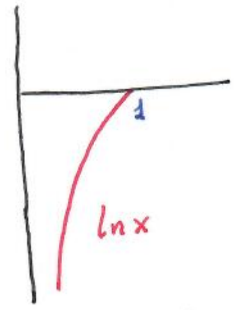
INTEGRALES IMPROPIAS

PROBLEMA 1:

a)



$\int_0^1 \ln x \, dx =$

$= \lim_{s \to 0} \int_s^1 \ln x \, dx \underset{\text{partes}}{=} \lim_{s \to 0} \left[x \ln x \Big|_s^1 - \int_s^1 dx \right] =$

$= \lim_{s \to 0} \left[-s \ln s - x \Big|_s^1 \right] = 0 - (-1) = 1.$

LA INTEGRAL IMPROPIA ES CONVERGENTE

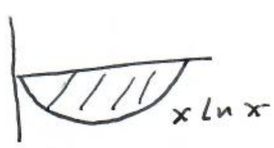
b) $\int_1^2 \frac{1}{x \ln x} \, dx = \ln \ln x \Big|_1^2 = \ln \ln 2 - \ln \ln 1 = \infty$

$(\ln x)' = \frac{1}{x}$

LA INTEGRAL IMPROPIA DIVERGE

c) $\int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{s \to 1^+} \int_s^2 \frac{dx}{\sqrt{x-1}} = \lim_{s \to 1^+} 2\sqrt{x-1} \Big|_s^2 = 2$

d) $\int_0^1 x \ln x \, dx$



$\lim_{x \to 0} x \ln x = 0$

NO ES UNA INTEGRAL IMPROPIA

$\int_0^1 x \ln x \, dx \underset{\text{partes}}{=} \frac{x^2}{2} \ln x \Big|_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{1}{x} \, dx =$

$= 0 - \frac{x^2}{4} \Big|_0^1 = -\frac{1}{4}$

INTEGRALES IMPROPIAS

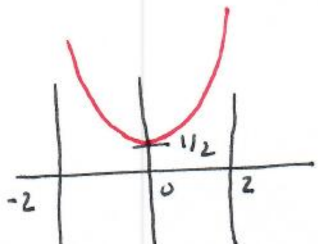
PROBLEMA 2: $\displaystyle\int_a^{\infty} u'v = \lim_{r\to\infty}\int_a^{r} u'v =$ (↓ PARCIAL)

$$\lim_{r\to\infty}\left[uv\Big|_a^{r} - \int_a^{r} uv' \right] =$$

$$= \lim_{r\to\infty}\left[uv(r) - uv(a) \right] - \lim_{r\to\infty}\int_a^{r} uv' =$$

$$= uv\Big|_a^{\infty} - \int_a^{\infty} uv'$$

PROBLEMA 3: b) $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}}$



$f(x) = \dfrac{1}{\sqrt{4-x^2}}$ ES PAR, ASÍ $f(x) = f(-x)$,

POR TANTO $\displaystyle\int_0^{r} f(x)\,dx = \int_{-r}^{0} f(x)\,dx.$

SI EXISTE LA INTEGRAL IMPROPIA $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}} =$

$$= 2\int_0^{2} \frac{dx}{\sqrt{4-x^2}} = 2\lim_{r\to 2^-}\int_0^{r} \frac{dx}{\sqrt{4-x^2}} =$$

AHORA $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(\frac{x}{2}\right)^2}} =$

$\frac{x}{2} = \cos y$

$dx = 2(-\operatorname{sen} y)\,dy$

$$= \frac{1}{2}\int \frac{-2\operatorname{sen} y}{\sqrt{1-\cos^2 y}}\,dy = -y = -\operatorname{Arc\,cos}\frac{x}{2}$$

$$= 2\lim_{r\to 2^-} -\operatorname{Arc\,cos}\frac{x}{2}\Big|_0^{r} = 2\left[-\operatorname{Arc\,cos} 1 + \operatorname{Arc\,cos} 0\right] =$$

$$= 2\left[0 + \frac{\pi}{2}\right] = \pi.$$

INTEGRALES IMPROPIAS

PROBLEMA 3:

e) $\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = 1$

$\lim_{x\to\infty} -\frac{1}{x} = 0$

f) $\int_0^\infty e^{-x}\,\text{sen}\,x\,dx = \lim_{r\to\infty}\int_0^r e^{-x}\,\text{sen}\,x\,dx =$

