

# EL TEOREMA DE BANACH-STEINHAUS

OTRA MANERA USANDO EL TEOREMA DE LA JERARQUÍA DE LAS NORMAS (7.15)

EL SIGUIENTE RESULTADO SURTIÓ ACUTACIÓN DE OPERACIONES TIENE MUCHAS APLICACIONES. ANTES DE LA OTRA SE NECESITA EL SIGUIENTE RESULTADO LÓGICO.

LEMA DE BAIER SEA  $X$  UN ESPACIO MÉTRICO

COMPLETO. SEA  $(C_n)_{n \geq 1}$  UNA SUCECIÓN DE CERRADOS DE  $X$ . SÚBSEGUENCIA QUE

$$\text{Int } C_n = \tilde{C}_n = \emptyset \quad \forall n \geq 1$$

Entonces  $\text{Int}(\bigcup_{n=1}^{\infty} C_n) = \emptyset$ .

COROLARIO (QUE SE USARÁ) SI EN UN ESPACIO MÉTRICO COMPLETO  $X$ ,

$$\bigcup_{n=1}^{\infty} C_n = X \quad \text{con } C_n \text{ CERRADO Y EN } X,$$

Entonces  $\exists n_0 : \tilde{C}_{n_0} \neq \emptyset$ .

PRUEBA SEA  $G_n = X \setminus C_n$ ,  $G_n$  ES UN ABIERTO DENSO DE  $X$

$\forall x \in C_n$  y  $\forall \epsilon > 0 \exists (x, \epsilon) \cap G_n \neq \emptyset$  YA QUE  $\tilde{C}_n = \emptyset$ .

TEENEMOS QUE VERIFICAR QUE  $G = \bigcap_{n=1}^{\infty} G_n$  ES DENSO EN  $X$ .  $G^c = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} C_n$ . DE INTERSECCIONES VACÍAS, QUE

SEA  $A$  UN ABIERTO VACÍO DE  $X$ , TENEMOS QUE  $A \cap G \neq \emptyset$ . ES LO QUE QUEREMOS VER.

SEA  $x_0 \in A$ ,  $\exists r_0 > 0$  CON  $\bar{B}(x_0, r_0) \subset A$

SEA  $x_1 \in \bar{B}(x_0, r_0) \cap G_1 = \emptyset$  Y  $r_1 > 0$  CON

$$\bar{B}(x_1, r_1) \subset \bar{B}(x_0, r_0) \cap G_1$$

$$\text{con } 0 < r_1 < \frac{r_0}{2}$$

LO CUAL ES POSIBLE POR SER  $G_1$  ABIERTO Y DENSO.

PROCEDIMIENTO POR INDUCCIÓN SE CONSTROYE UNA SUCECIÓN DE

$$(x_n)_{n=1}^{\infty} \text{ Y } (r_n)_{n=1}^{\infty} \text{ CON}$$

$$\bar{B}(x_{n+1}, r_{n+1}) \subset \bar{B}(x_n, r_n) \cap G_{n+1} \quad \forall n \geq 0$$

$$0 < r_{n+1} < \frac{r_n}{2}$$

ASI LA SUCECIÓN  $(x_n)_{n=1}^{\infty}$  ES UNA SUCECIÓN DE CAUCHY EN  $X$

QUE ES COMPLETO, LUEGO  $\exists x = \lim_{n \rightarrow \infty} x_n$ . COMO  $x_{n+p} \in \bar{B}(x_n, r_n)$

$\forall n \geq 0$  Y  $\forall p > 0$  SE OBTIENE QUE  $x \in \bar{B}(x_n, r_n) \quad \forall n \geq 0$ , EN

PARTICULAR  $x \in A \cap G$ .

TEOREMA DE BANACH-STEINHAUS (O DE LA ACOTACION)  
 UNIFORME 1432

SEA  $X$  UN ESPACIO DE BANACH " $E$ " Y UN ESPACIO  
 NORMADO. SEA  $(T_\alpha)_{\alpha \in I} \subseteq \mathcal{B}(X, Y)$  UNA  
 FAMILIA DE OPERADORES LINEALES Y CONTINUOS  
 DE  $X$  A  $Y$ . SI

$$\sup_{\alpha \in I} \|T_\alpha x\| < \infty \quad \forall x \in X$$

ENTONCES  $\sup_{\alpha \in I} \|T_\alpha\| < \infty$

(E.O. LA FAMILIA  $(T_\alpha)_{\alpha \in I}$  ESTÁ ACOTADA EN  $\mathcal{B}(X, Y)$ .)

