

EXTENSIONS OF CONVEX FUNCTIONS WITH PRESCRIBED SUBDIFFERENTIALS

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ABSTRACT. Let E be an arbitrary subset of a Banach space X , $f : E \rightarrow \mathbb{R}$ be a function, and $G : E \rightrightarrows X^*$ be a set-valued mapping. We give necessary and sufficient conditions on f, G for the existence of a continuous convex extension $F : X \rightarrow \mathbb{R}$ of f such that the subdifferential ∂F of F coincides with G on E .

1. INTRODUCTION AND MAIN RESULTS

In this paper we are concerned with the following nonsmooth extension problem.

Let X be a Banach space and E be an arbitrary subset of X . If $f : E \rightarrow \mathbb{R}$ is a function and $G : E \rightrightarrows X^$ is a set-valued mapping, what necessary and sufficient conditions on f, G will guarantee the existence of a continuous convex extension F of f to all of X such that $\partial F(x) = G(x)$ for every $x \in E$?*

As a consequence of the results of K. Schulz and B. Schwartz in [14], a partial answer to this question was given in the finite-dimensional setting; more precisely, if E is convex and G is the subdifferential mapping of f on E , then there exists a finite convex extension F of f to all of \mathbb{R}^n with $G(x) \subset \partial F(x)$ for every $x \in E$ if and only if, for every pair of sequences $(y_k)_k \subset E$, $(y_k^*)_k$ with $y_k^* \in G(y_k)$ and $\lim_k |y_k^*| = +\infty$, one has that $\lim_k \frac{\langle y_k, y_k^* \rangle - f(y_k)}{|y_k^*|} = +\infty$ too. Moreover, whenever this condition is satisfied, the function

$$F(x) = \sup\{f(y) + \langle y^*, x - y \rangle : y \in E, y^* \in G(y)\}, \quad x \in \mathbb{R}^n;$$

defines such an extension. Nevertheless the subdifferential $\partial F(x)$ of F at points $x \in E$ does not necessarily coincide with $G(x)$.

Very recently, a related problem has been solved for convex functions of the classes $C^1(\mathbb{R}^n)$ and $C^{1,\omega}(X)$ (for a Hilbert space X) in the situation where the mapping G is single-valued and one additionally requires that the extension F be of class $C^1(\mathbb{R}^n)$ (which amounts to asking that $\partial F(x)$ be a singleton for every $x \in \mathbb{R}^n$) or of class $C^{1,\omega}(X)$; see [1, 2, 3]. A solution to a similar problem for general (not necessarily convex) functions was given in [11, Theorem 5], characterizing the pairs $f : E \rightarrow \mathbb{R}$, $G : E \rightrightarrows \mathbb{R}^n$ with f continuous and G upper semicontinuous and nonempty, compact and convex-valued which admit a (generally nonconvex) extension F of f whose Fréchet subdifferential is upper semicontinuous on \mathbb{R}^n and extends G from E .

Let us also mention the results by B. Mulansky and M. Neamtu [12] which prove that any finite subset of data on \mathbb{R} or \mathbb{R}^2 which is *strictly convex* in an appropriate sense can be interpolated by a convex polynomial and the work by O. Bucicovschi and J. Lebl [7] which study the continuity and regularity of extensions of functions defined on compact subset K of \mathbb{R}^n to the convex hull of K .

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In the infinite-dimensional setting, several characterizations of those pairs of Banach spaces $Y \subset X$ for which every continuous convex function defined on Y admits a continuous convex extension to the superspace X have been established by J. Borwein, S. Fitzpatrick, V. Montesinos and J. Vanderwerff in [4, 5] and by C. A. De Bernardi and L. Veselý in [8, 9, 10]. A consequence of these results is that, if X is a normed space and Y is a closed subspace of X such that X/Y is separable, then every continuous convex function on Y can be extended to a continuous convex function on X . On the other hand, if $X = \ell_\infty$ and $Y = c_0$ or ℓ_p with $1 < p < \infty$, there are examples of continuous convex functions on Y which have no continuous convex extensions to all of X . Also [8, Theorem 3.1] asserts that if A is an open convex subset of a topological vector space X , Y is a subspace of X and $f : A \cap Y \rightarrow \mathbb{R}$ is a continuous convex function, then f admits a continuous convex extension $F : A \rightarrow \mathbb{R}$ if and only if $A = \bigcup_n A_n$ for some increasing sequence $\{A_n\}_n$ of open convex subsets of A such that f is bounded on each $A_n \cap Y$. See also the paper [15] by L. Veselý and L. Zajíček for results on the extension of delta-convex functions.

The following theorem is one of the main results of this paper and provides a complete answer to the problem in finite-dimensional spaces.

Theorem 1.1. *Let $E \subset \mathbb{R}^n$ be arbitrary, $f : E \rightarrow \mathbb{R}$ be a function and $G : E \rightrightarrows \mathbb{R}^n$ be a set-valued mapping such that $G(x)$ is compact, convex and nonempty for every $x \in E$. There exists a convex function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $F(x) = f(x)$ and $\partial F(x) = G(x)$ for every $x \in E$ if and only if the following conditions are satisfied.*

(C) : $f(x) \geq f(y) + \langle y^*, x - y \rangle$ for every $x, y \in E$, $y^* \in G(y)$.

(EX) : If $(y_k)_k \subset E$, $(y_k^*)_k \subset \mathbb{R}^n$ are such that $y_k^* \in G(y_k)$ for every k and $\lim_k |y_k^*| = +\infty$, then

$$\lim_k \frac{\langle y_k, y_k^* \rangle - f(y_k)}{|y_k^*|} = +\infty.$$

(SCS) : If $x \in E$ and $(y_k)_k \subset E$, $(y_k^*)_k \subset \mathbb{R}^n$ are such that $y_k^* \in G(y_k)$ for every k , $(y_k^*)_k$ is bounded, and

$$\lim_k (f(x) - f(y_k) - \langle y_k^*, x - y_k \rangle) = 0,$$

then there exists a sequence $(x_k^*)_k \subset G(x)$ such that $\lim_k |x_k^* - y_k^*| = 0$.

