AM

HOJA 2:

PROBLEMA 11:

$$|a_n \cos nx + b_n \operatorname{sen} nx| \leq |a_n| + |b_n| \quad \forall x \in [-\pi, \pi]$$

Como $\sum_{n=1}^{\infty} |a_n| + |b_n| = \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| < \infty$,

así por el criterio M-Weierstrass

la serie $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \operatorname{sen} nx$

converge uniformemente a su límite puntual en $[-\pi, \pi]$. Si este límite puntual es f se tiene la inversa completa.

PROBLEMA 12: Si $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$, existe su transformada de Fourier. $\hat{f}$. Por ser

- f par la función $f(t) \operatorname{sen} st$ es una función impar sobre $\mathbb{R}$, así

$$\int_{-\infty}^{\infty} f(t) \operatorname{sen} st\, dt = \int_{-\infty}^{0} f(t) \operatorname{sen} st\, dt + \int_{0}^{\infty} f(t) \operatorname{sen} st\, dt =$$

$$= -\int_{\infty}^{0} f(-x) \operatorname{sen} s(-x)(-1)\, dx + \int_{0}^{\infty} f(t) \operatorname{sen} st\, dt = 0.$$

$t = -x$, $dt = -dx$ cambio en la primera integral

Por tanto $\hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-ist}\, dt =$

$$= \int_{-\infty}^{\infty} f(t) \cos st\, dt + i \int_{-\infty}^{\infty} f(t) \operatorname{sen} st\, dt = \int_{-\infty}^{\infty} f(t) \cos st\, dt \in \mathbb{R}$$

- Si f es impar, entonces $f(t) \cos st$ es una función impar sobre $\mathbb{R}$ y así $\int_{-\infty}^{\infty} f(t) \cos st\, dt = 0$,

Por tanto $\hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-ist}\, dt = -i \int_{-\infty}^{\infty} f(t) \operatorname{sen} st\, dt$.

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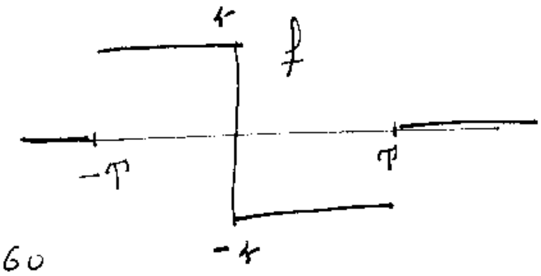
HOJA 2:

PROBLEMA 13:

c) $f(t) = \begin{cases} k & \text{si } -T \le t < 0 \\ -k & \text{si } 0 \le t < T \\ 0 & \text{si } t \notin [-T,T] \end{cases}$



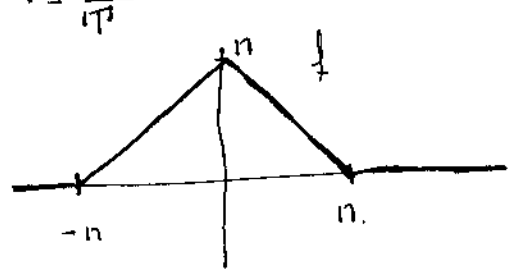
f es una función impar, luego por el ejercicio anterior

$$\hat{f}(\lambda) = -i\int_{-T}^{T} f(t)\,\text{sen}\,\lambda t\,dt = -i\left[\int_{-T}^{0} k\,\text{sen}\,\lambda t\,dt + \int_{0}^{T} -k\,\text{sen}\,\lambda t\,dt\right] =$$

$$= -ik\left[-\frac{\cos\lambda t}{\lambda}\Big|_{-T}^{0} + \frac{\cos\lambda t}{\lambda}\Big|_{0}^{T}\right] = -ik\left[-\frac{1}{\lambda} + \frac{\cos\lambda T}{\lambda} + \frac{\cos\lambda T}{\lambda} - \frac{1}{\lambda}\right] =$$

$$= \frac{2ik}{\lambda}\cdot\left[1 - \cos\lambda T\right] = \begin{cases} \xrightarrow[\lambda\to\pm\infty]{} 0 \\ \xrightarrow[\lambda\to 0]{} 0 \\ \xrightarrow[\lambda\to\pm\frac{\pi}{T}]{} \frac{4ik}{\lambda} \end{cases}$$