ASI  $\exists C > 0$  C.M.  $\|T_\alpha x\| \leq C \|x\| \quad \forall x \in X$  Y  $\alpha \in I$ .

DEM. PARA CADA  $n \geq 1$  SEA

$$C_n = \{x \in X : \forall \alpha \in I \quad \|T_\alpha x\| \leq n\} = \bigcap_{\alpha \in I} \{x \in X : \|T_\alpha x\| \leq n\}$$

$$\text{C.M. } h_\alpha = \|T_\alpha\| \phi_\alpha$$

ES UN CONJUNTO CERRADO PUES CADA  $h_\alpha$  ES CONTINUA.

Y POR HIPOTESIS  $(\sup_{\alpha \in I} \|T_\alpha x\| < \infty \quad \forall x \in X) \quad \bigcup_{n=1}^{\infty} C_n = X$

POR EL LEMA DE BAIRES  $\exists C_{n_0}$  CON INTERIOR NO VACIO

SEA  $x_0 \in X$ ,  $r > 0$  C.M.  
 $B(x_0, r) \subseteq C_{n_0}$

SE TIENE QUE  $\|T_\alpha(x_0 + z)\| \leq n_0 \quad \forall \alpha \in I$  Y  $\forall z \in B(0, r)$ .

SON EN  $\phi_\alpha$  LINEAL

$$\sup_{\alpha \in I} \{ \|T_\alpha z\| : z \in B(0, r) \} \leq n_0 + \|T_\alpha x_0\| \quad \forall \alpha \in I$$

$$\|T_\alpha\| \leq \frac{n_0 + \|T_\alpha x_0\|}{r} \quad \forall \alpha \in I$$

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COROLARIO SEA  $(T_\alpha)_{\alpha \in I} \subseteq \mathcal{B}(X, Y)$ ,  $X$  BANACH E  $Y$  ESPACIO NORMADO. SI  $(T_\alpha)_{\alpha \in I} \equiv \infty$  ES UN CONJUNTO ACOTADO DE  $\mathcal{B}(X, Y)$ , ENTONCES EXISTE UN CONJUNTO  $D \subseteq X$  DENSO TAL QUE  $\sup \{ \|T_\alpha\| : \alpha \in I \} = \infty \quad \forall \alpha \in I$ .

DEJA (SIN INVERSA SE USA EN LAS APLICACIONES)

SEA  $S(x) = \sup \{ \|T_\alpha(x)\| : \alpha \in I \}$  (S NO TIENE PUNTO SUP FINITO)

SEA  $E_n = \{ x \in \overline{B}(0, 1) : S(x) > n \}$

ENTONCES  $E_n = \bigcup_{\alpha \in I} \{ x \in \overline{B}(0, 1) : \|T_\alpha(x)\| > n \}$

COMO CADA  $T_\alpha$  ES CONTINUA  $\{ x \in \overline{B}(0, 1) : \|T_\alpha(x)\| > n \}$  ES UN SUBCONJUNTO DE  $\overline{B}(0, 1)$

ASI CADA  $E_n$  ES UN SUBCONJUNTO DE  $\overline{B}(0, 1)$  Y EN  $\overline{B}(0, 1)$

ES UN ESPACIO METRICO COMPLETO, ASI SI CADA  $E_n$  ES DENSO EN  $\overline{B}(0, 1)$ , ENTONCES LA DEMOSTRACION DEL LEMA DE BARTLE

SE TIENE BIEN QUE  $\bigcap_{n=1}^{\infty} E_n$  ES DENSO EN  $\overline{B}(0, 1)$

Y POR TANTO  $S(x) = \infty \quad \forall x \in \bigcap_{n=1}^{\infty} E_n$

COMO  $S(kx) = |k| S(x)$ , ASI  $S(x) = \infty$

$\forall x \in D = \{ kx : 0 \neq k \in \mathbb{K} \text{ y } x \in \bigcap_{n=1}^{\infty} E_n \}$  EL CUAL ES OBVIAMENTE DENSO EN  $X$

