INTEGRALES IMPROPIAS II

PROBLEMA 1: $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \infty$

ASÍ DADO $M>0$, EXISTE $N<0$ TAL QUE SI $x<N$

ENTONCES $\frac{f(x)}{g(x)} > M$.

SI $f(x), g(x) > 0 \Rightarrow f(x) > M\, g(x) \quad \forall x<N$

SI $f(x), g(x) < 0 \Rightarrow f(x) < M\, g(x) \quad \forall x<N$

POR HIPÓTESIS $f(x) > M\, g(x) > 0 \quad \forall x<N$

ASÍ $\int_{-\infty}^{N} f(x)\,dx \geq M \int_{-\infty}^{N} g(x)\,dx$

LUEGO LA CONVERGENCIA DE $\int_{-\infty}^{T} f(x)\,dx = \int_N^T f + \int_{-\infty}^N f$

IMPLICA LA CONVERGENCIA DE $\int_{-\infty}^N g$; $\int_N^T g$

EXISTE POR SER g CONTINUA.

PROBLEMA 2: $\lim_{x\to\infty} \frac{f(x)\,x^2}{1+\operatorname{sen}^2 x} = 0$

$\forall \varepsilon>0$ EXISTE $M>0$ TAL QUE SI $x>M$

$$\left| \frac{f(x)\,x^2}{1+\operatorname{sen}^2 x} \right| < \varepsilon$$

LUEGO $|f(x)| \leq \frac{\varepsilon(1+\operatorname{sen}^2 x)}{x^2} \leq \frac{2\varepsilon}{x^2}$

ASÍ $\int_M^\infty |f(x)|\,dx \leq \int_M^\infty \frac{2\varepsilon}{x^2}\,dx < \varepsilon$, LUEGO

$\int_M^\infty f(x)\,dx$ ES ABSOLUTAMENTE INTEGRABLE. POR SER f CONTINUA

LA INTEGRAL $\int_0^M f(x)\,dx$ EXISTE

PROBLEMA 3: SEA $f(x) = \frac{1}{x^2}$; $f\,[1,\infty) \to (0,\infty)$

ES CONTINUA Y $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leq \frac{1}{x^2y^2}|y^2-x^2| \leq$

$\leq \frac{x+y}{x^2y^2}|x-y| \leq \left(\frac{1}{xy^2}+\frac{1}{x^2y}\right)|x-y|$

$\leq 2\,|x-y|$. f ES UNIF. CONTINUA

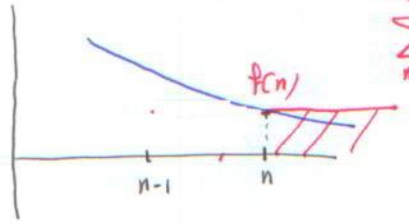
ADEMÁS $\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = 1$.

INTEGRALES IMPROPIAS II

PROBLEMA 4:) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^k}$

SEA $f(x)=\dfrac{1}{(\ln x)^k}$ CONTINUA Y DECRECIENTE

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^k} \geq \int_2^{\infty} \frac{1}{(\ln x)^k}\,dx$$



AHORA COMPARAMOS CON $\frac{1}{x}$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{(\ln x)^k}} = \lim_{x\to\infty} \frac{(\ln x)^k}{x} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to\infty} \frac{k(\ln x)^{k-1}}{x} \underset{\text{L'HÔPITAL}}{=} \dots$$

$$\dots = \lim_{x\to\infty} \frac{k!\,\ln x}{x} = \lim_{x\to\infty} k!\,\frac{1}{x} = 0$$

LUEGO PARA $\varepsilon>0$ $\exists\, M>0$ TAL QUE SI $x>M$

SE TIENE QUE $\frac{1}{x} \leq \varepsilon \frac{1}{(\ln x)^k}$

COMO $\int_2^{\infty} \frac{1}{x}\,dx$ DIVERGE TAMBIÉN LO HACE $\int_2^{\infty}\frac{1}{(\ln x)^k}dx$

Y POR TANTO LA SERIE.

