation of the border of D. Each side s is associated with the only group element g_s so that $s = D \cap g_s(D)$.

Note that in order to find the generators in Theorem 4.2 we search for coset representatives g of the stabilizer G_0 of j which are such that the bisector between j

Table 4
The sides

s	g_s	s	g_s
P	i	R	j
C_l	$(2 + i - w + iw)j$	C_r	$(1 - 2i + w + iw)j$
S_l	$iw + (-2 + i - iw)j$	S_r	$-i + iw + (2 + iw)j$
V_l	$-i + iw + (2i - w)j$	V_r	$iw + (1 + 2i - w)j$
W_l	$-2i + (1 - w + iw)j$	W_r	$-2i + (-i + w + iw)j$
Ch_l	$-i - iw + (-2i + iw)j$	Ch_r	$2i - iw - (i + iw)j$
D_l	$2i - iw + (1 + w)j$	D_r	$i + iw + (-2 + w)j$

and $g(j)$ contributes to one side of the fundamental domain D; clearly, these elements are not uniquely determined. However, for Poincaré's method we have to be more careful, since it could happen that $D \cap g(D)$ is a side while $D \cap gh(D)$ is not a side for $h \in G_0$. We, therefore, need to look for sides of the form $D \cap gh(D)$ with $h \in G_0$ and g in a transversal of G_0. In the Mathematica program used to do the calculations for Theorem 4.2 we have all this data already available. After some more calculations, fourteen elements (generators) $g_1 h_1, \ldots, g_{14} h_{14}$ (as above) are found such that the sides $D \cap g_i h_i(D)$ cover the boundary of D. Let us make a few remarks on the actual calculations. First, we observe that in order to look for the elements $g_i h_i$ we need to look amongst the elements that map one of the bisectors that form the border of D to another bisector. Secondly, two of these elements, the public and the roof, are easily found by looking geometrically at the action of i and j on D. Thirdly, for all but two bisectors one finds a group element that maps the bisector to itself.

After some calculations with Mathematica, we find that in our example the sides coincide with the bisectors except for the conductor C and the chorus Ch, each of which have to be divided in two sides: $C_l = C \cap (X \leq 0)$ and $C_r = C \cap (X \geq 0)$ for the conductor, and $Ch_l = Ch \cap (X \leq 0)$ and $Ch_r = Ch \cap (X \geq 0)$ for the chorus. Table 4 gives the pairing between the sides and the associated group elements.

Hence we obtain

Theorem 4.3. *Let* $R = \mathbb{Z} \left[\dfrac{1 + \sqrt{-7}}{2} \right]$. *The group* $SL_1(\mathbb{H}(R))$ *is generated by the following* 14 *elements*:

$$g_P = i \qquad\qquad g_R = j$$
$$g_{C_l} = (2 + i - w + iw)j \qquad g_{C_r} = (1 - 2i + w + iw)j$$
$$g_{S_l} = iw + (-2 + i - iw)j \qquad g_{S_r} = -i + iw + (2 + iw)j$$
$$g_{V_l} = -i + iw + (2i - w)j \qquad g_{V_r} = iw + (1 + 2i - w)j$$
$$g_{W_l} = -2i + (1 - w + iw)j \qquad g_{W_r} = -2i + (-i + w + iw)j$$
$$g_{Ch_l} = -i - iw + (-2i + iw)j \qquad g_{Ch_r} = 2i - iw - (i + iw)j$$
$$g_{D_l} = 2i - iw + (1 + w)j \qquad g_{D_r} = i + iw + (-2 + w)j$$

We now we have a set of generators of cardinality fourteen. In view of Theorem 4.2, this set may seem redundant. Nevertheless we note that $g_{C_r} = i^3 g_{C_l}, \quad g_{Ch_r} = g_{Ch_l} j.$

We also note that Poincaré's method allows us to calculate relations for discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ and from these we may easily derive presentations for the corresponding subgroups of $\mathrm{SL}_2(\mathbb{C})$. For example, the next step in Poincaré's method is to obtain a pairing between the sides, so that if s_1 and s_2 are two sides associated then $g_{s_1} = g_{s_2}^{-1}$. The first type of relations, the reflections, then correspond to the sides that are associated with themselves. In our example, this happens with all sides as they all have order two in $\mathrm{PSL}_2(\mathbb{C})$. Indeed it is easily verified that for every side s, $g_s^2 = -1$.

