NÚMEROS COMPLEJOS

PROBLEMA 1:

a) $Re(1+i) = 1$ c) $\overline{3-2i} = 3+2i$

d) $(1+i)(3-2i) = 3+3i-2i-2i^2 = 5+i$

$i^2 = -1$

f) $\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1-2i+i^2}{1^2+1^2} = \frac{-2i}{2} = -i$

h) $(1+\sqrt{3}i)^3 \underset{\downarrow}{=} 1+3\sqrt{3}i+3(\sqrt{3}i)^2+(\sqrt{3}i)^3 =$

BINOMIO DE NEWTON $= 1+3\sqrt{3}i - 9 - i\sqrt{27} =$

$= -8+(3\sqrt{3}-\sqrt{27})i$

PROBLEMA 2: a) $x+yi = \frac{1-\sqrt{3}i}{2} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

Luego $x = 1/2$ e $y = -\frac{\sqrt{3}}{2}$.

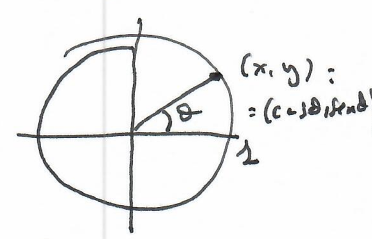
b) $x+yi = |x-yi| = \sqrt{x^2+y^2} \Rightarrow \begin{cases} y=0 \\ x=|x| \iff \underline{\underline{x \geq 0}} \end{cases}$

c) $\sum_{k=0}^{100} i^k = |x-yi|$.

$i^0+i^1+i^2+i^3 = 1+i-1-i = 0$ luego

$\sum_{k=0}^{100} i^k = i^{100} = (i^2)^{50} = 1$

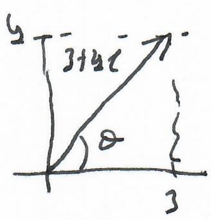
$1 = \sqrt{x^2+y^2}$



101 | 4
21 25
1

$|x-yi| = \sqrt{x^2+y^2}$

PROBLEMA 3: a) $3+4i$.



$|3+4i| = \sqrt{9+16} = 5$

$\tan\theta = \frac{3}{4}$

$\text{Arctan}\ \frac{3}{4} = \text{Arg}(3+4i)$

e) $\frac{1+i}{1-i} \underset{\downarrow}{=} -i$ ASÍ $\left|\frac{1+i}{1-i}\right| = 1$ y $\text{Arg}\,\frac{1+i}{1-i} = \frac{3\pi}{2}$

PROBLEMA 1:

PROBLEMA 4:

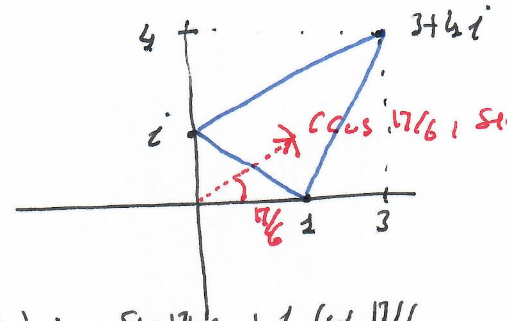
a) $z^2 = 3 - 4i \Rightarrow z = \sqrt{3-4i}$ SALEN 2 RAÍCES!

$z = 5\left(\cos\left(\text{Arctn}\frac{3}{-4}\right)/2 + i\,\text{sen}\left(\left(\text{Arctn}\frac{3}{-4}\right)/2\right)\right)$ ó

$5\left(\cos\left(\text{Arctn}\frac{3}{-4}/2 + \pi\right) + i\,\text{sen}\left(\text{Arctn}\frac{3}{-4}/2 + \pi\right)\right)$.

c) $z^4 - 2z^2 + 4 = (z^2 - 2)^2 = 0 \Rightarrow z^2 = 2 \iff z = \pm\sqrt{2}$.