VERIFICAR QUE TODOS LOS  $E_n$  SON DENSO EN  $\overline{B}(0, 1)$

SI NO  $\exists E_{n_0} \equiv \emptyset$  DENSO EN  $\overline{B}(0, 1)$  Y  $\exists a \in \overline{B}(0, 1)$  Y  $\exists \epsilon > 0$  CON  $\|T_\alpha b\| \leq n_0 \quad \forall \alpha \in I$  Y  $\forall b \in \overline{B}(a, \epsilon) \cap \overline{B}(0, 1)$

PUES QUE  $\overline{B}(0, 1)$  ES ACOTADO, SEA  $r = \frac{\epsilon}{\|a\| + 1}$

ENTONCES  $\forall z \in \overline{B}(0, 1) \quad a + rz \in \overline{B}(a, \epsilon)$  Y ASI  $\|T_\alpha(a + rz)\| \leq n_0 \quad \forall \alpha \in I$  Y  $\forall z \in \overline{B}(0, 1)$

LUEGO  $\|T_\alpha(z)\| \leq n_0 + \|T_\alpha(a)\| < \infty \quad \forall \alpha \in I$  Y  $\forall z \in \overline{B}(0, 1)$

LO QUE PROHIBIRIA QUE  $\|T_\alpha\| \equiv \frac{1}{r}(n_0 + n_0) \quad \forall \alpha \in I$  LO QUE CONTRADICE LA HIPOTESIS

COROLARIO (1927 - BANACH-STEINHAUS).

SEA  $X$  UN ESPACIO DE BANACH Y  $Y$  UN ESPACIO NORMATIVO. SEA  $(T_n)_{n=1}^{\infty} \in \mathcal{B}(X, Y)$ .

UNA SUCESSION DE OPERADORES TALES QUE

$\forall x \in X \quad T_n x \rightarrow T x \in Y$ , ENTONCES

a)  $\sup_n \|T_n\| < \infty$

b)  $T \in \mathcal{B}(X, Y)$ .

c)  $\|T\| = \liminf_{n \rightarrow \infty} \|T_n\|$

DEM

a) LAS MISURAS DE ESTE TEOREMA SON UN CASO PARTICULAR DEL TEOREMA DE BANACH-STEINHAUS, LUEGO

$\sup_n \|T_n\| < \infty$ .

b) EMBO COMO  $\|T_n\| \leq M \quad \forall n$

ASÍ  $\|T_n(x)\| \leq M \|x\| \quad \forall x \in X$  y  $\forall n$

LUEGO  $\|T(x)\| \leq M \|x\| \quad \forall x \in X$  ASÍ  $T \in \mathcal{B}(X, Y)$

ACOTADA; COMO ES UN CASO (EJEMPLO) QUE  $T$  ES LINEAL SE SIGUE QUE  $T \in \mathcal{B}(X, Y)$

c)  $\|T_n x\| \leq \|T_n\| \|x\| \quad \forall x \in X$

TOMANDO LIMITES INFERIORES,

$\liminf_{n \rightarrow \infty} \|T_n x\| (= \lim_{n \rightarrow \infty} \|T_n(x)\|) = \|T x\| \leq \|x\| \liminf_{n \rightarrow \infty} \|T_n\|$ .

ASÍ  $\|T\| \in \liminf_{n \rightarrow \infty} \|T_n\|$ . c.q.d.

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we write

$$(X_1 \oplus X_2 \oplus \cdots)_0 = \{(x_n) : x_n \in X_n \text{ and } (\|x_n\|)_{n=1}^\infty \in c_0\}.$$

Please note that in each case we have defined  $(X_1 \oplus X_2 \oplus \cdots)_p$  to be a proper subspace of the formal sum  $X_1 \oplus X_2 \oplus \cdots$ . In particular, we will no longer be able to claim that  $(X_1 \oplus X_2 \oplus \cdots)_p$  and  $(X_1 \oplus X_2 \oplus \cdots)_q$  are isomorphic for  $p \neq q$ . Notice, for example, that  $(\mathbb{R} \oplus \mathbb{R} \oplus \cdots)_p = \ell_p$ .