As we mentioned above, condition (EX) was considered in [14] in order to ensure the existence of convex extensions F of f such that $G \subseteq \partial F$ on E . It is condition (SCS) which allows us to obtain the exact identity $\partial F = G$ on E . If E is compact, then condition (EX) trivially holds and, assuming that G is upper semicontinuous (which we can fairly do since the subdifferential of a continuous convex function on \mathbb{R}^n is upper semicontinuous), condition (SCS) can be simplified by replacing sequences with points. Recall that if Y, Z are two topological spaces, a set-valued mapping $G : Y \rightrightarrows Z$ is said to be upper semicontinuous on Y provided that for every $y \in Y$ and every open set W in Z containing $G(y)$, there exists a neighbourhood V of y such that $G(V) \subset W$. For any undefined terms in Convex Analysis that we may use in this paper, we refer to the books [6, 13, 16].

Theorem 1.2. *Let $E \subset \mathbb{R}^n$ be compact, $f : E \rightarrow \mathbb{R}$ be a function and $G : E \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous set-valued mapping such that $G(x)$ is compact, convex and nonempty for every $x \in E$. There exists a convex function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $F(x) = f(x)$ and $\partial F(x) = G(x)$ for every $x \in E$ if and only if the following conditions are satisfied.*

(C) : $f(x) \geq f(y) + \langle y^*, x - y \rangle$ for every $x, y \in E$, $y^* \in G(y)$.

(PCS) : If $x \in E$ and $y \in E$, $y^* \in G(y)$, then

$$f(x) = f(y) + \langle y^*, x - y \rangle \implies y^* \in G(x).$$

However, in the case that E is unbounded (and even if E is closed), condition (PCS) cannot replace (SCS) , as the following example shows.

Example 1.3. Let $g(x, y) = |x| + \theta(x)$, $(x, y) \in \mathbb{R}^2$, where $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is a symmetric convex function with $\theta(t) = t^2$ for $t \in [-1, 1]$, and θ affine on $[1, +\infty)$. Let $E = \{(0, 0)\} \cup \{(x, y) \in \mathbb{R}^2 : |x| \geq \min\{1, e^y\}\}$, and define f and G on E by

$$f = g \quad \text{on } E,$$

$$G(x, y) = \begin{cases} \{\nabla g(x, y)\} & \text{if } (x, y) \in E \setminus \{(0, 0)\} \\ \{(0, 0)\} & \text{if } (x, y) = (0, 0). \end{cases}$$

Then $G(E)$ is bounded, G is upper semicontinuous and it can be checked that (f, G) satisfies conditions (C) and (PCS) on E , but there is no convex function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $F = f$ and $\partial F = G$ on E , because for every convex function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\varphi = f$ on E we must have $\varphi(x, y) = |x| + \theta(x)$ on \mathbb{R}^2 , and in particular $\partial\varphi(0, 0) = [-1, 1] \times \{0\}$. As a matter of fact, for every pair of convex functions $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\eta : \mathbb{R} \rightarrow \mathbb{R}$, we have that if $\psi(x, y) = \eta(x)$ for all $(x, y) \in E$ then $\psi(x, y) = \eta(x)$ for all $(x, y) \in \mathbb{R}^2$. This is an easy consequence of [3, Theorem 1.11], but next we also provide a direct proof of this assertion for the reader's convenience. We first claim that for every $(x_0, y_0) \in \mathbb{R}^2$ and every $(a, b) \in \partial\psi(x_0, y_0)$ we have $b = 0$. Indeed, by convexity we have

$$\psi(x, y) \geq \psi(x_0, y_0) + a(x - x_0) + b(y - y_0)$$

for all $(x, y) \in \mathbb{R}^2$. Taking (x, y) of the form $(x(t), y(t)) = (2, t)$, $t \in \mathbb{R}$, and noting that $(2, t) \in E$ for all $t \in \mathbb{R}$, we get

$$\eta(2) = \psi(2, t) \geq \psi(x_0, y_0) + a(2 - x_0) + b(t - y_0)$$

for all $t \in \mathbb{R}$, which is impossible unless $b = 0$. Thus we have that $\partial\psi(x, y) \subset \mathbb{R} \times \{0\}$ for all $(x, y) \in \mathbb{R}^2$, which implies that, for each $x \in \mathbb{R}$, the function $\mathbb{R} \ni y \mapsto \psi(x, y) \in \mathbb{R}$ does not depend on y . Since for every $(x, y) \in \mathbb{R}^2$ there exists some y_0 with $(x, y_0) \in E$ we deduce that $\psi(x, y) = \psi(x, y_0) = \eta(x)$. Thus $\psi(x, y) = \eta(x)$ for all $(x, y) \in \mathbb{R}^2$.

However, note that if we set $G(0, 0) = [-1, 1] \times \{0\}$ (instead of just $\{(0, 0)\}$) then the set-valued jet (f, G) does satisfy all the hypotheses of Theorem 1.1, and therefore there exists a convex function $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $(\psi, \partial\psi) = (f, G)$ on E (of course this function must be $\psi(x, y) = |x| + \theta(x)$).

Theorem 1.1 actually is a corollary of the following more general result for functions that are bounded on bounded subsets of a separable Banach space.

