AVR PRÁCTICA-21

Nombre y apellidos..

1.- Sea función la $f(x)$ límite uniforme de una serie de funciones $\sum_{n=0}^{\infty} a_n \cos nx + b_n \operatorname{sen} nx$ sobre $[-\pi, \pi]$.

1_1.- Cálcula $\int_{-\pi}^{\pi} \cos^2 nx dx$ y $\int_{-\pi}^{\pi} \operatorname{sen}^2 nx dx$.

$\cos^2 nx = \frac{1+\cos 2nx}{2}$, ASI

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2} dx = \frac{x}{2} + \frac{\operatorname{sen} 2nx}{4n}\Big|_{-\pi}^{\pi} = \pi.$$

$$\int_{-\pi}^{\pi} \operatorname{sen}^2 nx \, dx = \int_{-\pi}^{\pi} 1 - \cos^2 nx \, dx = 2\pi - \pi = \pi.$$

1_2.- Cálcula $\int_{-\pi}^{\pi} \cos nx \cos mx dx$, para $n \neq m$.

Por partes

$$\boxed{\int_{-\pi}^{\pi} \cos nx \cos mx \, dx} = \frac{\operatorname{sen} nx}{n} \cos mx \Big|_{-\pi}^{\pi} + \frac{m}{n}\int_{-\pi}^{\pi} \operatorname{sen} nx \operatorname{sen} mx \, dx =$$

Por partes

$$\frac{m}{n}\left[-\frac{\cos nx}{n} \operatorname{sen} mx \Big|_{-\pi}^{\pi} + \frac{m}{n}\int_{-\pi}^{\pi} \cos nx \cos mx \, dx \right] = \boxed{\frac{m^2}{n^2}\int_{-\pi}^{\pi} \cos nx \cos mx \, dx}$$

Como $\frac{m^2}{n^2} \neq 1 \Rightarrow \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0$.

De mismo modo $\int_{-\pi}^{\pi} \cos nx \operatorname{sen} mx \, dx = 0$ y $\int_{-\pi}^{\pi} \operatorname{sen} nx \operatorname{sen} mx \, dx = 0$ $n \neq m$.

1_3.- Prueba que $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$.

$$\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a_n \cos nx + b_n \operatorname{sen} nx \, dx =$$

Convergencia uniforme

$$= \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \operatorname{sen} nx \, dx =$$

$\int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \operatorname{sen} nx \, dx = 0$ $n \geq 1$

$$= a_0 \int_{-\pi}^{\pi} 1 \, dx = a_0 \, 2\pi.$$

Despejando $\quad a_0 = \frac{\int_{-\pi}^{\pi} f(x) \, dx}{2\pi}.$

1_4.- Prueba que $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx, \qquad n \geq 1.$

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \int_{-\pi}^{\pi} \left(\sum_{k=0}^{\infty} a_k \cos kx + b_k \operatorname{sen} kx \right) \cos nx \, dx =$$

Convergencia uniforme.

Nota $f_k \to f$ uniform.; $|f_k \cos nx - f \cos nx| = |f_n - f| |\cos nx| \leq |f_n - f|$.

$$= \sum_{k=0}^{\infty} a_k \int_{-\pi}^{\pi} \cos kx \cos nx \, dx + b_k \int_{-\pi}^{\pi} \operatorname{sen} kx \cos nx \, dx =$$

$1_2)$

$$= a_n \int_{-\pi}^{\pi} \cos^2 nx \, dx = a_n \pi.$$

Despejando

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

(COMO EN EL CASO ANTERIOR)

1_5.- Prueba que $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen} nx dx, \qquad n \geq 1.$

$$\int_{-\pi}^{\pi} f(x) \operatorname{sen} nx \, dx = \int_{-\pi}^{\pi} \left(\sum_{k=0}^{\infty} a_k \cos kx + b_k \operatorname{sen} kx \right) \operatorname{sen} nx \, dx \underset{\text{CONVERGENCIA UNIFORME}}{=}$$

$$\sum_{k=0}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kx \operatorname{sen} nx \, dx + b_k \int_{-\pi}^{\pi} \operatorname{sen} kx \operatorname{sen} nx \, dx \right) \underset{1 \text{ y } 2}{=}$$

$$= b_n \int_{-\pi}^{\pi} \operatorname{sen}^2 nx \, dx = b_n \pi$$ Y DESPEJANDO

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen} nx \, dx.$$

2.- Dada la función $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)!}$, determina su dominio. Prueba que f es derivable y calcula su derivada.

SI USAMOS EL CRITERIO DEL COCIENTE, PARA LA SERIE EN VALOR ABSOLUTO

$$\lim_{n\to\infty} \frac{\frac{|x|^{2(n+1)+1}}{(2(n+1))!}}{\frac{|x|^{2n+1}}{(2n)!}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0 < 1,$$

ASÍ LA SERIE ES ABSOLUTAMENTE CONVERGENTE $\forall x \in \mathbb{R}$

$\operatorname{Dom} f = \mathbb{R}$

$$S_N = \sum_{n=0}^{N} \frac{x^{2n+1}}{(2n)!} \longrightarrow f(x) \text{ PUNTUALMENTE}$$

CADA S_N ES DERIVABLE $(S_N)'(x) = \sum_{n=0}^{N} \frac{(2n+1)x^{2n}}{(2n)!}$

LA SERIE DE FUNCIONES $\sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n)!}$ CONVERGE
(LA SERIE DE DERIVADAS)
UNIFORMEMENTE EN $[-M, M]$, $\forall M > 0$, CLARO

$$\left| \frac{(2n+1)x^{2n}}{(2n)!} \right| \leq \frac{(2n+1)M^{2n}}{(2n)!} \quad \forall x \in [-M, M]$$ Y LA SERIE

$\sum \frac{(2n+1)M^{2n}}{(2n)!}$ ES CONVERGENTE: LA PRUEBA M-WEIERSTRASS NOS DA LA CONVERGENCIA UNIFORME.

ASÍ POR LOS RESULTADOS SOBRE CONVERGENCIA Y DERIVADAS

$$f'(x) = \left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)!} \right)' = \sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{(2n)!} \right)' = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n)!}$$