# EL TEOREMA DE BANACH-STEINHAUS

OTRA PRUEBA USANDO EL MÉTODO DE LA JOROBA DESLIZANTE (7bis)

EL SIGUIENTE RESULTADO SOBRE ACOTACIÓN DE OPERADORES TIENE MUCHAS APLICACIONES. ANTES DE LA PRUEBA SE NECESITA EL SIGUIENTE RESULTADO TOPOLÓGICO.

**LEMA DE BAIRE** SEA $X$ UN ESPACIO MÉTRICO COMPLETO. SEA $(C_n)_{n\geq 1}$ UNA SUCESIÓN DE CERRADOS DE $X$. SUPONGAMOS QUE

$$\operatorname{Int} C_n = \mathring{C}_n = \emptyset \quad \forall n\geq 1$$

ENTONCES $\operatorname{Int}\left(\bigcup_{n=1}^{\infty} C_n\right) = \emptyset$.

**COROLARIO** (QUE SE USARÁ) SI EN UN ESPACIO MÉTRICO COMPLETO $X$, $\bigcup_{n=1}^{\infty} C_n = X$ CON $C_n$ CERRADO $\forall n$, ENTONCES $\exists n_0$: $\mathring{C}_{n_0} \neq \emptyset$.

**DEM** SEA $G_n = X \setminus C_n$, $G_n$ ES UN ABIERTO DENSO DE $X$ $\forall x \in C_n$ Y $\varepsilon>0$ $B(x,\varepsilon)\cap G_n \neq \emptyset$ YA QUE $\mathring{C}_n = \emptyset$.

TENEMOS QUE PROBAR QUE $G = \bigcap_{n=1}^{\infty} G_n$ ES DENSO EN $X$ $G^c = \bigcup G_n^c = \bigcup C_n$. DE INTERIOR VACÍO, QUE ES LO QUE QUEREMOS VER

SEA $A$ UN ABIERTO NO VACÍO DE $X$, VEAMOS QUE $A\cap G \neq \emptyset$.

SEA $x_0 \in A$, $\exists r_0>0$ CON $\overline{B}(x_0,r_0) \subset A$

SEA $x_1 \in B(x_0,r_0)\cap G_1$ Y $r_1>0$ CON

$$\overline{B}(x_1,r_1) \subseteq B(x_0,r_0)\cap G_1$$

$$\text{CON} \quad 0<r_1<\frac{r_0}{2}$$

LO CUAL ES POSIBLE POR SER $G_1$ ABIERTO Y DENSO. PROCEDIENDO POR INDUCCIÓN SE CONSTRUYEN DOS SUCESIONES $(x_n)_{n=1}^{\infty}$ Y $(r_n)_{n=1}^{\infty}$ CON

$$\begin{cases} \overline{B}(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n)\cap G_{n+1} & \forall n\geq 0 \\ 0<r_{n+1}<\frac{r_n}{2} \end{cases}$$

ASÍ LA SUCESIÓN $(x_n)_{n=1}^{\infty}$ ES UNA SUCESIÓN DE CAUCHY EN $X$, QUE ES COMPLETO, LUEGO $\exists x = \lim_{n\to\infty} x_n$. COMO $x_{n+p} \in B(x_n,r_n)$ $\forall n\geq 0$ Y $p\geq 0$ SE OBTIENE QUE $x \in \overline{B}(x_n,r_n)$ $\forall n\geq 0$, EN PARTICULAR $x \in A\cap G$.

# TEOREMA DE BANACH-STEINHAUS (O DE LA ACOTACIÓN UNIFORME 1932)

Sea $X$ un espacio de Banach "$E$" y $Y$ un espacio normado. Sea $(T_i)_{i\in I} \subseteq \mathcal{B}(X,Y)$ una familia de operadores lineales y continuos de $X$ a $Y$. Si

$$\sup_{i\in I}\|T_i x\|<\infty \quad \forall x\in X$$

entonces $\sup_{i\in I}\|T_i\|<\infty$

(e.d. la familia $(T_i)_{i\in I}$ está acotada en $\mathcal{B}(X,Y)$.
Así $\exists C\geq 0$ con $\|T_i x\|\leq C\|x\|$ $\forall x\in X$ y $\forall i\in I$.