Theorem 1.4. *Let E be an arbitrary subset of a separable Banach space X , let $f : E \rightarrow \mathbb{R}$ be a function and let $G : E \rightrightarrows X^*$ be a set-valued mapping such that $G(x)$ is a nonempty convex w^* -compact subset of X^* for every $x \in E$. There exists a convex function $F : X \rightarrow \mathbb{R}$, bounded on bounded subsets and such that $F(x) = f(x)$ and $\partial F(x) = G(x)$ for every $x \in E$, if and only if the following conditions are satisfied.*

(C) : $f(x) \geq f(y) + y^*(x - y)$ for every $x, y \in E$, $y^* \in G(y)$.

(EX) : If $(y_k)_k \subset E$, $(y_k^*)_k \subset X^*$ are such that $y_k^* \in G(y_k)$ for every k and $\lim_k \|y_k^*\|_* = +\infty$, then

$$\lim_k \frac{y_k^*(y_k) - f(y_k)}{\|y_k^*\|_*} = +\infty.$$

(SCS) : If $x \in E$ and $(y_k)_k \subset E$, $(y_k^*)_k \subset X^*$ are such that $y_k^* \in G(y_k)$ for every k and $(y_k^*)_k$ is bounded, then

$$\lim_k (f(x) - f(y_k) - y_k^*(x - y_k)) = 0 \implies \exists (x_k^*)_k \subset G(x) \quad \text{such that} \quad w^*\text{-}\lim_k (y_k^* - x_k^*) = 0.$$

Without any restriction on the Banach space X , but still focusing on the special class of convex functions which are bounded on bounded subsets of X , we have the following result.

Theorem 1.5. *Let X be a Banach space, let $E \subset X$ be arbitrary, $f : E \rightarrow \mathbb{R}$ be a function and $G : E \rightrightarrows X^*$ be a set-valued mapping such that $G(x)$ is a nonempty convex w^* -compact subset of X^* for every $x \in E$. There exists a convex function $F : X \rightarrow \mathbb{R}$ that is bounded on bounded subsets and such that $F(x) = f(x)$ and $\partial F(x) = G(x)$ for every $x \in E$ if and only if the following conditions are satisfied.*

(C) : $f(x) \geq f(y) + y^*(x - y)$ for every $x, y \in E$, $y^* \in G(y)$.

(EX) : If $(y_k)_k \subset E$, $(y_k^*)_k \subset X^*$ are such that $y_k^* \in G(y_k)$ for every k and $\lim_k \|y_k^*\|_* = +\infty$, then

$$\lim_k \frac{y_k^*(y_k) - f(y_k)}{\|y_k^*\|_*} = +\infty.$$

(CS) : If $x \in E$ and $(y_k)_k \subset E$, $(y_k^*)_k \subset X^*$ are such that $y_k^* \in G(y_k)$ for every k and $(y_k^*)_k$ is bounded, then

$$\lim_k (f(x) - f(y_k) - y_k^*(x - y_k)) = 0 \implies G(x) \cap \overline{\{y_k^*\}_k}^{w^*} \neq \emptyset.$$

Finally, a complete solution to our problem for functions that are not necessarily bounded on bounded sets is given in the following theorem.

Theorem 1.6. *Let X be a Banach space, let $E \subset X$ be arbitrary, $f : E \rightarrow \mathbb{R}$ be a function and $G : E \rightrightarrows X^*$ be a set-valued mapping such that $G(x)$ is a nonempty convex w^* -compact subset of X^* for every $x \in E$. There exists a continuous convex function $F : X \rightarrow \mathbb{R}$ such that $F(x) = f(x)$ and $\partial F(x) = G(x)$ for every $x \in E$ if and only if the following conditions are satisfied.*

(C) : $f(x) \geq f(y) + y^*(x - y)$ for every $x, y \in E$, $y^* \in G(y)$.

(GEX) : If $(y_k)_k \subset E$, $(y_k^*)_k \subset X^*$ are such that $y_k^* \in G(y_k)$ for every k and $\lim_k \|y_k^*\|_* = +\infty$, then

$$\lim_k (y_k^*(y_k - x) - f(y_k)) = +\infty \quad \text{for every } x \in X.$$

(CS) : If $x \in E$ and $(y_k)_k \subset E$, $(y_k^*)_k \subset X^*$ are such that $y_k^* \in G(y_k)$ for every k and $(y_k^*)_k$ is bounded, then

$$\lim_k (f(x) - f(y_k) - y_k^*(x - y_k)) = 0 \implies G(x) \cap \overline{\{y_k^*\}_k}^{w^*} \neq \emptyset.$$

In fact, we will see that in the preceding theorems, an extension F is given by the formula

$$F(x) = \sup\{f(y) + y^*(x - y) : y \in E, y^* \in G(y)\}, \quad x \in X.$$

This extension has the property that, for every continuous convex function H on X such that $H = f$ and $\partial H \supset G$ on E , we have $F \leq H$ on X . Therefore F is the minimal continuous convex extension of the datum $(f, G) : E \rightarrow \mathbb{R} \times 2^{X^*}$.