We next have to find the cycle relations that are associated with the edges of the fundamental domain. The fundamental domain has thirty four edges (we identify each edge with a pair of adjacent sides):

$(D_l, S_l); (D_l, W_l); (D_l, P); (D_l, R); (D_r, S_r); (D_r, W_r); (D_r, P); (D_r, R);$

$(S_l, V_l); (S_l, W_l); (S_l, C_l); (S_l, P); (V_l, V_r); (V_l, W_l); (V_l, C_l); (V_l, \mathrm{Ch}_l); (V_r, S_r);$

$(V_r, W_r); (V_r, C_r); (V_r, \mathrm{Ch}_r); (S_r, W_r); (S_r, C_r); (S_r, P); (W_l, \mathrm{Ch}_l); (W_l, R);$

$(W_r, \mathrm{Ch}_r); (W_r, R); (C_l, C_r); (C_l, P); (C_r, P); (\mathrm{Ch}_r, \mathrm{Ch}_l); (\mathrm{Ch}_r, R); (\mathrm{Ch}_l, R); (P, R).$

Recall that a cycle is a list of even length

$$[e_1, g_1, e_2, g_2, e_3, \ldots, e_n, g_n], \tag{7}$$

where for each $i = 1, \ldots, n$, e_i is an edge, g_i is a generator or the inverse of a generator (that is, one of the group elements of the form g_s with s a side), $g_i(e_i) = e_{i+1}$ and $e_{n+1} = e_1$). Each cycle gives rise to a relation $(g_n \ldots g_1)^k = 1$ where k is the order of $g_n \ldots g_1$.

In principle, one can construct infinitely many cycles. However, some of the relations obtained from cycles can be dropped according to the following two principles. First, cyclic permutations of even order of the cycles clearly give rise to new cycles with equivalent associated relations. Therefore these two cycles are considered as equal. Secondly, we only need irreducible lists, that is, g_{i+1} should be different from g_i^{-1} for each i. Of course, merging cycles with the same starting edges results into new cycles. Hence, in first instance we will only consider cycles such that $e_i \neq e_1$ for $i = 2, \ldots, n$. The irreducible cycles satisfying this last condition are called minimal cycles.

Notice that if e is an edge of the side $s = D \cap g^{-1}(D)$, then $g(e)$ is an edge of the side $g(s)$. If f is another edge such that $g(f)$ is embedded in D, then the convex closure of $g(e)$ and $g(f)$ is embedded in $g(s) = D \cap g(D)$. Hence, the convex closure containing e and f is embedded in the side s, and therefore f is one of the edges of s. In other words, if g is a generator and e is an edge such that $g(e)$ is another edge, then

e is one of the edges of the side, $s = D \cap g(D)$, that is $g = g_s$. Thus, each g_i in a cycle should be one of the two generators associated to a side containing e_i.

In our case all the generators have order two and this has two consequences. First, each generator maps the associated side to itself and therefore two consecutive edges of the list $(e_1, \dots, e_n)$ belong to the same side. Second, one generator cannot be taken twice consecutively in the same list. Thus, if $g_i = g_s$ for the side s, then s contains the edge e_i and g_i maps s into itself. Therefore $e_{i+1} = g_i(e_i)$ is also an edge of s. This implies that $g_{i+1} = g_{s_1}$ where $e_{i+1} = s \cap s_1$. Consequently, a minimal cycle as in (7) is determined by the pair (e_1, g_1).