Dem: Para cada $n\geq 1$ sea

$$C_n = \{x\in X : \forall i\in I \ \|T_i x\|\leq n\} = \bigcap_{i\in I} h_i^{-1}([0,n])$$

con $h_i = \|\ \|\circ T_i$.

es un conjunto cerrado pues cada $h_i$ es continua.

y por hipótesis ($\sup_{i\in I}\|T_i x\|<\infty \ \forall x\in X$) $\bigcup_{n=1}^{\infty} C_n = X$

por el lema de Baire $\exists C_{n_0}$ con interior no vacío

sea $x_0\in X$, $r>0$ con

$$B(x_0,r)\subseteq C_{n_0}$$

se tiene que

$$\|T_i(x_0+z)\|\leq n_0 \quad \forall i\in I \text{ y } \forall z\in B(0,r).$$

por ser $T_i$ lineal

$$\sup\{\|T_i z\| : z\in B(0,r)\}\leq n_0+\|T_i x_0\| \quad \forall i\in I$$

y como $\exists k$ con $\|T_i x_0\|\leq k$ $\forall i\in I$ se sigue que

$$\|T_i\|\leq \frac{n_0+k}{r} \quad \forall i\in I.$$

Lo que prueba el resultado.

**COROLARIO** Sea $(T_i)_{i\in I} \subseteq \mathcal{B}(X,Y)$, $X$ Banach e $Y$ espacio normado. Si $(T_i)_{i\in I}$ <u>no</u> es un conjunto acotado de $\mathcal{B}(X,Y)$, entonces existe un conjunto $D \subseteq X$ denso tal que

$$\sup\{\|T_i a\| : i\in I\} = \infty \quad \forall a\in D.$$

**DEM** (Sin prueba se usa en las aplicaciones)

Sea $S(x) = \sup\{\|T_i(x)\| \ \ i\in I\}$ (S no tiene porque ser finita)

Sea $E_n = \{x\in \overline{B(0,1)} : S(x) > n\}$.

Entonces

$$E_n = \bigcup_{i\in I}\{x\in \overline{B(0,1)} : \|T_i(x)\| > n\}$$

Como cada $T_i$ es continua $\{x\in \overline{B(0,1)} : \|T_i(x)\| > n\}$ es un abierto de $\overline{B(0,1)}$ $\forall n\in\mathbb{N}$

Así cada $E_n$ es un abierto de $\overline{B(0,1)}$

$\overline{B(0,1)}$ es un espacio métrico completo, así <u>si</u> cada $E_n$ es denso en $\overline{B(0,1)}$, entonces por la demostración del lema de Baire se <u>tendría</u> que $\bigcap_{n=1}^{\infty} E_n$ es densa en $\overline{B(0,1)}$

y por tanto $S(x) = \infty \quad \forall x\in \bigcap_{n=1}^{\infty} E_n$

Como $S(kx) = |k|\, S(x)$, así $S(x) = \infty$

$\forall x\in D = \{kx : 0\neq k\in\mathbb{K} \text{ y } x\in \bigcap_{n=1}^{\infty} E_n\}$ el cual es obviamente denso en $X$

Veamos que todos los $E_n$ son densos en $\overline{B(0,1)}$

Si no $\exists\, E_{n_0}$ <u>no</u> denso en $\overline{B(0,1)}$ y $\exists\, a\in \overline{B(0,1)}$ y $1>\varepsilon>0$

con $\|T_i b\| \le n_0 \quad \forall i\in I$ y $\forall b\in B(a,\varepsilon)\cap \overline{B(0,1)}$

Puesto que $\overline{B(0,1)}$ es acotado, sea $r = \dfrac{\varepsilon}{\|a\|+1}$

Entonces $\forall z\in \overline{B(0,1)}$ $\quad a + rz \in B(a,\varepsilon)$ y así

$$\|T_i(a+rz)\| \le n_0 \quad \forall i\in I \text{ y } \forall z\in \overline{B(0,1)}$$

Luego $\quad r\|T_i(z)\| \le n_0 + \|T_i(a)\| < \infty \quad \forall i$ y $\forall z\in \overline{B(0,1)}$

lo que probaría que $\|T_i\| \le \frac{1}{r}(n_0 + n_0) \quad \forall i\in I$ lo que contradice la hipótesis

## COROLARIO (1927 - BANACH-STEINHAUS).

Sea $X$ un espacio de Banach e $Y$ un espacio normado. Sea $(T_n)_{n=1}^{\infty} \subseteq \mathcal{B}(X,Y)$ una sucesión de operadores tales que $\forall x \in X$ $\quad T_n x \xrightarrow[n\to\infty]{} Tx \in Y$, entonces

a) $\sup_n \|T_n\| < \infty$

b) $T \in \mathcal{B}(X,Y)$.