$\int e^{-x}\,\text{sen}\,x\,dx = -e^{-x}\,\text{sen}\,x + \int e^{-x}\cos x\,dx =$

PARTES

$= -e^{-x}\,\text{sen}\,x + \left[-e^{-x}\cos x - \int e^{-x}\,\text{sen}\,x\,dx\right]$

PARTES

DESPEJANDO $2\int e^{-x}\,\text{sen}\,x\,dx = -e^{-x}[\text{sen}\,x + \cos x]$

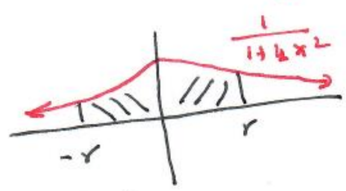
$= \lim_{r\to\infty} \frac{-e^{-x}[\text{sen}\,x + \cos x]}{2}\Big|_0^r =$

$= \lim_{r\to\infty}\left[-\frac{e^{-r}[\text{sen}\,r + \cos r]}{2} + \frac{1}{2}\right] = 1/2$

$e^{-r} \xrightarrow[r\to\infty]{} 0$

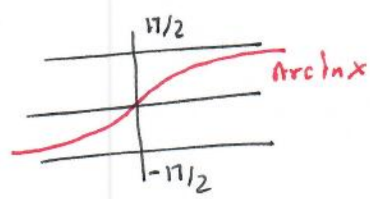
y $|\text{sen}\,r + \cos r| \le 2$.

g) $\int_{-\infty}^\infty \frac{dx}{1+4x^2} =$

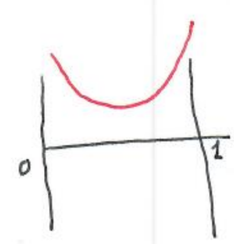


FUNCIÓN PAR

$= 2\int_0^\infty \frac{1}{1+\left(\frac{x}{\frac{1}{2}}\right)^2} = \text{Arctn}\,2x\Big|_0^\infty = \pi/2$



h) $\int_0^1 \frac{dx}{\sqrt{-x^2+x}} = \int_{1/2}^1 \frac{dx}{\sqrt{-x^2+x}} + \int_0^{1/2}\frac{dx}{\sqrt{-x^2+x}}$



$= \lim_{r\to 1}\int_{1/2}^r \frac{dx}{\sqrt{-x^2+x}} + \lim_{s\to 0}\int_s^{1/2}\frac{dx}{\sqrt{-x^2+x}}$

$\int \frac{1}{\sqrt{-x^2+x}}\,dx = \int\frac{dx}{\sqrt{-x^2+x-\frac{1}{4}+\frac{1}{4}}} = \int\frac{dx}{\sqrt{-(x-1/2)^2+\frac{1}{4}}} =$

INTEGRALES IMPROPIAS

3:] k) CONTINUACIÓN

$$= \int \frac{dx}{\sqrt{\frac{1}{4}\left(1 - 4(x-1/2)^2\right)}} = 2\int \frac{dx}{\sqrt{1-(2x-1)^2}} =$$

$2x-1 = \cos y$

$x = \frac{1+\cos y}{2}$

$dx = -\frac{1}{2}\sin y\, dy$

$$= 2\int \frac{-1/2 \sin y}{\sqrt{1-\cos^2 y}}\, dy = -\int dy = -y = -\operatorname{Arc}\cos(2x-1)$$

ASÍ

$$\lim_{r\to 1} \int_{1/2}^{1} \frac{dx}{\sqrt{-x^2+x}} = \lim_{r\to 1} -\operatorname{Arc}\cos(2x-1)\Big|_{1/2}^{r} =$$

$$-\operatorname{Arc}\cos(r) + \operatorname{Arc}\cos 0 = -\frac{\pi}{4} + 1 = 1 - \frac{\pi}{4}$$

$$\lim_{s\to 0} \int_{s}^{1/2} \frac{dx}{\sqrt{-x^2+x}} = \lim_{s\to 0} -\operatorname{Arc}\cos(2x-1)\Big|_{s}^{1/2} =$$

$$= \lim_{s\to 0} -1 + \operatorname{Arc}\cos(-1) = -1 + \pi.$$

Luego $\int_0^1 \frac{dx}{\sqrt{x-x^2}} = 1 - \pi/4 - 1 + \pi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

m) $\int_0^\infty x^n e^{-x}\, dx = -x^n e^{-x}\Big|_0^\infty + n\int x^{n-1} e^{-x}\, dx =$

