OPERACIONES SOBRE L_1

DEFINICIÓN SEAN $f, g \; \mathbb{R} \to \mathbb{R}$ DOS FUNCIONES.

a) SE DEFINE LA CONVOLUCIÓN DE f Y g POR.

$$f * g(x) = \int_{\mathbb{R}} f(x-y)\,g(y)\,dy \qquad \forall x \in \mathbb{R}.$$

LINEAL EN g

SIEMPRE Y CUANDO LA INTEGRAL ANTERIOR EXISTA.

b) SE DEFINE LA TRANSFORMADA DE FOURIER DE f POR.

(*)

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\,e^{i\lambda x}\,dx \qquad \forall \lambda \in \mathbb{R}$$

LINEAL EN f

(PARA CALCULARLAS SE EMPLEAN TÉCNICAS DE ANÁLISIS COMPLEJO)

SIEMPRE Y CUANDO LA INTEGRAL ANTERIOR EXISTA

OBSERVACIÓN a) $f * g \quad \mathbb{R} \longrightarrow \mathbb{R}.$

b) $\hat{f} \quad \mathbb{R} \longrightarrow \mathbb{C}$

LAS OPERACIONES ANTERIORES TRANSFORMAN FUNCIONES EN FUNCIONES. AMBAS OPERACIONES TIENEN IMPORTANTES PROPIEDADES QUE SE USAN EN ANÁLISIS CLÁSICO (VER LAS ASIGNATURAS ANÁLISIS DE FOURIER, ANÁLISIS REAL Y EDP). VEAMOS CUANDO PODEMOS TRABAJAR CON TALES OPERACIONES Y QUE PROPIEDADES TIENEN.

TEOREMA (DESIGUALDAD DE YOUNG). SEA $g \in L_1(\mathbb{R})$ Y $f \in L_p(\mathbb{R})$ $1 \leq p \leq \infty$, ENTONCES PARA CASI TODO $x \in \mathbb{R}$ ESTÁ DEFINIDA LA CONVOLUCIÓN $f * g(x) = \int_{\mathbb{R}} f(x-y)\,g(y)\,dy$. ADEMÁS $f * g \in L^p(\mathbb{R})$ Y

$$\|f * g\|_p \leq \|f\|_p \, \|g\|_1.$$

DEM SI $f \in L_\infty(\mathbb{R})$ $\left| \int_{\mathbb{R}} f(x-y)\,g(y)\,dy \right| \leq \int_{\mathbb{R}} |f(x-y)|\,|g(y)|\,dy \leq$
$\leq \|f\|_\infty \int_{\mathbb{R}} |g(y)|\,dy = \|f\|_\infty \cdot \|g\|_1.$

LOS OTROS CASOS SON UN POCO MÁS COMPLICADOS.

(*) EN OTROS LIBROS

$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx$ Ó $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx$

EN CUALQUIER CASO LOS PROCEDIMIENTOS Y RESULTADOS NO CAMBIAN

OBSERVACIÓN SI $f, g \in L_1(\mathbb{R})$, ENTONCES $f * g \in L_1(\mathbb{R})$.

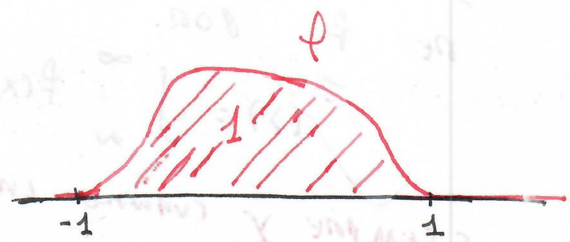
Y ASÍ LA CONVOLUCIÓN ES UNA OPERACIÓN CERRADA EN $L_1(\mathbb{R})$.

LA CONVOLUCIÓN DE FUNCIONES TIENE PROPIEDADES INTERESANTES POR SÍ MISMA (SIRVE PARA REGULARIZAR FUNCIONES) Y ES ESPECIALMENTE ÚTIL JUNTO A LA TRANSFORMADA DE FOURIER.

TEOREMA SEA $f: \mathbb{R} \to \mathbb{R}$ TAL QUE

a) $f \geq 0$ b) $f \in C_0^\infty(\mathbb{R})$ c) $\|f\|_1 = 1$

d) $\text{sopp } f = \{x : |x| \leq 1\}$



SEAN $f_k(x) = k f(kx)$.

$k = 1, 2, \dots$

OBSERVACIÓN $f_k \in C_0^\infty(\mathbb{R})$ Y $\text{sopp } f_k = \{x \in \mathbb{R} : |x| \leq 1/k\}$

SEA f UNA FUNCIÓN $f: \mathbb{R} \to \mathbb{R}$ QUE VERIFICA ALGUNA DE ESTAS DOS HIPÓTESIS.

1) $f \in C_0(\mathbb{R})$.

ó 2) $f \in L_p(\mathbb{R})$, $1 \leq p < \infty$, Y f SE ANULA FUERA DE UN COMPACTO DE $\mathbb{R}$ [illegible]

SEA $f_k = f * f_k$.

ENTONCES $(f_k) \subseteq C_0^\infty(\mathbb{R})$, PARA k SUFICIENTEMENTE GRANDE;

Y $f_k \longrightarrow f$ UNIFORMEMENTE SI $f \in C_0(\mathbb{R})$

Ó $f_n \xrightarrow{\|\cdot\|_p} f$ SI $f \in L^p(\mathbb{R})$ Y TIENE SOPORTE COMPACTO

PROPOSICIÓN

SEA $f \in L_1(\mathbb{R})$, ENTONCES

$$\hat{f} \in L_\infty(\mathbb{R}) \quad \text{Y ASÍ} \quad \begin{aligned} L_1(\mathbb{R}) &\xrightarrow{\wedge} L_\infty(\mathbb{R}) \\ f &\longrightarrow \hat{f} \end{aligned}$$

ES UN OPERADOR LINEAL Y ACOTADO

DEM

$$|\hat{f}(s)| = \left| \int_{-\infty}^{\infty} f(x) e^{isx} dx \right| \leq$$

$$\leq \int_{-\infty}^{\infty} |f(x)| |e^{isx}| dx = \|f\|_1$$

ASÍ $\|\hat{f}\|_\infty \leq \|f\|_1$

PROPIEDADES DE LA TRANSFORMADA DE FOURIER

a) SEA $f: \mathbb{R} \to \mathbb{R}$ Y $n \in \mathbb{N}$ CON:

i) $\exists\, f', f'', \ldots f^{n)}$

ii) $f^{k}(x) = \int_0^x f^{k+1)} dx$

iii) $f, f', \ldots f^{n)} \in L_1(\mathbb{R})$

ENTONCES $\widehat{f^{k)}}(s) = (-is)^k \hat{f}(s)$.