It should also be pointed out that the order of the factors  $X_1, X_2, \dots$  in an  $\ell_p$  sum does not matter; that is, if  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  is any permutation, then

$$(X_1 \oplus X_2 \oplus \cdots)_p = (X_{\pi(1)} \oplus X_{\pi(2)} \oplus \cdots)_p,$$

where “=” means “is isometric to.” (Why?)

Although this may sound terribly complicated, all that we need for now is one very simple observation: We always have

$$(\ell_p \oplus \ell_p \oplus \cdots)_p = \ell_p \quad \text{and} \quad (c_0 \oplus c_0 \oplus \cdots)_0 = c_0,$$

for any  $1 \leq p < \infty$ . And why should this be true? The proof, in essence, is one sentence:  $\mathbb{N}$  can be written as the union of infinitely many, pairwise disjoint, infinite subsets. (How does this help?)

Given this notation, the proof of Pełczyński’s theorem is just a few lines.

**Theorem 5.10.** *Let  $X$  be one of the spaces  $\ell_p$ ,  $1 \leq p < \infty$ , or  $c_0$ . Then, every infinite-dimensional complemented subspace of  $X$  is isomorphic to  $X$ .*

*Proof.* For simplicity of notation, let’s consider  $X = \ell_p$  for some  $1 \leq p < \infty$ . The proof in case  $X = c_0$  is identical.

If  $Y$  is an infinite-dimensional complemented subspace of  $\ell_p$ , then we can write  $\ell_p = Y \oplus Z$  for some Banach space  $Z$ . And, from Corollary 5.7, we can also write  $Y = X_1 \oplus W$ , where  $W$  is some Banach space and where  $X_1 \approx \ell_p$ . In brief,  $Y \approx \ell_p \oplus W$ . Thus,

$$\ell_p \oplus Y \approx \ell_p \oplus (\ell_p \oplus W) \approx (\ell_p \oplus \ell_p) \oplus W \approx \ell_p \oplus W \approx Y$$

since  $\ell_p \oplus \ell_p \approx \ell_p$ . Now for some prestidigitiation:

$$\begin{aligned} \ell_p \oplus Y &= (\ell_p \oplus \ell_p \oplus \cdots)_p \oplus Y \\ &\approx ((Y \oplus Z) \oplus (Y \oplus Z) \oplus \cdots)_p \oplus Y \\ &\approx (Z \oplus Z \oplus \cdots)_p \oplus (Y \oplus Y \oplus \cdots)_p \oplus Y \\ &\approx (Z \oplus Z \oplus \cdots)_p \oplus (Y \oplus Y \oplus \cdots)_p \\ &\approx ((Y \oplus Z) \oplus (Y \oplus Z) \oplus \cdots)_p = \ell_p. \end{aligned}$$

Hence,  $Y \approx \ell_p \oplus Y \approx \ell_p$ .  $\square$

**Notes and Remarks**

Essentially all of the results in this chapter can be attributed to Pełczyński [15, 113], who might fairly be called the father of modern Banach space theory. After the devastation of the Polish school during World War II, the study of linear functional analysis was slow to recover. Aleksander Pełczyński and Joram Lindenstrauss resurrected the lost arts in the late 1950s and early 1960s and went on to form new centers in Poland and in Israel, respectively. Along with Robert James in America, they founded a new school of Banach space theory. Needless to say, all three names will be cited frequently in these notes.

The various gliding hump arguments presented in this chapter are typical of the genre and, it seems, based on ideas that are both old and of some historical curiosity (for more on this, see [134] and [135]). The first known appearance of a gliding hump argument is due to Lebesgue in 1905 [90] (but see also [91]), who used it to show that if  $(f_n)$  is a weakly convergent sequence in  $L_1$ , then  $(\|f_n\|_1)$  is bounded. A similar application of the gliding hump technique merits inclusion here:

**Theorem 5.11 (The Uniform Boundedness Theorem).** *Let  $(T_\alpha)_{\alpha \in A}$  be a family of linear maps from a Banach space  $X$  into a normed space  $Y$ . If the family is pointwise bounded, then, in fact, it is uniformly bounded. That is, if*

$$\sup_{\alpha \in A} \|T_\alpha x\| < \infty,$$

for each  $x \in X$ , then,

$$\sup_{\alpha \in A} \|T_\alpha\| = \sup_{\|x\|=1} \sup_{\alpha \in A} \|T_\alpha x\| < \infty.$$