It is now easy to compute the minimal cycles. Up to a cyclic permutation of even order, the following table lists them all (we have divided the table in several parts in order to emphasize the symmetry).

$$
\begin{array}{|l|}
\hline
\left[(D_l, S_l), g_{D_l}, (D_l, W_l), g_{W_l}, (W_l, Ch_l), g_{Ch_l}, (V_l, Ch_l), g_{V_l}, (V_l, C_l), g_{C_l}, (S_l, C_l), g_{S_l}\right] ; \\
\left[(D_r, S_r), g_{D_r}, (D_r, W_r), g_{W_r}, (W_r, Ch_r), g_{Ch_r}, (V_r, Ch_r), g_{V_r}, (V_r, C_r), g_{C_r}, (S_r, C_r), g_{S_r}\right] ; \\
\hline
\left[(D_l, P), g_{D_l}, (D_l, R), g_R, (D_r, R), g_{D_r}, (D_r, P), g_P\right] ; \\
\hline
\left[(S_l, V_l), g_{S_l}, (S_l, W_l), g_{W_l}, (V_l, W_l), g_{V_l}\right] ; \\
\left[(V_r, S_r), g_{V_r}, (V_r, W_r), g_{W_r}, (S_r, W_r), g_{S_r}\right] ; \\
\hline
\left[(C_l, C_r), g_{C_l}, (C_l, P), g_P, (C_r, P), g_{C_r}\right] ; \\
\left[(Ch_r, Ch_l), g_{Ch_r}, (Ch_r, R), g_R, (Ch_l, R), g_{Ch_r}\right] ; \\
\hline
\left[(S_l, P), g_{S_l}\right] ; \qquad \left[(S_l, P), g_P, (S_r, P), g_{S_r}, (S_r, P), g_P\right] ; \\
\left[(S_r, P), g_{S_r}\right] ; \qquad \left[(S_r, P), g_P, (S_l, P), g_{S_l}, (S_l, P), g_P\right] ; \\
\hline
\left[(W_l, R), g_{W_l}\right] ; \qquad \left[(W_l, R), g_R, (W_r, R), g_{W_r}, (W_r, R), g_R\right] ; \\
\left[(W_r, R), g_{W_r}\right] ; \qquad \left[(W_r, R), g_R, (W_l, R), g_{W_l}, (W_l, R), g_R\right] ; \\
\hline
\left[(V_l, V_r), g_{V_l}\right] ; \qquad \left[(V_l, V_r), g_{V_r}\right] ; \\
\hline
[(P, R), g_P] ; \qquad\quad [(P, R), g_R] \\
\hline
\end{array}
$$

These minimal cycles yield us the following extra relations amongst the generators:

$$
\begin{array}{|l|l|}
\hline
g_{S_l} g_{C_l} g_{V_l} g_{Ch_l} g_{W_l} g_{D_l} = -I & g_P g_{D_r} g_R g_{D_l} = I \\
g_{S_r} g_{C_r} g_{V_r} g_{Ch_r} g_{W_r} g_{D_r} = I & \\
\hline
g_{V_l} g_{W_l} g_{S_l} = -I & g_{C_r} g_P g_{C_l} = I \\
g_{S_r} g_{W_r} g_{V_r} = -I & g_{Ch_r} g_R g_{Ch_l} = -I \\
\hline
(g_{S_l})^2 = -I & (g_{W_l})^2 = -I \\
(g_P g_{S_r} g_P)^2 = -I & (g_R g_{W_r} g_R)^2 = -I \\
(g_{S_r})^2 = -I & (g_{W_r})^2 = -I \\
(g_P g_{S_l} g_P)^2 = -I & (g_R g_{W_l} g_R)^2 = -I \\
\hline
(g_{V_l})^2 = -I & (g_P)^2 = -I \\
(g_{V_r})^2 = -I & (g_R)^2 = -I \\
\hline
\end{array}
$$

Finally, we still have to consider cycles that can be obtained by merging minimal cycles with the same initial edge. There are six pairs of such "mergeable" cycles:

