# HOJA 4:

PROBLEMA 1:

$u(x,y) = xy \qquad \frac{\partial u}{\partial x} = y \quad \frac{\partial^2 u}{\partial x^2} = 0$

$$\frac{\partial u}{\partial y} = x \quad \frac{\partial^2 u}{\partial y^2} = 0$$

CLARAMENTE $u$ ES ARMÓNICA.

SI $f = u + vi$, $u$ y $v$ SATISFACEN LAS CONDICIONES DE CAUCHY RIEMANN: $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

Y ASÍ, ARGUMENTANDO COMO EN TEORÍA,

$$v(x,y) = \int_0^y \frac{\partial u}{\partial x}(x,t)\,dt - \int_0^x \frac{\partial u}{\partial y}(s,0)\,ds =$$

$$= \int_0^y t\,dt - \int_0^x s\,ds = \frac{y^2}{2} - \frac{x^2}{2}$$

$f(x,y) = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right) \in H(\mathbb{C})$ YA QUE $u$ y $v \in C^1(\mathbb{R}^2)$ Y VERIFICAN LAS CONDICIONES DE CAUCHY-RIEMANN

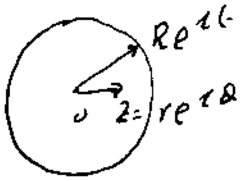
# HOJA 4:

PROBLEMA 3:

a) $P(r, t-\theta) = \dfrac{R^2 - r^2}{R^2+r^2 - 2Rr\cos(t-\theta)} = \dfrac{R^2-r^2}{|Re^{it} - re^{i\theta}|^2} > 0.$

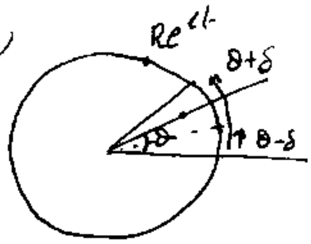
b) $P(r, t-\theta) = \dfrac{R^2-r^2}{R^2+r^2-2Rr\cos(t-\theta)}$ como $R^2+r^2-2Rr\cos(t-\theta) =$

$= |Re^{it}-re^{i\theta}|^2 \neq 0$




$r < R$ $\quad P$ es continua.

c)




$|Re^{it} - Re^{i\theta}| = |Re^{i(\theta+\delta+k)} - Re^{i\theta}| =$

$= R\,|e^{i(\delta+k)} - 1| \underset{\delta>0,\,k>0}{>} R\,|e^{i\delta} - 1|$

Así $\dfrac{R^2-r^2}{|Re^{it}-re^{i\theta}|^2} \underset{r \to R}{\leq} \dfrac{R^2-r^2}{R^2|e^{i\delta}-1|^2} \xrightarrow[r \to R]{} 0$

independientemente de $t \in [0, 2\pi] - (\theta-\delta, \theta+\delta)$

d) $P(r, t-\theta) = \dfrac{R^2-r^2}{|Re^{it} - re^{i\theta}|^2} = \dfrac{Re^{it}}{Re^{it} - re^{i\theta}} + \dfrac{re^{-i\theta}}{Re^{-it} - re^{-i\theta}} =$

$= \dfrac{s}{s-z} + \dfrac{\bar{z}}{\bar{s}-\bar{z}} = \mathrm{Re}\left(\dfrac{s+z}{s-z}\right)$ si $s = Re^{it}$, $z = re^{i\theta}$

como $\dfrac{s+z}{s-z} \in H(D(0,R))$, así $P(r, t-\theta)$ es armónica en $z = re^{i\theta}$.

e) $P(r, t-\theta) = \dfrac{R^2-r^2}{R^2+r^2-2Rr\cos(t-\theta)}$ es claramente $2\pi$-periódica en $\theta - t$

f) $P(0, t-\theta) = \dfrac{R^2}{R^2} = 1$

g) Si $f \equiv 1 \in H(\mathbb{C})$, su parte real $u(r,\theta) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P(r, t-\theta)\, u(R,t)\, dt$