PROBLEMA 5:



$i(\cos\pi/6 + i\,\text{sen}\,\pi/6) = -\text{sen}\,\pi/6 + i\cos\pi/6$

$1(\cos\pi/6 + i\,\text{sen}\,\pi/6) = \cos\pi/6 + i\,\text{sen}\,\pi/6$

} NUEVOS VÉRTICES!

$(3+4i)(\cos\pi/6 + i\,\text{sen}\,\pi/6) = \left(3\cos\frac{\pi}{6} - 4\,\text{sen}\frac{\pi}{6}\right) + i\left(4\cos\frac{\pi}{6} + 3\,\text{sen}\frac{\pi}{6}\right)$.

PROBLEMA 6:

a) $\sqrt[3]{-8} = \sqrt[3]{8}\left(\cos\frac{2k\pi}{3} + i\,\text{sen}\frac{2k\pi}{3}\right)$ $k = 0, 1$ y 2.

b) $\sqrt[3]{-2+2i}$ $-2+2i = \sqrt{8}\left(\cos\frac{3\pi}{4} + i\,\text{sen}\frac{3\pi}{4}\right)$

$\downarrow$

$\text{Arctn}(-1) = \frac{3\pi}{4}$.

ASÍ $\sqrt[3]{-2+2i} = \sqrt[3]{\sqrt{8}}\left(\cos\frac{3\pi}{12} + \frac{2k\pi}{3} + i\,\text{sen}\frac{3\pi}{12} + \frac{2k\pi}{3}\right)$

$k = 0, 1, 2$.

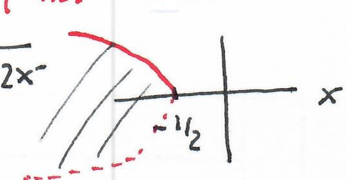
PROBLEMA 7:

a) $|z| < 1 - \text{Re}\,z$ SI $z = x + iy$ $\sqrt{x^2+y^2} < 1 - x$ (ASÍ $x < 1$).

$\iff x^2 + y^2 < (1-x)^2 = 1 - 2x + x^2$

LUEGO $y^2 < 1 - 2x$ $\Rightarrow \begin{cases} |y| < \sqrt{1-2x} \\ x < -1/2 \end{cases}$

$x < \frac{1}{2}$



PROBLEMA 7º]

e) $\left|\dfrac{z-3}{z+3}\right| = 2$

$$\frac{z-3}{z+3} = \frac{(z-3)^2}{(z+3)(z-3)} = \frac{z^2-6z+9}{10} = \frac{1}{10}z^2 - \frac{3}{5}z + \frac{9}{10}$$

O TAMBIEN

$$\left|\frac{z-3}{z+3}\right| = \frac{|z-3|}{|z+3|} = 2 \iff |z-3| = 2|z+3|$$

SI $z = x+yi$. $\iff \sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$

$$\iff (x-3)^2+y^2 = 4\left((x+3)^2+y^2\right)$$

$$\iff (x-3)^2-2^2(x+3)^2 = 3y^2$$

$$\iff (x-3+2x+6)(x-3-2x-6) = 3y^2$$

$$\iff (3x+3)(-x-9) = -3x^2-30x-27 = 3y^2$$

$$\iff x^2+10x+9 = -y^2$$

$$\iff y = \pm\sqrt{-x^2-10x-9} =$$

$$= \pm\sqrt{-x^2-2\cdot 5x-25+16} =$$

$$= \pm\sqrt{16-(x-5)^2} \implies x \in (1,9).$$

PROBLEMA 8º]

a) $\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\,\mathrm{sen}\,\frac{2k\pi}{n} \quad k=0,1,\dots n-1.$

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\left(\frac{\mathrm{Arg}\,z}{n}+\frac{2k\pi}{n}\right) + i\,\mathrm{sen}\left(\frac{\mathrm{Arg}\,z}{n}+\frac{2k\pi}{n}\right)\right) =$$

$$= \sqrt[n]{|z|}\left(\cos\frac{\mathrm{Arg}\,z}{n} + i\,\mathrm{sen}\,\frac{\mathrm{Arg}\,z}{n}\right)\cdot\left(\cos\frac{2k\pi}{n} + i\,\mathrm{sen}\,\frac{2k\pi}{n}\right)$$