In order to help the reader get acquainted more quickly with the conditions of the above theorems, let us say a few words as to why we chose to label them (C), (EX), etc. Condition (C) refers to *convexity* of the set-valued 1-jet, meaning that all the function data lie above each of the putative supporting hyperplanes arising from the subdifferential data. Condition (EX) and (GEX) are related to the *existence* of at least one continuous convex function $F : X \rightarrow \mathbb{R}$ satisfying $F(x) = f(x)$ and $G(x) \subseteq \partial F(x)$ for all $x \in E$. In finite dimensions we only need the existence condition (EX), but in the infinite-dimensional setting a *generalized existence condition* such as (GEX) becomes necessary. These conditions are not sufficient to ensure the equality $G(x) = \partial F(x)$ for all $x \in E$, and that is why we need to introduce other conditions like (CS), (SCS), (PCS), which refer to how *convexity* forces *subdifferentials* to behave, both asymptotically and pointwise. For instance, (PCS) tells us that in order that a convex extension F with prescribed subdifferential $\partial F = G$ exists, the given data must

satisfy the property that if a point $(x, f(x))$ of the graph of f is touched by a putative supporting hyperplane at some other point y , then that hyperplane must also be one of the putative hyperplanes of F at x . When E is not compact or G is not upper semicontinuous this easy *pointwise* condition is no longer sufficient, and even in the finite-dimensional situation we must consider sequences instead of points, as in condition (SCS) , which tells us that if a sequence of putative hyperplanes at points (y_k) asymptotically touches a point $(x, f(x))$ in the graph of f then there is a corresponding sequence of putative supporting hyperplanes at x with a similar asymptotic behavior. In the nonseparable case the situation becomes even more complicated, and we need condition (CS) in place of the weaker condition (SCS) .

Let us finish this introduction by examining three variations of an example of Borwein, Montesinos and Vanderwerff [5, Example 4.2] in the light of Theorem 1.6. We wish to use these examples to illustrate the role of conditions (GEX) , (EX) and (CS) in an infinite-dimensional setting.

Example 1.7. (1) Let $E = B_{\ell^2}$ be the open unit ball of ℓ^2 . The function $f : E \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} x_n^{2n}, \quad x = (x_n)_{n \geq 1} \in E,$$

is convex, bounded and differentiable (in fact real-analytic). The same formula defines a real-analytic convex function on c_0 , so in particular f can be extended to a continuous convex function F on c_0 with $\partial F = \partial f$ on E , and hence $(f, \partial f)$ satisfies condition (GEX) in Theorem 1.6 with $X = c_0$. However f has no continuous convex extension from B_{ℓ^2} to all of ℓ^∞ . Indeed, let us assume that there exists a continuous convex function F on ℓ^∞ with $F = f$ on E . Then $(f, \partial F)$ must satisfy condition (GEX) of Theorem 1.6 on the set E with $X = \ell^\infty$. For every $k \in \mathbb{N}$, define $y_k := r_k e_k$, where r_k is such that $\frac{3}{4} = r_k^{2k-1}$, and let $y_k^* \in \partial F(y_k) \subset \ell^\infty$. Then $y_k \in E$ and, since $F = f$ on E , it is clear that the linear functionals y_k^* and $Df(y_k) = 2kr_k^{2k-1}e_k^*$ coincide on ℓ^2 . Moreover, by the density of ℓ^2 in c_0 and the continuity of y_k^*, e_k^* with respect to the norm $\|\cdot\|_\infty$, the linear functionals y_k^* and $2kr_k^{2k-1}e_k^*$ must also agree on c_0 . That is, we have

$$(1.1) \quad y_k^* = 2kr_k^{2k-1}e_k^* = \frac{3}{2}ke_k^* \quad \text{on } c_0, \quad \text{for every } k \in \mathbb{N}.$$

Because F is continuous at 0, there exists $\delta > 0$ such that $F(x) \leq \frac{3}{4}$ whenever $\|x\|_\infty \leq \delta$. Then

$$\frac{3}{4} \geq F(x) \geq f(y_k) + y_k^*(x - y_k) = r_k^{2k} + y_k^*(x) - 2kr_k^{2k} \quad \text{whenever } \|x\|_\infty \leq \delta.$$

This implies that

$$\|y_k^*\|_* \leq \delta^{-1} \left(\frac{3}{4} + (2k-1)r_k^{2k} \right) = \frac{3}{4}\delta^{-1} (1 + (2k-1)r_k) \leq \frac{3k}{2\delta} \quad \text{for every } k \in \mathbb{N}.$$

The preceding inequality together with (1.1) yield

$$(1.2) \quad \frac{3}{2}k \leq \|y_k^*\|_* \leq \frac{3k}{2\delta} \quad \text{for every } k \in \mathbb{N}.$$

If for every $x \in \ell^\infty$, the sequence $(\frac{2}{3}k^{-1}y_k^*(x))_k$ converges to 0, then, using (1.1) and (1.2), we deduce that the mapping $\ell^\infty \ni x \mapsto (\frac{2}{3}k^{-1}y_k^*(x))_k$ defines a bounded linear projection onto c_0 , which is absurd since c_0 is not complemented in ℓ^∞ . Thus there exist $x \in \ell^\infty$ and a subsequence $(k_n)_n$ such that $\frac{2}{3}k_n^{-1}y_{k_n}^*(x) \geq 1$ for all n . Bearing in mind the first inequality in (1.2) we have

$$y_{k_n}^*(y_{k_n} - x) - f(y_{k_n}) \leq 2k_n r_{k_n}^{2k_n} - \frac{3}{2}k_n - r_{k_n}^{2k_n} = \frac{3}{2}k_n r_{k_n} - \frac{3}{2}k_n - \frac{3}{4}r_{k_n} \leq 0 \quad \text{for every } n.$$

This shows that condition (GEX) of Theorem 1.6 with $X = \ell^\infty$ fails for $(f, \partial F)$ on E , a contradiction. We conclude that there is no continuous convex extension of f to ℓ^∞ .

(2) If E and f are as in (1), there is no convex function $F : \ell^2 \rightarrow \mathbb{R}$ which is bounded on bounded subsets and such that $F = f$ on E . Indeed, let F be a continuous convex function $F : \ell^2 \rightarrow \mathbb{R}$ with $F = f$ on E . We learnt from (1) that F must satisfy $y_k^* = \frac{3}{2}ke_k^*$ on ℓ_2 for every $y_k^* \in \partial F(y_k)$ and every k . We have that $\lim_k \|y_k^*\|_* = \infty$ while

$$\lim_k \frac{y_k^*(y_k) - f(y_k)}{\|y_k^*\|_*} = \lim_k \frac{2kr_k^{2k} - r_k^{2k}}{2kr_k^{2k-1}} = \lim_k \frac{2k-1}{2k} r_k = 1.$$

This shows that condition (EX) of Theorem 1.5 is not fulfilled for $(F, \partial F)$ and therefore F is not bounded on bounded subsets.