$\Longleftrightarrow \quad 1 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P(r, t-\theta)\, dt.$

HOJA 4:

PROBLEMA 1:)

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r, t-\theta)\, F(t)\,dt =$$

$$F(t) = \begin{cases} 1 & \text{si } t \in [0,\pi] \\ 0 & \text{si } t \notin [0,\pi] \end{cases}$$

$$= \frac{1}{2\pi}\int_0^{\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(t-\theta)}\,dt = \quad u = t-\theta, \quad du = dt$$

$$= \frac{R^2-r^2}{2\pi}\int_{-\theta}^{\pi-\theta} \frac{1}{R^2+r^2-2Rr\cos u}\,du =$$

$$= \frac{R^2-r^2}{2\pi}\cdot\frac{1}{R^2+r^2}\int_{-\theta}^{\pi-\theta} \frac{1}{1-\frac{2Rr}{R^2+r^2}\cos u}\,du.$$

SI $a = \frac{2Rr}{R^2+r^2} \in (0,1)$ YA QUE $(R-r)^2 = R^2+r^2-2Rr > 0$

$$\int \frac{1}{1-a\cos u}\,du \underset{x = \tg u/2}{=} \int \frac{1}{1-a\frac{(1-x^2)}{1+x^2}}\cdot\frac{2}{1+x^2}\,dx = \int \frac{2}{(1+x^2)-a+ax^2}\,dx$$

$2\operatorname{Arctg} x = u$

$du = \frac{2}{1+x^2}\,dx$

$\cos u = \frac{1-x^2}{1+x^2}$

$$= \int \frac{2}{(1-a)+x^2(1+a)}\,dx =$$

$$= \frac{2}{1-a}\int \frac{1}{1+\left(\sqrt{\frac{1+a}{1-a}}\,x\right)^2} = \sqrt{\frac{1-a}{1+a}}\,\frac{2}{1-a}\operatorname{Arctg}\sqrt{\frac{1+a}{1-a}}\,x =$$

$$= \sqrt{\frac{1-a}{1+a}}\,\frac{2}{1-a}\operatorname{Arctg}\sqrt{\frac{1+a}{1-a}}\,\tg\left(\frac{u}{2}\right).$$

ASI
$$u(r,\theta) = \frac{R^2-r^2}{2\pi}\,\frac{1}{R^2+r^2}\left[\sqrt{\frac{1-a}{1+a}}\,\frac{2}{1-a}\operatorname{Arctg}\left(\sqrt{\frac{1+a}{1-a}}\,\tg\frac{u}{2}\right)\Big|_{-\theta}^{\pi-\theta}\right]$$

# HOJA 4ª

## PROBLEMA 6:)

**LEMA** SI $x = r\cos\theta$
$y = r\,\mathrm{sen}\,\theta$

$$\Delta u(x,y) = 0 \iff r^2 u_{rr}(r,\theta) + r u_r(r,\theta) + u_{\theta,\theta}(r,\theta) = 0$$

**DEM** SEA $u(x,y)$ Y SEA

$$g(r,\theta) = u(r\cos\theta, r\,\mathrm{sen}\,\theta).$$

$$\frac{\partial g}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\mathrm{sen}\,\theta$$

$$\frac{\partial^2 g}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y\,\partial x}\cos x\,\mathrm{sen}\,x + \frac{\partial^2 u}{\partial x\,\partial y}\cos\theta\,\mathrm{sen}\,\theta + \frac{\partial^2 u}{\partial y^2}\mathrm{sen}^2 x =$$

$$= \frac{\partial^2 u}{\partial x^2}\cos^2 x + \frac{\partial^2 u}{\partial y^2}\mathrm{sen}^2 x + 2\frac{\partial^2 u}{\partial x\,\partial y}\cos\theta\,\mathrm{sen}\,\theta$$