(PARCIAL) ($\underbrace{}_{0}$)

$$= n\int_0^\infty x^{n-1} e^{-x}\, dx = \ldots = n! \int_0^\infty e^{-x}\, dx =$$

(PARCIAL $n-1$ VECES)

$$= n!\left[-e^{-x}\Big|_0^\infty\right] = n!$$

INTEGRALES IMPROPIAS

PROBLEMA 4: a) $\int_1^\infty \frac{\operatorname{Arctn} x}{x^2}\,dx \underset{\text{PARTES}}{=}$

$$= -\frac{1}{x}\operatorname{Arctn} x\Big|_1^\infty + \int_1^\infty \frac{1}{x(1+x^2)}\,dx =$$

$\lim_{x\to\infty} \frac{\operatorname{Arctn} x}{x} = 0$

$\operatorname{Arctn} 1 = \frac{\pi}{4}$

$$= \frac{\pi}{4} + \int_1^\infty \frac{1}{x(1+x^2)}\,dx$$

AHORA $\int_1^\infty \frac{dx}{x(1+x^2)} = \int_1^\infty \frac{1}{x} - \frac{x}{1+x^2}\,dx =$

$$= \left[\ln x - \frac{1}{2}\ln(1+x^2)\Big|_1^\infty\right] = \ln\left(\frac{x}{\sqrt{1+x^2}}\right)\Big|_1^\infty =$$

$\lim_{x\to\infty} \frac{x}{\sqrt{1+x^2}} = 1$

$$= -\ln\left[\frac{1}{\sqrt{2}}\right] = \ln\sqrt{2}.$$

PROBLEMA 5: $\int_2^\infty \left(\frac{cx}{x^2+1}\right) - \frac{1}{2x+1}\,dx =$

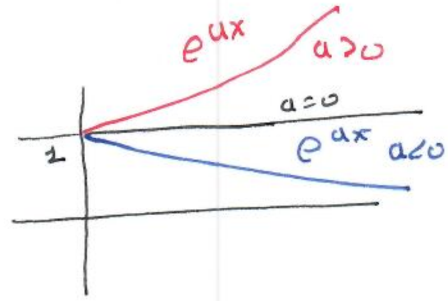
$$= \left[\frac{c}{2}\ln x^2+1 - \frac{1}{2}\ln 2x+1\right|_2^\infty =$$

$$= \frac{1}{2}\ln\frac{(x^2+1)^c}{2x+1}\Big|_2^\infty =$$

$$\left(\lim_{x\to\infty} \ln\frac{(x^2+1)^c}{2x+1} = \begin{cases} \infty & \text{SI } c > \frac{1}{2} \\ \ln\frac{1}{2} & \text{SI } c = 1/2 \\ -\infty & \text{SI } c < 1/2 \end{cases}\right)$$

$$\underset{c=1/2}{=} \frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{\sqrt{26}}{5} = \frac{1}{2}\ln\frac{\frac{1}{2}}{\frac{\sqrt{26}}{5}} = \ln\sqrt{\frac{5}{2\sqrt{26}}}$$

INTEGRALES IMPROPIAS



PROBLEMA 6:) c) $\int_0^\infty e^{ax}\,dx$

La integral existe si $a<0$;
en otro caso no

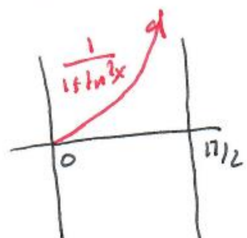
d) $\int_1^\infty \frac{x^2}{\sqrt{2x^4-x+1}}\,dx = \int_1^\infty \frac{1}{\sqrt{2-\frac{1}{x^3}+\frac{1}{x^4}}}\,dx$