d) $f(x) = \begin{cases} x+\pi & \text{si } -\pi \le x \le 0 \\ \pi - x & \text{si } 0 \le x \le \pi \\ 0 & \text{en otro caso} \end{cases}$



f es una función par, por tanto

$$\hat{f}(\lambda) = \int_{-\pi}^{\pi} f(x)\cos\lambda x\,dx = \int_{-\pi}^{0}(x+\pi)\cos\lambda x\,dx + \int_{0}^{\pi}(\pi - x)\cos\lambda x\,dx =$$

$$= \pi\left(\int_{-\pi}^{0}\cos\lambda x\,dx + \int_{0}^{\pi}\cos\lambda x\,dx\right) + x\left(\int_{-\pi}^{0}\cos\lambda x\,dx - \int_{0}^{\pi}\cos\lambda x\,dx\right) =$$

Como $\cos\lambda x$ es una función par

$$= 2\pi\int_{0}^{\pi}\cos\lambda x\,dx = 2\pi\left(\frac{\text{sen}\,\lambda x}{\lambda}\Big|_{0}^{\pi}\right) = 2\pi\,\frac{\text{sen}\,\lambda\pi}{\lambda}$$

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PRÁCTICA 2ª

PROBLEMA 14:

$$f_n(x) = \cos(2\pi\alpha x)\,\chi_{[-\frac{n}{\alpha},\frac{n}{\alpha}]}(x).$$

OBSERVACIÓN: EL PERIODO DE f_n ES $\frac{1}{\alpha}$ POR TANTO LA FRECUENCIA ES α.

LA TRANSFORMADA DE FOURIER DE f_n ES

$$\hat{f}_n(\lambda) = \mathcal{F}\{f_n\}(\lambda) = \int_{-\frac{n}{\alpha}}^{\frac{n}{\alpha}} \cos(2\pi\alpha x)\, e^{-i\lambda x}\,dx \underset{f_n \text{ ES PAR}}{=}$$

$$= \int_{-\frac{n}{\alpha}}^{\frac{n}{\alpha}} \cos(2\pi\alpha x)\cos\lambda x\,dx \underset{\cos A\cos B = \frac12\{\cos(A+B)+\cos(A-B)\}}{=} \frac12\int_{-\frac{n}{\alpha}}^{\frac{n}{\alpha}} \cos(2\pi\alpha x+\lambda x)+\cos(2\pi\alpha x-\lambda x)\,dx =$$

$$= \frac12\,\frac{\operatorname{sen}(2\pi\alpha x+\lambda x)}{2\pi\alpha+\lambda}\Big|_{-\frac{n}{\alpha}}^{\frac{n}{\alpha}} + \frac12\,\frac{\operatorname{sen}(2\pi\alpha x-\lambda x)}{2\pi\alpha-\lambda}\Big|_{-\frac{n}{\alpha}}^{\frac{n}{\alpha}}$$

$$= \frac{1}{\cancel{2}}\frac{1}{2\pi\alpha+\lambda}\Big[\cancel{2}\operatorname{sen}\left(2\pi n+\frac{\lambda}{\alpha}n\right)\Big] + \frac{1}{\cancel{2}}\frac{1}{2\pi\alpha-\lambda}\Big[\cancel{2}\operatorname{sen}\left(2\pi n-\lambda\frac{n}{\alpha}\right)\Big]$$

$$= \frac{1}{4\pi^2\alpha^2-\lambda^2}\Big[(2\pi\alpha-\lambda)\operatorname{sen}\frac{\lambda}{\alpha}n - (2\pi\alpha+\lambda)\operatorname{sen}\lambda\frac{n}{\alpha}\Big] =$$

$$= \frac{-2\lambda}{4\pi^2\alpha^2-\lambda^2}\operatorname{sen}\frac{\lambda}{\alpha}n$$

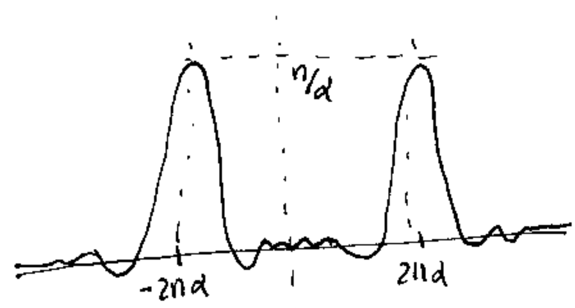
ESTA FUNCIÓN ES PAR.