OBSÉRVESE QUE ES UNA PROPIEDAD PARECIDA A LA DE LA TRANSFORMADA DE LAPLACE Y SE UTILIZA TANTO PARA RESOLVER EDO COMO EDP LINEALES.

b) (TEOREMA DE INVERSIÓN) SI $f \in L_1(\mathbb{R})$ Y $\hat{f} \in L_1(\mathbb{R})$

SEA $g(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{-ixt} dt$ (TRANSFORMADA INVERSA)

ENTONCES $g(x) \xrightarrow[x \to \pm\infty]{} 0$ Y $f(x) = \frac{1}{2\pi} g(x)$ EN μ-c.t.p.

c) SI $f, g \in L_1(\mathbb{R})$ Y $h = f * g \in L_1(\mathbb{R}) \Rightarrow \hat{h}(s) = \hat{f}(s) \cdot \hat{g}(s)$.

DEM SOLO VEREMOS c). (a) ES ASEQUIBLE Y b) MUY DIFÍCIL SE USA LA ESTRUCTURA DE BANACH DE $L_1(\mathbb{R})$

SI $f, g \in L_1(\mathbb{R})$, POR LA DESIGUALDAD DE YOUNG $h \in L_1(\mathbb{R})$ Y ASÍ EXISTE $\hat{h}(s) = \int_{-\infty}^{\infty} e^{isx} \left(\int_{-\infty}^{\infty} f(x-y) g(y) dy \right) dx =$

$$\underset{\text{TEOREMA DE FUBINI}}{=} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{is(x-y)} f(x-y)\, dx \right) e^{isy} g(y)\, dy =$$

$$= \int_{-\infty}^{\infty} \hat{f}(s) e^{isy} g(y) dy = \hat{f}(s) \cdot \hat{g}(s).$$

APLICACIONES

① $\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} \\ u(x,0) = u_0(x). \end{cases}$

$$\widehat{\frac{\partial^2 u}{\partial x^2}}(\lambda,t) = -\lambda^2 \hat{u}(\lambda,t)$$

$$\hat{u}_t(\lambda,t) = \int_{-\infty}^{\infty} u_t(x,t) e^{i\lambda x} dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t) e^{i\lambda t} dx = \frac{\partial \hat{u}}{\partial t}(\lambda,t)$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial t}(\lambda,t) = -\lambda^2 \hat{u}(\lambda,t)$$

HEMOS PASADO DE UNA EDP A UNA E.D.O. ; ASÍ

$$\hat{u}(\lambda,t) = e^{-\lambda^2 t}\hat{u}_0(\lambda)$$

AHORA SE VUELVE PARA ATRÁS CON EL TEOREMA DE INVERSIÓN

② SI F ES UNA FUNCIÓN DE DISTRIBUCIÓN DE DENSIDAD f, $F(x) = \int_{-\infty}^{x} f(t)\,dt$, ENTONCES $\hat{f} = \varphi_F$ FUNCIÓN CARACTERÍSTICA DE LA DISTRIBUCIÓN F

SI F_1 y F_2 SON DOS FUNCIONES DE DISTRIBUCIÓN Y φ_1 y φ_2 SUS CORRESPONDIENTES FUNCIONES CARACTERÍSTICAS. ENTONCES

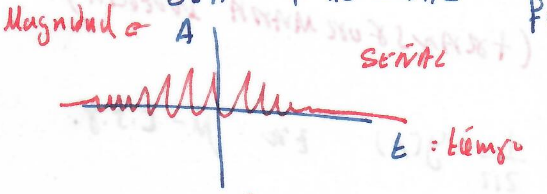
$$\varphi_{F_1 * F_2} = \varphi_1 \cdot \varphi_2.$$

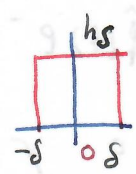
DONDE SE PRUEBA QUE $F_1 * F_2$ ES OTRA FUNCIÓN DE DISTRIBUCIÓN.

EJEMPLO $F(s) = \int_{-\infty}^{\lambda} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\theta)^2}{2\sigma^2}} dx \Rightarrow \varphi_F(t) = e^{\left(-\frac{\sigma t^2}{2} + it\theta\right)}$

(CON TÉCNICAS DE VARIABLE COMPLEJA)

③ SEA $f: \mathbb{R} \to \mathbb{R}$ $f \in L_1(\mathbb{R})$ $\hat{f}(\lambda)$. DOMINIO LA FRECUENCIA



(*) 

$$\hat{f}(\lambda)\cdot h_S(\lambda) = \hat{f}(\lambda)\cdot \widehat{\frac{\operatorname{sen}\delta t}{\pi t}}(\lambda) = \widehat{f * g}$$

FILTRADO EN FRECUENCIA

$\boxed{\text{FILTRO IDEAL, NO CAUSAL}}$

LUEGO FILTRAR EN FRECUENCIA ES LO MISMO QUE CONVOLUCIONAR CON g

EJEMPLO $\widehat{\frac{\operatorname{sen}\delta t}{\pi t}} = \chi_{[-\delta,\delta]}(\lambda)$.

VISTO EN VARIABLE COMPLEJA

(*) HAY CIRCUITOS ELÉCTRICOS QUE SIMULAN h_S Y OTROS QUE SIMULAN A LA TRANSFORMADA DE FOURIER

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On L_1-convergence of Fourier series

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ABSTRACT

In this paper we consider the trigonometric Fourier series with the β-general monotone coefficients. Necessary and sufficient conditions of L_1-convergence for such a series, that is $\|f - S_n\| = o(1)$, are obtained in terms of coefficients.