$\displaystyle\sum_{n=2}^{\infty} (\ln n)^{-n}$ SEA $f(x)=\dfrac{1}{(\ln x)^x}$ DECRECIENTE

$$\sum_{n=3}^{\infty} \int \frac{1}{(\ln n)^n}\,dx \leq \int_2^{\infty} \frac{1}{(\ln x)^x}\,dx$$

COMPARAMOS $\frac{1}{(\ln x)^x}$ CON $\frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{\frac{1}{(\ln x)^x}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{e^{x\ln(\ln x)}} \underset{\text{L'HÔPITAL}}{=} \lim_{x\to\infty} \frac{2x}{e^{x\ln(\ln x)}\left[\ln\ln x + \cancel{x}\frac{1}{\ln x}\cdot\frac{1}{\cancel{x}}\right]} =$$

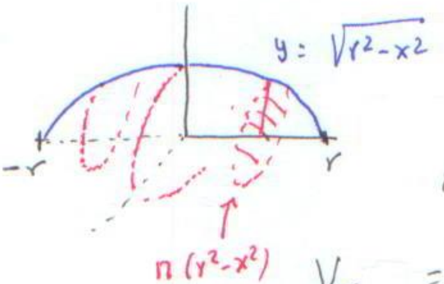
$$\underset{\text{L'HÔPITAL}}{=} \lim_{x\to\infty} \frac{2}{e^{x\ln(\ln x)}\left[\ln\ln x + \frac{1}{\ln x}\right]^2 + e^{x\ln\ln x}\left[\frac{1}{\ln x}\cdot\frac{1}{x} + \frac{1}{\ln^2 x}\cdot\frac{1}{x^2}\right]} = 0$$

LUEGO $\int_3^{\infty} \frac{1}{(\ln x)^x}\,dx$ CONVERGE, YA QUE LO HACE $\int_3^{\infty}\frac{1}{x^2}\,dx$

Y POR TANTO LA SERIE

INTEGRALES IMPROPIAS II

PROBL. 5º]



LA ESFERA ES UN VOLUMEN DE REVOLUCIÓN ASÍ

$$V_{S,r} = \int_{-r}^{r} \pi \left(\sqrt{r^2-x^2}\right)^2 dx =$$

$$= \pi \int_{-r}^{r} r^2 dx - \pi \int_{-r}^{r} x^2 dx =$$

$$= 2\pi r^3 - \pi \left[\frac{x^3}{3} \Big|_{-r}^{r} \right] = 2\pi r^3 - \pi \left[2\frac{r^3}{3} \right] =$$

$$= 2\pi \left[r^3 - \frac{r^3}{3} \right] =$$

$$= \frac{2\pi r^3}{3}$$

PROBLEMA 6:]

$$\int_0^\infty e^{-\frac{(x-a)^2}{2}} dx = \int_0^\infty e^{-y^2} \sqrt{2}\, dy =$$

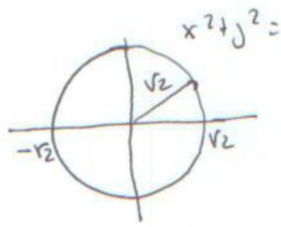
$y = \frac{x-a}{\sqrt{2}}$

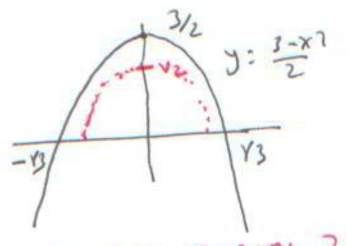
$dy = \frac{1}{\sqrt{2}} dx$

$$= \sqrt{2}\, \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{2}}.$$

PROBLEMA 7:]

a)





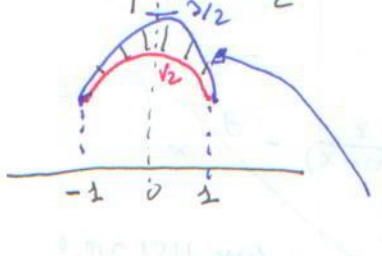
$\sqrt{3} > \sqrt{2}$

¿CÓMO SE RELACIONAN AMBAS CURVAS?