Como $\frac{1}{\sqrt{2-\frac{1}{x^3}+\frac{1}{x^4}}} \sim \frac{1}{\sqrt{2}}$ si x es grande

no puede existir la integral

f) $\int_0^1 \frac{\ln x}{x}\,dx = \frac{\ln^2 x}{2}\Big|_0^1 = \infty.$

j) $\int_0^{\pi/2} \frac{dx}{1+\tan^2 x} =$



$= \int_0^{\pi/2} \frac{dx}{\frac{1}{\cos^2 x}} = \int_0^{\pi/2} \cos^2 x\,dx = \frac{x}{2} + \frac{\operatorname{sen} 2x}{4}\Big|_0^{\pi/2} = \frac{\pi}{4}.$

k) $\int_0^\infty \frac{\cos x}{x}\,dx = \int_0^1 \frac{\cos x}{x}\,dx + \int_1^\infty \frac{\cos x}{x}\,dx$ no existe

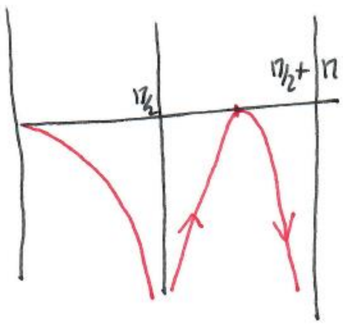
•) $\cos x \geq \delta > 0$ para $x \in [0,1]$ y $\int_0^1 \frac{1}{x}\,dx$ no existe

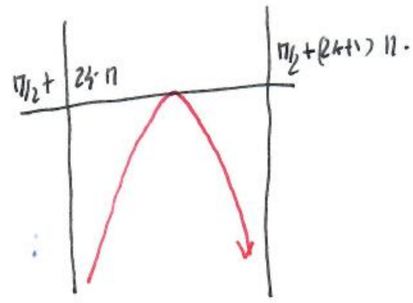
••) $\int_1^\infty \frac{\cos x}{x}\,dx$ como $\left|\int_1^r \cos x\,dx\right| \leq 2$ y $\frac{1}{x} \downarrow 0$ ($x\to\infty$)
el criterio de Dirichlet nos dice que existe la integral

por tanto $\int_0^\infty \frac{\cos x}{x}\,dx$ no existe.

INTEGRALES IMPROPIOS

PROBLEMA 6: l) $\int_0^\infty e^{-x} \ln(\cos^2 x)\, dx =$





$$\int_0^{\pi/2} e^{-x}\ln(\cos^2 x)\,dx + \sum_{k=0}^{\infty} \int_{\frac{\pi}{2}+k\pi}^{\pi/2+(k+1)\pi} e^{-x} \ln(\cos^2 x)\, dx$$

OBSERVEMOS QUE $\frac{\cos x}{x-\pi/2} \xrightarrow[x\to\pi/2]{} 1$

LUEGO $\lim_{x\to\pi/2} \frac{\ln \cos^2 x}{\ln(x-\pi/2)^2} \underset{\text{L'HOSPITAL}}{=} \lim_{x\to\pi/2} \frac{-2\sin x \cos x}{2(x-\pi/2)} \cdot \frac{(x-\frac{\pi}{2})^2}{\cos^2 x} = 1$

AHORA $\int_0^{\pi/2} \ln(x-\pi/2)^2 \underset{\substack{y = x-\pi/2 \\ dy = dx}}{=} \int_{-\pi/2}^{0} \ln y^2\, dy \underset{\text{PARTES}}{=}$

$$= y \ln y^2 \Big|_{-\pi/2}^{0} - \int_{-\pi/2}^{0} y \frac{2y}{y^2}\, dy =$$

$$= \frac{\pi}{2} \ln\left(\frac{\pi}{2}\right)^2 - \pi \quad \text{FINITO}$$

SEA $A = \int_{\pi/2+k\pi}^{\pi/2+k\pi+\frac{\pi}{2}} \ln(\cos^2 x)\, dx = \int_{\pi/2+k\pi+\pi/2}^{\pi/2+(k+1)\pi} \ln(\cos^2 x)\, dx$

ASÍ $\left| \int_0^\infty e^{-x} \ln(\cos^2 x)\, dx \right| \le |A| + \sum_{k=0}^{\infty} \int_{\frac{\pi}{2}+k\pi}^{\pi/2+(k+1)\pi} |e^{-x} \ln(\cos^2 x)|\, dx$

$$\le |A| + \sum_{k=0}^{\infty} e^{-\frac{\pi}{2}-k\pi} \cdot \int_{\pi/2+k\pi}^{\pi/2+(k+1)\pi} |\ln(\cos^2 x)|\, dx \le$$

$$\le |A| + \sum_{k=0}^{\infty} 2|A|\, e^{-\pi/2} (e^{-\pi})^k\, dx < \infty$$

SERIE GEOMÉTRICA

INTEGRALES IMPROPIAS

PROBLEMA 6: m) $\int_0^1 \operatorname{sen}\frac{1}{x}\,dx$

$\operatorname{sen}\frac{1}{x}$ $x\in[0,1]$ es continua en todo [0,1] salvo quizás en $x=0$. Luego $\operatorname{sen}\frac{1}{x}$ es integrable Riemann en $[0,1]$.