1. Introduction

Let a measurable 2π-periodic function f be L_1-integrable, i.e.,

$$\|f\| \equiv \|f\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |f|\,dx < \infty.$$

Let also the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

or the sine series

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{1.2}$$

be its Fourier series. Then the partial sums $S_n(f,x)$ are given by

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \quad \text{or} \quad \sum_{k=1}^{n} a_k \sin kx,$$

respectively. Series (1.1) (or (1.2)) converges in L_1 if

$$\|f(x) - S_n(f,x)\| = o(1) \quad \text{as } n \to \infty. \tag{1.3}$$

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The L_1-convergence problem is to find the criterion for (1.3) to hold in terms of coefficients $\{a_k\}$. In many cases the L_1-convergence class is described as

$$|a_n|\ln n = o(1) \quad \text{as } n \to \infty. \tag{1.4}$$

This is true for series (1.1) (or (1.2)) with monotone decreasing coefficients ($\{a_k\} \in M$) [19], or quasi-monotone coefficients ($\{a_k\} \in QM$) [9,19], or O-regularly quasi-monotone coefficients ($\{a_k\} \in ORVQM$) [16,24]. Here,

$$QM = \{a_n \in \mathbf{R}_+ \colon \exists \tau > 0 \text{ such that } \{n^{-\tau} a_n\}\downarrow\}$$

and the $ORVQM$-class is given by

$$ORVQM = \left\{a_n \in \mathbf{R}_+ \colon \exists\{\lambda_n\}\uparrow,\ \lambda_{2n} \leqslant C\lambda_n \text{ such that } \left\{\frac{a_n}{\lambda_n}\right\}\downarrow\right\}.$$

Recent investigations on L_1-convergence problem can be found in e.g. [1–4,8,11,18,20,25,27] (for the history of the topic see the survey [23]; for the related L_1-integrability results see [12]).

Our main idea is to consider the behavior of

$$\alpha_n = \sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}|.$$

We give conditions on α_n, which guaranties the accuracy of the criterion

$$\|f(x) - S_n(f,x)\| = o(1) \quad (\text{or } O(1)) \quad \text{iff} \quad |a_n|\ln n = o(1) \quad (\text{or } O(1)). \tag{1.5}$$

Similar ideas were also considered in [4,11,20,27]. For example, the following classes of coefficients were studied.

The class of general monotone coefficients GM, is defined as

$$GM = \left\{a_n \in \mathbf{C} \colon \sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \leqslant C|a_n|\right\}.$$

It turns out that for the series with GM-coefficients one can prove three convergence criteria for trigonometric series in L_p: for $p = \infty$, $p = 1$, and $1 < p < \infty$ (see the history and results in [20]).

The $GBVS$ [11] and $NBVS$-classes [27] are defined as

$$GBVS = \left\{a_n \in \mathbf{C} \colon \sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \leqslant C \max_{n\leqslant\nu\leqslant n+N} |a_\nu| \text{ for some integer } N\right\}$$

and

$$NBVS = \left\{a_n \in \mathbf{C} \colon \sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \leqslant C\big(|a_n| + |a_{2n}|\big)\right\},$$

respectively.

For mentioned above classes the following embeddings are true:

$$M \subsetneq QM \subsetneq ORVQM \subsetneq GM \subsetneq GBVS \cup NBVS. \tag{1.6}$$

For the $GBVS$ and $NBVS$-classes criterion (1.5) was proved in [11] and [27], respectively.

For a more general class

$$\left\{a_n \in \mathbf{C} \colon \sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \leqslant C \sum_{\nu=[n/c]}^{[cn]} \frac{|a_\nu|}{\nu} \text{ for some integer } c > 1\right\} \tag{1.7}$$

criterion (1.5) was considered in [26].

The novelty of the present paper is that we obtain the L_1-convergence result for the series with general behavior of α_n. First, if this expression is bounded by some non-negative sequence β_n such that $\beta_n \ln n = o(1)$, then this allows us to obtain the sufficient conditions in the problem of L_1-convergence, that is (1.4) $\Rightarrow$ (1.3). If the behavior of coefficients is rather regular, that is if, e.g.,

$$\sum_{k=0}^{n-1} \frac{\beta_{k+n}}{k+1} \leqslant C \sum_{k=0}^{n-1} \frac{|a_{k+n}|}{k+1}, \tag{1.8}$$

or if

$$\sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \leqslant C\beta_n \equiv C\left(\ln^{-1} n \max_{m\geqslant n} \frac{\ln m}{m} \sum_{\nu=m}^{2m} |a_\nu|\right), \tag{1.9}$$

then this allows us to write criterion (1.5). The last result simply follows from a general theory of the series with β-general monotone coefficients (Corollary 3.3.2). Since the class defined by (1.9) is wider[1] than any of the classes considered in (1.6) and (1.7), we generalize these results.

The rest of the paper is organized as follows. In Section 2, we introduce the notion of β-general monotone sequences and give some useful properties. In Sections 3 and 4, we investigate the L_1-convergence results for the Fourier series with β-general monotone coefficients. Section 4 gives some conditions on $\{a_n\}$ which follow from (1.3). In many cases this just implies necessary conditions of (1.3). Our main idea here is to obtain a general result that will depend on the structure of the spectrum of f. We conclude with Section 6, where we provide a few remarks.

Finally, we would like to mention the nice technique by Belov [2–4] that encouraged us to obtain some results of the present paper.

2. General monotone coefficients

We start by presenting our main definition (see also [20–22]).

¿ {βn} monotona?

Definition. Let $\beta = \{\beta_n\}_{n=1}^\infty$ be a non-negative sequence. The sequence of complex numbers $a = \{a_n\}_{n=1}^\infty$ is said to be *β-general monotone*, or *general monotone with the bound β*, or $a \in GM(\beta)$, if the relation

$$\sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \leqslant C\beta_n$$

holds for all integers n, where the constant C is independent of n.

The main examples of the sequence β_n are

$$(1)\qquad {}_1\beta_n = |a_n|,$$

$$(2)\qquad {}_2\beta_n = \sum_{k=n}^{n+N} |a_k| \quad \text{for some integer } N,$$

$$(3)\qquad {}_3\beta_n = \sum_{\nu=0}^{N} |a_{c^\nu n}| \quad \text{for some integers } N \text{ and } c > 1,$$

$$(4)\qquad {}_4\beta_n = |a_n| + \sum_{\nu=n+1}^{[cn]} \frac{|a_\nu|}{\nu} \quad \text{for some } c > 1,$$

$$(5)\qquad {}_5\beta_n = \sum_{\nu=[n/c]}^{[cn]} \frac{|a_\nu|}{\nu} \quad \text{for some } c > 1,$$

$$(6)\qquad {}_6\beta_n = \ln^{-1} n\left(\max_{m\geqslant[n/c]} \frac{\ln m}{m} \sum_{\nu=m}^{2m} |a_\nu|\right) \quad \text{for some } c > 1.$$

We have [22] (see also (2.4))

$$GM({}_1\beta + {}_2\beta + {}_3\beta + {}_4\beta + {}_5\beta) \equiv GM({}_5\beta) \subsetneq GM({}_6\beta). \tag{2.1}$$

We also note that $GM \equiv GM({}_1\beta)$, $GBVS \equiv GM({}_2\beta)$, and $NBVS \subseteq GM({}_3\beta)$. An example showing $GM({}_6\beta)\backslash GM({}_5\beta) \neq \emptyset$ is given in Section 6.