$$\left\{ \begin{array}{l} x^2+y^2=2 \\ y = \frac{3-x^2}{2} \end{array} \right| \iff x^2 + \left(\frac{3-x^2}{2}\right)^2 = 2 \iff x^2 + \frac{9}{4} - \frac{6x^2}{4} + \frac{x^4}{4} = 2$$

$$\iff \frac{x^4}{2} - \frac{1}{2}x^2 + \frac{1}{4} + 2 = 2$$

$$\iff \left(\frac{x^2}{2} - \frac{1}{2}\right)^2 = 0 \iff x = \pm 1$$



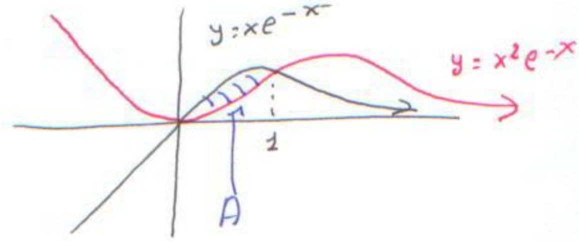
$$A = \int_{-1}^{1} \frac{3-x^2}{2} - \sqrt{2-x^2}\, dx$$

INTEGRALES IMPROPIAS II

PROBLEMA 7:

b) $y = xe^{-x}$

$y = x^2e^{-x}$



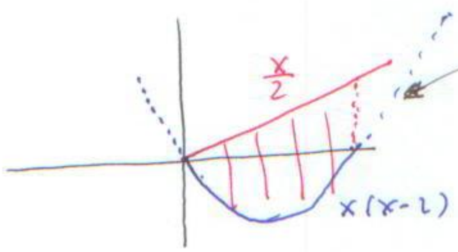
SI $x=0$ ó $x=1$ AMBAS CURVAS COINCIDEN.

PARA $x\in[0,1]$, $x > x^2$, LUEGO $xe^{-x} > x^2e^{-x}$, ASÍ

$$A = \int_0^1 xe^{-x} - x^2e^{-x} = \int_0^1 e^{-x}(x-x^2)\,dx =$$

$\downarrow$ PARTES

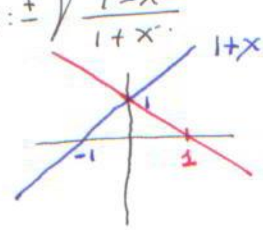
c) $f(x) = x(x-2)$ y ; $y(x) = x/2$ $\qquad x\in[0,2]$



$$A = \int_0^2 \frac{x}{2} - x(x-2)\,dx =$$

$$= \int_0^2 -x^2 + \frac{5}{2}x\,dx = \dots$$

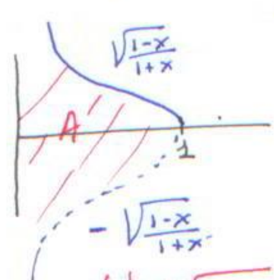
d) $y^2 = \dfrac{1-x}{1+x}$ $\qquad x = -1$ $\qquad f(x) = \pm\sqrt{\dfrac{1-x}{1+x}}$



Dom $f = (-1,1]$ sig $(1-x)$ = sig $(1+x)$

$$\lim_{x\to -1^+} \frac{1-x}{1+x} = \infty$$

$f(1)=0$



$$A = \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} - \left(-\sqrt{\frac{1-x}{1+x}}\right)dx = 2\int_{-1}^1 \sqrt{\frac{1-x}{1+x}}\,dx$$ INTEGRAL IMPROPIA

$$= 2\int_\infty^0 \frac{-4u^2}{(u^2+1)^2}\,du = 8\int_0^\infty \frac{u^2}{(u^2+1)^2}\,du = \dots$$

$u^2 = \dfrac{1-x}{1+x}$

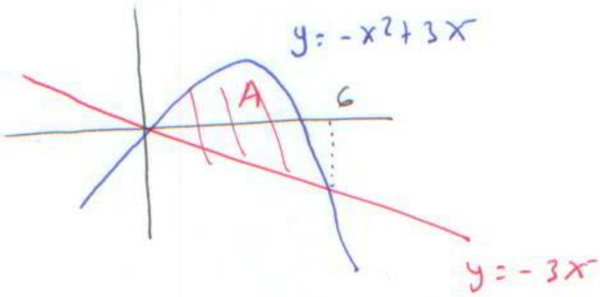
$u^2 + u^2x = 1-x$

$\Longleftrightarrow (u^2+1)x = 1-u^2$

$x = \dfrac{1-u^2}{u^2+1}$ $\qquad dx = \dfrac{-2u(u^2+1) - (1-u^2)2u}{(u^2+1)^2} = \dfrac{-4u}{(u^2+1)^2}$