*Proof.* Suppose that  $\sup_\alpha \|T_\alpha\| = \infty$ . We will extract a sequence of operators  $(T_n)$  and construct a sequence of vectors  $(x_n)$  such that

- (a)  $\|x_n\| = 4^{-n}$ , for all  $n$ , and
- (b)  $\|T_n x\| > n$ , for all  $n$ , where  $x = \sum_{n=1}^\infty x_n$ .

To better understand the proof, consider

$$T_n x = T_n(x_1 + \cdots + x_{n-1}) + T_n x_n + T_n(x_{n+1} + \cdots).$$

The first term has its norm bounded by  $M_{n-1} = \sup_\alpha \|T_\alpha(x_1 + \cdots + x_{n-1})\|$ . We’ll choose the central term so that

$$\|T_n x_n\| \approx \|T_n\| \|x_n\| > M_{n-1}.$$

We'll control the last term by choosing  $x_{n+1}, x_{n+2}, \dots$  to satisfy

$$\left\| \sum_{k=n+1}^{\infty} x_k \right\| \leq \frac{1}{3} \|x_n\|.$$

It is now time for some details. Suppose that  $x_1, \dots, x_{n-1}$  and  $T_1, \dots, T_{n-1}$  have been chosen. Set

$$M_{n-1} = \sup_{\alpha \in A} \|T_{\alpha}(x_1 + \dots + x_{n-1})\|.$$

Choose  $T_n$  so that

$$\|T_n\| > 3 \cdot 4^n (M_{n-1} + n).$$

Next, choose  $x_n$  to satisfy

$$\|x_n\| = 4^{-n} \quad \text{and} \quad \|T_n x_n\| > \frac{2}{3} \|T_n\| \|x_n\|.$$

This completes the inductive step.

It now follows from our construction that

$$\|T_n x_n\| > \frac{2}{3} \|T_n\| \|x_n\| > 2(M_{n-1} + n)$$

and

$$\begin{aligned} \|T_n(x_{n+1} + \dots)\| &\leq \|T_n\| \sum_{k=n+1}^{\infty} 4^{-k} \\ &= \|T_n\| \cdot \frac{1}{3} \cdot 4^{-n} \\ &= \frac{1}{3} \|T_n\| \|x_n\| \\ &< \frac{1}{2} \|T_n x_n\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|T_n x\| &\geq \|T_n x_n\| - \|T_n(x_1 + \dots + x_{n-1})\| - \|T_n(x_{n+1} + \dots)\| \\ &> \frac{1}{2} \|T_n x_n\| - \|T_n(x_1 + \dots + x_{n-1})\| \\ &> (M_{n-1} + n) - M_{n-1} = n. \quad \square \end{aligned}$$

The original proof of this theorem, due to Banach and Steinhaus in 1927 [10], is lost to us I'm sorry to report. As the story goes, Saks, the referee of their paper, suggested an alternate proof – the one that you see in most modern textbooks – using the Baire category theorem (see [32] and [151]). I am told by Joe Diestel that their original manuscript is thought to have been lost during

the war. It's unlikely that we'll ever know their original method of proof, but it's a fair guess that their proof was very similar to the one given above. This is not based on idle conjecture: For one, the gliding hump technique was quite well known to Banach and Steinhaus and had already surfaced in their earlier work. More importantly, the technique was well known to many authors at the time; in particular, this is essentially the same proof given by Hausdorff in 1932 [67].

Curiously, this proof resurfaces every few years (in the *American Mathematical Monthly*, for example) under the label “a nontopological proof of the uniform boundedness theorem.” See, for example, [51] and [68]. Apparently, the proof using Baire's theorem (itself an elusive result) is memorable, whereas the gliding hump proof (based solely on first principles) is not.

Theorem 5.8 (in the form of Exercise 5) is due to H. R. Pitt [120]. See Lindenstrauss and Tzafriri [94] for more on strictly singular operators and, more importantly, much more on the subspace structure of  $\ell_p$  and  $c_0$ .