We now give the following important properties of $GM(\beta)$-sequences.

Lemma 2.1. *If $a = \{a_n\}_{n=1}^\infty \in GM(\beta)$, then one has for any integer n*

$$|a_\nu| \leqslant C\beta_n + |a_m| \quad \textit{for any } \nu, m = n, \ldots, 2n, \tag{2.2}$$

$$|a_\nu| \leqslant C\beta_n + \frac{1}{n}\sum_{j=n+1}^{2n} |a_j| \quad \textit{for any } \nu = n, \ldots, 2n, \tag{2.3}$$

[1] The embedding is proper; see (2.1) and Section 6.

$$|a_n| \leqslant \frac{C}{n}\left(\sum_{\nu=[n/2]}^{n-1} \beta_\nu + \sum_{j=n}^{2n-1} |a_j|\right), \tag{2.4}$$

$$|a_n|\left(\sum_{\nu=1}^{[n/2]} d_\nu\right) \leqslant C\left(\sum_{\nu=1}^{[n/2]} d_\nu \beta_{\nu+[n/2]} + \sum_{\nu=1}^{[n/2]} d_\nu a_{2(\nu+[n/2])}\right) \quad \text{for any } d_\nu \geqslant 0. \tag{2.5}$$

Proof. Indeed, using two simple identities ($\Delta a_s = a_s - a_{s+1}$)

$$a_\nu = \sum_{s=\nu}^{j-1} \Delta a_s + a_j, \quad j = \nu+1, \ldots, 2n \quad \text{and} \quad a_\nu = -\sum_{s=i}^{\nu-1} \Delta a_s + a_i, \quad i = n, \ldots, \nu-1,$$

we get

$$|a_\nu| \leqslant \sum_{s=\nu}^{j-1} |\Delta a_s| + |a_j| \leqslant \sum_{s=n}^{2n-1} |\Delta a_s| + |a_j| \quad \text{and} \quad |a_\nu| \leqslant \sum_{s=i}^{\nu-1} |\Delta a_s| + |a_i| \leqslant \sum_{s=n}^{2n-1} |\Delta a_s| + |a_i|.$$

Therefore, (2.2) follows. Further, summing up on j and i, we write

$$(2n-\nu)|a_\nu| \leqslant C\beta_n(2n-\nu) + \sum_{j=\nu+1}^{2n} |a_j| \quad \text{and} \quad (\nu-n)|a_\nu| \leqslant C\beta_n(\nu-n) + \sum_{i=n+1}^{\nu} |a_i|,$$

which implies (2.3).

To prove (2.5), we use (2.2):

$$|a_n| \leqslant C\beta_{\nu+[n/2]} + |a_{2\nu+2[n/2]}|, \quad \text{for any } \nu = 1, \ldots, [n/2].$$

Now we add up these inequalities with weights $\{d_\nu\}$.

Finally, (2.4) immediately follows from (2.5) with $d_\nu \equiv 1$. This completes the proof. □

3. L_1-convergence of trigonometric series

Before giving a result on L_1-convergence (or boundedness) classes of trigonometric series, that is

$$\left\| f(x) - S_n(f,x) \right\|_1 = o(1) \quad (\text{or } O(1)), \tag{3.1}$$

we remark that if

$$\sum_{k\in\mathbf{Z}} c_k e^{ikx}$$

is the Fourier series of $f \in L_1$, then (3.1) is equivalent to

$$\left\| V_n(x) - S_n(x) \right\|_1 = o(1) \quad (\text{or } O(1)), \tag{3.2}$$

where $V_n(x)$ is $(C,1)$-means of $S_n(x) = S_n(f,x)$, i.e.,

$$V_n(x) = \frac{1}{n+1}\sum_{\nu=0}^{n} S_\nu(x) = \frac{1}{n+1}\sum_{\nu=0}^{n}\left(\sum_{|k|\leqslant\nu} c_k e^{ikx}\right).$$

Therefore, we will further study only condition (3.2) for the sequences $\{c_k\}_{k\in\mathbf{Z}}$. For the sequence $\{c_k\}_{k\in\mathbf{Z}}$ condition (3.2) is true if and only if the same condition holds for $\{c_k\}_{k>0} \equiv \{\ldots, 0, c_1, c_2, \ldots\}$ and $\{c_{-k}\}_{k>0} \equiv \{\ldots, 0, c_{-1}, c_{-2}, \ldots\}$. For the cosine series ($c_k = c_{-k} = a_k/2$)

$$\frac{a_0}{2} + \sum_{k\geqslant 1} a_k \cos kx,$$

the accuracy of condition (3.2) for $\{c_k\}_{k>0}$ is equivalent to

$$\left\| V_n(h,x) - S_n(h,x) \right\|_1 = o(1) \quad (\text{or } O(1)), \tag{3.3}$$

where $S_n(h,x) = a_0/2 + \sum_{\nu=1}^{n} a_\nu \cos \nu x$ and $V_n(h,x) = \frac{1}{n+1}\sum_{\nu=0}^{n} S_\nu(h,x)$. Similar argumentation is also applied to the sine series ($c_k = -c_{-k} = -ia_k/2$).

Therefore, for simplicity, in the rest of the paper we study the accuracy of condition (3.2) for the sequences $\{c_n\}_{n>0}$.

Secondly, we note that [2]

$$\|V_n(x)-S_n(x)\|_1 \leqslant C\left(\frac{1}{n+1}\sum_{j=1}^{n}\|S_j(x)-S_{[j/2]}(x)\|_1+\max_{k=[n/2],\dots,n}\|S_k(x)-S_{[n/2]}(x)\|_1\right), \tag{3.4}$$

which simply follows from

$$S_n(x)-V_n(x)=\frac{1}{n+1}\sum_{j=1}^{n} jC_j,\quad C_j=c_je^{ijx}+c_{-j}e^{-ijx}$$

and

$$\sum_{j=1}^{n} jC_j=\sum_{j=1}^{n}\bar{C}_j+\sum_{j=[n/2]+1}^{n}(2j-n-1)C_j,\quad \bar{C}_j=\sum_{l=[j/2]+1}^{j}C_l.$$

Let us now present sufficient conditions for relation (3.2) to hold in terms of β_n. We start with a simple result that immediately follows from Lemma 2.1 and (3.4).

Theorem 3.1. *Let $c=\{c_n\}_{n=1}^{\infty}\in GM(\beta)$, where a non-negative sequence $\beta=\{\beta_n\}_{n=1}^{\infty}$ satisfies*

$$\sum_{j=n}^{2n-1}\frac{\beta_j+|c_j|}{j-n+1}=o(1)\quad (\text{or } O(1)). \tag{3.5}$$

Then the sequence $\{c_n\}_{n=1}^{\infty}$ satisfies (3.2).