c) $\|T\| \le \liminf_{n\to\infty} \|T_n\|$

DEM

a) Las hipótesis de este teorema son un caso particular del teorema de Banach-Steinhaus, luego

$$\sup_n \|T_n\| < \infty.$$

b) $\exists M$ con $\|T_n\| \le M$ $\forall n \in \mathbb{N}$

Así $\|T_n(x)\| \le M \|x\|$ $\quad \forall x \in X$ y $\forall n \in \mathbb{N}$

luego $\downarrow n\to\infty$ $\quad \|T(x)\| \le M\|x\|$ $\quad \forall x \in X$ así $T$ es acotada; como es obvio (ejercicio) que $T$ es lineal se sigue que $T \in \mathcal{B}(X,Y)$

c) $\|T_n x\| \le \|T_n\| \, \|x\|$ $\quad \forall x \in X$

tomando límites inferiores,

$$\liminf_{n\to\infty} \|T_n x\| \left(= \lim_{n\to\infty} \|T_n(x)\|\right) = \|Tx\| \le$$

$$\le \|x\| \liminf_{n\to\infty} \|T_n\|.$$

Así $\|T\| \le \liminf_{n\to\infty} \|T_n\|$. c.q.d.

we write

$$(X_1 \oplus X_2 \oplus \cdots)_0 = \{(x_n) : x_n \in X_n \text{ and } (\|x_n\|)_{n=1}^\infty \in c_0\}.$$

Please note that in each case we have defined $(X_1 \oplus X_2 \oplus \cdots)_p$ to be a proper subspace of the formal sum $X_1 \oplus X_2 \oplus \cdots$. In particular, we will no longer be able to claim that $(X_1 \oplus X_2 \oplus \cdots)_p$ and $(X_1 \oplus X_2 \oplus \cdots)_q$ are isomorphic for $p \neq q$. Notice, for example, that $(\mathbb{R} \oplus \mathbb{R} \oplus \cdots)_p = \ell_p$.

It should also be pointed out that the order of the factors $X_1, X_2, \ldots$ in an $\ell_p$ sum does not matter; that is, if $\pi : \mathbb{N} \to \mathbb{N}$ is any permutation, then

$$(X_1 \oplus X_2 \oplus \cdots)_p = (X_{\pi(1)} \oplus X_{\pi(2)} \oplus \cdots)_p,$$

where "$=$" means "is isometric to." (Why?)

Although this may sound terribly complicated, all that we need for now is one very simple observation: We always have

$$(\ell_p \oplus \ell_p \oplus \cdots)_p = \ell_p \quad \text{and} \quad (c_0 \oplus c_0 \oplus \cdots)_0 = c_0,$$

for any $1 \le p < \infty$. And why should this be true? The proof, in essence, is one sentence: $\mathbb{N}$ *can be written as the union of infinitely many, pairwise disjoint, infinite subsets*. (How does this help?)

Given this notation, the proof of Pełczyński's theorem is just a few lines.

**Theorem 5.10.** *Let $X$ be one of the spaces $\ell_p$, $1 \le p < \infty$, or $c_0$. Then, every infinite-dimensional complemented subspace of $X$ is isomorphic to $X$.*

*Proof.* For simplicity of notation, let's consider $X = \ell_p$ for some $1 \le p < \infty$. The proof in case $X = c_0$ is identical.

If $Y$ is an infinite-dimensional complemented subspace of $\ell_p$, then we can write $\ell_p = Y \oplus Z$ for some Banach space $Z$. And, from Corollary 5.7, we can also write $Y = X_1 \oplus W$, where $W$ is some Banach space and where $X_1 \approx \ell_p$. In brief, $Y \approx \ell_p \oplus W$. Thus,

$$\ell_p \oplus Y \approx \ell_p \oplus (\ell_p \oplus W) \approx (\ell_p \oplus \ell_p) \oplus W \approx \ell_p \oplus W \approx Y$$

since $\ell_p \oplus \ell_p \approx \ell_p$. Now for some prestidigitation:

$$\begin{aligned}
\ell_p \oplus Y &= (\ell_p \oplus \ell_p \oplus \cdots)_p \oplus Y \\
&\approx ((Y \oplus Z) \oplus (Y \oplus Z) \oplus \cdots)_p \oplus Y \\
&\approx (Z \oplus Z \oplus \cdots)_p \oplus (Y \oplus Y \oplus \cdots)_p \oplus Y \\
&\approx (Z \oplus Z \oplus \cdots)_p \oplus (Y \oplus Y \oplus \cdots)_p \\
&\approx ((Y \oplus Z) \oplus (Y \oplus Z) \oplus \cdots)_p = \ell_p.
\end{aligned}$$