$$\lim_{\lambda\to\pm 2\pi\alpha} \frac{-2\lambda}{4\pi^2\alpha^2-\lambda^2}\operatorname{sen}\frac{\lambda}{\alpha}n \underset{\text{L'HÔPITAL}}{=}$$

$$= \lim_{\lambda\to\pm 2\pi\alpha} \frac{-2\operatorname{sen}\frac{\lambda}{\alpha}n - 2\lambda\frac{n}{\alpha}\cos\left(\lambda\frac{n}{\alpha}\right)}{-2\lambda} =$$

$$= \frac{2\pi\alpha\, n/\alpha}{2\pi\alpha} = \frac{n}{\alpha}.$$

ADEMÁS $\lim_{\lambda\to\pm\infty}\hat{f}_n(\lambda) = 0$



POR OTRO LADO $\lim_{n\to\infty} f_n(x) =$

$$= \lim_{n\to\infty}\cos 2\pi\alpha x\,\chi_{[-\frac{n}{\alpha},\frac{n}{\alpha}]} = \cos 2\pi\alpha x$$

$f(x) = \cos 2\pi\alpha x$, $\hat{f}(x) = \lim_{n\to\infty}\int_{-n/\alpha}^{n/\alpha}\cos 2\pi\alpha x\, e^{-i x\lambda}\,dx =$

$$= \lim_{n\to\infty}\frac{-2\lambda\operatorname{sen}(\lambda\frac{n}{\alpha})}{4\pi\alpha^2-\lambda^2} = \begin{cases}\infty & \text{SI } \lambda = \pm 2\pi\alpha\\ \text{Y NO EXISTE SI } \lambda \neq \pm 2\pi\alpha.\end{cases}$$

PROBLEMA 15°] SI $h(t) = A e^{-\alpha t} \chi_{[0,\infty)}(t)$

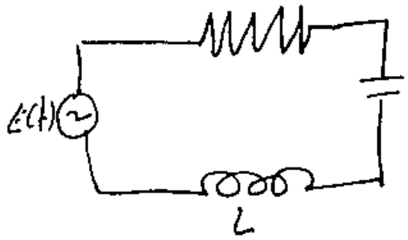
ENTONCES $\hat{h}(s) = \int_{-\infty}^{\infty} h(t)\, e^{-ist}\, dt = \int_0^\infty A e^{-\alpha t} e^{-ist}\, dt =$

$$= \int_0^\infty A\, e^{-t[\alpha + is]}\, dt = A \left. \frac{e^{-t[\alpha+is]}}{-[\alpha+is]} \right|_0^\infty =$$

$$= A \frac{1}{\alpha + is}.$$ (FILTRO DE BUTTERWORTH)

$\lim_{x \to \infty} e^{-x} = 0$

LA FUNCIÓN DE TRANSFERENCIA DE UN CIRCUITO "RLC"



ES $H(s) = \frac{1}{1 + RCis + (is)^2 LC}$

SI $L=0$ (NO HAY INDUCTANCIA; ES DECIR EN UN CIRCUITO RC)

$$= \frac{RC}{\frac{1}{RC} + is}$$

SI TOMAMOS $\frac{1}{RC}$ DE MODO QUE $\frac{1}{RC} = \alpha$

Y SI LA SEÑAL DE ENTRADA $E(t)$ LA AMPLIFICAMOS αA (e.d. $A\alpha E(t)$), ~~antes~~ ANTES DE PASARLA POR EL FILTRO RC)

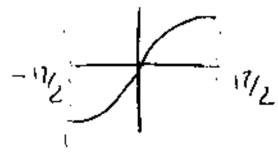
ENTONCES $E(t) \longrightarrow A\alpha E(t)$ Y AL PASARLA POR EL FILTRO OBTENEMOS $\widehat{A\alpha E(t)} \cdot \frac{RC}{\frac{1}{RC} + is} = A\alpha\, \widehat{E(t)} \cdot \frac{\frac{1}{\alpha}}{\alpha + is} =$

$$= \widehat{E(t)}\, \frac{A}{\alpha + is}.$$

HOJA 2:

PROBLEMA 16:]

b) $g(x) = \operatorname{sen} x \, \chi_{[-\pi/2, \pi/2]}(x)$



g ES UNA FUNCIÓN IMPAR, ASÍ QUE EL PROBLEMA 1:]

$$\hat{g}(\lambda) = -i \int_{-\pi/2}^{\pi/2} \operatorname{sen} x \operatorname{sen} \lambda x \, dx \underset{\downarrow}{=}$$

$\operatorname{sen} a \operatorname{sen} b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

$$= -i \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[\cos x(1-\lambda) - \cos x(1+\lambda)\right] dx =$$

$$= -\frac{i}{2}\left[\frac{\operatorname{sen} x(1-\lambda)}{1-\lambda}\Big|_{-\pi/2}^{\pi/2} - \frac{\operatorname{sen} x(1+\lambda)}{1+\lambda}\Big|_{-\pi/2}^{\pi/2} \right] = -i\left[\frac{\operatorname{sen} \frac{\pi}{2}(1-\lambda)}{1-\lambda} - \frac{\operatorname{sen} \frac{\pi}{2}(1+\lambda)}{1+\lambda}\right] =$$

$$= -i\left[\frac{\cos \frac{\pi}{2}\lambda}{1-\lambda} - \frac{\cos \frac{\pi}{2}\lambda}{1+\lambda}\right] = -\frac{i}{1-\lambda^2}\left[(1+\lambda)\cos \tfrac{\pi}{2}\lambda - (1-\lambda)\cos \tfrac{\pi}{2}\lambda\right] =$$

$$= -\frac{i}{1-\lambda^2} 2\lambda \cos \tfrac{\pi}{2}\lambda = \hat{g}(\lambda)$$ ES UNA FUNCIÓN IMPAR

ASÍ $$g(x) = \mathcal{F}^{-1}[\hat{g}(\lambda)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{i}{1-\lambda^2} 2\lambda \cos \tfrac{\pi}{2}\lambda \cdot e^{i\lambda x} d\lambda =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\lambda}{1-\lambda^2}\cos \tfrac{\pi}{2}\lambda \operatorname{sen} \lambda x \, d\lambda \underset{\downarrow}{=}$$

COMO $\frac{\lambda}{1-\lambda^2}\cos \frac{\pi}{2}\lambda \operatorname{sen} \lambda x$ ES PAR

$$= \frac{2}{\pi}\int_0^{\infty} \frac{\lambda}{1-\lambda^2}\cos \tfrac{\pi}{2}\lambda \operatorname{sen} \lambda x \, d\lambda$$

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HOJA 7:

PROBLEMA 18º Supongamos que a y b son positivos y que $0 < a < b$; sea $h(t) = f(at)$ y $g(t) = f(bt)$

Sea $y > 0$ $\quad h * g(y) = \int_{-\infty}^{\infty} h(t)\, g(y-t)\, dt =$

$$= \int_{-\infty}^{\infty} e^{-at} \chi_{[0,\infty)}(at)\, e^{-b(y-t)} \cdot \chi_{[0,\infty)}(b(y-t))\, dt$$

Como $\chi_{[0,\infty)}(t) = \begin{cases} 1 & \text{si } t > 0 \\ 0 & \text{si } t < 0 \end{cases}$

por tanto
$$\chi_{[0,\infty)}(b(y-t)) \underset{b>0}{=} \begin{cases} 1 & t < y \\ 0 & t > y \end{cases}$$

$$\underset{(*)}{=} \int_0^y e^{-at} e^{bt} e^{-by}\, dt = e^{-by} \int_0^y e^{-t(a-b)}\, dt =$$

$$= e^{-by} \left\{ -\frac{e^{-t(a-b)}}{a-b} \right|_0^y = e^{-by} \left[\frac{e^{-y(a-b)} - 1}{b-a} \right] =$$

$$= \frac{e^{-ya} - e^{-by}}{b-a} = \frac{f(ay) - f(by)}{b-a}.$$

Si $a = b$ y $y > 0$, continuando en $(*)$ se sigue que $\int_0^y e^{-ay}\, dt = y\, e^{-ay} = y\, f(ay)$