$k=0,1,\dots,n-1$

b) ¿SI $z^n = 1$ y $w^n = 1$, entonces

$$(z\cdot w)^n = z^n\cdot w^n = 1\cdot 1 = 1.$$

c) SI $z^n = 1 \implies \left(\frac{1}{z}\right)^n = \frac{1}{z^n} = \frac{1}{1} = 1.$

PROBLEMA 9º] USAR LAS DE CUERPO DE $\mathbb{C}$

PROBLEMA 10:

b) $z^n = \bar{z}$ $\quad z=0$ es solución

Si $z \neq 0$ $\quad |z^n| = |z|^n$ y $|\bar{z}| = |z|$ luego

$|z|^n = |z| \iff |z| = 1$

$z = (\cos\theta + i\,\mathrm{sen}\,\theta)$ ASÍ $z^n = (\cos n\theta + i\,\mathrm{sen}\,n\theta) = \cos\theta - i\,\mathrm{sen}\,\theta$

$$\iff \begin{cases} \cos n\theta = \cos\theta \\ \mathrm{sen}\,n\theta = -\mathrm{sen}\,\theta \end{cases} \iff \underline{\underline{\theta = 0}}$$

PROBLEMA 11: $\quad z \neq 1 \quad |z| = 1$

a) ¿ $\mathrm{Re}\,\dfrac{1+z}{1-1/z} = 0$? $\qquad \dfrac{1+z}{1-\frac{1}{z}} = \dfrac{z(1+z)}{z-1} = \dfrac{z(1+z)(\overline{z-1})}{|z-1|^2}$

Luego $\mathrm{Re}\,\dfrac{1+z}{1-1/z} = 0 \iff \mathrm{Re}\; z(1+z)(\overline{z-1}) = 0$

Ahora $(z+z^2)(\bar{z}-1) = z\cdot\bar{z} + z^2\bar{z} - z - z^2 =$

$= |z|^2 + z|z|^2 - z - z^2 \underset{|z|=1}{=} 1 - z^2$ Ahora $\mathrm{Re}\,(1-z^2) = 0 \iff z = 1$ o -1.

$\cancel{= |z|^2(1+z) - z(1+z) =}$

$\cancel{= (|z|^2 - z)(1+z)}$

$\cancel{\mathrm{Re}\,((|z|^2 - z)(1+z)) = 0 \Rightarrow (x^2+y^2-x)(1+x) + y(1+x) + y^2 = 0}$

$\cancel{z = x + yi.}$

Sólo es cierto si $z=-1$; no es otro caso

b) $\mathrm{Im}(z + z^{-1}) = 0$ $\qquad z + \dfrac{1}{z} = \dfrac{z^2+1}{z} = \dfrac{\bar{z}(z^2) + \bar{z}}{|z|^2} \underset{|z|=1}{=}$

$= z + \bar{z} \in \mathbb{R}$. ¡cierto!

PROBLEMA 12: $\quad \dfrac{1}{z} = \dfrac{\bar{z}}{z\cdot\bar{z}} = \dfrac{\bar{z}}{|z|^2}$ ASÍ

$\mathrm{Re}\,\dfrac{1}{z} = \mathrm{Re}\,\dfrac{\bar{z}}{|z|^2} > 0$ $\qquad$ como $\mathrm{Re}\,\dfrac{z}{|z|^2} = \mathrm{Re}\,\dfrac{\bar{z}}{|z|^2} > 0$

por hipótesis

Se tiene que $\boxed{\mathrm{Re}\,z > 0}$

PROBLEMA 13:

$$\sqrt[5]{7i} = \sqrt[5]{7}\left(\cos\left(\frac{\pi}{10}+\frac{2k\pi}{5}\right) + i\,\mathrm{sen}\left(\frac{\pi}{10}+\frac{2k\pi}{5}\right)\right)$$