Hence, $Y \approx \ell_p \oplus Y \approx \ell_p$. $\square$



## Notes and Remarks

Essentially all of the results in this chapter can be attributed to Pełczyński [15, 113], who might fairly be called the father of modern Banach space theory. After the devastation of the Polish school during World War II, the study of linear functional analysis was slow to recover. Aleksander Pełczyński and Joram Lindenstrauss resurrected the lost arts in the late 1950s and early 1960s and went on to form new centers in Poland and in Israel, respectively. Along with Robert James in America, they founded a new school of Banach space theory. Needless to say, all three names will be cited frequently in these notes.

The various gliding hump arguments presented in this chapter are typical of the genre and, it seems, based on ideas that are both old and of some historical curiosity (for more on this, see [134] and [135]). The first known appearance of a gliding hump argument is due to Lebesgue in 1905 [90] (but see also [91]), who used it to show that if $(f_n)$ is a weakly convergent sequence in $L_1$, then $(\|f_n\|_1)$ is bounded. A similar application of the gliding hump technique merits inclusion here:

**Theorem 5.11 (The Uniform Boundedness Theorem).** *Let $(T_\alpha)_{\alpha\in A}$ be a family of linear maps from a Banach space $X$ into a normed space $Y$. If the family is pointwise bounded, then, in fact, it is uniformly bounded. That is, if*

$$\sup_{\alpha\in A} \|T_\alpha x\| < \infty,$$

*for each $x \in X$, then,*

$$\sup_{\alpha\in A} \|T_\alpha\| = \sup_{\alpha\in A} \sup_{\|x\|=1} \|T_\alpha x\| < \infty.$$

*Proof.* Suppose that $\sup_\alpha \|T_\alpha\| = \infty$. We will extract a sequence of operators $(T_n)$ and construct a sequence of vectors $(x_n)$ such that

(a) $\|x_n\| = 4^{-n}$, for all $n$, and
(b) $\|T_n x\| > n$, for all $n$, where $x = \sum_{n=1}^\infty x_n$.

To better understand the proof, consider

$$T_n x = T_n(x_1 + \cdots + x_{n-1}) + T_n x_n + T_n(x_{n+1} + \cdots).$$

The first term has its norm bounded by $M_{n-1} = \sup_\alpha \|T_\alpha(x_1 + \cdots + x_{n-1})\|$. We'll choose the central term so that

$$\|T_n x_n\| \approx \|T_n\|\|x_n\| >> M_{n-1}.$$

We'll control the last term by choosing $x_{n+1}, x_{n+2}, \ldots$ to satisfy

$$\left\| \sum_{k=n+1}^{\infty} x_k \right\| \le \frac{1}{3}\|x_n\|.$$

It is now time for some details. Suppose that $x_1, \ldots, x_{n-1}$ and $T_1, \ldots, T_{n-1}$ have been chosen. Set

$$M_{n-1} = \sup_{\alpha \in A} \|T_\alpha(x_1 + \cdots + x_{n-1})\|.$$

Choose $T_n$ so that

$$\|T_n\| > 3 \cdot 4^n (M_{n-1} + n).$$

Next, choose $x_n$ to satisfy

$$\|x_n\| = 4^{-n} \quad \text{and} \quad \|T_n x_n\| > \frac{2}{3}\|T_n\|\|x_n\|.$$

This completes the inductive step.