(3) Let $r \in (0, 1)$ and set $E = rB_{\ell_2}$, $X = \ell^\infty$. Define $f : E \rightarrow \mathbb{R}$, $G : E \rightrightarrows X^*$ by

$$f(x) = \sum_{n=1}^{\infty} x_n^{2n}, \quad G(x) = \left\{ g(x) := \sum_{n=1}^{\infty} 2nx_n^{2n-1}e_n^* \right\}, \quad x = (x_n)_{n \geq 1} \in E.$$

Then f admits a Lipschitz convex extension F to all of X such that $\partial F = G$ on E . Indeed, an easy calculation shows that $\|g(x)\|_* \leq \sum_{n=1}^{\infty} r^{2n-1} = 2r(1-r^2)^{-2}$ for every $x \in E$. Besides, (f, G) satisfies condition (C) on E and thus the formula

$$F(x) = \sup_{y \in E} \{f(y) + g(y)(x - y)\}, \quad x \in X,$$

defines a Lipschitz convex function with $F = f$ on E and $g(x) \in \partial F(x)$ for every $x \in E$. In order to see that $\partial F = G$ on E , let us check that (f, G) satisfies condition (CS) of Theorem 1.5. Assume that $x = (x_n)_n \in E$, $(y_k = (y_n^k)_n)_k \subset E$, and $y_k^* = g(y_k)$ are such that

$$\lim_k \sum_{n=1}^{\infty} \left(x_n^{2n} - (y_n^k)^{2n} - 2n(y_n^k)^{2n-1}(x_n - y_n^k) \right) = \lim_k (f(x) - f(y_k) - g(y_k)(x - y_k)) = 0.$$

This implies that

$$\lim_k \left(\varphi_n(x_n) - \varphi_n(y_n^k) - \varphi_n'(y_n^k)(x_n - y_n^k) \right) = 0, \quad \text{where } \varphi_n(t) = t^{2n}, \quad t \in \mathbb{R}, n \in \mathbb{N}.$$

Since each $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ is strictly convex we must have $\lim_k y_n^k = x_n$ for every $n \in \mathbb{N}$. Now observe that

$$2n|(y_n^k)^{2n-1} - x_n^{2n-1}| \leq 4nr^{2n-1} \quad \text{for every } k \in \mathbb{N}, \quad \text{where } \sum_{n=1}^{\infty} 4nr^{2n-1} < \infty.$$

Thus, for every $u \in X$, the Dominated Convergence Theorem gives us

$$\lim_k |(y_k^* - g(x))(u)| \leq \lim_k \left(\sum_{n=1}^{\infty} 2n|(y_n^k)^{2n-1} - x_n^{2n-1}| \right) \|u\| = \|u\| \sum_{n=1}^{\infty} \lim_k 2n|(y_n^k)^{2n-1} - x_n^{2n-1}| = 0,$$

which shows that $g(x) \in \overline{\{g(y_k)\}_k}^{w^*}$. Thus condition (CS) of Theorem 1.5 is satisfied for (f, G) on E .

2. PROOFS OF THEOREMS 1.6, 1.5 AND 1.4

Throughout this section $(X, \|\cdot\|)$ will be a Banach space and $\|\cdot\|_*$ will denote the dual norm on X^* . We will start with the proof of the most general results of the paper, that is, Theorems 1.6 and 1.5. Then, a small observation (see Remark 2.6 below) will allow us to deduce Theorem 1.4 for separable spaces.

2.1. Theorems 1.6 and 1.5. Only if part. Let $F : X \rightarrow \mathbb{R}$ be a continuous convex function. We obviously have $F(x) \geq F(y) + y^*(x - y)$ for every $x, y \in X$, $y^* \in \partial F(y)$. Thus condition (C) is necessary in Theorems 1.6 and 1.5.

Let us now prove that (GEX) is a necessary condition in Theorem 1.6. Let $x \in X$ and let $(y_k)_k \subset X$, $(y_k^*)_k \subset X^*$ with $y_k^* \in \partial F(y_k)$ and $\lim_k \|y_k^*\|_* = +\infty$. Assume, for the sake of contradiction, that $\lim_k (y_k^*(y_k - x) - F(y_k)) \neq +\infty$. Then, after passing to a subsequence, we may find $M > \max\{F(x), 0\}$ such that

$$(2.1) \quad y_k^*(y_k - x) - F(y_k) \leq M \quad \text{for every } k.$$

Now we consider, for every k , a vector $v_k \in X$ with

$$(2.2) \quad \|v_k\| \leq 1 \quad \text{and} \quad y_k^*(v_k) \geq \frac{1}{2} \|y_k^*\|_*.$$

If we define $z_k = x + \frac{4M}{\|y_k^*\|_*} v_k$, then we get

$$0 \leq F(z_k) - F(y_k) + y_k^*(y_k - z_k) \leq F(z_k) + M - \frac{4M}{\|y_k^*\|_*} y_k^*(v_k) \leq F(z_k) - M,$$

where the first inequality follows from the convexity of F , the second one from (2.1), and the third one from (2.2). Since $(z_k)_k$ converges to x , the continuity of F at x gives that $M \leq \lim_k F(z_k) = F(x)$, contradicting the choice of M .