Proof. First, following [2] and using inequality (3.4), we note that (3.2) is implied by

$$\max_{n\leqslant m\leqslant 2n}\|S_m(x)-S_{n-1}(x)\|_1=o(1)\quad (\text{or } O(1)). \tag{3.6}$$

Let us now show that our condition on $\{c_n\}$ guaranties the accuracy of (3.6). Indeed,

$$\begin{aligned}
\|S_m(x)-S_{n-1}(x)\|_1 &= \left\|\sum_{j=n}^{m}c_je^{ijx}\right\|_1=\left\|\sum_{j=n}^{m-1}\Delta c_j\sum_{k=n}^{j}e^{ikx}+c_m\sum_{k=n}^{m}e^{ikx}\right\|_1\\
&\leqslant C\left(\sum_{j=n}^{m-1}|\Delta c_j|\ln[(j-n)+2]+|c_m|\ln[(m-n)+2]\right)\\
&\leqslant C\left(\sum_{j=n}^{m-1}|\Delta c_j|\sum_{k=n}^{j}\frac{1}{k-n+1}+\sum_{k=n}^{m-1}\frac{1}{k-n+1}\left(|c_k|+\sum_{\nu=k}^{m-1}|\Delta c_\nu|\right)\right)\\
&\leqslant C\sum_{k=n}^{2n-1}\frac{\beta_k+|c_k|}{k-n+1}.
\end{aligned}$$

Thus, we obtain (3.6), which yields (3.2). The proof is now complete. □

This theorem immediately implies the following result (see also [8,10]).

Corollary 3.1.1. *Let $c=\{c_n\}_{n=1}^{\infty}\in GM(\beta)$ such that*

$$(\beta_n+|c_n|)\ln n=o(1)\quad (\text{or } O(1)). \tag{3.7}$$

Then condition (3.2) holds true.

In particular, we obtain the following sufficient condition for (3.2) to hold.

Corollary 3.1.2. *Let $c=\{c_n\}_{n=1}^{\infty}\in GM({}_6\beta)$. Then*

$$|c_n|\ln n=o(1)\quad (\text{or } O(1)) \tag{3.8}$$

implies condition (3.2).

We also describe some subclasses of β-general monotone coefficients such that only some conditions on $\{c_n\}$ (instead of (3.5) or (3.7)) guaranty the accuracy of (3.2).

Corollary 3.1.3. *Let $c = \{c_n\}_{n=1}^{\infty} \in GM(\beta)$, where*

$$\sum_{k=0}^{n-1} \frac{\beta_{k+n}}{k+1} \leqslant C \sum_{k=0}^{[ln]-1} \frac{|c_{k+[ln]}|}{k+1} \equiv C I_n(\{c\}, l) \tag{3.9}$$

for some $l > 0$. Then condition

$$\sum_{k=n}^{2n} \frac{|c_k|}{k-n+1} = o(1) \quad (\text{or } O(1)) \tag{3.10}$$

implies (3.2).

In particular, (3.8) $\Rightarrow$ (3.2).

We now show that condition (3.9) also provides the necessity part: (3.2) $\Rightarrow$ (3.8).

Theorem 3.2. *Let $c = \{c_n\}_{n=1}^{\infty} \in GM(\beta)$, where (3.9) holds for some $l > 0$. Then conditions (3.2), (3.8), and (3.10) are equivalent.*

Proof. By the previous corollary, it is sufficient to show that (3.2) $\Rightarrow$ (3.8). Indeed, by Lemma 2.1 (see (2.5) with $d_\nu = 1/\nu$), we get

$$C|c_n| \ln n \leqslant C|c_n| \sum_{\nu=1}^{[n/2]-1} \frac{1}{\nu} \leqslant C\left(\sum_{\nu=1}^{[n/2]-1} \frac{\beta_{\nu+[n/2]}}{\nu} + \sum_{\nu=1}^{[n/2]-1} \frac{|c_{2(\nu+[n/2])}|}{\nu} \right) \leqslant C\big(I_{[n/2]}(\{c\}, l) + I_n(\{c\}, 1) + I_{n-1}(\{c\}, 1)\big).$$

Finally, we use known fact [4] (see also Section 5) that (3.2) always implies $I_n(\{c\}, 1) = o(1)$ or $(O(1))$. $\square$

Remark 3.1. If $c = \{c_n\}_{n=1}^{\infty} \in GM(\beta)$, where

$$\sum_{k=0}^{n-1} \frac{\beta_{k+n}}{k+1} \leqslant C T_n(\{I\}) \tag{3.11}$$

for some regular linear summability method T_n, then conditions (3.2), (3.8), and (3.10) are equivalent.

We now present one more result for the series with β-general monotone coefficients, when condition (3.2) implies condition (3.8).

Theorem 3.3. *Let $c = \{c_n\}_{n=1}^{\infty} \in GM(\beta)$, where*

$$\sum_{k=[n/2]}^{n-1} \beta_k \leqslant C \sum_{k=[n/c]}^{[cn]} c_k \tag{3.12}$$

for some $c > 1$. Let also condition (3.2) hold true. Then $|c_n| \ln n = o(1)$ or $(O(1))$, i.e., (3.8) holds.

Proof. Indeed, by Lemma 2.1 (see (2.4)), we obtain that $\{c_n\}$ is a weak-monotone sequence, that is

$$n|c_n| \leqslant C \sum_{\nu=[n/l]}^{[ln]} |c_\nu|, \quad l > 1. \tag{3.13}$$

Further, we apply [28, Ch. 7, (8.11)] (see also [4, (36)]): if $\{c_n\}$ is a weak-monotone, then (3.2) $\Rightarrow$ (3.8). The proof is complete. $\square$

Comparing this result with Corollary 3.1.2 and taking into account that the $GM(_5\beta)$-sequences satisfy (3.12) and $GM(_5\beta) \subset GM(_6\beta)$, we have the following

Corollary 3.3.1. *Let $c = \{c_n\}_{n=1}^{\infty} \in GM(_5\beta)$. Then (3.2) $\Leftrightarrow$ (3.8).*

However, a more general result is also true.