It now follows from our construction that

$$\|T_n x_n\| > \frac{2}{3}\|T_n\|\|x_n\| > 2(M_{n-1} + n)$$

and

$$\begin{aligned} \|T_n(x_{n+1} + \cdots)\| &\le \|T_n\| \sum_{k=n+1}^{\infty} 4^{-k} \\ &= \|T_n\| \cdot \frac{1}{3} \cdot 4^{-n} \\ &= \frac{1}{3}\|T_n\|\|x_n\| \\ &< \frac{1}{2}\|T_n x_n\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|T_n x\| &\ge \|T_n x_n\| - \|T_n(x_1 + \cdots + x_{n-1})\| - \|T_n(x_{n+1} + \cdots)\| \\ &> \frac{1}{2}\|T_n x_n\| - \|T_n(x_1 + \cdots + x_{n-1})\| \\ &> (M_{n-1} + n) - M_{n-1} = n. \quad \square \end{aligned}$$

Historia Curiosa

The original proof of this theorem, due to Banach and Steinhaus in 1927 [10], is lost to us I'm sorry to report. As the story goes, Saks, the referee of their paper, suggested an alternate proof – the one that you see in most modern textbooks – using the Baire category theorem (see [32] and [151]). I am told by Joe Diestel that their original manuscript is thought to have been lost during



the war. It's unlikely that we'll ever know their original method of proof, but it's a fair guess that their proof was very similar to the own given above. This is not based on idle conjecture: For one, the gliding hump technique was quite well known to Banach and Steinhuas and had already surfaced in their earlier work. More importantly, the technique was well known to many authors at the time; in particular, this is essentially the same proof given by Hausdorff in 1932 [67].

Curiously, this proof resurfaces every few years (in the *American Mathematical Monthly*, for example) under the label "a nontopological proof of the uniform boundedness theorem." See, for example, [51] and [68]. Apparently, the proof using Baire's theorem (itself an elusive result) is memorable, whereas the gliding hump proof (based solely on first principles) is not.

Theorem 5.8 (in the form of Exercise 5) is due to H. R. Pitt [120]. See Lindenstrauss and Tzafriri [94] for more on strictly singular operators and, more importantly, much more on the subspace structure of $\ell_p$ and $c_0$.

Corollary 5.7 and Theorem 5.10 might lead you to believe that the $\ell_p$ spaces have a rather simple subspace structure. Once we drop the word "complemented," however, the situation changes dramatically: *For $p \ne 2$, the space $\ell_p$ contains infinitely many mutually nonisomorphic subspaces* (cf., e.g., [94, 95]).

Pełczyński's decomposition method (Theorem 5.10) has one obvious practical disadvantage: It's virtually impossible to write down an explicit isomorphism! From this point of view, it's at best an existence proof.

We might paraphrase the conclusion of Pełczyński's theorem by saying that the spaces $c_0$ and $\ell_p$, $1 \le p < \infty$ are *prime* because they have no nontrivial "factors." It's also true that $\ell_\infty$ is prime, but the proof is substantially harder.

The heart of Pełczyński's method is that an infinite direct sum $Y \oplus Y \oplus \cdots$ is able to "swallow up" one more copy of $Y$. The decomposition method will generalize, under the right circumstances, leading to a Schröder–Bernstein-like theorem for Banach spaces. Please take note of the various ingredients used in the proof:

(i) $Y$ embeds complementably in $X$, and $X$ embeds complementably in $Y$,
(ii) $X = (X \oplus X \oplus \cdots)_X$, and
(iii) $X \oplus X \approx X$ (which may *fail* for certain spaces but actually follows from (ii) in this case).

There are several good survey articles on complemented subspaces and Schröder–Bernstein-like theorems for Banach spaces. Highly recommended are the articles by Casazza [20, 21, 22, 23] and Mascioni [99].

## APLICACIÓN: EXISTENCIA DE FUNCIONES CONTINUAS CUYA SERIE DE FOURIER DIVERGE AL MENOS EN UN PUNTO.

SEA $C[-\pi,\pi] = \{ f : [-\pi,\pi] \to \mathbb{R} : f$ continua y $f(-\pi) = f(\pi) \}$.

SEA $S_m f \underset{(*)}{=} \dfrac{a_0}{2} + \sum_{k=1}^{m} a_k \cos kx + b_k \operatorname{sen} kx$.

DONDE $a_0, a_k, b_k$ SON LOS COEFICIENTES (CLÁSICOS) DE FOURIER DE $f$ CON REFERENCIA AL SISTEMA ORTOGONAL $\{1, \cos nx, \operatorname{sen} nx\}_{n=1}^{\infty}$.