$\big[(S_l,P),g_{S_l}\big],$	$\big[(S_l,P),g_P,(S_r,P),g_{S_r},(S_r,P),g_P\big]$;
$\big[(S_r,P),g_{S_r}\big],$	$\big[(S_r,P),g_P,(S_l,P),g_{S_l},(S_l,P),g_P\big]$;
$\big[(W_l,R),g_{W_l}\big],$	$\big[(W_l,R),g_R,(W_r,R),g_{W_r},(W_r,R),g_R\big]$;
$\big[(W_r,R),g_{W_r}\big],$	$\big[(W_r,R),g_R,(W_l,R),g_{W_l},(W_l,R),g_R\big]$;
$\big[(V_l,V_r),g_{V_l}\big],$	$\big[(V_l,V_r),g_{V_r}\big]$;
$[(P,R),g_P]$	$[(P,R),g_R]$

Two elements in the same row can be merged to produce new cycles and henceforth we obtain new relations.

$(g_{S_l}g_Pg_{S_r}g_P)^2 = -I$	$(g_{W_l}g_Rg_{W_r}g_R)^2 = -I$
$(g_{S_r}g_Pg_{S_l}g_P)^2 = -I$	$(g_{W_r}g_Rg_{W_l}g_R)^2 = -I$
$(g_{V_l}g_{V_r})^2 = -I$	$(g_Pg_R)^2 = -I$

Longer cycles are possible, but it is clear from the relations obtained so far that the relations obtained from these are a consequence of the former.

To state a presentation it is convenient to introduce an extra generator J which corresponds with the element $-I$. To simplify notation we will write a generator g_S simply as S. Hence we have obtained a presentation with fifteen (including J) and 27 relations (including $J^2 = 1$ and J central).

Replacing S_l, S_r and Ch_r by their respective inverses, the relations take a nicer form. It is this presentation that is stated in the following theorem.

Theorem 4.4. *Let $R = \mathbb{Z}\big[\frac{1+\sqrt{-7}}{2}\big]$. The group $\mathrm{SL}_1(\mathbb{H}(R))$ has the presentation defined by the generators*

$$J, P, R, C_l, C_r, S_l, S_r, V_l, V_r, W_l, W_r, \mathrm{Ch}_l, \mathrm{Ch}_r, D_l, D_r$$

and the relations

$$P^2 = R^2 = C_l^2 = C_r^2 = S_l^2 = S_r^2 = V_l^2 = V_r^2 = W_l^2 = W_r^2$$

$$= \mathrm{Ch}_l^2 = \mathrm{Ch}_r^2 = D_l^2 = D_r^2 = J,$$

$$S_lC_lV_l\mathrm{Ch}_lW_lD_l = S_rC_rV_r\mathrm{Ch}_rW_rD_r = PD_rRD_l = V_lW_lS_l = S_rW_rV_r$$

$$= C_rPC_l = \mathrm{Ch}_rR\mathrm{Ch}_l = 1,$$

$$(S_lPS_rP)^2 = (W_lRW_rR)^2 = (V_lV_r)^2 = (PR)^2 = J \quad and \quad J^2 = 1.$$

Of course, some of the defining generators are redundant. For example, one can eliminate V_l, D_l, V_r, D_r, P and R. So the group $\mathrm{SL}_1(\mathbb{H}(R))$ can be generated by nine elements.

5. The orthogonal group

Let K be a number field, R its maximal order, $\mathbb{H}(K) = \left(\frac{-1,-1}{K}\right)$ the classical quaternion algebra over K and $\tau(y)$ is the orthogonal 3×3-matrix (with respect to the norm n (1) and the standard basis i, j, k of the imaginary part $\mathbb{H}_0(K)$ of $\mathbb{H}(K)$) associated to the isometry $\tau_y : \mathbb{H}_0(K) \to \mathbb{H}_0(K)$ with $\tau_y(x) = yxy^{-1}$.

A well-known Theorem of Cartan–Dieudonné says that the sequence

$$1 \to K^* \to \mathbb{H}(K)^* \xrightarrow{\tau} SO_3(K) \to 1$$

is exact. If we restrict ourselves to $R[1/2]$ and we denote by $\mathrm{Pic}_2(R[1/2])$ the subgroup of the class group $\mathrm{Pic}(R[1/2])$ formed by the elements of order two, we obtain the following sequence:

$$1 \to R[1/2]^* \to \mathbb{H}(R[1/2])^* \xrightarrow{\tau} SO_3(R[1/2]) \to \mathrm{Pic}_2(R[1/2]),$$

also known to be exact (see for example [6, 7.2.20]).

