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Characterizations of strictly singular operators on Banach lattices

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Abstract

New characterizations of strictly singular operators between Banach lattices are given. It is proved that, for Banach lattices X and Y such that X has finite cotype and Y satisfies a lower 2-estimate, an operator $T : X \to Y$ is strictly singular if and only if it is disjointly strictly singular and ℓ_2-singular. Moreover, if T is regular then the same equivalence holds provided that Y is just order continuous. Furthermore, it is shown that these results fail if the conditions on the lattices are relaxed.

Introduction

Strictly singular operators were introduced by Kato [**18**] in connection with the perturbation theory of Fredholm operators. Recall that an operator $T : X \to Y$ between Banach spaces is *strictly singular* if it is not an isomorphism when restricted to any infinite-dimensional (closed) subspace of X. Strictly singular operators constitute a closed two-sided operator ideal that contains the ideal of compact operators. Moreover, an operator $T : X \to Y$ is strictly singular if and only if, for every infinite-dimensional subspace M of X, there exists an infinite-dimensional subspace N of M such that the restriction $T|_N$ is compact.

In the context of Banach lattices, a weaker notion is the following: given a Banach lattice X, a Banach space Y, and an operator $T : X \to Y$, we say that T is *disjointly strictly singular* if it is not an isomorphism when restricted to the closed linear span of any disjoint sequence in X. This notion is quite a useful tool in the study of strictly singular operators on Banach lattices, for example, in the context of domination problems for positive operators (cf. [**10**]), and for comparing structures of rearrangement invariant spaces (cf. [**13, 14**]). Several properties of disjointly strictly singular operators have been studied in [**8, 9, 11**].

In this paper, we are interested in giving characterizations of the strict singularity of operators acting between Banach lattices. Since strictly singular operators are disjointly strictly singular, we are mainly interested in converse statements.

Our motivation stems from the following facts. First, it is well known that an endomorphism of $L_p = L_p[0,1]$, with $1 \leqslant p < \infty$, is strictly singular if and only if it is ℓ_p-singular and ℓ_2-singular [**22, 25**]. In other words, an endomorphism T on L_p is strictly singular if and only if it is disjointly strictly singular and ℓ_2-singular. Recall that an operator between Banach spaces is called *ℓ_p-singular* for some $1 \leqslant p \leqslant \infty$ if it is not an isomorphism when restricted to any subspace isomorphic to ℓ_p. For recent results on ℓ_p-singular operators, we refer to [**16**].

Given an order continuous Banach lattice X, if an operator $T : X \to Y$ is disjointly strictly singular and ℓ_p-singular for *every* $1 \leqslant p \leqslant 2$, then T is strictly singular. This can be seen using the Kadeč–Pełczyński disjointification method and Aldous' theorem on subspaces of L_1 (see [**2**]). Furthermore, in the special case of X (or Y) being a Banach lattice with type 2,

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