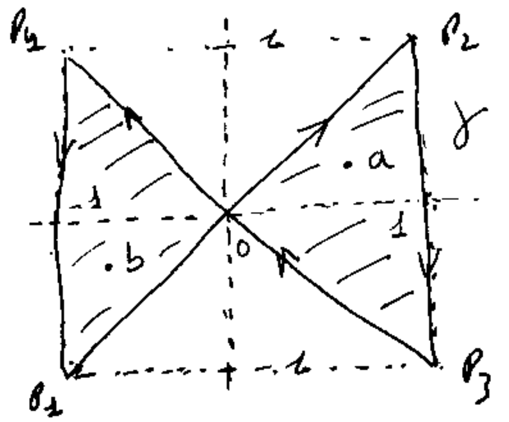
HOJA 2.

PROBLEMA 2:]



SEA $P_1 = -1-i$

$P_2 = 1+i$

$P_3 = 1-i$

$P_4 = -1+i$

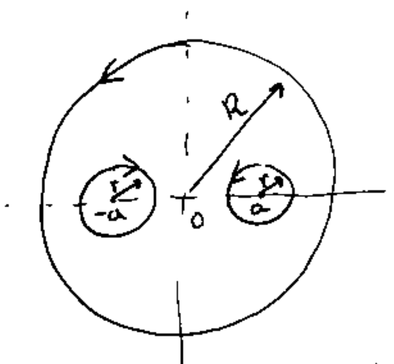
$\operatorname{Ind}_\gamma(a) = -1$

$\operatorname{Ind}_\gamma(b) = 1$

SI C NO PERTENECE A LOS DOS TRIANGULOS SEÑALADOS

$\operatorname{Ind}_\gamma(c) = 0$

PROBLEMA 3:] SEA Γ COMO EN EL DIBUJO



$\operatorname{Ext}_\Gamma = \mathbb{C} \setminus \overline{D(0,R)} \cup D(-a,r)$

NO CONEXO

$\operatorname{Int}_\Gamma = D(0R) \setminus \overline{D(-a,r)} \cup D(a,r)$

NO CONEXO

$\operatorname{Int}_\Gamma \subseteq D(0,R) \setminus \overline{D(a,r)} \cup D(a,r)$

UNIÓN DE ABIERTOS, NO VACIOS Y DISJUNTOS

HOJA 2ª

PROBLEMA 4:

a) $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\cdot\gamma'(t)}{f(\gamma(t))}\,dt =$

$= \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{z}\,dz = \mathrm{Ind}_{f\circ\gamma}(0)$. "NÚMERO DE VECES QUE LA CURVA $f\circ\gamma$ RODEA AL CERO".

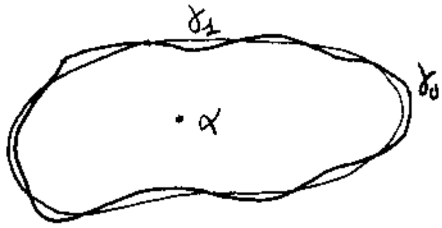
b) $\mathrm{Ind}_{g\circ\gamma}(0) = \frac{1}{2\pi i}\int_{g\circ\gamma}\frac{1}{z}\,dz = \frac{1}{2\pi i}\int_a^b \frac{g'(\gamma(t))\cdot\gamma'(t)}{g(\gamma(t))}\,dt =$

$\underset{g(z)=z^n}{=} \frac{1}{2\pi i}\int_a^b \frac{n(\gamma(t))^{n-1}\gamma'(t)}{(\gamma(t))^n}\,dt = \frac{n}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}\,dt \underset{\gamma(t)\neq 0}{=}$

$= \frac{n}{2\pi i}\int_\gamma \frac{1}{z}\,dz = n\,\mathrm{Ind}_\gamma(0)$

HOJA 2:

PROBLEMA 5:



$\gamma_1(s) \neq \alpha \quad \forall s \in [0,1]$

Y $\gamma_0(s) \neq \alpha \quad \forall s \in [0,1]$ Y YA

QUE $|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \quad \forall s \in [0,1]$

LUEGO LA CURVA $\gamma(s) = \dfrac{\gamma_1(s) - \alpha}{\gamma_0(s) - \alpha}$ ESTA BIEN DEFINIDA

$$\frac{\gamma_1(s) - \alpha}{\gamma_0(s) - \alpha} = \frac{\gamma_1(s) - \gamma_0(s)}{\gamma_0(s) - \alpha} + \frac{\gamma_0(s) - \alpha}{\gamma_0(s) - \alpha}$$

