FUNCIONES ELEMENTALES. SENO Y COSENO

PROBLEMA 1º) a) $f(x) = \operatorname{arctn}(\operatorname{tn} x \cdot \operatorname{arctn} x)$.

APLICANDO LA REGLA DE LA CADENA Y LA REGLA DE DERIVACIÓN DE UN PRODUCTO

$$f'(x) = \operatorname{arctn}'(\operatorname{tn}x \cdot \operatorname{arctn} x)\left[\operatorname{tn}x \cdot \operatorname{arctn}x\right]' =$$

$$= \frac{1}{1+\operatorname{tn}^2x \cdot \operatorname{arctn}^2 x}\left[(1+\operatorname{tn}^2x)\operatorname{arctn}x + \frac{\operatorname{tn}x}{1+x^2}\right].$$

PROBLEMA 2º) a) $$\lim_{x\to 0} \frac{\cos x - 1 + (x^2/2)}{x^4} \underset{\text{L'HÔPITAL}}{=}$$

$$= \lim_{x\to 0} \frac{-\operatorname{sen}x + x}{4x^3} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to 0} \frac{-\cos x + 1}{12x^2} \underset{\text{L'HÔPITAL}}{=}$$

$$= \lim_{x\to 0} \frac{\operatorname{sen}x}{24x} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to 0} \frac{\cos x}{24} = 1/24.$$

PROBLEMA 3º) $f(x) = \dfrac{\operatorname{sen}x}{x}$, $f(0) = 1$.

Como $\lim_{x\to 0} \frac{\operatorname{sen}x}{x} = 1$, f es continua.

− f', $f'(x) = \dfrac{x\cos x - \operatorname{sen}x}{x^2}$ si $x \neq 0$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\frac{\operatorname{sen}x}{x} - 1}{x} =$$

$$= \lim_{x\to 0} \frac{\operatorname{sen}x - x}{x^2} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to 0} \frac{\cos x - 1}{2x} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to 0} \frac{-\operatorname{sen}x}{2} = 0$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{x\cdot\cos x - \operatorname{sen}x}{x^2} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to 0} \frac{\cancel{\cos x} - \cancel{x}\operatorname{sen}x - \cancel{\cos x}}{2\cancel{x}} = 0$$

f' es continua también en $x = 0$. Veamos si existe $f''(0)$

− $$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x\to 0} \frac{x\cos x - \operatorname{sen}x}{x^3} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to 0} \frac{-\cancel{x}\operatorname{sen}x}{3x^{\cancel{2}}} = -1/3.$$

FUNCIONES SENO Y COSENO

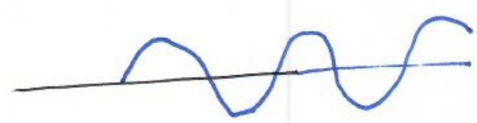
PROBLEMA 4:] a) $\lim_{x\to\infty} x\,\text{sen}(1/x) = \lim_{x\to\infty} \dfrac{\text{sen}\,1/x}{\frac{1}{x}} \underset{\text{L'HÔPITAL}}{=}$

$$= \lim_{x\to\infty} \frac{\cancel{\left(-\frac{1}{x^2}\right)} \cos\frac{1}{x}}{\cancel{\left(-\frac{1}{x^2}\right)}} = 1.$$

b) $\lim_{x\to 0} \dfrac{\int_0^{x^2} \cos(t^{1/2})\,dt}{\text{sen}^2 x} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to 0} \dfrac{\cos x \cdot \cancel{2}x}{\cancel{2}\,\text{sen}x\,\cancel{\cos x}} = 1.$

PROBLEMA 5:] $\lim_{x\to\infty} \dfrac{\text{sen}x}{x}$ como $\left|\dfrac{\text{sen}x}{x}\right| \le \dfrac{1}{x} \xrightarrow[x\to\infty]{} 0$,

el límite anterior es cero



NO EXISTE $\lim_{x\to\infty} \text{sen}x$

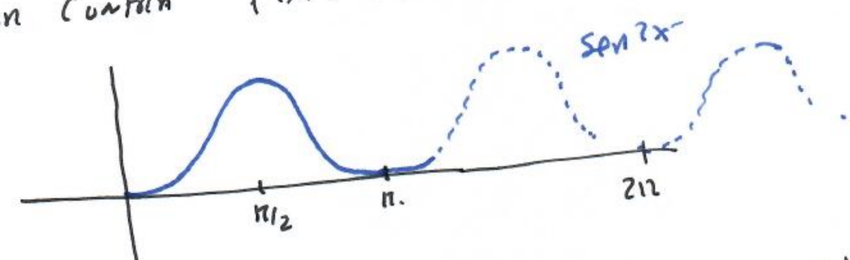
ASÍ SI $x = 2k\pi \Rightarrow \text{sen}\,2k\pi = 0 \quad \forall k \in \mathbb{N}$.