Corollary 3.3.2. *Let $c = \{c_n\}_{n=1}^{\infty} \in GM({}_6\beta)$. Then (3.2) $\Leftrightarrow$ (3.8).*

Proof. By Corollary 3.1.2, (3.8) implies (3.2). To prove the reverse implication, we notice that (3.2) always gives [4]

$$\frac{\ln n}{n}\sum_{\nu=n}^{2n}|a_\nu| = o(1) \quad (\text{or } O(1))$$

and therefore

$$({}_6\beta_n)\ln n = o(1) \quad (\text{or } O(1)).$$

Using Lemma 2.1, we write

$$|c_n|\ln n \leqslant C\ln n({}_6\beta_n) + \frac{\ln n}{n}\sum_{j=n+1}^{2n}|c_j|,$$

which gives the required result. □

We finally remark that for any $c = \{c_n\}_{n=1}^{\infty} \in GM(\beta)$ such that $\beta_n \ln n = o(1)$ (or $O(1)$), (3.2) implies (3.8).

4. L_1-convergence of trigonometric series: Second approach

In this section we introduce one more definition of the general monotone type sequences [21].

Definition. Let $\beta = \{\beta_n\}_{n=1}^{\infty}$ be a non-negative sequence and let $k = \{k_j\}_{j=1}^{\infty}$ be a positive sequence such that $1 = k_1 < k_2 < \cdots$. The sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in \mathbb{C}$ is said to be in the $GM(\beta, k) \equiv GM(\{\beta_n\}, \{k_j\})$-class if either

$$\sum_{s=n}^{k_{j+1}-1}|a_s - a_{s+1}| \leqslant C\beta_n \quad \text{for } n \in (k_j, k_{j+1}) \tag{4.1}$$

or

$$\sum_{s=k_j+1}^{n}|a_s - a_{s+1}| \leqslant C\beta_n \quad \text{for } n \in (k_j, k_{j+1}). \tag{4.2}$$

In this way if the behavior of $\{\{\beta_n\}_{n\in(k_j,k_{j+1}]}\}_j$ is regular in some sense, then the $GM(\beta, k)$-class consists of sequences that have a similar structure in each interval $(k_j, k_{j+1}]$. Note that $\{\beta_{k_j}\}_j$ does not depend on $\{a_s\}$ and so can be any appropriate sequence for a given situation. We also have

$$GM(\beta) \subset GM(\beta, \{2^j\}).$$

A definition of this type for an important case when $\beta_n = |a_n|$ was first studied in [4].

Let us now investigate the L_1-convergence problem for trigonometric series with the Fourier coefficients from the $GM(\beta, k)$-class.

Theorem 4.1. *Let $\beta = \{\beta_n\}_{n=1}^{\infty}$ be a non-negative sequence and $k = \{k_j\}_{j=1}^{\infty}$, $k_j \in \mathbf{N}$ satisfy*

$$\sum_{j=1}^{m} k_j \leqslant Ck_m, \quad m \in \mathbf{N}. \tag{4.3}$$

Let also $a = \{a_n\}_{n=1}^{\infty} \in GM(\{\beta_n\}, \{k_j\})$, where a sequence $\beta = \{\beta_n\}_{n=1}^{\infty}$ satisfies

$$\sum_{j=[n/2]}^{2n-1}\frac{\beta_j + |a_j|}{|j-n|+1} = o(1) \quad (\text{or } O(1)). \tag{4.4}$$

Then the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies (3.2).

Remark 4.1. Condition (4.3) characterizes the class of sequences $k = \{k_j\}_{j=1}^{\infty}$, $k_j \in \mathbf{N}$ that can be represented as a finite union of lacunary sequences in the Hadamard sense, that is

$$\exists l \in \mathbf{N} \text{ such that } \frac{k_{s+l}}{k_s} \geqslant \lambda > 1 \quad \text{for any } s \in \mathbf{N}. \tag{4.5}$$

Corollary 4.1.1. *Let $\beta = \{\beta_n\}_{n=1}^{\infty}$ be a non-negative sequence and $k = \{k_j\}_{j=1}^{\infty}$, $k_j \in \mathbf{N}$ satisfy (4.3). Let also $a = \{a_n\}_{n=1}^{\infty} \in GM(\{\beta_n\}, \{k_j\})$, where a sequence $\beta = \{\beta_n\}_{n=1}^{\infty}$ satisfies*

$$\sum_{j=[n/2]}^{2n-1} \frac{\beta_j}{|j-n|+1} \leqslant C \sum_{j=[n/2]}^{2n-1} \frac{|a_j|}{|j-n|+1}. \tag{4.6}$$

Then (3.2) *is equivalent*

$$\sum_{j=[n/2]}^{2n-1} \frac{|a_j|}{|j-n|+1} = o(1) \quad (\text{or } O(1)). \tag{4.7}$$

In the case of $\beta_n \equiv |a_n|$ this result is known (see [4, Theorem 1]). Corollary 4.1.1 also implies [25, Theorem 1] and results of the paper [18].

The proof of Theorem 4.1 is similar to the proof presented above. We only write two key relations:

$$\max_{k_j+1 \leqslant \cdots \leqslant s \leqslant \cdots \leqslant k_{j+1}} \left\| \sum_{m=k_j+1}^{s} a_m e^{imx} \right\| = o(1) \quad (\text{or } O(1)) \quad \Rightarrow \quad (3.2)$$

and

$$\max_{k_j+1 \leqslant \cdots \leqslant s \leqslant \cdots \leqslant k_{j+1}} \left\| \sum_{m=k_j+1}^{s} a_m e^{imx} \right\| \leqslant C \left(\sum_{m=k_j+1}^{k_{j+1}} \frac{|a_m| + \beta_m}{k_{j+1} - m + 1} + \sum_{m=k_j+1}^{k_{j+1}} \frac{|a_m| + \beta_m}{m - k_j} \right).$$

To prove Corollary 4.1.1, we remark that (3.2) always implies (4.7) (see the next section).

We now give some conditions on the $GM(\{\beta_n\}, \{k_j\})$-sequences, under which criterion (1.5) is also true. Let us now assume that $\beta_n \equiv |a_n|$ and (4.5) holds with $l = 1$, i.e., the sequence $\{k_j\}$ is lacunary.

Theorem 4.2. *Let $a = \{a_n\}_{n=1}^{\infty} \in GM(\{\beta_n\}, \{k_j\})$ such that*

(1) *if*

$$\sum_{s=m}^{k_{j+1}-1} |a_s - a_{s+1}| \leqslant C\beta_m \quad \text{for} \quad m \in (k_j, k_{j+1}), \tag{4.8}$$

then $|a_{k_j+1}| \leqslant C|a_{[\lambda_0 k_j]}|$ for some $\lambda_0 \in (1, \lambda)$;

(2) *if*

$$\sum_{s=k_j+1}^{m} |a_s - a_{s+1}| \leqslant C\beta_m \quad \text{for } m \in (k_j, k_{j+1}), \tag{4.9}$$

then $|a_{k_{j+1}-1}| \leqslant C|a_{[k_{j+1}/\lambda_0]}|$ for some $\lambda_0 \in (1, \lambda)$.