In order to see that (EX) is necessary in Theorem 1.5, assume further that F is bounded on bounded subsets. Suppose that $(y_k^*)_k$ is a sequence such that $\lim_k \|y_k^*\|_* = +\infty$ and $y_k^* \in \partial F(y_k)$ for every k but there exists $M > 0$ with

$$(2.3) \quad y_k^*(y_k) - F(y_k) \leq M \|y_k^*\|_* \quad \text{for every } k.$$

Consider, for every k , a vector $v_k \in X$ with

$$(2.4) \quad \|v_k\| \leq 1 \quad \text{and} \quad y_k^*(v_k) \geq \frac{1}{2} \|y_k^*\|_*.$$

If we define $z_k = 4M v_k$, we can write

$$\begin{aligned} 0 &\leq F(z_k) - F(y_k) - y_k^*(z_k - y_k) = F(z_k) + (y_k^*(y_k) - F(y_k)) - y_k^*(z_k) \\ &\leq F(z_k) + M \|y_k^*\|_* - 2M \|y_k^*\|_* = F(z_k) - M \|y_k^*\|_*, \end{aligned}$$

where the first inequality follows from the fact that $y_k^* \in \partial F(y_k)$, and the second one from (2.3) and (2.4). Since $(F(z_k))_k$ is a bounded sequence and $\lim_k \|y_k^*\|_* = \infty$, the last chain of inequalities yields a contradiction. This proves that $\lim_k \frac{y_k^*(y_k) - f(y_k)}{\|y_k^*\|_*} = +\infty$.

Finally, in order to show that (CS) is a necessary condition in Theorems 1.6 and 1.5, let us first prove the following fact.

Fact 2.1. Let $h : X \rightarrow \mathbb{R}$ be a continuous convex function and let $x, y \in X$ be two points such that $h(x) = h(y) + y^*(x - y)$ for some $y^* \in \partial h(y)$. Then $y^* \in \partial h(x)$.

Proof. Because $y^* \in \partial h(y)$, we can write, for every $z \in X$,

$$h(z) \geq h(y) + y^*(z - y) = h(y) + y^*(x - y) + y^*(z - x) = h(x) + y^*(z - x),$$

that is $y^* \in \partial h(x)$. □

Now assume, seeking a contradiction, that (CS) is not satisfied. Then we can find $x \in X$ and sequences $(y_k)_k, (y_k^*)_k$ such that $y_k^* \in \partial F(y_k)$, $(y_k^*)_k$ is bounded and

$$(2.5) \quad \lim_k (F(x) - F(y_k) - y_k^*(x - y_k)) = 0,$$

but $\partial F(x) \cap \overline{\{y_k^*\}_k}^{w^*} = \emptyset$. Since $(y_k^*)_k$ is bounded, we can find a subnet $(y_{k_\alpha}^*)_{\alpha \in D}$ of $(y_k^*)_k$ w^* -convergent to $\xi \in X^*$. Obviously, ξ does not belong to $\partial F(x)$ and, since $\partial F(x)$ is w^* -closed, we can apply the Hahn-Banach Theorem for (X^*, w^*) in order to find $v \in X$ with

$$(2.6) \quad \xi(v) > \sup_{x^* \in \partial F(x)} x^*(v).$$

For every $\alpha \in D$, the number $r_\alpha = F(x) - F(y_{k_\alpha}) - y_{k_\alpha}^*(x - y_{k_\alpha})$ is strictly positive (as otherwise, $y_{k_\alpha}^* \in \partial F(x)$ by Fact 2.1) and $\lim_\alpha r_\alpha = 0$ by (2.5). We now write

$$(2.7) \quad \begin{aligned} F(x + \sqrt{r_\alpha}v) - F(x) &\geq F(y_{k_\alpha}) + y_{k_\alpha}^*(x + \sqrt{r_\alpha}v - y_{k_\alpha}) - F(x) = -r_\alpha + y_{k_\alpha}^*(v)\sqrt{r_\alpha} \\ &= -r_\alpha + (y_{k_\alpha}^* - \xi)(v)\sqrt{r_\alpha} + \xi(v)\sqrt{r_\alpha}, \end{aligned}$$

for every $\alpha \in D$. Let us consider a net $(z_\alpha^*)_{\alpha \in D}$ such that $z_\alpha^* \in \partial F(x + \sqrt{r_\alpha}v)$ for each α . Since the net $\{x + \sqrt{r_\alpha}v\}_{\alpha \in D}$ strongly converges to x , for any $\varepsilon > 0$, the $\|\cdot\|$ - w^* -upper semicontinuity of ∂F (see [6, Proposition 6.1.1]) gives $\alpha_\varepsilon \in D$ such that for every $\alpha \in D$ with $\alpha_\varepsilon \leq \alpha$, we can find $x_{\varepsilon, \alpha}^* \in \partial F(x)$ with $|(z_\alpha^* - x_{\varepsilon, \alpha}^*)(v)| \leq \varepsilon$. Let $\varepsilon > 0$ and $\alpha \in D$ with $\alpha_\varepsilon \leq \alpha$. From the convexity of F and the fact that $z_\alpha^* \in \partial F(x + \sqrt{r_\alpha}v)$, it follows that

$$F(x + \sqrt{r_\alpha}v - \sqrt{r_\alpha}v) - F(x + \sqrt{r_\alpha}v) \geq z_\alpha^*(-\sqrt{r_\alpha}v),$$

and so

$$(2.8) \quad F(x + \sqrt{r_\alpha}v) - F(x) \leq z_\alpha^*(v)\sqrt{r_\alpha} \leq \left((z_\alpha^* - x_{\varepsilon, \alpha}^*)(v) + \sup_{x^* \in \partial F(x)} x^*(v) \right) \sqrt{r_\alpha} \leq \left(\varepsilon + \sup_{x^* \in \partial F(x)} x^*(v) \right) \sqrt{r_\alpha}$$

Dividing by $\sqrt{r_\alpha}$ in (2.7) and (2.8), we obtain

$$\varepsilon + \sup_{x^* \in \partial F(x)} x^*(v) \geq -\sqrt{r_\alpha} + (y_{k_\alpha}^* - \xi)(v) + \xi(v) \quad \text{for all } \alpha \in D, \alpha_\varepsilon \leq \alpha.$$

Because the net $\{-\sqrt{r_\alpha} + (y_{k_\alpha}^* - \xi)(v)\}_{\alpha \in D}$ converges to 0, we obtain from the preceding inequality that $\xi(v) \leq \varepsilon + \sup_{x^* \in \partial F(x)} x^*(v)$. Since $\varepsilon > 0$ is arbitrary, this contradicts (2.6).