$k = 0, 1, 2, 3, 4.$

SI $k=0$: $\frac{\pi}{10}$

SI $k=1$: $\boxed{\frac{\pi}{10}+\frac{2\pi}{5} = \frac{5\pi}{10} = \frac{\pi}{2}}$

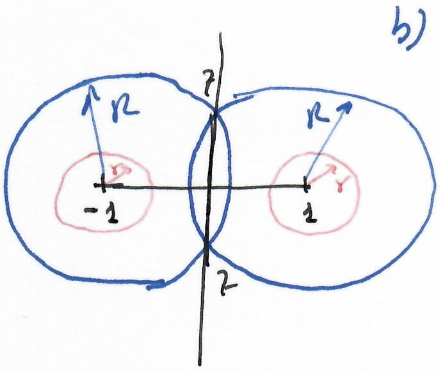
SI $k=2$: $\frac{\pi}{10}+\frac{4\pi}{5} = \frac{9}{10}\pi$

SI $k=3$: $\frac{\pi}{10}+\frac{6\pi}{5} = \frac{13}{10}\pi$

SI $k=4$: $\frac{\pi}{10}+\frac{8\pi}{5} = \frac{17\pi}{10}$

PROBLEMA 14:

b) $|z-1| = |z+1|$ $z \in \mathbb{C}$ tal que dista lo mismo de -1 y 1.



La solución es $\{ri : r \in \mathbb{R}\}$.

PROBLEMA 15:

a) $||z|-|w|| \leq |z-w|$ y $||z|-|w|| \leq |z+w|$ Como en $\mathbb{R}$.

b) $|z-w|^2 + |z+w|^2 = \langle z-w, \overline{z-w}\rangle + \langle z+w, \bar{z}+\bar{w}\rangle$

$= \langle z,\bar{z}\rangle + \langle w,\bar{w}\rangle \cancel{- \langle z,\bar{w}\rangle - \langle w,\bar{z}\rangle}$

$+ \langle z,\bar{z}\rangle + \langle w,\bar{w}\rangle \cancel{+ \langle z,\bar{w}\rangle + \langle w,\bar{z}\rangle} = 2(|z|^2+|w|^2)$

PROBLEMA 16: a) $|zv| = |z|\,|v| \leq |w|\,|v| = |wv|$.

$|z| \leq |w|$ hipótesis

b) $|z+v| \leq |w+v|$.   Claramente no.

$z=1 < 2=w$ si $v=-2$.

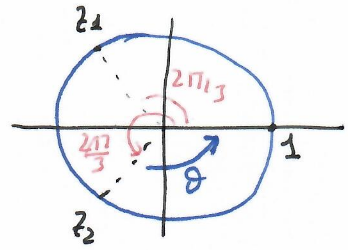
$|1-2| = 1 > |2-2| = 0.$

PROBLEMA 17: c) $\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right) + i\,\mathrm{sen}\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right) =$

$\theta = \mathrm{Arg}\, z$

$= \sqrt[n]{|z|}\left(\cos\frac{\theta}{n} + i\,\mathrm{sen}\frac{\theta}{n}\right)\left(\cos\frac{2k\pi}{n} + i\,\mathrm{sen}\frac{2k\pi}{n}\right)$ $k=0,1,\ldots,n-1.$

PROBLEMA 19:



$\sqrt[3]{1} = \{1, z_1, z_2\}$

$\theta = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$

LUEGO $\alpha = \cos\frac{2\pi}{3} + i\,\text{sen}\,\frac{2\pi}{3} = z_1$ VERIFICA

QUE $\alpha z_2 = z_1 z_2 = 1$

(OBSERVEMOS QUE $z_2 = \overline{z_1}$.)