Then criterion (1.5) *holds true.*

Remark 4.2. This theorem was proved in [25] for the case when $\{a_m\}_{m=k_j+1}^{k_{j+1}}$ keeps its sign. In the case of $a_s - a_{s+1} = 0$ for $s \in (k_j, k_{j+1})$ this result was proved in [17].

To prove Theorem 4.2, we use Corollary 4.1.1 and have only to show that

$$|a_n| \ln n = o(1) \quad (\text{or } O(1)) \quad \Leftrightarrow \quad \sum_{j=[n/2]}^{2n-1} \frac{|a_j|}{|j-n|+1} = o(1) \quad (\text{or } O(1)). \tag{4.10}$$

The nontrivial part ($\Leftarrow$) can be proved using similar arguments to [25, Theorems 2 and 4] and the following inequalities: in the case (1)

$$|a_{n+s}| \leqslant C|a_n|, \quad n, n+s \in (k_j, k_{j+1}) \quad \text{and} \quad |a_n| \leqslant C|a_{[k_j\lambda_0]}|, \quad n \in (k_j, [k_j\lambda_0]),$$

and in the case (2)

$$|a_n| \leqslant C|a_{n+s}|, \quad n, n+s \in (k_j, k_{j+1}) \quad \text{and} \quad |a_n| \leqslant C|a_{[k_{j+1}/\lambda_0]}|, \quad n \in ([k_{j+1}/\lambda_0], k_{j+1}).$$

5. Some consequences of $\|f - S_n\| = o(1)$

This section is devoted to implications of condition

$$\left\|V_n(x) - S_n(x)\right\|_1 = o(1) \quad (\text{or } O(1)). \tag{5.1}$$

Here for simplicity we consider $S_n(x) = \sum_{k=1}^{n} c_\nu e^{i\nu x}$ and $V_n(x) = 1/(n+1)\sum_{k=1}^{n} S_k(x)$. The following result is sensible to the structure of the spectrum of these polynomials. In the proof we follow Belov's argument [2].

Theorem 5.1. *Let $q \geqslant 1$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$ and*

$$\{b_k\} \in \begin{cases} l_r, & \text{if } 1 \leqslant q < 2; \\ l_\infty, & \text{if } 2 \leqslant q < \infty. \end{cases}$$

Let also $\delta > 0$ and $\{n_k\} \in \mathbf{N}$ satisfy

$$\frac{n_{k+1}}{n_k} \geqslant 1 + \delta |b_k|^q.$$

Then

$$\max_{n-1 \leqslant k \leqslant 2n} \left\|V_k(x) - S_k(x)\right\|_1 \geqslant C\left(\sum_{1 \leqslant s \leqslant l} |b_s|^q |c_{n_s}|^q\right)^{\frac{1}{q}}, \tag{5.2}$$

where $n = n_1 \leqslant n_2 \leqslant \cdots \leqslant n_l \leqslant 2n$. Particularly, if $n_k = k$, we have

$$\max_{n-1 \leqslant k \leqslant 2n} \left\|V_k(x) - S_k(x)\right\|_1 \geqslant C\left(\sum_{l=n}^{2n-1} \frac{|c_l|^q}{l-n+1}\right)^{\frac{1}{q}}. \tag{5.3}$$

Proof. First, we note that the identity

$$S_m - S_{n-1} = \frac{m+1}{m}(S_n - V_n) + (V_{n-1} - S_{n-1}) + \sum_{k=n}^{m-1} \frac{S_k - V_k}{k}$$

implies $(n \leqslant m \leqslant 2n)$

$$\max_{n \leqslant k \leqslant m} \left\|S_k(x) - S_{n-1}(x)\right\|_1 \leqslant C \max_{n-1 \leqslant k \leqslant m} \left\|V_k(x) - S_k(x)\right\|_1. \tag{5.4}$$

Thus, by (5.4),

$$\left\|\sum_{k=1}^{n+1} c_{k+n-1} e^{ikx}\right\| = \left\|\sum_{k=n}^{2n} c_k e^{ikx}\right\| \leqslant C \max_{n-1 \leqslant k \leqslant 2n} \left\|V_k(x) - S_k(x)\right\|_1.$$

Finally, inequality (5.2) simply follows from the Hardy-type inequality [15, Theorem 4] (see also [7]).

If $n_k = k$, then we set $b_s^q = s^{-1}$ and

$$\left(\sum_{k=1}^{n+1} \frac{|c_{k+n-1}|^q}{k}\right)^{\frac{1}{q}} \leqslant C \max_{n-1 \leqslant k \leqslant 2n} \left\|V_k(x) - S_k(x)\right\|_1,$$

which implies (5.3). The proof is now complete. □

In a general situation

$$S_n(x) = \sum_{|k| \leqslant n} c_k e^{ikx} \quad \text{and} \quad V_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} S_\nu(x)$$

one can use the identity

$$2i \sum_{k=n}^{m} k c_k e^{ikx} = \left(S_m'(x) - S_{n-1}'(x)\right) + i\left(\widetilde{S}_m'(x) - \widetilde{S}_{n-1}'(x)\right). \tag{5.5}$$

Then by the Bernstein–Szegő inequality [28, Ch.10, (3.25)], (5.5) implies

$$2\left\|\sum_{k=n}^{m} kc_k e^{ikx}\right\| \leqslant \left\|S'_m - S'_{n-1}\right\| + \left\|\widetilde{S}'_m - \widetilde{S}'_{n-1}\right\| \leqslant 2m\|S_m - S_{n-1}\|.$$

Further,

$$m\left\|\sum_{k=n}^{m} c_k e^{ikx}\right\| \leqslant \left\|\sum_{k=n}^{m} kc_k e^{ikx}\right\| + \left\|\sum_{k=n}^{m}(m-k)c_k e^{ikx}\right\| \leqslant m\left\|\sum_{k=n}^{m} c_k e^{ikx}\right\| + \left\|\sum_{k=n}^{m}(m-k)c_k e^{ikx}\right\|.$$

To estimate the second term, we write

$$\left\|\sum_{k=n}^{m}(m-k)c_k e^{ikx}\right\| = \left\|\sum_{k=0}^{m-n} kc_{m-k} e^{-ikx}\right\| \leqslant (m-n)\left\|\sum_{k=n}^{m} c_k e^{i(k-m)x}\right\| \leqslant (m-n)\left\|\sum_{k=n}^{m} c_k e^{ikx}\right\|.$$

Therefore,

$$\left\|\sum_{k=n}^{2n} c_k e^{ikx}\right\| \leqslant 2\|S_{2n} - S_{n-1}\|.$$

Similarly, we get

$$\left\|\sum_{k=n}^{2n} c_{-k} e^{-ikx}\right\| + \left\|\sum_{k=n}^{m} c_k e^{ikx}\right\| \leqslant C \max_{n-1\leqslant k\leqslant 2n} \left\|V_k(x) - S_k(x)\right\|_1,$$

which by the classical (not generalized) Hardy inequality [28, Ch. 7, (8.7)] implies [3]

$$\sum_{j=[n/2]}^{2n} \frac{|c_j| + |c_{-j}|}{|j-n|+1} \leqslant C \max_{[n/2]-1\leqslant k\leqslant 2n} \left\|V_k(x) - S_k(x)\right\|_1.$$

This estimate can be improved (see (5.2)) by using the generalized Hardy inequality [15].