INTEGRALES IMPROPIAS II

PROBLEMA 7:] e) $A = \{(x,y) \in \mathbb{R}^2 : -3x \le y \le -x^2+3x\}$



$$\left.\begin{array}{l} y = -x^2+3x \\ y = -3x \end{array}\right| \iff$$

$$-3x = -x^2+3x$$

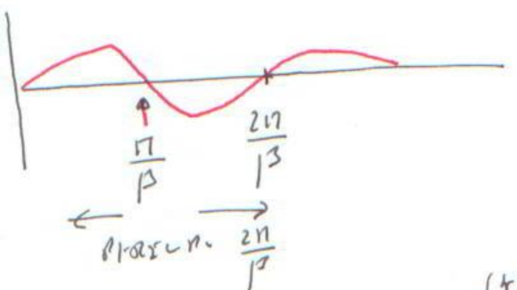
$$\iff x(6-x) = 0$$

$$x = 0 \text{ y } x = 6$$

LUEGO $A = \displaystyle\int_0^6 -x^2+3x-(-3x)\,dx$

$$= \int_0^6 -x^2+6x\,dx = -\frac{x^3}{3}+\frac{6x^2}{2}\Big|_0^6 = -72+108 = 36.$$

PROBLEMA 8:]



$$\int_0^\infty e^{-\alpha x}\operatorname{sen}\beta x\,dx = \sum_{k=0}^{\infty}\int_{k\frac{2\pi}{\beta}}^{(k+1)\frac{2\pi}{\beta}} e^{-\alpha x}\operatorname{sen}\beta x\,dx.$$

$$\int_{k\frac{2\pi}{\beta}}^{(k+1)\frac{2\pi}{\beta}} e^{-\alpha x}\operatorname{sen}\beta x\,dx \underset{\text{PARTES}}{=} \underbrace{\frac{e^{-\alpha x}}{-\alpha}\operatorname{sen}\beta x\Big|_{k\frac{2\pi}{\beta}}^{(k+1)\frac{2\pi}{\beta}}}_{0} + \frac{\beta}{\alpha}\int_{k\frac{2\pi}{\beta}}^{(k+1)\frac{2\pi}{\beta}} e^{-\alpha x}\cos\beta x\,dx =$$

$$\underset{\text{PARTES}}{=} \frac{\beta}{\alpha}\left[-\frac{e^{-\alpha x}}{\alpha}\cos\beta x\Big|_{k\frac{2\pi}{\beta}}^{(k+1)\frac{2\pi}{\beta}} - \frac{\beta}{\alpha}\int_{k\frac{2\pi}{\beta}}^{(k+1)\frac{2\pi}{\beta}} e^{-\alpha x}\operatorname{sen}\beta x\,dx\right]$$

DESPEJANDO $\displaystyle\left(1+\frac{\beta^2}{\alpha^2}\right)\int_{k\frac{2\pi}{\beta}}^{(k+1)\frac{2\pi}{\beta}} e^{-\alpha x}\operatorname{sen}\beta x\,dx = \frac{\beta}{\alpha}\left[-e^{-\alpha\left((k+1)\frac{2\pi}{\beta}\right)}+e^{-\alpha\frac{2\pi}{\beta}k}\right]$

ASÍ $\displaystyle\int_{k\frac{2\pi}{\beta}}^{(k+1)\frac{2\pi}{\beta}} e^{-\alpha x}\operatorname{sen}\beta x\,dx = \frac{\alpha^2}{\alpha^2+\beta^2}\cdot\frac{\beta}{\alpha}\,e^{-\alpha k\frac{2\pi}{\beta}}\left[1-e^{-\alpha\frac{2\pi}{\beta}}\right] =$