If R is a unique factorization domain, then $\mathrm{Pic}_2(R[1/2])$ is trivial, and hence we have the exact sequence

$$1 \to R[1/2]^* \to \mathbb{H}(R[1/2])^* \xrightarrow{\tau} SO_3(R[1/2]) \to 1 \qquad (8)$$

and, by further restriction to R,

$$1 \to R^* \to \mathbb{H}(R)^* \xrightarrow{\tau} SO_3(R),$$

where, as a consequence of a general theorem on arithmetic groups (see for example [14]), the image $\tau(\mathbb{H}(R)^*)$ of $\mathbb{H}(R)^*$ under τ is a subgroup of finite index in $SO_3(R)$.

An element $x = r_0 + r_1 i + r_2 j + r_3 k \in \mathbb{H}(R)$ is said to be reduced if $n(x) \neq 0$ and the ideal of R generated by the elements r_0, r_1, r_2 and r_3 is R. From the Theorem of Cartan–Dieudonné we deduce that if R is a unique factorization domain, then every element in $SO_3(R)$ is of the form $\tau(x)$ for some reduced element x of $\mathbb{H}(R)$. For this reason, it is useful to know which reduced elements x verify that $\tau(x) \in SO_3(R)$.

In the following two lemmas we use that if P is a prime ideal of R containing the prime integer p and both the ramification degree and the residual degree of P over p are 1, then the embedding of $\mathbb{Z}$ in R induces an isomorphism $\mathbb{Z}/p^n\mathbb{Z} \simeq R/P^nR$ for every positive integer n. In particular, if $p = 2$, then either $x \in P$ or $x^2 - 1 \in P^3$ for every $x \in R$.

Lemma 5.1. *Assume that 2 splits completely in R and let x be a reduced element of $\mathbb{H}(R)$. Then $\tau(x) \in SO_3(R)$ if and only if $4 \in n(x)R$.*

Proof. Write $x = x_0 + x_1 i + x_2 j + x_3 k$ with $x_i \in R$. Then

$$\tau(x) = n(x)^{-1} M_x,$$

where

$$M_x = (m_{ij}) = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_0 x_3 - x_1 x_2) & 2(x_0 x_2 - a x_1 x_3) \\ 2(x_0 x_3 + x_1 x_2) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_0 x_1 + b x_2 x_3) \\ -2(x_0 x_2 + a x_1 x_3) & -2(x_0 x_1 - b x_2 x_3) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}.$$

Assume now that $\tau(x) \in \mathrm{SO}_3(R)$, that is, suppose each $m_{ij} \in Rn(x)$. It is not difficult to see that this implies that $4 x_i x_j \in Rn(x)$ for all $i \neq j$. Moreover,

$$n(x) + m_{11} + m_{22} + m_{33} = 4 x_0^2$$

$$n(x) + m_{11} - m_{22} - m_{33} = 4 x_1^2$$

$$n(x) - m_{11} + m_{22} - m_{33} = 4 x_2^2$$

$$n(x) - m_{11} - m_{22} + m_{33} = 4 x_3^2$$

and hence $4 x_i^2 \in n(x) R$ for all i. Since $R = \sum_{i=1}^4 R x_i$, $R = \sum_{i,j} R x_i x_j$ and then $4 \in n(x) R$. This proves one implication of the lemma.