$$\frac{\partial g}{\partial \theta} = \frac{\partial u}{\partial x}(-r\,\mathrm{sen}\,\theta) + \frac{\partial u}{\partial y} r\cos\theta$$

$$\frac{\partial^2 g}{\partial^2 \theta} = \left[\frac{\partial^2 u}{\partial x^2}(-r\,\mathrm{sen}\,\theta) + \frac{\partial^2 u}{\partial y\,\partial x} r\cos\theta\right](-r\,\mathrm{sen}\,\theta) + \frac{\partial u}{\partial x}(-r\cos\theta)$$

$$+\left[\frac{\partial^2 u}{\partial x\,\partial y}(-r\,\mathrm{sen}\,\theta) + \frac{\partial^2 u}{\partial y^2} r\cos\theta\right] r\cos\theta + \frac{\partial u}{\partial y}(-r\,\mathrm{sen}\,\theta)$$

$$= \frac{\partial^2 u}{\partial x^2} r^2\,\mathrm{sen}^2\theta + \frac{\partial^2 u}{\partial y^2} r^2\cos^2\theta - 2\frac{\partial^2 u}{\partial x\,\partial y} r^2\,\mathrm{sen}\,\theta\cos\theta$$

$$+ \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\,\mathrm{sen}\,\theta$$

AHORA.

$$[1] = r^2\frac{\partial^2 g}{\partial r^2}(r,\theta) + r\frac{\partial g}{\partial r}(r,\theta) + \frac{\partial^2 g}{\partial \theta^2}(r,\theta) =$$

$$= r^2\left[\frac{\partial^2 u}{\partial x^2}\cos^2 x + \frac{\partial^2 u}{\partial y^2}\mathrm{sen}^2 x + \cancel{2\frac{\partial^2 u}{\partial x\,\partial y}\cos\theta\,\mathrm{sen}\,\theta}^{\textcircled{1}}\right] +$$

$$+ r\left\{\cancel{\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\mathrm{sen}\,\theta}^{\textcircled{2}}\right]$$

$$+\left\{\frac{\partial^2 u}{\partial x^2} r^2\,\mathrm{sen}^2\theta + \frac{\partial^2 u}{\partial y^2} r^2\cos^2\theta \cancel{- 2\frac{\partial^2 u}{\partial x\,\partial y} r^2\,\mathrm{sen}\,\theta\cos\theta}^{\textcircled{1}}\right.$$

$$\left. + \cancel{\frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\,\mathrm{sen}\,\theta)}^{\textcircled{2}}\right] =$$

$$= r^2\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = [2].$$

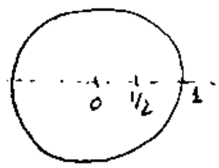
ASI SI $[1] = 0 \implies [2] = 0$

SI $[2] = 0 \implies [1] = 0$ CQD

# HOJA 4:

## PROBLEMA 7º)

a)




SEGÚN LAS DESIGUALDADES DE HARNACK

$$\frac{1-1/2}{1+1/2}\, u(0) \leq u(1/2) \leq \frac{1+1/2}{1-1/2}\, u(0)$$

$$\Longleftrightarrow \quad \frac{1/2}{3/2} \leq u(1/2) \leq \frac{3/2}{1/2} \quad \Longleftrightarrow \quad 1/3 \leq u(1/2) \leq 3.$$

b) ES UNA SIMPLE APLICACIÓN DE LO ANTERIOR.

SEAN $R, r > 0$ CON $R > r$

$$\frac{R-r}{R+r}\, u(0) \leq u(z) \leq \frac{R+r}{R-r}\, u(0) \quad \forall z \in \overline{D(0,r)}$$

$$\downarrow R\to\infty \qquad\qquad \downarrow R\to\infty$$

$$u(0) \leq u(z) \leq u(0).$$