2.2. Theorems 1.6 and 1.5. If part. Let us assume that (f, G) satisfies conditions (C) , (GEX) and (CS) of Theorem 1.6 and define

$$F(x) = \sup_{y \in E} \sup_{y^* \in G(y)} \{f(y) + y^*(x - y)\}, \quad x \in X.$$

Claim 2.2. F is finite everywhere in X .

Proof. Consider $x \in X$ and sequences $(y_k)_k \subset E$, $(y_k^*)_k$ with $y_k^* \in G(y_k)$ for every k , such that

$$(2.9) \quad F(x) = \lim_k (f(y_k) + y_k^*(x - y_k)).$$

If we take some $z_0 \in E$, condition (C) yields $f(z_0) \geq f(y_k) + y_k^*(z_0 - y_k)$, hence

$$f(y_k) + y_k^*(x - y_k) \leq f(z_0) + y_k^*(x - z_0).$$

This shows that $F(x)$ will be finite as soon as we prove that $(y_k^*)_k$ is bounded. Assume, for the sake of contradiction, that $(y_k^*)_k$ is unbounded. Then condition (GEX) tells us that, possibly after passing to a subsequence, $\lim_k (f(y_k) + y_k^*(x - y_k)) = -\infty$. This contradicts (2.9), since obviously $F(x) > -\infty$. \square

Claim 2.3. Assuming further that (f, G) satisfies condition (EX) of Theorem 1.5, F is bounded on bounded subsets of X .

Proof. Since condition (EX) is stronger than (GEX) of Theorem 1.6, we already know that F is finite everywhere in this case, by virtue of Claim 2.2. Let us fix $z_0 \in E$ and $z_0^* \in G(z_0)$. Assume, seeking a contradiction, that B is a bounded subset of X for which $F|_B$ is unbounded. Then we can find $(x_k)_k \subset B$ such that $\lim_k F(x_k) = +\infty$. By the definition of $F(x_k)$, we can find sequences $(y_k)_k \subset E$, $(y_k^*)_k$ with $y_k^* \in G(y_k)$ for every k , such that

$$(2.10) \quad F(x_k) \leq f(y_k) + y_k^*(x_k - y_k) + \frac{1}{k} \quad \text{for every } k.$$

Moreover, since $f(z_0) + z_0^*(x_k - z_0)$ is one of the expressions considered in the definition of $F(x_k)$, the sequences $(y_k)_k$, $(y_k^*)_k$ can be even selected so that

$$(2.11) \quad f(z_0) + z_0^*(x_k - z_0) \leq f(y_k) + y_k^*(x_k - y_k) \quad \text{for every } k.$$

Condition (C) implies that

$$f(y_k) + y_k^*(x_k - y_k) \leq f(z_0) + y_k^*(y_k - z_0) + y_k^*(x_k - y_k) = f(z_0) + y_k^*(x_k - z_0).$$

Since $(x_k)_k$ is bounded, the preceding inequality shows that $(y_k^*)_k$ must be unbounded, as otherwise (2.10) would give that $(F(x_k))_k$ is bounded, a contradiction. Passing to a subsequence we may assume that $\lim_k \|y_k^*\|_* = +\infty$, and then condition (EX) tells us that $\lim_k \frac{y_k^*(y_k) - f(y_k)}{\|y_k^*\|_*} = +\infty$. Using (2.11) we easily obtain

$$\frac{y_k^*(y_k) - f(y_k)}{\|y_k^*\|_*} \leq \frac{1}{\|y_k^*\|_*} y_k^*(x_k) - \frac{f(z_0) + z_0^*(x_k - z_0)}{\|y_k^*\|_*},$$

where the last term is bounded above. This yields a contradiction and shows that F is bounded on B . \square

Claim 2.4. F is continuous and convex on X , $F = f$ and $G \subset \partial F$ on E .

Proof. The function F , being the supremum of a family of lower semicontinuous convex functions, is convex and lower semicontinuous as well. Moreover, we learnt from Claim 2.2 that $\text{dom}(F) = X$. Since X is a Banach space, every lower semicontinuous convex function on X is continuous on $\text{int}(\text{dom}(F))$ (see for instance [6, Proposition 4.1.5, p. 129]), hence F is continuous on X .

The inequality $F \geq f$ on E is obvious by definition of F , and the converse inequality follows immediately from condition (C) . Finally, for every $x \in E, z \in X, x^* \in G(x)$, the definition of F and the equality $F = f$ on E give

$$F(z) \geq f(x) + x^*(z - x) = F(x) + x^*(z - x),$$

and then $x^* \in \partial F(x)$. \square

To conclude the proof of Theorems 1.6 and 1.5 it only remains to prove the following.

Claim 2.5. $\partial F = G$ on E .