PROBLEMA 7: a) $\int_0^\infty \frac{\operatorname{sen}x}{1+x^2}\,dx$

$$\int_0^\infty \left|\frac{\operatorname{sen}x}{1+x^2}\right| dx \le \int_0^\infty \frac{1}{1+x^2}\,dx = \operatorname{Arctg}x\Big|_0^\infty = \pi/2$$

Por el criterio de comparación es absolutamente convergente.

c) $\int_0^1 \frac{e^{-x}\cos\frac{1}{x}}{\sqrt{x}}$ $\qquad \left|e^{-x}\cos\frac{1}{x}\right| \le 1 \quad \forall x\in[0,1]$

Luego $\int_0^1 \left|\frac{e^{-x}\cos\frac{1}{x}}{\sqrt{x}}\right| dx \le \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_0^1 = 2.$

Por el criterio de comparación la integral es absolutamente convergente.

PROBLEMA 8: Como existe $\int_0^\infty f(x)\,dx$, por el criterio de Cauchy

$$0 = \lim_{x\to\infty}\int_{x/2}^{x} f(s)\,ds \ge \lim_{x\to\infty}\int_{x/2}^{x} f(x)\,ds$$

f es decreciente

$$= \lim_{x\to\infty} \frac{x}{2} f(x). \text{ Así } \lim_{x\to\infty} x f(x) = 0.$$

PROBLEMA 9: a) $\int_0^\infty \frac{1}{x^\sigma}\,dx = \int_0^1 \frac{1}{x^\sigma}\,dx + \int_1^\infty \frac{1}{x^\sigma}\,dx$

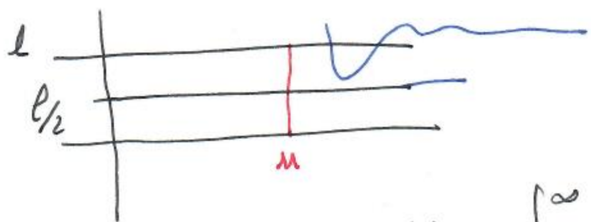
$\int_0^1 \frac{1}{x^\sigma}\,dx$ diverge si $\sigma \ge 1$ y $\int_1^\infty \frac{1}{x^\sigma}\,dx$ diverge si $\sigma \le 1$.

b) Sea $\frac{1}{x+1}$ continua en $[0,\infty)$; $\frac{1}{x+1} \xrightarrow[x\to\infty]{} 0$ y no existe $\int_0^\infty f(x)\,dx.$

INTEGRALES IMPROPIAS

PROBLEMA 10.-) a) $\lim_{x\to\infty} f(x) = \ell$

Supongamos $\ell > 0$, el otro caso $\ell < 0$ se hace de forma análoga.

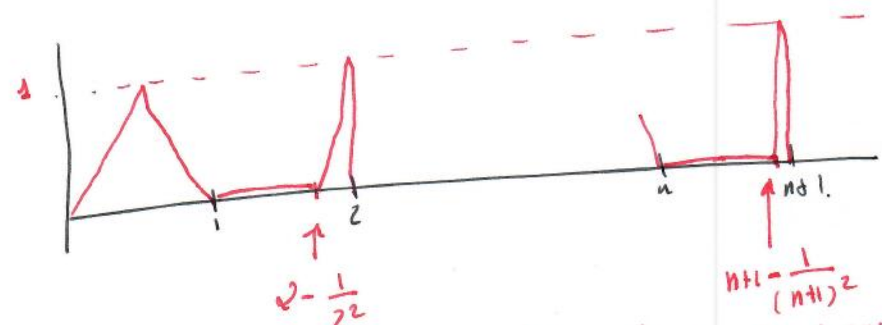


Existe $M>0$ con $f(x) > \ell/2$ para todo $x > M$

Así $\int_0^\infty f = \int_0^M f + \int_M^\infty f(x)\,dx$
$\in \mathbb{R}$ $\quad \in \mathbb{R} \quad \in \mathbb{R}$ ¡!