Corollary 5.7 and Theorem 5.10 might lead you to believe that the  $\ell_p$  spaces have a rather simple subspace structure. Once we drop the word “complemented,” however, the situation changes dramatically: For  $p \neq 2$ , the space  $\ell_p$  contains infinitely many mutually nonisomorphic subspaces (cf., e.g., [94, 95]).

Pelczyński's decomposition method (Theorem 5.10) has one obvious practical disadvantage: It's virtually impossible to write down an explicit isomorphism! From this point of view, it's at best an existence proof.

We might paraphrase the conclusion of Pelczyński's theorem by saying that the spaces  $c_0$  and  $\ell_p$ ,  $1 \leq p < \infty$  are *prime* because they have no nontrivial “factors.” It's also true that  $\ell_{\infty}$  is prime, but the proof is substantially harder.

The heart of Pelczyński's method is that an infinite direct sum  $Y \oplus Y \oplus \dots$  is able to “swallow up” one more copy of  $Y$ . The decomposition method will generalize, under the right circumstances, leading to a Schröder–Bernstein-like theorem for Banach spaces. Please take note of the various ingredients used in the proof:

- (i)  $Y$  embeds complementably in  $X$ , and  $X$  embeds complementably in  $Y$ ,
- (ii)  $X = (X \oplus X \oplus \dots)_X$ , and
- (iii)  $X \oplus X \approx X$  (which may fail for certain spaces but actually follows from (i) in this case).

There are several good survey articles on complemented subspaces and Schröder–Bernstein-like theorems for Banach spaces. Highly recommended are the articles by Casazza [20, 21, 22, 23] and Mascioni [99].

Алгоритм  
Судья

APLICACIÓN: EXISTENCIA DE FUNCIONES CONTINUAS  
 CUYA SERIE DE FOURIER NIVELA AL  
 MENOS EN UN PUNTO.

SEA  $C[-\pi, \pi] \rightarrow \mathbb{R} : f$  CONTINUA Y  $f(-\pi) = f(\pi)$ .

SEA  $S_m f = \frac{a_0}{2} + \sum_{k=1}^m a_k \cos kx + b_k \sin kx$ .

SONTOS  $a_0, a_k, b_k$  SON LOS COEFICIENTES ECLASICO DE  
 FOURIER DE  $f$  (COEFICIENTES AL SISTEMA ORTOGONAL  
 $\{1, \cos nx, \sin nx\}_{n \in \mathbb{N}}$ ).

SAIBE-MOI QUE:  
 $S_m f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin(m+1/2)t}{2 \sin(t/2)} dt =$   
 $= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(t) f(x+t) dt$

SONTOS  $D_m(t) = \begin{cases} \frac{\sin(m+1/2)t}{2 \sin(t/2)} & \text{SI } t \neq 0 \\ m+1/2 & \text{SI } t = 0 \end{cases}$

SONTOS  $D_m$  NIVELA EN  $\mathbb{R}$  (CONTINUA EN  $[-\pi, \pi]$ )

$\lim_{t \rightarrow 0} \frac{\sin(m+1/2)t}{2 \sin(t/2)} = \lim_{t \rightarrow 0} \frac{(m+1/2)t}{(t/2)} = m+1/2$

ES CLARO QUE:

$S_m C[-\pi, \pi] \rightarrow \mathbb{R}$   
 $f \rightarrow S_m f(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_m(t) dt$

ES UNA SUCESIÓN DE FUNCIONES LINEALES SOBRE  
 $C[-\pi, \pi]$  CONTINUA.

$|S_m f| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)| |D_m(t)| dt \leq \|f\|_{\infty} \frac{1}{2\pi} \|D_m\|_1$

POR SER  $D_m$  CONTINUA  $\Rightarrow D_m \in L_1[-\pi, \pi]$  Y ASI  $\|D_m\|_1$  ES FINITO.