ASI $\left| \dfrac{\gamma_1(s) - \alpha}{\gamma_0(s) - \alpha} - 1 \right| = \left| \dfrac{\gamma_1(s) - \gamma_0(s)}{\gamma_0(s) - \alpha} \right| < 1$

POR HIPÓTESIS

ASÍ $\gamma(s) \in D(1,1) \quad \forall s \in [0,1]$, COMO $0 \notin D(1,1)$

TIENE SENTIDO CALCULAR $\mathrm{Ind}_\gamma(0) = 0$ YA QUE

$0 \in \mathbb{C} - D(1,1) \subseteq \mathbb{C} - \gamma^*$ Y POR TANTO 0 PERTENECE

A LA COMPONENTE CONEXA NO ACOTADA DE $\mathbb{C} - \gamma^*$

POR OTRO LADO

$$\mathrm{Ind}_\gamma(0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z}\, dz = \frac{1}{2\pi i} \int_0^1 \frac{1}{\gamma(s)} \cdot \gamma'(s)\, ds =$$

$$= \frac{1}{2\pi i} \int_0^1 \frac{\gamma_0(s) - \alpha}{\gamma_1(s) - \alpha} \cdot \left[\frac{\gamma_1'(s)[\gamma_0(s) - \alpha] - \gamma_0'(s)[\gamma_1(s) - \alpha]}{(\gamma_0(s) - \alpha)^2} \right] ds =$$

$$= \frac{1}{2\pi i} \int_0^1 \frac{\gamma_1'(s)}{\gamma_1(s) - \alpha} - \frac{\gamma_0'(s)}{\gamma_0(s) - \alpha}\, ds = \frac{1}{2\pi i} \left[\int_{\gamma_1} \frac{1}{z - \alpha}\, dz - \int_{\gamma_0} \frac{1}{z - \alpha}\, dz \right] =$$

$= \mathrm{Ind}_{\gamma_1}(\alpha) - \mathrm{Ind}_{\gamma_0}(\alpha)$ ASI $\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_0}(\alpha)$.

HOJA 2:

PROBLEMA 8:

a) $\frac{z+1}{z} = \frac{1}{z} + 1$

b) $\frac{z}{z^2+1} = \frac{z}{(z-i)(z+i)} = \frac{1/2}{z-i} + \frac{1}{2}\,\frac{1}{z+i}$

SI ESCRIBIMOS $f(z) = \frac{1}{z+i} = \sum_{n=0}^{\infty} \frac{f^{n)}(i)}{n!}(z-i)^n = *$

COMO
$$\left.\begin{aligned} f'(z) &= \frac{-1}{(z+i)^2} \\ f''(z) &= \frac{2}{(z+i)^3} \\ f'''(z) &= \frac{-2\,3}{(z+i)^4} \end{aligned}\right\} (*)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2i)^{n+1}}(z-i)^n.$$

Y POR INDUCCION

$f^{n)}(z) = \frac{(-1)^n\, n!}{(z+i)^{n+1}}$; ASI $f^{n)}(i) = \frac{(-1)^n n!}{(2i)^{n+1}}$

c) $\frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1} = \frac{1}{z} - \sum_{n=0}^{\infty}(-1)^n z^n.$

SE PROCEDE COMO EN (*)

d) $\frac{z}{z+1} = z\left(\frac{1}{z+1}\right) = z\sum_{n=0}^{\infty}(-1)^n z^n = \sum_{n=0}^{\infty}(-1)^n z^{n+1}$

e) $\frac{e^z}{z^2} = \frac{\sum_{k=0}^{\infty}\frac{z^k}{k!}}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{k=0}^{\infty}\frac{z^k}{(k+2)!}$

f) $\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left[\frac{1}{z-1}\cdot\frac{1}{z-2}\right] = \frac{1}{z}\left[\frac{-1/3}{z-1} + \frac{1/3}{z-2}\right]$ EN SERIES DE TAYLOR

HAY QUE DESARROLLAR $\frac{1}{z-1}$ Y $\frac{1}{z-2}$

CENTRADAS EN $z=0$

$\frac{1}{z(z-1)(z-2)} = \frac{1}{z-1}\left[\frac{1}{z}\cdot\frac{1}{z-2}\right] = \frac{1}{z-1}\left[\frac{-1/2}{z} + \frac{1/2}{z-2}\right]$ HAY QUE

DESARROLLAR $\frac{1}{z}$ Y $\frac{1}{z-2}$ EN SERIES DE TAYLOR

CENTRADAS EN $z=1$.

g) $\operatorname{sen}\frac{1}{z} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\left(\frac{1}{z}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}\cdot\frac{1}{z^{2n+1}}.$