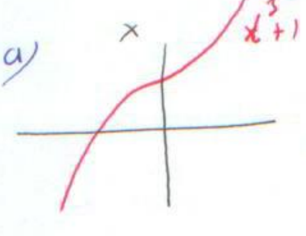
$$= \frac{\alpha\beta}{\alpha^2+\beta^2}\left[1-e^{-\frac{\alpha}{\beta}2\pi}\right]\left(e^{-\frac{\alpha}{\beta}2\pi}\right)^k$$

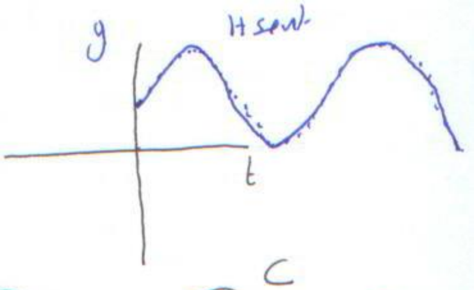
COMO $e^{-\frac{\alpha}{\beta}2\pi} < 1$, LA RAZÓN DE LA SERIE GEOMÉTRICA,

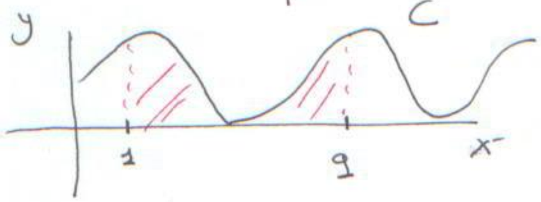
LA SERIE ES CONVERGENTE

INTEGRALES IMPROPIAS II

PROBLEMA 9: a)







si $t=0 \Rightarrow x=1$

si $t=2 \Rightarrow x=9$

AREA $\int_1^9 y(x)\,dx = \int_0^2 (1+\operatorname{sen}t)\,3t^2\,dt =$

$x = t^3+1$

$(y(x) = 1+\operatorname{sen}t)$

$dx = 3t^2\,dt$

$x=1 \Rightarrow t=0$

$x=9 \Rightarrow t=2$

$= \int_0^2 3t^2\,dt + \int_0^2 3t^2\operatorname{sen}t\,dt = \ldots$ etc.

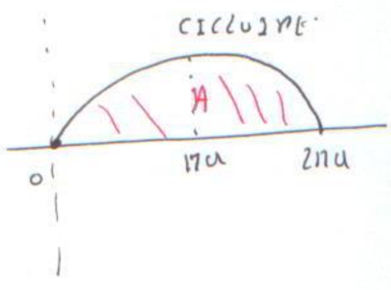
PROBLEMA 10: $x = a(t-\operatorname{sen}t)$

$y = a(1-\cos t)$

$x'(t) = a(1-\cos t) \geq 0$ creciente

$y'(t) = a\operatorname{sen}t \begin{cases} \geq 0 & \text{si } t\in[0,\pi] \\ <0 & \text{si } t\in(\pi,2\pi] \end{cases}$

$t=0$ ó 2π $\quad y(t)=0$



$A = \int_0^{2\pi a} y(x)\,dx = \int_0^{2\pi} a^2(1-\cos t)^2\,dt =$

$x = a(t-\operatorname{sen}t)$

$y(x) = a(1-\cos t)$

$dx = a(1-\cos t)$

$x=0 \Rightarrow t=0$

$x=2\pi a \Rightarrow t=2\pi$

$= \int_0^{2\pi} a^2 - 2a^2\cos t + a^2\cos^2 t\,dt = a^2\,2\pi + \underbrace{\left[-2a^2\operatorname{sen}t\right]_0^{2\pi}}_{=0} + a^2\left(\frac{\operatorname{sen}2t}{4}+\frac{t}{2}\right)\Big|_0^{2\pi} =$