SI $x = \frac{\pi}{2} + 2k\pi \quad \text{sen}(\pi/2 + 2k\pi) = 1 \quad \forall k \in \mathbb{N}$

PROBLEMA 6:] a) $f(x) = \text{sen}^2 x$

OBSERVAMOS QUE SI $g(x) = \text{sen}x \Rightarrow g'(x) = \cos x$ y $\begin{cases} g'(0) = 1 \\ \text{y} \\ g'(\pi) = -1 \end{cases}$

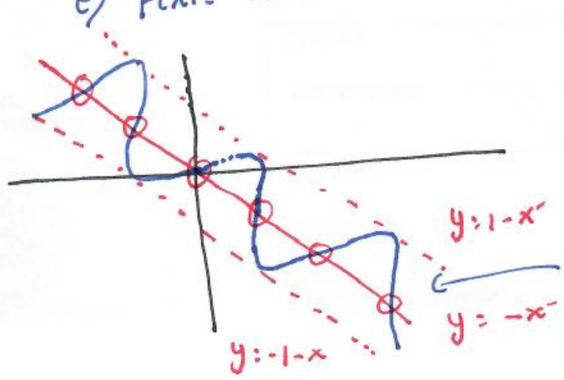
Sin contar $f'(x) = 2\,\text{sen}x\cos x$ y $f'(0) = f'(\pi) = 0$



e) $f(x) = \text{sen}x - x$

como $-1 \le \text{sen}x \le 1 \Rightarrow$

$\Rightarrow -1 - x \le \text{sen} - x \le 1 - x$



$f(x) = \text{sen}x - x$

$O = (k\pi, \text{sen}k\pi - k\pi)$

$k \in \mathbb{Z}$.

FUNCIONES SENO Y COSENO

PROBLEMA 7:] MIRAR TEORÍA.

PROBLEMA 8:] SI $\cos(x+y) = \cos x \cos y - \operatorname{sen} x \operatorname{sen} y$

ENTONCES DERIVANDO EN x

$$\cos'(x+y) = -\operatorname{sen}(x+y)$$

$$(\cos x \cos y - \operatorname{sen} x \operatorname{sen} y)' = -\operatorname{sen} x \cos y - \cos x \operatorname{sen} y$$

IGUALANDO $\boxed{\operatorname{sen}(x+y) = \operatorname{sen} x \cos y + \cos x \operatorname{sen} y}$

AHORA $\boxed{\cos(2x) = \cos(x+x) = \cos^2 x - \operatorname{sen}^2 x}$ *

$$\operatorname{sen}(2x) = \operatorname{sen}(x+x) = 2 \operatorname{sen} x \cos x.$$

$$\cos 3x = \cos(2x+x) = \cos 2x \cos x - \operatorname{sen} 2x \operatorname{sen} x =$$
$$= (\cos^2 x - \operatorname{sen}^2 x)\cos x - (2\operatorname{sen} x \cos x)\operatorname{sen} x =$$
$$= \cos^3 x - \operatorname{sen}^2 x \cos x - 2\operatorname{sen}^2 x \cos x =$$
$$= \cos^3 x - 3\operatorname{sen}^2 x \cos x.$$

$$\operatorname{sen} 3x = \operatorname{sen} 2x + x = \operatorname{sen} 2x \cos x + \cos 2x \operatorname{sen} x =$$
$$= 2\operatorname{sen} x \cos^2 x + \cos^2 x \operatorname{sen} x - \operatorname{sen}^3 x$$
$$= 3\cos^2 x \operatorname{sen} x - \operatorname{sen}^3 x.$$

$$\boxed{\operatorname{sen}(x+\pi/2)} = \operatorname{sen} x \underbrace{\cos \pi/2}_{=0} + \cos x \underbrace{\operatorname{sen} \pi/2}_{=1} = \boxed{\cos x}$$

PROBLEMA 9:) a) $\cos^2 x \underset{\text{EJER 8}}{=} \cos 2x + \operatorname{sen}^2 x = \cos 2x + (1 - \cos^2 x)$