SABEMOS QUE:

$$S_m f(x) \underset{(*)}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\operatorname{sen}(m+1/2)t}{2\operatorname{sen}(t/2)} \, dt =$$

(*) VER (8 bis)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(t) f(x+t) \, dt$$

DONDE $D_m(t) = \begin{cases} \dfrac{\operatorname{sen}(m+1/2)t}{2\operatorname{sen}(t/2)} & \text{si } t \neq 0 \\ m + 1/2 & \text{si } t = 0 \end{cases}$

SON LOS NÚCLEOS DE DIRICHLET (CONTINUOS EN $[-\pi,\pi]$)

$$\lim_{t\to 0} \frac{\operatorname{sen}(m+1/2)t}{2\operatorname{sen}(t/2)} \underset{t\to 0}{=} \frac{(\cos(m+1/2)t)(m+1/2)}{\cos(t/2)} = m + 1/2$$

ES CLARO QUE

$$S_m : C[-\pi,\pi] \longrightarrow \mathbb{R}$$
$$f \longrightarrow S_m f(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_m(t) \, dt$$

ES UNA SUCESIÓN DE FUNCIONALES LINEALES SOBRE $C[-\pi,\pi]$ CONTINUOS.

$$|S_m f| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|\,|D_m(t)| \, dt \le \|f\|_\infty \frac{1}{2\pi} \|D_m\|_1$$

POR SER $D_m$ CONTINUA $\Rightarrow D_m \in L_1[-\pi,\pi]$ Y ASÍ $\|D_m\|_1$ ES FINITA.

**LEMA** $\|S_m\| = \dfrac{\|D_m\|_1}{2\pi}$.

SEA $E = \{ t \in [-\pi,\pi] : D_m(t) \ge 0 \}$. CONJUNTO CERRADO, POR TANTO COMPACTO.

$d(\cdot, E)$ CONTINUA

Y SEA $f_n(t) = \dfrac{1 - n\, d(t,E)}{1 + n\, d(t,E)}$, $t \in [-\pi,\pi]$, $n \in \mathbb{N}$

CLARAMENTE $\|f_n\|_\infty \le 1$, $f_n(t) = 1$ $\forall t \in E$ ($d(t,E) = 0$ EN ESTE CASO)

Y $f_n(t) \xrightarrow[n\to\infty]{} -1$ SI $t \notin E$

$f_n$ ESTÁ BIEN DEFINIDA YA QUE $E$ ES CERRADO ($E = D_m^{-1}([0,\infty)) \cap [-\pi,\pi]$). Y COMO LA DISTANCIA A UN CERRADO ES UNA FUNCIÓN CONTINUA, $f_n$ ES CONTINUA Y $|f_n| \le 1$.

ES FÁCIL VER QUE $f_n(\pi) = f_n(-\pi)$ (POR SER $D_m$ PAR)

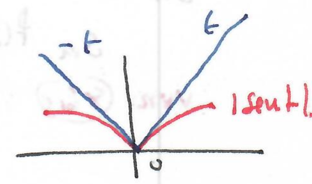
ASÍ POR EL TEOREMA DE LA CONVERGENCIA DOMINADA

$$S_m f_n(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\, D_m(t)\,dt \xrightarrow[n\to\infty]{} \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_m(t)|\,dt$$

ESTO PRUEBA QUE $\|S_m\| \ge \frac{\|D_m\|_1}{2\pi}$ ($\|f_n\|_\infty \le 1$).

LEMA $\|D_m\|_1 \xrightarrow[m\to\infty]{} \infty$.

DEM PUESTO QUE $|\mathrm{sen}\, t| \le |t|$ $\forall t \in \mathbb{R}$




ASÍ $$\int_{-\pi}^{\pi} |D_m(t)|\,dt \underset{D_m \text{ ES PAR}}{\ge} \cancel{2}\int_0^{\pi} \frac{|\mathrm{sen}(m+1/2)t|}{\cancel{2}|t/2|}\,dt =$$

$$\underset{\text{CAMBIO } (m+\frac{1}{2})t = s}{=} 2\int_0^{(m+1/2)\pi} \frac{|\mathrm{sen}\, s|}{s}\,ds \ge 2\sum_{n=1}^{m}\int_{(n-1)\pi}^{n\pi} \frac{|\mathrm{sen}\, s|}{n\pi}\,ds =$$

$$dt = \frac{1}{m+1/2}$$

$$= \frac{4}{\pi}\sum_{n=1}^{m}\frac{1}{n} \xrightarrow[m\to\infty]{} \infty$$

$$\int_0^{\pi} \mathrm{sen}\,x\,dx = -\cos x\Big|_0^{\pi} = 2.$$

POR LO TANTO $\sup\{\|S_m\| : m \in \mathbb{N}\} = \infty$

POR EL TEOREMA DE BANACH-STEINHAUS (1: COROLARIO) EXISTE $D \subseteq C[-\pi,\pi]$ DENSO TAL QUE $\forall f \in D$

$$\sup_n \{|S_m f(0)|\} = \infty$$

LUEGO PARA CADA $f \in D$, LA SERIE DE FOURIER DE $f$ EN $x = 0$ ES DIVERGENTE.