$= 2\pi a^2 + a^2\frac{\cancel{2}\pi}{\cancel{2}} = 3a^2\pi.$

INTEGRALES IMPROPIAS II

PROBLEMA 11: $x^3+y^3=axy$

Si $y=tx$, entonces

$$x^3+t^3x^3=axxt \quad \text{así}$$

$$(1+t^3)x^3=ax^2t \quad \text{luego}$$

$$x=\frac{at}{1+t^3}$$

y por tanto

$$y=\frac{at^2}{1+t^3}$$

Parametrización de la curva dada en implícitas, $x^3+y^3=axy$

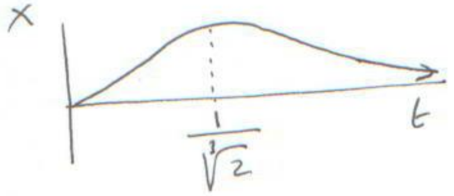
Representamos esta curva

$x=\frac{at}{1+t^3}$ $\quad t=0 \quad x=0$ $\quad \lim_{t\to\infty}\frac{at}{1+t^3}=0$

$$x'(t)=\frac{a(1+t^3)-at\,3t^2}{(1+t^3)^2}=\frac{a-2at^3}{(1+t^3)^2}\begin{cases}>0 & \text{si } t<\frac{1}{\sqrt[3]{2}}\\ =0 & \text{si } t=\frac{1}{\sqrt[3]{2}}\\ <0 & \text{si } t>\frac{1}{\sqrt[3]{2}}\end{cases}$$

$x'(0)=a$

$x'(\infty)=0$



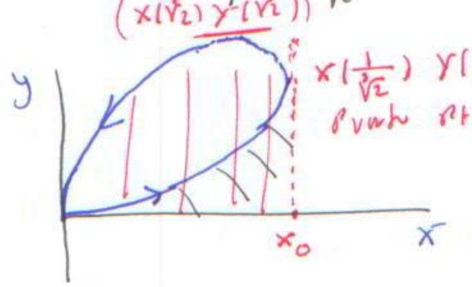
$y(t)=\frac{at^2}{1+t^3}\geq 0 \quad \lim_{t\to\infty}\frac{at^2}{1+t^3}=0 \quad y'(t)=\frac{2at(1+t^3)-3at^4}{(1+t^3)^2}=\frac{2at-at^4}{(1+t^3)^2}=0$ si $t=\sqrt[3]{2}$

$y'(0)=0$

$y'(\infty)=0$

$$\lim_{t\to\infty}\frac{y'(t)}{x'(t)}=\lim_{t\to\infty}\frac{2at-at^4}{a-2at^3}=\infty$$

Así



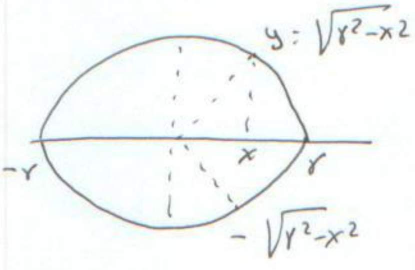
$$\int_{\frac{1}{\sqrt[3]{2}}}^{\infty} y(x(t))\,x'(t)\,dt-\int_0^{\frac{1}{\sqrt[3]{2}}} y(x(t))\,x'(t)\,dt$$

$$=\int_{\frac{1}{\sqrt[3]{2}}}^{\infty}\frac{at^2}{(1+t^3)}\cdot\frac{a-2at^3}{(1+t^3)^2}\,dt-\int_0^{\frac{1}{\sqrt[3]{2}}}\frac{at^2}{(1+t^3)}\cdot\frac{a-2at^3}{(1+t^3)^2}\,dt=\ ---$$

INTEGRALES IMPROPIAS II

PROBLEMA 12: a) VER TEORÍA.

b)



$$\ell = \int_{-r}^{r} \sqrt{1+\left(\frac{-2x}{2\sqrt{r^2-x^2}}\right)^2}\,dx + \int_{-r}^{r} \sqrt{1+\left(\frac{2x}{2\sqrt{r^2-x^2}}\right)^2}\,dx$$

$$= 2\int_{-r}^{r} \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\,dx$$

$$= 2r\int_{-r}^{r} \frac{1}{\sqrt{r^2-x^2}}\,dx = 2r\int_{-r}^{r} \frac{1}{r\sqrt{1-\left(\frac{x}{r}\right)^2}}\,dx$$