DESPEJANDO $2\cos^2 x = \cos 2x + 1 \Rightarrow \boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$

$$\boxed{\operatorname{sen}^2 x = 1 - \cos^2 x = 1 - \left(\frac{1+\cos 2x}{2}\right) = \frac{1-\cos 2x}{2}}$$

b) $\cos^2 \frac{x}{2} = \frac{1+\cos x}{2} \Rightarrow \cos \frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$

FUNCIONES SENO Y COSENO

PROBLEMA 10: $A\,\text{sen}(x+B) = A(\text{sen}\,x \cos B + \cos x\,\text{sen}\,B) =$

$$= (A\cos B)\,\text{sen}\,x + (A\,\text{sen}\,B)\cos x$$

Es suficiente con tomar $a = A\cos B$ y $b = A\,\text{sen}\,B$..

PROBLEMA 11: $\tan(x+y) = \dfrac{\text{sen}(x+y)}{\cos(x+y)} =$

$x+y \neq \pi/2 + k\pi$

$$= \frac{\text{sen}\,x\cos y + \cos x\,\text{sen}\,y}{\cos x\cos y - \text{sen}\,x\,\text{sen}\,y} = \frac{\cancel{\cos x \cos y}\left(\frac{\text{sen}\,x}{\cos x} + \frac{\text{sen}\,y}{\cos y}\right)}{\cancel{\cos x \cos y}\left(1 - \frac{\text{sen}\,x\,\text{sen}\,y}{\cos x\cos y}\right)} =$$

Esto es

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

PROBLEMA 12: ¿$\text{sen}(\text{Arctan}\,x)$?

$$x = \tan(\text{Arctan}\,x) = \frac{\text{sen}(\text{Arctan}\,x)}{\cos(\text{Arctan}\,x)} = \frac{\text{sen}(\text{Arctan}\,x)}{\sqrt{1-\text{sen}^2 \text{Arctan}\,x}}$$

Así $x = \dfrac{A}{\sqrt{1-A^2}}$ luego $x^2 = \dfrac{A^2}{1-A^2} \Rightarrow$

$x^2 - x^2A^2 = A^2$ luego $x^2 = (1+x^2)A^2$ y

despejando $A = \text{sen}(\text{Arctan}\,x) = \dfrac{x}{\sqrt{1+x^2}}$

Del mismo modo

$$x = \tan(\text{Arctan}\,x) = \frac{\text{sen}(\text{Arctan}\,x)}{\cos(\text{Arctan}\,x)} = \frac{\sqrt{1-\cos^2(\text{Arctan}\,x)}}{\cos(\text{Arctan}\,x)}$$

Así $x = \dfrac{\sqrt{1-B^2}}{B} \Rightarrow x^2B^2 = 1-B^2$ luego $B^2(1+x^2) = 1$

Así $B = \sqrt{\dfrac{1}{1+x^2}}$

y $B = \cos(\text{Arctan}\,x) = \sqrt{\dfrac{1}{1+x^2}}$

FUNCIONES SENO Y COSENO

PROBLEMA 13:) SI $u = \tan\frac{x}{2}$, encuentre

$u = \tan\frac{x}{2}$ ASÍ $x = 2\,\mathrm{Arctan}\,u$.

LUEGO $\boxed{\operatorname{sen} x} = \operatorname{sen}(2\,\mathrm{Arctan}\,u) = 2\operatorname{sen}\mathrm{Arctan}\,u \cos\mathrm{Arctan}\,u =$

Prob 8 / Problema 12:)

$$= 2\frac{u}{\sqrt{1+u^2}}\cdot\frac{1}{\sqrt{1+u^2}} = \boxed{\frac{2u}{1+u^2}}$$

— · —

$u = \tan\frac{x}{2}$ ASÍ $x = 2\,\mathrm{Arctan}\,u$

LUEGO $\cos x = \cos(2\,\mathrm{Arctan}\,u) = \cos^2(\mathrm{Arctan}\,u) - \operatorname{sen}^2(\mathrm{Arctan}\,u) =$

Prob 8 / Problema 12:

$$= \frac{1}{1+u^2} - \frac{u^2}{1+u^2} = \frac{1-u^2}{1+u^2}$$

PROBLEMA 14:)

a) $|\operatorname{sen} x - \operatorname{sen} y| = |\cos\xi||x-y| \leq |x-y|$

T VALOR MEDIO $\quad 0 \leq |\cos\xi| \leq 1$

$\xi \in (x, y)$

b) SI $0 \leq x, y \leq \pi$, $\xi \in (0,\pi)$ y $|\cos\xi| < 1$.