LEMMA  $\|S_m\| = \frac{\|D_m\|_1}{2\pi}$

SEA  $\epsilon > 0$   $t \in [-\pi, \pi] : D_m(t) \geq 0$ . (CONJUNTO CERRADO, POR TANTO COMPACTO)

Y SEA  $f_n(t) = \frac{1 - n d(t, \epsilon)}{1 + n d(t, \epsilon)}$ ,  $t \in [-\pi, \pi]$ ,  $n \in \mathbb{N}$

CLARAMENTE  $\|f_n\|_{\infty} \leq 1$ ,  $f_n(t) = 1 \quad \forall t \in \epsilon$  ( $d(t, \epsilon) = 0$  EN ESTE CASO)

Y  $f_n(t) \rightarrow -1$  SI  $t \notin \epsilon$

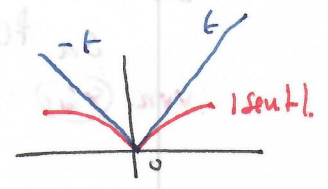
$f_n$  ESTÁ BIEN DEFINIDA Y-A QUE L'ES CERRADO  
 ( $t = D_m^{-1}([0, \pi]) \cap [-\pi, \pi]$ ). Y COMO LA DISTANCIA  
 A UN CERRADO ES UNA FUNCIÓN CONTINUA,  $f_n$   
 ES CONTINUA Y  $|f_n| \leq 1$ .  
 ES FÁCIL VER QUE  $f_n(\pi) = f_n(-\pi)$  (POR SER  $D_m$  PAR)  
 ASÍ SON EC TEOREMA DE LA CONVERGENCIA PUNTOA

$$S_m f_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(t) D_m(t) dt \xrightarrow{m \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(t) dt$$

ESTO PROVEBA QUE  $\|S_m\| \geq \frac{\|D_m\|_1}{2\pi}$  ( $\|f_n\|_{\infty} \leq 1$ ).

LEMA  $\|D_m\|_1 \rightarrow \infty$   
 $m \rightarrow \infty$ .

LEMMA PROVEBA QUE  $|\text{sen } t| \leq |t| \quad \forall t \in \mathbb{R}$



$$\text{ASÍ} \int_{-\pi}^{\pi} |D_m(t)| dt \geq 2 \int_0^{\pi} \frac{|\text{sen}(m + \frac{1}{2})t|}{|t/2|} dt =$$

$$\downarrow \text{D}_m \text{ ES PAR}$$

$$2 \int_0^{(m+\frac{1}{2})\pi} \frac{|\text{sen } s|}{s} ds \geq 2 \sum_{n=1}^m \int_{(n-1)\pi}^{n\pi} \frac{|\text{sen } s|}{n} ds =$$

$$\downarrow \text{CAMBIO } (m + \frac{1}{2})t = s$$

$$dt = \frac{1}{m + \frac{1}{2}}$$

$$= \frac{4}{\pi} \sum_{n=1}^m \frac{1}{n} \xrightarrow{m \rightarrow \infty} \infty$$

$$\int_0^{\pi} \text{sen } x dx = -\cos x \Big|_0^{\pi} = 2.$$

CON LO ANTERIOR  $\sup \{ \|S_m\| \mid m \in \mathbb{N} \} = \infty$

CON EL TEOREMA DE WEIERSTRASS-STEINHAUS (1: (1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14) (15) (16) (17) (18) (19) (20) (21) (22) (23) (24) (25) (26) (27) (28) (29) (30) (31) (32) (33) (34) (35) (36) (37) (38) (39) (40) (41) (42) (43) (44) (45) (46) (47) (48) (49) (50) (51) (52) (53) (54) (55) (56) (57) (58) (59) (60) (61) (62) (63) (64) (65) (66) (67) (68) (69) (70) (71) (72) (73) (74) (75) (76) (77) (78) (79) (80) (81) (82) (83) (84) (85) (86) (87) (88) (89) (90) (91) (92) (93) (94) (95) (96) (97) (98) (99) (100) (101) (102) (103) (104) (105) (106) (107) (108) (109) (110) (111) (112) (113) (114) (115) (116) (117) (118) (119) (120) (121) (122) (123) (124) (125) (126) (127) (128) (129) (130) (131) (132) (133) (134) (135) (136) (137) (138) (139) (140) (141) (142) (143) (144) (145) (146) (147) (148) (149) (150) (151) (152) (153) (154) (155) (156) (157) (158) (159) (160) (161) (162) (163) (164) (165) (166) (167) (168) (169) (170) (171) (172) (173) (174) (175) (176) (177) 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EXISTE  $D \subset C[-\pi, \pi]$  TAL QUE  $\forall f \in D$   
 $\sup_n \{ \|S_m f(x)\| \} = \infty$   
 LUEGO PARA CADA  $f \in C[-\pi, \pi]$  EXISTE  $x \in \mathbb{R}$  TAL QUE  
 $\sup_n \{ \|S_m f(x)\| \} = \infty$