Ahora $\int_M^\infty f(x)\,dx \geq \int_M^\infty \ell/2 = \infty$

b) NO



$\int_0^\infty f = \sum_{n=1}^\infty \frac{1}{2n^2} < \infty$, pero $\lim_{x\to\infty} f(x)$ no existe.

c) El ejemplo anterior no es uniformemente continua.

Si existe el límite, por a) es nulo.

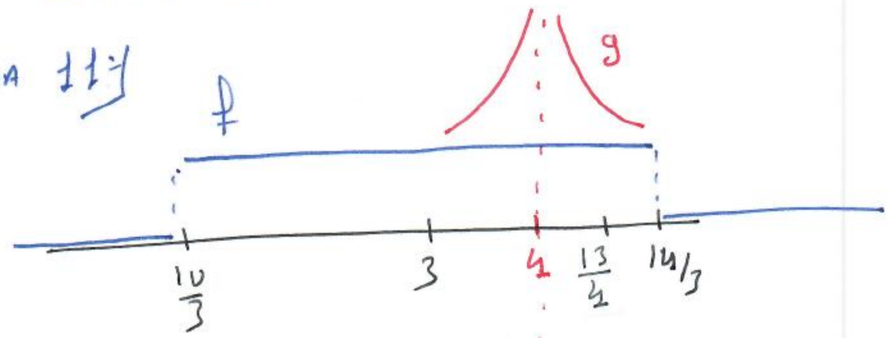
Si suponemos que no existe $\lim_{x\to\infty} f(x)$, existiría ℓ y $x_n \uparrow \infty$ con $\ell > 0$ (si $\ell < 0$, se hace igual) y $f(x_n) \geq \ell$. Sea δ_n tal que $f(x) \geq \ell/2$ $\forall x \in (x_n-\delta_n, x_n+\delta_n)$. $\delta_n \to 0$ (en otro caso como existe $\int_0^\infty f$,

$$\sum_n \int_{x_n-\delta_n}^{x_n+\delta_n} f \geq \sum_{n=1}^{\infty} \delta_n \,\ell/2 = \infty$$ ¡contradice el criterio de Cauchy!

Pero si $\delta_n \to 0$, la función no es uniformemente continua.

INTEGRALES IMPROPIOS

PROBLEMA 11:



$$\int_3^\infty f\cdot g = \int_3^{14/3} g = \int_3^{4} g + \int_4^{14/3} g$$

DIVERGE — DIVERGE

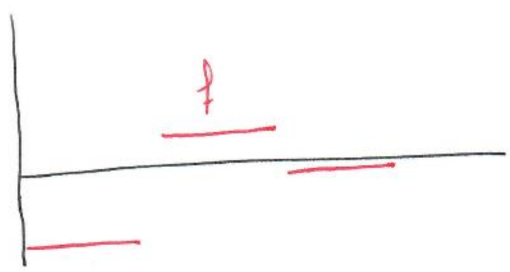
$$\int_{13/3}^\infty f\cdot g = \int_{13/3}^{14/3} g \quad \text{convergente}$$

$$\int_{14/3}^\infty f\cdot g \equiv 0$$

Si $a < 4$ $\quad \int_a^\infty f\cdot g$ DIVERGE.

PROBLEMA 12: Sea $f\ (1,\infty) \longrightarrow \mathbb{R}$

$x \longrightarrow f(x) \equiv \frac{(-1)^n}{\sqrt{n}}$ si $x \in [n, n+1]$

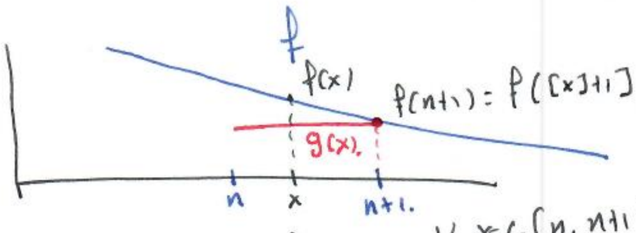


$$\int_1^\infty f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \text{convergente}$$

por el criterio de Abel para series

$$\int_1^\infty f^2(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{divergente}$$