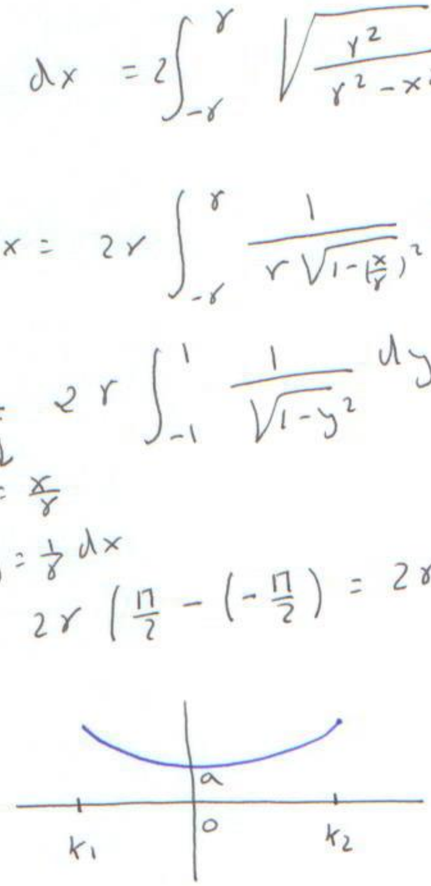
$$= 2\int_{-r}^{r} \frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}\,dx = 2r\int_{-1}^{1} \frac{1}{\sqrt{1-y^2}}\,dy =$$

$y = \frac{x}{r}$

$dy = \frac{1}{r}\,dx$

$$= 2r \operatorname{Arc\,sen} y \Big|_{-1}^{1} = 2r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2r\pi.$$

c) $y = a\cosh\frac{x}{a}$ $a>0$

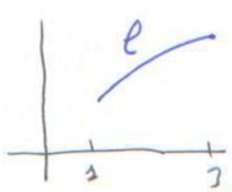


LA LONGITUD $\ell = \int_{k_1}^{k_2} \sqrt{1+(y')^2}\,dx =$

$$= \int_{k_1}^{k_2} \sqrt{1+\operatorname{senh}^2\frac{x}{a}}\,dx = \int_{k_1}^{k_2} \cosh\frac{x}{a} = a\operatorname{senh}\frac{x}{a}\Big|_{k_1}^{k_2}$$

$(\cosh^2 x - \operatorname{senh}^2 x = 1)$

d) 1) $f(x) = \sqrt{2x}$ $x \in [1,3]$



$$\ell = \int_1^3 \sqrt{1+\left(\frac{2}{2\sqrt{2x}}\right)^2}\,dx = \int_1^3 \sqrt{1+\frac{1}{2x}}\,dx =$$

$$= \int_1^3 \sqrt{\frac{2x+1}{2x}}\,dx =$$

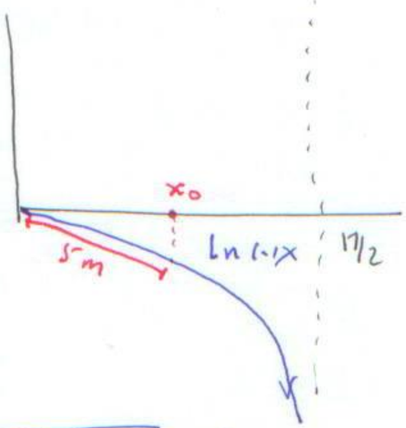
$y = \sqrt{\frac{2x+1}{2x}} \iff 2xy^2 = 2x+1 \iff x = \frac{1}{2(y^2-1)}$

$$= \int_{\sqrt{3/2}}^{\sqrt{7/6}} y \cdot \frac{-4y}{\cancel{4}(y^2-1)^2}\,dy = \ldots \text{etc}$$

$dx = \frac{-4y}{4(y^2-1)^2}$

INTEGRALES IMPROPIAS II

PROBLEMAS 13=)



$$\boxed{S = \int_0^{x} \sqrt{1+\left((\ln \cos x)'\right)^2}\, dx =}$$

$$= \int_0^{x_0} \sqrt{1 + \frac{\operatorname{sen}^2 x}{\cos^2 x}}\, dx = \int_0^{x_0} \frac{1}{\cos x}\, dx =$$