To prove the converse, assume $4 \in n(x) R$. Let $p_1, \ldots, p_k$ be the prime ideals of R containing $n(x)$. So $n(x) R = p_1^{r_1} \cdots p_k^{r_k}$ for some $r_i = 0, 1, 2$. We need to prove that all $m_{uv} \in n(x) R$ or equivalently $m_{uv} \in p_i^{r_i}$ for every i. Let $p = p_i$ and $r = r_i$. So we have to prove that every $m_{uv} \in p^r R$. If $r = 0$ then this is obvious. If $r = 1$, then since $2 \in p$, it is clear that $m_{uv} \in p$ for $u \neq v$, and $m_{uu} = n(x) - 2(x_{u'}^2 + x_{v'}^2) \in p$ (for some u', v'). If $r = 2$, then, since x is reduced, $x_u - 1 \in p$ for at least one u and from the fact that $x_0^2 + x_1^2 + x_2^2 + x_3^2 \in p^2$ one deduces that $x_u - 1 \in p$ for all u (recall that $R/p^2 R \simeq \mathbb{Z}/4\mathbb{Z}$). Hence, $2(x_u^2 + x_v^2) \in p^2$ for all $u \neq v$ and $(x_u x_v - x_{u'} x_{v'}) \in p^2$. It follows that $m_{uv} \in p^2$ as wanted. $\square$

In order to compare $\tau(\mathbb{H}(R)^*)$ with $\tau(\mathbb{H}(R[1/2])^*)$ we consider the algebra of the Hurwitz quaternions over R, that is

$$\mathbb{H}u(R) = \left\{ \frac{x_0 + x_1 i + x_2 j + x_3 k}{2} \mid x_i \in R \text{ and } x_i - x_j \in 2R \right\}.$$

Note that if $x \in \mathbb{H}u(R)$ then $n(y) \in R$. Hence, it is clear that $y \in \mathbb{H}(R)^*$ if and only if $n(y) \in R^*$.

We denote by v_P the discrete valuation associated to a prime ideal P of R, and by R_P, the discrete valuation ring associated to v_P. Note that if $2 \notin P$, then $\mathbb{H}(R_P) = \mathbb{H}u(R_P)$ and if P is a divisor of $2R$ with ramification index 1, then an element $x = x_0 + x_1 i + x_2 j + x_3 k$ belongs to $\mathbb{H}(R_P)$ if and only if $v_P(x_r) \geqslant 0$ for every

$r \in \{0, 1, 2, 3\}$, and $x \in \mathbb{H}u(R_P) \backslash \mathbb{H}(R_P)$ if and only if $v_P(x_r) = -1$ for every $r \in \{0, 1, 2, 3\}$.

Lemma 5.2. *Let p be a prime ideal in R such that both its ramification and residual degree over 2 are one. Assume $v \in \mathbb{H}u(R)$ is such that $n(v) = p$. Then $x \in \mathbb{H}u(R)v$ if and only if $n(x) \in Rp$.*

Proof. One implication is obvious. For the converse, assume that $n(x) \in Rp$. Let $w = xv^{-1} = x\bar{v}/p = w_0 + w_1 i + w_2 j + w_3 k$. We have to prove that $w \in \mathbb{H}u(R)$. For every prime ideal Q in R different from pR we have that $w \in \mathbb{H}u(R_Q)$. Thus, it is sufficient to show that $w \in \mathbb{H}u(R_{pR})$. Let $m_r = -v_p(w_r)$. Since $x\bar{v} \in \mathbb{H}u(R)$, either $m_r \leqslant 1$ for every r or $m_r = 2$ for every r. Let $m = \max\{m_r \mid 1 \leqslant r \leqslant 4\}$ and write $w_r = \frac{y_r}{p^{m_r}} = \frac{x_r}{p^m}$, with $x_r \in R$. Since $n(w) \in R$ we obtain that

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 \in p^{2m} R$$

If $m \leqslant 0$, then $w \in \mathbb{H}(R_{pR})$ and we are done. Thus we assume that $m \geqslant 1$ and hence $x_r \notin pR$ for some $r \in \{0, 1, 2, 3\}$ and

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 \in p^2 R$$

Using the isomorphism $\mathbb{Z}/4\mathbb{Z} \simeq R/p^2 R$, we deduce that $x_r^2 - 1 \in p^2 R$ for every r and hence $m = m_r$ for every r. Next, using the isomorphism $\mathbb{Z}/8\mathbb{Z} \simeq R/p^3 R$ one deduces that $x_0^2 + x_1^2 + x_2^2 + x_3^2 - 4 \in Rp^3$ and therefore $m = 1$. We conclude that $w \in \mathbb{H}u(R_{pR})$ as desired. □