PROBLEMA 20:

$\cos 3t = \cos(t+2t) = \cos t \cos 2t - \text{sen}\,t\,\text{sen}\,2t =$

USAMOS: $\cos(a+b) = \cos a \cos b - \text{sen}\,a\,\text{sen}\,b$

$= \cos t(\cos^2 t - \text{sen}^2 t) - \text{sen}\,t\,(2\cos t\,\text{sen}\,t) =$

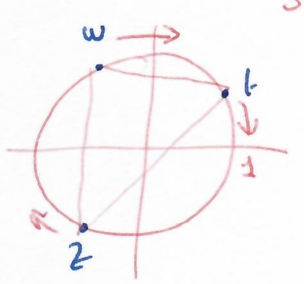
$= \cos^3 t - \cos t\,\text{sen}^2 t - 2\cos t\,\text{sen}^2 t =$

$= \cos^3 t - 3\cos t\,\text{sen}^2 t = \cos^3 t - 3\cos t(1-\cos^2 t) =$

$= 4\cos^3 t - 3\cos t.$

PROBLEMA 21: $|z| = |w| = |t| = 1$

SALVO UN GIRO, PODEMOS SUPONER QUE $t = 1$



ASÍ $z + w + 1 = 0$ LUEGO $z = 1 - w$

LUEGO $\text{Im}\,z + \text{Im}\,w = 0$ ASÍ $\text{Im}\,z = -\text{Im}\,w$

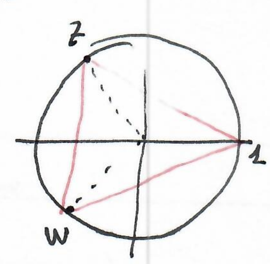
COMO $|z| = |w| = 1 \Rightarrow |\text{Re}\,z| = |\text{Re}\,w| = \sqrt{1 - \text{Im}^2 z} =$

$= \sqrt{1 - \text{Im}^2 w}$

COMO FORMAN PARTE DE UN TRIÁNGULO NO PUEDE OCURRIR QUE $\text{Re}\,z = -\text{Re}\,w$ (EN ESE CASO $z + w = 0$ Y ASÍ $z + w + 1 \neq 0$). LUEGO $\text{Re}\,z = \text{Re}\,w$.

COMO $\text{Re}\,z + \text{Re}\,w = 1 \Rightarrow \text{Re}\,z = \text{Re}\,w = -\frac{1}{2}$

$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}$ $\quad w = -\frac{1}{2} - \frac{\sqrt{3}}{2}$ $\quad t = 1$



AHORA $|z - 1| = \left|-\frac{3}{2} + \frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$

$|w - 1| = \left|-\frac{3}{2} - \frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$

$|z - w| = \left|2\frac{\sqrt{3}}{2}i\right| = \sqrt{3}$ EFECTIVAMENTE EL TRIÁNGULO ES EQUILÁTERO.

PROBLEMA 22:

a) ¿ $z-1 \mid z^n-1$? Como $z=1$ es raíz de z^n-1, se tiene que $z-1$ divide a z^n-1

b) $z^{n-1}+z^{n-2}+ \quad +z+1=0$ Sea $w\neq 1$ con $w^n=1$

Así $w^n-1=0$ luego w es raíz de z^n-1.

Ahora $z^n-1=(z-1)(z^{n-1}+z^{n-2}+ \quad +z+1)$ ya que

$= z^n+z^{n-1}+ \quad +z-z^{n-1}-z^{n-2}- \quad -1 = z^n-1$

Como $w\neq 1$ $(w-1)\neq 0$ luego si $w^n-1=0 \Rightarrow$

$\Rightarrow w^{n-1}+w^{n-2}+ \quad +w+1=0.$

c) Si $n=km$, dividiendo

$$\begin{array}{l|l} x^{km} \qquad -1 & x^m-1 \\ \hline -x^{km}+x^{(k-1)m} & x^{(k-1)m}+x^{(k-2)m} \quad +x^m+1 \\ \hline x^{(k-1)m} \quad -1 & \\ \quad \ddots & \\ \qquad x^m-1 & \\ \qquad x^m+1 & \\ \hline \qquad 0. & \end{array}$$

PROBLEMA 23: Usamos el problema 22/b)