PROBLEMA 13:



Así $f(n+1) = f([x]+1) = g(x) \le f(x) \quad \forall x \in [n, n+1]$

Luego $f(x) \geq g(x) \quad \forall x \in \mathbb{R}$

$$\int_0^\infty f(x)\,dx \geq \int_0^\infty g(x)\,dx = \sum_{n\geq 1}^{\infty} f(n)..$$

Si f es de términos positivos, si $\int_0^\infty f(x)$ con[verge] la serie converge

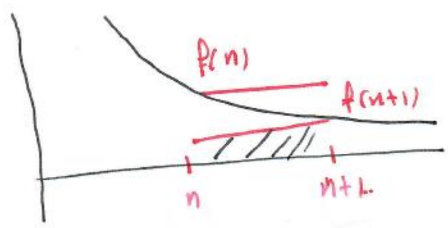
INTEGRALES IMPROPIAS

PROBLEMA 13:) CONTINUACIÓN

1) $\sum_{n=2}^{\infty} \frac{1}{n^k}$ para $k > 0$

Si $k > 1$, $\int_1^{\infty} \frac{1}{x^k}$ converge y por tanto la serie.

Si $k \leq 1$ observamos que $\frac{1}{x^k}$



Así $\int_1^{\infty} \frac{1}{x^k} \leq \sum_{n=1}^{\infty} \frac{1}{(n+1)^k}$

divergente $k \leq 1$ — serie divergente.

2) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ observamos que $\frac{1}{x(\ln x)^3}$ decrece y positiva en $x > 1$

Ahora $\int_2^{\infty} \frac{1}{x(\ln x)^3} = \left. \frac{-1}{2(\ln x)^2} \right|_2^{\infty} = \frac{1}{2(\ln 2)^2}$

convergente, luego la serie $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ es convergente.

PROBLEMA 14:)

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} = \begin{cases} \infty & \text{si } p \geq 1 \\ \cdot & \\ \infty & \text{si } p < 1 \end{cases}$$

Observamos que si $p < 1$, $\int_0^{\infty} \frac{1}{x^{1-p}(1+x)} dx$ no converge o en "0" o en "∞".

INTEGRALES IMPROPIAS

PROBLEMA 15:

a) $\int_{-1}^{\infty} \frac{e^{-x^2}}{x}\, dx$ PROBLEMA $x \sim 0$

c) $\int_0^1 \frac{1}{x^2-1}\, dx$ PROBLEMA EN $x \sim 1$.

d) $\int_0^{\infty} \operatorname{sen} x\, dx = -\cos x \Big|_0^{\infty}$ y no existe $\lim_{x\to\infty} \cos x$.

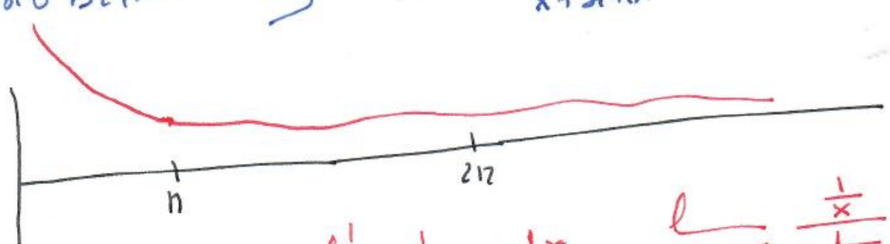
b) $\int_2^{\infty} \frac{(\sqrt{x}+1)^3}{x^3+2x+1}\, dx$

$$\frac{(\sqrt{x}+1)^3}{x^3+2x+1} \leq \frac{(2\sqrt{x})^3}{x^3+2x+1} = \frac{8x^{3/2}}{x^3+2x+1}.$$

$\sqrt{x}>1$

y $\lim_{x\to\infty} \frac{\frac{8x^{3/2}}{x^3+2x+1}}{\frac{1}{x^{3/2}}} = 8$. Por el criterio de comparación por límite, como $\int_2^{\infty} \frac{1}{x^{3/2}} < \infty$, también converge $\int \frac{8x^{3/2}}{x^3+2x+1}\, dx$ y también por el criterio de comparación también converge $\int_2^{\infty} \frac{(\sqrt{x}+1)^3}{x^3+2x+1}\, dx$