CURSO	N.º DE MATRÍCULA
ASIGNATURA	GRUPO
NOMBRE	D.N.I. n.º
APELLIDOS	

Ejercicios del ALUMNO

FACULTAD DE CIENCIAS MATEMÁTICAS  
 DE MADRID  
 UNIVERSIDAD COMPLUTENSE



FECE-1413 211-REVISÃO DE CÁLCULO

$$S_m f(x) = \frac{a_0}{2} + \sum_{k=1}^m a_k \cos kx + b_k \sin kx =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{k=1}^m \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt dt \cos kx + \sum_{k=1}^m \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt dt \sin kx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ \frac{1}{2} + \sum_{k=1}^m \cos kt \cos kx + \sin kt \sin kx \right] dt =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ \frac{1}{2} + \sum_{k=1}^m \cos(k(t-x)) \right] dt. \quad (*)$$

USANDO  
 $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(LEMA DIRICHLET) PARA  $u \in ]-\pi, \pi[$   $\frac{1}{2} + \sum_{k=1}^m \cos ku = \begin{cases} \frac{\sin(m+1/2)u}{2 \sin u/2} & u \neq 0 \\ m+1/2 & u = 0 \end{cases}$

OBSERVAÇÃO:  $\lim_{u \rightarrow 0} \frac{\sin(m+1/2)u}{2 \sin u/2} = m+1/2$

DEFIN: Se  $e^{iu} = \cos u + i \sin u$ ;  $(e^{iu})^n = e^{inu} = \cos nu + i \sin nu$

ASS  $\cos nu = \operatorname{Re} \{ (e^{iu})^n \}$   $y$   
 $\frac{1}{2} + \cos u + \dots + \cos nu = -\frac{1}{2} + (1 + \cos u + \dots + \cos nu) = -\frac{1}{2} + \operatorname{Re} \left\{ \sum_{k=0}^m (e^{iu})^k \right\}$

SE  $z = e^{iu}$ , LA SOMA  $\sum_{k=0}^m (e^{iu})^k = \frac{1 - (e^{iu})^{m+1}}{1 - e^{iu}}$   
 SOMA GEOMÉTRICA

ASS  $\frac{1}{2} + \sum_{k=1}^m \cos ku = -\frac{1}{2} + \operatorname{Re} \left\{ \frac{1 - e^{i(m+1)u}}{1 - e^{iu}} \right\} =$

AMARRA  $\operatorname{Re} \left\{ \frac{1 - e^{i(m+1)u}}{1 - e^{iu}} \right\} = \operatorname{Re} \left\{ \frac{e^{-iu/2} - e^{i(m+1)u/2}}{e^{-iu/2} - e^{iu/2}} \right\} =$

$= \operatorname{Re} \left\{ \frac{e^{-iu/2} - e^{i(m+1)u/2}}{-2i \sin u/2} \right\} = \frac{\sin u/2 + \sin(m+1/2)u}{2 \sin u/2}$

$= -\frac{1}{2} + \frac{\sin u/2 + \sin(m+1/2)u}{2 \sin u/2} = \frac{\sin(m+1/2)u}{2 \sin u/2}$

(\*)  $\frac{1}{\pi} \int_{-\pi-x}^{\pi-x} f(u+x) \left[ \frac{1}{2} + \sum_{k=1}^m \cos ku \right] du =$   
 211-REVISÃO DE CÁLCULO

CAMBIO  $u = t-x$   
 $t = u+x$   
 $dt = du$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) \left[ \frac{1}{2} + \sum_{k=1}^m \cos ku \right] du =$   
 uF-FUNÇÃO  
 Y RE (LEMA)

$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u+x) \left[ \frac{\sin(m+1/2)u}{2 \sin u/2} \right] du.$