SI $0 \leq x \leq \pi < y \leq 2\pi$.

$|\operatorname{sen} x - \operatorname{sen} y| \leq |\operatorname{sen} x - \operatorname{sen}\pi| + |\operatorname{sen}\pi - \operatorname{sen} y| <$

$< |x-\pi| + |\pi - y| = \pi - x + y - \pi = |x-y|$.

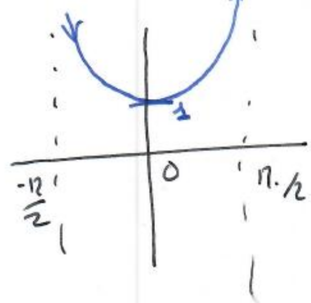
por a)

FUNCIONES SENO Y COSENO

PAR, POR SER $\cos x$ PAR

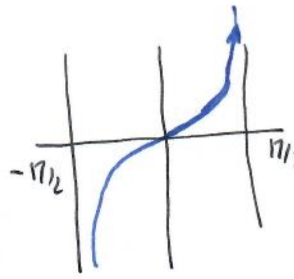
PROBLEMA 15: a) $\sec x = \frac{1}{\cos x}$ $\quad x \in (-\pi/2, \pi/2)$

$\lim_{x\to \pi/2^-} \sec x = \infty$, $\lim_{x\to -\pi/2^+} \sec x = \infty$

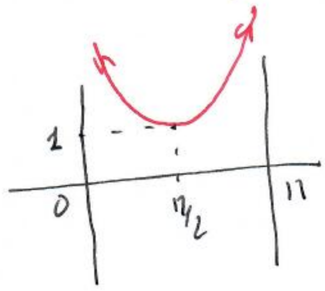


$(\sec x)' = \frac{\sin x}{\cos^2 x} \begin{cases} >0 & x>0 \\ =0 & \text{si } x=0 \\ <0 & \text{si } x<0 \end{cases}$

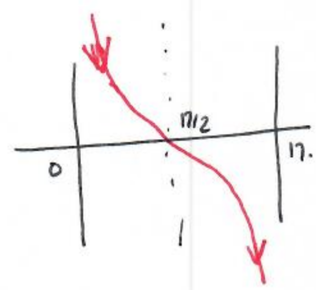
b) $\tan x$



c) $\operatorname{cosec} x = \frac{1}{\operatorname{sen} x} \geq 0 \quad x \in (0,\pi)$; mínimo en $x=\pi/2$ ya que $\operatorname{sen} \pi/2 \geq \operatorname{sen} x \quad \forall x \in [0,\pi]$.



d) $\cot x = \frac{\cos x}{\operatorname{sen} x} \quad x \in (0,\pi)$.



PROBLEMA 16: a) $\operatorname{sen} x - \operatorname{sen} 0 = \cos \xi \,(x-0) < x$

T. VALOR MEDIO $\qquad \cos\xi < 1$ si $\xi \in (0,\pi/2)$

$\Rightarrow \operatorname{sen} x < x \iff \frac{\operatorname{sen} x}{2} < \frac{x}{2}$ si $x \in (0,\pi/2)$

— · —

$\tan x - \tan 0 = (\tan \xi)'(x-0) = \frac{1}{\cos^2 \xi} x > x$

$0 < \cos^2\xi < 1 \quad \xi \in (0,\pi/2)$

$\Rightarrow \tan x > x \iff \frac{\operatorname{sen} x}{2\cos x} > x/2$ si $x \in (0,\pi/2)$

Luego $\frac{\operatorname{sen} x}{2} < \frac{x}{2} < \frac{\tan x}{2}$

b) Si $x \in (0,\pi/2)$, por a) $\frac{\operatorname{sen} x}{x} < \frac{x}{x} = 1$

y $x < \frac{\operatorname{sen} x}{\cos x} \iff \cos x < \frac{\operatorname{sen} x}{x}$ ($x>0$, $\cos x>0$)

— · —

$\cos x < \frac{\operatorname{sen} x}{x} < 1$

$x\to 0$: $\cos x \to 1$, $\frac{\operatorname{sen} x}{x} \to 1$, $1 \to 1$

Así