$w= \cos\frac{2\pi}{n}+i\,\mathrm{sen}\frac{2\pi}{n}$ es raíz de $z^{n-1}+\dots+z+1=0$

es decir $w^{n-1}+ \quad +w=-1$

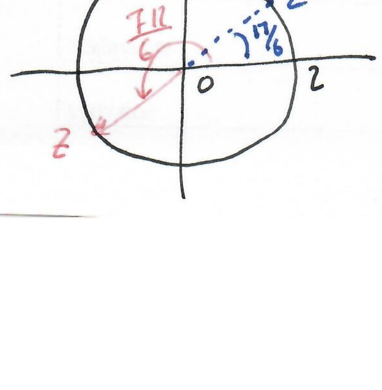
$w^k = \cos\frac{2k\pi}{n}+i\,\mathrm{sen}\frac{2k\pi}{n}.$

Luego $w^{n-1}+ \quad +w=-1 \iff$

$\left(\cos\frac{2(n-1)\pi}{n}+ \quad +\cos\frac{2\pi}{n}\right)+i\left(\mathrm{sen}\frac{2(n-1)\pi}{n}+ \quad +\mathrm{sen}\frac{\pi}{n}\right)=-1$

Igualando partes reales e imaginarias se tiene el resultado

PROBLEMA 24: $z\in\mathbb{C}$ tiene coordenadas polares $r=2$ y $\theta=\frac{7\pi}{6}$



$\pi < \frac{7\pi}{6} = \pi+\frac{1}{6}\pi < \frac{3\pi}{2}$

Parece que b) es la solución buscada

PROBLEMA 26:

$(z+6)$, $(z-i)$ y $(z-(1-2i))$ DIVIDEN A $P(x)$

$P(x) = (z+6)(z-i)(z-1+2i)\,Q(x)$.

$6(-i)(-1+2i) = 6i+12$.

LUEGO SI $P(x) = a_n x^n + \dots + a_0 = (z+6)(z-i)(z-1+2i)(b_m x^m + \dots + b_0)$

$a_0 = (6i+12)\,b_0$ LUEGO $2 \mid a_0$

PROBLEMA 27: a) SI PARA z: $z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$

CON $a_{n-1}, \dots, a_0 \in \mathbb{R}$, ENTONCES

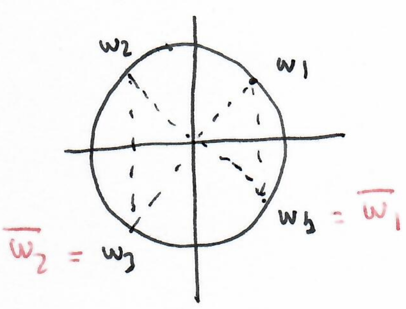
$$0 = \overline{z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0} = \bar{z}^n + a_{n-1}\bar{z}^{n-1} + \dots + a_1 \bar{z} + a_0$$

LUEGO $\bar{z}$ ES RAIZ DEL POLINOMIO

PROBLEMA 29: $P(x) = x^4+1$ LAS RAICES DE P

SON LAS CUATRO RAICES $\sqrt[4]{-1} = \cos\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{\pi}{4}+\frac{2k\pi}{4}\right)$

$k = 0, 1, 2, 3$



$$\boxed{P(x) = (x-w_1)(x-\overline{w_1})(x-w_2)(x-\overline{w_2})}$$

DESCOMPOSICION EN $\mathbb{C}$

$$P(x) = \left(x^2 - 2\,\mathrm{Re}\,w_1 x + |w_1|^2\right)\left(x^2 - 2\,\mathrm{Re}\,w_2 x + |w_2|^2\right)$$

$$= \left(x^2 - 2\tfrac{1}{\sqrt{2}}x + 1\right)\left(x^2 + 2\tfrac{1}{\sqrt{2}}x + 1\right)$$

DESCOMPOSICION EN $\mathbb{R}$

$P(x) = x^4+1$ NO ADMITE NINGUNA DESCOMPOSICION EN $\mathbb{Q}$.