Lucas 1842 - 1891 Prasolov Polynomials 2010

at it suffices to consider the case of polynomials with , if $p(z) = \sum a_n z^n$ is a polynomial with complex coef- the polynomial

$$= p(z)\overline{p(\overline{z})} = \left(\sum a_n z^n\right)\left(\sum \overline{a_n} z^n\right).$$

s of the roots of $\overline{p(\overline{z})}$ are the same as those of $p(z)$. nts of $p^*(z)$ are symmetric with respect to a_n and $\overline{a_n}$. coefficients of p^* are invariant under conjugation, that

et $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ *be a polynomial with* $(z) = z^m + b_1 z^{m-1} + \dots + b_m$, *where* $m = \frac{1}{2}n(n-1)$, *ose roots are all the sums of pairs of the roots of* p. *The le if and only if all the coefficients of the polynomials* p

that p is stable. To a negative root α of p there corre- $-\alpha$ with positive coefficients. To a pair of conjugate roots al part there corresponds the factor

$$\alpha - i\beta)(z - \alpha + i\beta) = z^2 - 2\alpha z + \alpha^2 + \beta^2$$

cients. Thus all the coefficients of p are positive.
oots of q fall into the pairs of conjugate roots because the real. Further, the real parts of all the roots of q are negative. nts as for p show that all the coefficients of q are positive. e coefficients of p and q be positive. In this case, all the real e negative. Therefore, if α is a real root of p, then $\alpha < 0$, and, f complex conjugate roots of p, then $2\alpha = (\alpha + i\beta) + (\alpha - i\beta)$ ce $2\alpha < 0$. □

... olynomial and of its derivative

Theorem 1.2.1 (Gauss-Lucas). *The roots of* P' *belong to the convex hull of the roots of the polynomial* P *itself.*

Proof. Let $P(z) = (z - z_1) \cdot \ldots \cdot (z - z_n)$. It is easy to verify that

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \dots + \frac{1}{z - z_n}. \tag{1}$$

Suppose that $P'(w) = 0$, $P(w) \neq 0$ and suppose on the contrary that w does not belong to the convex hull of the points $z_1, \dots, z_n$. Then one can draw a line through w that does not intersect the convex hull of $z_1, \dots, z_n$. Therefore the vectors $w - z_1, \dots, w - z_n$ lie in one half-plane determined by this line. Hence the vectors

$$\frac{1}{w - z_1}, \dots, \frac{1}{w - z_n}$$

also lie in one half-plane, since $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$. Hence,

$$\frac{P'(w)}{P(w)} = \frac{1}{w - z_1} + \dots + \frac{1}{w - z_n} \neq 0.$$

This is a contradiction, and hence w belongs to the convex hull of the roots of P. □

Relation (1) allows one to prove the following properties of the roots of P' for any polynomial P with real roots.

Theorem 1.2.2 ([An1]). *Let*

$$P(z) = (z - x_1) \cdot \ldots \cdot (z - x_n), \text{ where } x_1 < \dots < x_n.$$

If some root x_i *is replaced by* $x_i' \in (x_i, x_{i+1})$, *then all the roots of* P' *increase their value.*

Proof. Let $z_1 < z_2 < \dots < z_{n-1}$ be the roots of P', and let $x_1, \dots, x_n$ be the roots of P. Let $z_1' < z_2' < \dots < z_{n-1}'$ be the roots of Q' and let $x_1' = x_1, \dots, x_{i-1}' = x_{i-1}, x_i', x_{i+1}' = x_{i+1}, \dots, x_n' = x_n$ be the roots of Q. For the roots z_k and z_k', the relation (1) takes the form