Assume that R is a unique factorization domain and 2 splits completely in R. Thus $2 = p_1 \cdots p_n$, a product of non associate primes. Assume that for every $i = 1, \dots, n$ there is a reduced element $\pi_i \in \mathbb{H}(R)$ such that $n(\pi_i) = p_i$. If $x \in \mathbb{H}(R[1/2])^*$, then there are integers $\alpha_1, \dots, \alpha_n$ such that $y = p_1^{-\alpha_1} \cdots p_n^{-\alpha_n} x \in \mathbb{H}(R)$. Using Lemma 5.2 we deduce that there is an element $x \in \mathbb{H}u(R)^*$ and integers $\beta_1, \dots, \beta_n$ such that $y = x\pi_1^{\beta_1} \cdots \pi_n^{\beta_n}$. Hence

$$\mathbb{H}(R[1/2])^* = \{x\pi_1^{\beta_1} \cdots \pi_n^{\beta_n} p_1^{\alpha_1} \cdots p_n^{\alpha_n} \mid x \in \mathbb{H}u(R), \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{Z}\}$$

Moreover, by Lemma 5.1, $\tau(\pi_i) \in SO_3(R)$ and so $\tau(\mathbb{H}(R[1/2])^*) \subseteq SO_3(R)$. This implies in particular that $\mathbb{H}(R)$ is normalized by $\mathbb{H}(R[1/2])^*$.

From now on we consider again our example $R = \mathbb{Z}[\omega]$, with $\omega = \frac{1+\sqrt{-7}}{2}$. Note R is a unique factorization domain and $2 = \omega\bar{\omega}$, where ω and $\bar{\omega} = \frac{1-\sqrt{-7}}{2}$ are non-associated primes of R. Further, $n(\omega + i) = -\bar{\omega}$ and $n(\bar{\omega} + i) = -\omega$, so that both $\tau(\omega + i)$ and $\tau(\bar{\omega} + i)$ belong to $SO_3(R)$. Another element in $SO_3(R)$ is $\tau(\frac{1}{2}(1 + i + j + k))$. In the next proposition we show that these elements are enough to generate

$SO_3(R)/\tau(\mathbb{H}(R)^*)$, and a presentation of this quotient group is given. From this and Theorem 4.4, a presentation of $SO_3(R)$ follows.

Proposition 5.3. *Let* $R = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ *and let* $x = \tau(\omega + i)\mathbb{H}(R)^*$, $y = \tau(\bar{\omega} + i)\mathbb{H}(R)^*$, $u = \tau(\frac{1}{2}(1 + i + j + k))\mathbb{H}(R)^*$ *and* $v = x^{-1}ux$. *Then the following conditions are satisfied*:

1. $\mathbb{H}(R[1/2])^* = \{x(\omega + i)^n(\bar{\omega} + i)^m\omega^r\bar{\omega}^s \mid x \in \mathbb{H}u(R)^*, n, m, r, s \in \mathbb{Z}\}$,
2. $\mathbb{H}(R)^*$ *and* $\mathbb{H}u(R)^*$ *are normal subgroups of* $\mathbb{H}(R[1/2])^*$,
3. $SO_3(R)/\tau(\mathbb{H}u(R)^*)$ *is generated by the classes of* $\tau(\omega + i)$ *and* $\tau(\bar{\omega} + i)$ *and is isomorphic to the Klein group* $\mathbb{Z}_2^2$;
4. $\mathbb{H}u(R)^*/\mathbb{H}(R)^* \cong \tau(\mathbb{H}u(R)^*)/\tau(\mathbb{H}(R)^*) \simeq \mathbb{Z}_3^2$,
5. $SO_3(R)/\tau(\mathbb{H}(R)^*) = \langle x, y, u \rangle = \langle u, v \rangle \rtimes \langle x, y \rangle \simeq \mathbb{Z}_3^2 \rtimes \mathbb{Z}_2^2$ *and the action of* $\langle x, y \rangle$ *on* $\langle u, v \rangle$ *is given by* $u^x = v$, $v^x = u$, $u^y = v^2$ *and* $v^y = u^2$.