PROBLEMA 16: $f(x) = \frac{1}{x+\operatorname{sen} x}$



a) $\int_0^1 f(t)\, dt = \int_0^1 \frac{1}{x+\operatorname{sen} x}\, dx$ $\lim_{x\to 0} \frac{\frac{1}{x}}{\frac{1}{x+\operatorname{sen} x}} = \lim_{x\to 0} \frac{x+\operatorname{sen} x}{x} = 1$

no existe $\int_0^1 \frac{1}{x}$

b) $\int_1^{\infty} f(t)\, dt$ DIVERGE ; $\int_1^{\infty} \frac{1}{x}$ DIVERGE.

c) f es continua en $[\pi, 2\pi] \Rightarrow \exists \int_{\pi}^{2\pi} f(t) > 0$

f positiva

d) $\infty = \int_0^{1/2} f(t)\, dt > 1/3$

INTEGRALES IMPROPIAS

PROBLEMA 17:

a) $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx =$

CAMBIO

$y = \left(\frac{x-\mu}{\sigma}\right)$ $\quad x=\mu \Rightarrow y=0$

$x=-\infty \Rightarrow y=-\infty$

$dy = \frac{1}{\sigma} dx$ $\quad x=\infty \Rightarrow y=\infty$

$= \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{\infty} \dots dx =$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{1}{\sqrt{2}}y\right)^2} dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz =$

$z = \frac{1}{\sqrt{2}} y$

$dz = \frac{1}{\sqrt{2}} dy$

$= \frac{1}{\sqrt{\pi}}\, 2 \int_{0}^{\infty} e^{-z^2} dz = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1.$

e^{-z^2} PAR

PROBLEMA 18: a) $\int_0^1 e^{-t} t^{x-1} dt \leq \int_0^1 \frac{1}{t^{1-x}} dt < \infty$ $\quad 1-x < 1$, $x>0$

$\int_1^\infty e^{-t} t^{x-1} dt \leq \int_1^\infty e^{-t} t^n dt < \infty$ (por partes) $\quad x-1 < n$

b) $\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt = -e^{-t} t^x \Big|_0^\infty + x \int_0^\infty e^{-t} t^{x-1} dt = x\,\Gamma(x)$ (partes; $-e^{-t} t^x \Big|_0^\infty = 0$)

c) $\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 1$

ASÍ $\Gamma(2) = 1 \times \Gamma(1) = 1$, $\Gamma(3) = 2\,\Gamma(2) = 2$ $\quad \Gamma(4) = 3 \times \Gamma(3) = 3 \times 2 \quad \dots \Gamma(n) = (n-1)!$ inducción

d) $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \frac{1}{x} \int_0^\infty e^{-\sqrt[x]{u}} du.$

$u = t^x \Rightarrow t = \sqrt[x]{u}$

$du = x t^{x-1} dt$

ASÍ $\Gamma(1/2) = 2 \int_0^\infty e^{-u^2} du$; es decir $\int_0^\infty e^{-u^2} du = \frac{1}{2} \Gamma(1/2) =$

$= 1/2 \int_0^\infty e^{-t} t^{-1/2} dt.$

INTEGRALES IMPROPIAS

PROBLEMA 19:

a) $\mathcal{L}\{x\}s = \int_0^\infty x\, e^{-sx}\,dx = \frac{x\,e^{-sx}}{-s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-sx}\,dx$

↓ PARTES — $\frac{x\,e^{-sx}}{-s}\Big|_0^\infty = 0$, $s>0$

$= \frac{1}{s}\int_0^\infty e^{-sx}\,dx = \frac{1}{s}\left[\frac{e^{-sx}}{-s}\Big|_0^\infty\right] = \frac{1}{s^2}.$

b) $\mathcal{L}\, f'(s) = \int_0^\infty f'(x)\, e^{-sx}\,dx = f(x)\,e^{-sx}\Big|_0^\infty -$

↓ DEF ↓ PARTES

$+s\int_0^\infty f(x)\, e^{-sx}\,dx =$

$= \lim_{x\to\infty} f(x)\,e^{-sx} - f(0) + s\,\mathcal{L}\, f(s) =$

($\lim_{x\to\infty} f(x)\,e^{-sx} = 0$ por hipótesis)

$= s\,\mathcal{L}\, f(s) - f(0)$