Proof. Let $x \in E$ and suppose that there exists $\xi \in \partial F(x) \setminus G(x)$. Since $G(x)$ is w^* -closed and convex, the Hahn-Banach Theorem for (X^*, w^*) provides us with some $u \in X$ such that

$$(2.12) \quad \xi(u) > \sup_{x^* \in G(x)} x^*(u).$$

We now pick two sequences $(y_k)_k \subset E$, $(y_k^*)_k \subset X$ with $y_k^* \in G(y_k)$ such that

$$(2.13) \quad F(x + \frac{1}{k}u) \geq f(y_k) + y_k^*(x + \frac{1}{k}u - y_k) \geq F(x + \frac{1}{k}u) - \frac{1}{2k} \quad \text{for every } k.$$

The sequence $(y_k^*)_k$ must be bounded. Indeed, let us assume that $\lim_k \|y_k^*\|_* = +\infty$. By condition (GEX) we have that

$$(2.14) \quad \lim_k (f(y_k) + y_k^*(x + u - y_k)) = -\infty.$$

On the other hand, using the convexity of F in combination with (2.13) and taking some $x^* \in G(x)$, we have that

$$\begin{aligned} f(y_k) + y_k^*(x + \tfrac{1}{k}u - y_k) &\geq F(x + \tfrac{1}{k}u) - 2^{-k} \geq f(x) + x^*(x + \tfrac{1}{k}u - x) - 2^{-k} \\ &\geq f(y_k) + y_k^*(x - y_k) + x^*(\tfrac{1}{k}u) - 2^{-k}, \end{aligned}$$

which implies that

$$y_k^*(u) \geq x^*(u) - k2^{-k}$$

for every k . This shows that $(y_k^*(u))_k$ is bounded below and then, by virtue of (2.13) and (2.14), we obtain

$$F(x) = \lim_k (f(y_k) + y_k^*(x + \tfrac{1}{k}u - y_k)) = \lim_k (f(y_k) + y_k^*(x + u - y_k) - (1 - \tfrac{1}{k})y_k^*(u)) = -\infty,$$

which is absurd. Thus $(y_k^*)_k$ is a bounded sequence. Observe that the fact that $(y_k^*)_k$ is bounded together with (2.13) imply that

$$\lim_k (f(x) - f(y_k) - y_k^*(x - y_k)) = 0.$$

Hence, condition (CS) gives a point $x_0^* \in G(x) \cap \overline{\{y_k^*\}_k}^{w^*}$. Given $\varepsilon > 0$, we can find $k = k_\varepsilon \in \mathbb{N}$ such that $|(y_k^* - x_0^*)(u)| \leq \varepsilon$ and $k2^{-k} \leq \varepsilon$. Using that $\xi \in \partial F(x)$ and (2.13) we can write

$$\begin{aligned} \tfrac{1}{k}\xi(u) &\leq F(x + \tfrac{1}{k}u) - F(x) = F(x + \tfrac{1}{k}u) - f(x) \leq F(x + \tfrac{1}{k}u) - f(y_k) - y_k^*(x - y_k) \\ &\leq \tfrac{1}{k}y_k^*(u) + \tfrac{1}{2^k} = \tfrac{1}{k}(y_k^* - x_0^*)(u) + \tfrac{1}{k}x_0^*(u) + \tfrac{1}{2^k} \leq \tfrac{1}{k}(y_k^* - x_0^*)(u) + \tfrac{1}{k} \sup_{x^* \in G(x)} x^*(u) + \tfrac{1}{2^k}. \end{aligned}$$

We thus have that

$$\xi(u) \leq (y_k^* - x_0^*)(u) + \sup_{x^* \in G(x)} x^*(u) + \tfrac{k}{2^k} \leq 2\varepsilon + \sup_{x^* \in G(x)} x^*(u),$$

and letting $\varepsilon \rightarrow 0^+$ we obtain $\xi(u) \leq \sup_{x^* \in G(x)} x^*(u)$, which contradicts (2.12). \square

2.3. Proof of Theorem 1.4. It is enough to apply Theorem 1.5 in combination with the following remark.

Remark 2.6. If X is separable, condition (CS) in Theorems 1.5 and 1.6 is equivalent to condition (SCS) in Theorem 1.4.

Proof. Assume that (f, G) satisfies condition (CS) on a subset E and consider $x \in E$ and $(y_k)_k \subset E$, $(y_k^*)_k \subset X^*$ such that $y_k^* \in G(y_k)$ for every k , $(y_k^*)_k$ is bounded and

$$\lim_k (f(x) - f(y_k) - y_k^*(x - y_k)) = 0.$$

Since X is separable, the bounded subset $G(x) \cup \overline{\{y_k^*\}_k}^{w^*}$ of (X^*, w^*) is metrizable; let us denote a suitable distance by d . By the w^* -compactness of $G(x)$, we can find a sequence $(x_k^*)_k \subset G(x)$ such that

$$(2.15) \quad d(y_k^*, x_k^*) = \text{dist}(y_k^*, G(x)) := \inf\{d(y_k^*, x^*) : x^* \in G(x)\} \quad \text{for every } k.$$

Assume, for the sake of contradiction, that $d(y_k^*, x_k^*)$ does not tend to 0. Then we can find a subsequence $(k_j)_j$, a positive ε and $\xi \in X^*$ such that $(y_{k_j}^*)_j$ w^* -converges to ξ and $d(y_{k_j}^*, x_{k_j}^*) \geq \varepsilon$ for every j . Then

condition (CS) says that $G(x) \cap \{\xi, y_{k_j}^*\}_j \neq \emptyset$, which, because $\text{dist}(y_{k_j}^*, G(x)) \geq \varepsilon$, implies $\xi \in G(x)$, hence $d(y_{k_j}^*, \xi) \geq \varepsilon$ for every j by (2.15). This contradicts the fact that $w^*\text{-lim}_j y_{k_j}^* = \xi$ and therefore $w^*\text{-lim}(y_k^* - x_k^*) = 0$. \square

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