Proof. Statements 1 and 2 and the fact that $SO_3(R)/\tau(\mathbb{H}u(R)^*)$ is generated by the classes of $\tau(\omega + i)$ and $\tau(\bar{\omega} + i)$, are consequences of the arguments before the proposition. To prove that $SO_3(R)/\tau(\mathbb{H}u(R)^*) \simeq \mathbb{Z}_2^2$, it suffices to verify that $[\omega + i, \bar{\omega} + i] = 1$ and $(\omega + i)^2/\bar{\omega}$ and $(\bar{\omega} + i)^2/\omega$ are units of $\mathbb{H}(R)$. This proves statement 3.

It is easy to verify that $u, v \in \mathbb{H}u(R)^*$, $uvu^{-1}v^{-1} \in \mathbb{H}(R)^*$, $u^3 = -1$ and $v^3 = -1$. It follows that $\langle \tau(u), \tau(v) \rangle/\{1, -1\} \simeq \mathbb{Z}_3^2$. A direct verification shows that the nine elements so obtained form coset representatives for $\mathbb{H}(R)^*$ in $\mathbb{H}u(R)^*$ and statements 4 and 5 readily follow. $\quad\square$

References

[1] A.F. Beardon, The Geometry of Discrete Groups, Springer, Berlin, 1983.
[2] L. Bianchi, Sui gruppi de sostituzioni lineari con coeficienti appartenenti a corpi quadratici imaginari, Math. Ann. 40 (1892) 332–412.
[3] J. Elstrodt, F. Grunewald, J. Mennicke, Groups Acting on Hyperbolic Space, Harmonic Analysis and Number Theory, Springer, Berlin, 1998.
[4] B. Fein, B. Gordon, J.M. Smith, On the representation of -1 as a sum of two squares in an algebraic number field, J. Number Theory 3 (1971) 310–315.
[5] B. Fine, The Algebraic Theory of the Bianchi Groups, Marcel Dekker, New York, 1989.
[6] A.J. Hahn, O.T. O'Meara, The Classical Groups and K-Theory, Grundlehren der mathematischen Wissenschaften 291, Springer, Heidelberg, 1989.
[7] E. Jespers, Units in integral group rings: a survey, Proceedings of the International Conference on Methods in Ring Theory, Trento, 1997. Lecture Notes in Pure and Applied Mathematics, Vol. 198, Marcel Dekker, New York, 1998, pp. 141–169.
[8] E. Kleinert, Units in Skew Fields, Progress in Mathematics, 186, Birkhäuser Verlag, Basel, 2000.
[9] E. Kleinert, Units of classical orders: a survey, Enseign. Math. (2) 40 (3–4) (1994) 205–248.
[10] H. Poincaré, Mémoire sur les groupes kleinées, Acta. Math. 3 (1883) 49–92.
[11] R. Riley, Applications of a computer implementation of Poincaré's theorem on fundamental polyhedra, Math. Comp. 40 (162) (1983) 607–632.

[12] S.K. Sehgal, Units in integral group rings. Pitman Monographs, Surveys in Pure and Applied Mathematics, Vol. 69, Longman Scientific & Technical, Harlow, 1993.

[13] J.P. Serre, Local class field theory, in: J.W.S. Cassels, A. Frölich (Eds.), Algebraic Number Theory, Proceedings of an Instructional Conference Organized by the London Mathematics Society, Academic Press, New York, 1967, pp. 129–160.

[14] J.P., Serre, Arithmetic groups in Homological group theory, Proceedings of the Symposium, Durham, 1977, London Mathematical Society Lecture Note Series, 36, Cambridge University Press, Cambridge, New York, 1979, pp.105–136.

[15] R.G. Swan, Generators and relations for certain special linear groups, Adv. Math. 6 (1971) 1–77.