$u = \operatorname{tg} \frac{x}{2}$

$\cos x = \frac{1-u^2}{1+u^2}$

$du = \left(1+\operatorname{tg}^2 \frac{x}{2}\right) \frac{1}{2}\, dx$

$$= \int_0^{\operatorname{Arctg} 2u_0} \frac{\cancel{1+u^2}}{1-u^2} \, \frac{2}{\cancel{1+u^2}}\, du =$$

$$= \int_0^{\operatorname{Arctg} 2u_0} \frac{2}{(1+u)(1-u)}\, du = \int_0^{\operatorname{Arctg} 2u_0} \frac{1}{1+u} - \frac{1}{1-u}\, du =$$

$$= \ln\left(\frac{1+u}{1-u}\right)\Bigg|_0^{\operatorname{Arctg} 2u_0} \underset{u = \operatorname{tg}\frac{x}{2}}{=} \boxed{\ln\left(\frac{1+\operatorname{tg}\frac{\operatorname{Arctg} 2u_0}{2}}{1-\operatorname{tg}\frac{\operatorname{Arctg} 2u_0}{2}}\right).}$$

SE NOMBRA $\quad e^S = \frac{1+A}{1-A} \iff e^S - e^S A = 1+A$

$$\iff e^S - 1 = A(1+e^S)$$

$$A = \frac{e^S-1}{1+e^S}$$

LUEGO $\quad \operatorname{tg}\left(\frac{\operatorname{Arctg} 2u_0}{2}\right) = \frac{e^S-1}{1+e^S}$

ASÍ $\quad \operatorname{Arctg} 2u_0 = \operatorname{Arctg} \frac{e^S-1}{1+e^S}$

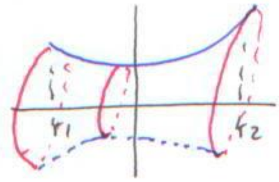
$$u_0 = \frac{1}{2} \operatorname{tg}\left(2 \operatorname{Arctg} \frac{e^S-1}{1+e^S}\right) = \dots$$

USAR

$\operatorname{tg} 2x = \dots$ etc.

INTEGRALES IMPROPIAS II

PROBLEMA 14: $y = a \cosh \frac{x}{a}$



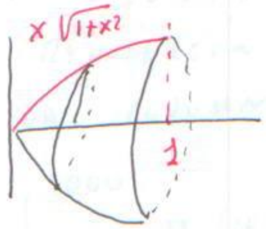
El volumen de revolución es

$$V = \int_{x_1}^{x_2} \pi a^2 \cosh^2 \frac{x}{a} \, dx =$$

$$= \pi a^2 \int_{x_1}^{x_2} \left(\frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2} \right)^2 dx = \pi a^2 \int_{x_1}^{x_2} \frac{e^{2\frac{x}{a}} + 2 + e^{-2\frac{x}{a}}}{4} \, dx =$$

$= \ldots$ etc.

PROBLEMA 15: b) $f(x) = x\sqrt{1+x^2}$ $\quad x \in [0,1]$



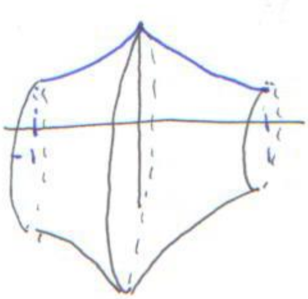
Volumen de revolución

$$\int_0^1 \pi x^2 (1+x^2) \, dx =$$

$$= \pi \int_0^1 x^2 + x^4 \, dx = \pi \left(\frac{x^3}{3} + \frac{x^5}{5} \right|_0^1 =$$

$$= \pi \left(\frac{1}{3} + \frac{1}{5} \right) = \frac{8\pi}{15}$$

PROBLEMA 16: $f(x) = \frac{1}{1+|x|}$ $\quad x \in [-1,1]$ función par



$$V = \int_{-1}^{1} \pi \left(\frac{1}{1+|x|} \right)^2 dx \underset{\text{par}}{=} 2\pi \int_0^1 \frac{1}{(1+x)^2} \, dx = 2\pi \left(\frac{-1}{1+x} \right|_0^1$$

$$= 2\pi \left(-\frac{1}{2} + 1 \right) = \pi.$$