$\lfloor f \in C[-\pi,\pi]$ $2\pi$- PERIÓDICA $\rfloor$

80 bis

$$S_m f(x) = \frac{a_0}{2} + \sum_{k=1}^{m} a_k \cos kx + b_k \operatorname{sen} kx :$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{k=1}^{m} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt \cos kx + \sum_{k=1}^{m}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\operatorname{sen} kt\,dt \operatorname{sen} kx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[ 1/2 + \sum_{k=1}^{m} \cos kt \cos kx + \operatorname{sen} kt \operatorname{sen} kx \Big] dt =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\Big[ 1/2 + \sum_{k=1}^{m} \cos(k(t-x))\Big] dt \underset{(*)}{=}$$

USANDO $\cos(A-B) = \cos A \cos B + \operatorname{sen} A \operatorname{sen} B$

(LEMA DIRICHLET) PARA $u \in [-\pi,\pi]$ $\quad \frac{1}{2} + \sum_{k=1}^{m} \cos ku = \begin{cases} \dfrac{\operatorname{sen}(m+1/2)u}{2\operatorname{sen} u/2} & u \neq 0 \\ m+1/2 & u=0 \end{cases}$

OBSERVACIÓN : $\lim_{u\to 0} \dfrac{\operatorname{sen}(m+1/2)u}{2\operatorname{sen} u/2} = m + 1/2$ ;

DEM SEA $e^{iu} = \cos u + i \operatorname{sen} u$ ; $(e^{iu})^n = e^{inu} = \cos nu + i \operatorname{sen} nu$.

ASÍ $\cos nu = \operatorname{Re}\{(e^{iu})^n\}$ Y

$1/2 + \cos u + \cdots + \cos mu = -1/2 + (1 + \cos u \cdots + \cos mu) = -1/2 + \operatorname{Re}\Big\{\sum_{k=0}^{m} (e^{iu})^k\Big\}$

SI $z = e^{iu}$, LA SUMA $\displaystyle\sum_{k=0}^{m} (e^{iu})^k = \frac{1-(e^{iu})^{m+1}}{1-e^{iu}}$

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ASÍ $\displaystyle \frac{1}{2} + \sum_{k=1}^{m} \cos ku = -1/2 + \operatorname{Re}\Big\{ \frac{1-e^{i(m+1)u}}{1-e^{iu}} \Big\} =$

AHORA $\displaystyle \operatorname{Re}\Big\{ \frac{1-e^{i(m+1)u}}{1-e^{iu}} \Big\} = \operatorname{Re}\Big\{ \frac{e^{-iu/2} - e^{i(m+1/2)u}}{e^{-iu/2} - e^{iu/2}} \Big\} =$

$$= \operatorname{Re}\Big\{ \frac{e^{-iu/2} - e^{i(m+1/2)u}}{-2i\operatorname{sen} u/2} \Big\} = \frac{\operatorname{sen} u/2 + \operatorname{sen}(m+1/2)u}{2\operatorname{sen} u/2}$$

$$= -1/2 + \frac{\operatorname{sen} u/2 + \operatorname{sen}(m+1/2)u}{2\operatorname{sen}(u/2)} = \frac{\operatorname{sen}(m+1/2)u}{2\operatorname{sen} u/2}$$

$(*)$ $\displaystyle \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(u+x)\underbrace{\Big[1/2 + \sum_{k=1}^{m} \cos(ku)\Big]}_{2\pi\text{- PERIÓDICA}} du =$

CAMBIO $u = t-x$
$t = u+x$
$dt = du$.

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\Big[1/2 + \sum_{k=1}^{m}\cos(ku)\Big] du =$$

$u \in [-\pi,\pi]$ Y EL LEMA

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\Big[\frac{\operatorname{sen}(m+1/2)u}{2\operatorname{sen} u/2}\Big] du.$$