Remark 5.1. One can also obtain some necessary conditions for $\|f - S_n\| = o(1)$ or $(O(1))$ and therefore for (5.1), using the following inequality by Bochkarev [5]

$$\left(\sum_{k=1}^{\infty}\left(\frac{\|\delta_k\|_p^p}{\|\delta_k\|_2^2}\right)^{\frac{2}{p-2}}\right)^{-\frac{1}{2}} \|f\|_2^2 \leqslant C(p)\|f\|_1, \quad 2 < p \leqslant \infty,$$

where $\delta_k = \delta_k(f,x) = \sum_{m=2^{k-1}+1}^{2^k}(a_m \cos mx + b_m \sin mx)$, $f \in L_1$, and $f \sim \sum_k \delta_k$.

6. Final remarks

1. Note that β-general monotone coefficients suit for the uniform and L_p-convergence criteria as well (see the history and results in [6,22]).

Theorem 6.1.

(A) *Suppose $\{a_n\}_{n=1}^{\infty} \in GM(\beta)$, where $n|a_n|$, $n\beta_n = o(1)$; then series (1.1) converges uniformly on $[0, 2\pi]$ iff $\sum_n a_n$ converges.*
(B) *Suppose $\{a_n\}_{n=1}^{\infty} \in GM(\beta)$, where $n|a_n|$, $n\beta_n = o(1)$; then series (1.2) converges uniformly on $[0, 2\pi]$.*

Theorem 6.2. *Let (1.1) be the Fourier series of a function $f \in L_1$. Let also $1 < p < \infty$.*

(A) *Suppose $\{a_n\}_{n=1}^{\infty} \in GM(\beta)$; then $\sum_n n^{p-2}\beta_n^p < \infty \Rightarrow f \in L_p$.*
(B) *Suppose a non-negative $\{a_n\}_{n=1}^{\infty} \in GM(_5\beta)$ or, more generally, $\{a_n\}$ satisfies*

$$\sum_{k=n}^{\infty}|\Delta a_k| \leqslant C \sum_{k=[n/c]}^{\infty} \frac{|a_k|}{k} \quad \text{for some } c > 1,$$

then $\sum_n n^{p-2}a_n^p < \infty \Leftrightarrow f \in L_p$.

The same results hold for series (1.2) and $\sum_{n=-\infty}^{\infty} a_n e^{inx}$.

2. Let us now construct an example of sequence $a=\{a_n\}$ such that $a\in GM({}_6\beta)$ but $a\notin GM({}_5\beta)$. We define $\varphi(k)=k\ln^\gamma k$, $\gamma\geqslant 1$, and $\psi_\delta(k)=\exp(k\ln^\delta k)$, $k\in\mathbf{N}$. By definition, put

$$N_{\xi+1}=N_\xi+\left[\psi_\delta(N_\xi)\right],\quad \delta>1.$$

Then we define $a=\{a_k\}$ as follows

$$a_k=\begin{cases}\frac{(-1)^k}{\varphi(N_\xi)}, & N_\xi\leqslant k<2N_\xi;\\ \frac{1}{\varphi(N_\xi)}, & 2N_\xi\leqslant k<2N_\xi+\psi_1(N_\xi);\\ 0, & N_\xi+\psi_1(N_\xi)\leqslant k<N_{\xi+1}.\end{cases}$$

Let us verify that $a\notin GM({}_5\beta)$. For any k we write $s\in\mathbf{N}$ such that $N_s\leqslant k<N_{s+1}$. Then for $k=N_s$ we have

$$\sum_{l=k}^{2k-1}|\Delta a_l|\geqslant C\frac{k}{\varphi(N_s)}\asymp\frac{N_s}{\varphi(N_s)}$$

but

$$\sum_{l=[k/c]}^{[ck]}\frac{|a_l|}{l}\leqslant\frac{C}{\varphi(N_s)}\sum_{l=[k/c]}^{[ck]}\frac{1}{l}\asymp\frac{1}{\varphi(N_s)},$$

which contradicts

$$\sum_{l=k}^{2k-1}|\Delta a_l|\leqslant C\sum_{l=[k/c]}^{[ck]}\frac{|a_l|}{l}.$$

Now we show that $a\in GM({}_6\beta)$. It is sufficient to consider $k\in(N_s-1,2N_s)$. Then

$$\sum_{l=k}^{2k-1}|\Delta a_l|\leqslant\sum_{l=N_s-1}^{2N_s}|\Delta a_l|\asymp\frac{N_s}{\varphi(N_s)}.$$

On the other hand,

$${}_6\beta_k=\ln^{-1}k\left(\max_{m\geqslant[k/c]}\frac{\ln m}{m}\sum_{\nu=m}^{2m}|a_\nu|\right)\geqslant\frac{C}{\ln N_s}\left(\max_{\xi\geqslant s}\sum_{\nu=(2N_\xi+\psi_1(N_\xi))/2}^{2N_\xi+\psi_1(N_\xi)}\frac{|a_\nu|\ln\nu}{\nu}\right)\geqslant\frac{C}{\ln N_s}\frac{\ln(2N_s+\psi_1(N_s))}{\varphi(N_s)}\geqslant C\frac{N_s}{\varphi(N_s)},$$

and therefore, $a\in GM({}_6\beta)$. We also remark that for this sequence one has

$$|a_k|\ln k\leqslant\frac{\ln(2N_s+\psi_1(N_s))}{\varphi(N_s)}=\begin{cases}O(1), & \text{if }\gamma=1,\\ o(1), & \text{if }\gamma>1.\end{cases}$$

3. Several open questions are of interest: to extend the L_1-convergence criteria, which we obtain in this paper (Corollary 3.3.2) to the multidimensional situation and the Walsh series (see also [13] and [14]).

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