REAL ANALYTIC APPROXIMATION OF LIPSCHITZ FUNCTIONS ON HILBERT SPACE AND OTHER BANACH SPACES

D. AZAGRA, R. FRY, AND L. KEENER

ABSTRACT. Let X be a separable Banach space with a separating polynomial. We show that there exists $C \geq 1$ such that for every Lipschitz function $f : X \to \mathbb{R}$, and every $\varepsilon > 0$, there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon$ and $\mathrm{Lip}(g) \leq C\mathrm{Lip}(f)$. This result is new even in the case when X is a Hilbert space. Furthermore, the proof of this results works for all Banach spaces X having a separating function Q (meaning a function $Q : X \to [0, +\infty)$ such that $Q(0) = 0$ and $Q(x) \geq \|x\|$ for $\|x\| \geq 1$) which has a Lipschitz, holomorphic extension $\widetilde{Q}$ to a uniformly wide neighborhood $\widetilde{V} = \{x + iy : x \in X, \|y\| < \delta\}$ of X in the complexification $\widetilde{X}$.

1. Introduction and main results

Not much is known about the natural question of approximating functions by real analytic functions on a real Banach space X. When X is finite dimensional, a famous paper of Whitney's [W] provides a completely satisfactory answer to this problem: a combination of integral convolutions with Gaussian kernels and real analytic approximations of partitions of unity allows to show that for every C^k function $f : \mathbb{R}^n \to \mathbb{R}^m$ and every continuous $\varepsilon : \mathbb{R}^n \to (0, +\infty)$ there exists a real analytic function g such that $\|D^j g(x) - D^j f(x)\| \leq \varepsilon(x)$ for all $x \in \mathbb{R}^n$ and $j = 1, ..., k$.

In an infinite dimensional Banach space X, the lack of a translation invariant measure which assigns finite, strictly positive volume to balls makes this question much more difficult to answer, as integral convolutions cannot be directly used in order to regularize a given function. By constructing a real analytic approximation of a partition of unity, Kurzweil was able to show in his classic paper [K] that for every Banach space X having a separating polynomial, for every Banach space Y, and for every continuous function $f : X \to Y$, $\varepsilon : X \to (0, +\infty)$, there exists a real analytic function $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ for all $x \in X$.

1991 *Mathematics Subject Classification.* Primary 46B20.

Key words and phrases. Real analytic, approximation, Lipschitz function, Banach space. The first named author is partly supported by grant Santander-Complutense 34/07-15844. The second named author is partly supported by NSERC (Canada).

In the special case of norms (or more generally convex funtions) defined on a Banach space X, Deville, Fonf and Hajek [DFH1, DFH1], by introducing specific formulae suited to this problem, have proved two important results: first, in l_p or L_p, with p an even integer, any equivalent norm can be uniformly approximated on bounded sets by analytic norms. Second, in $X = c_0$ or $X = C(K)$, with K a countable compact, any equivalent norm can be uniformly approximated by analytic norms on $X \setminus \{0\}$.

These results leave at least three important questions open. Question one, is Kurzweil's result improvable? Is it necessary for a Banach space X to have a separating polynomial in order to enjoy the property that every continuous function defined on X can be uniformly approximated by real analytic functions? Fry and, independently, Cepedello and Hajek have proved that every uniformly continuous function defined on c_0 (which fails to have a separating polynomial) [F1] and, more generally, a Banach space with a real analytic separating function [CH], can be uniformly approximated by real analytic functions. However, the approximating functions they construct (again, refining Kurzweil's technique, by employing a real analytic approximation of a partition of unity) are not uniformly continuous, and in any case the problem of approximating continuous functions (not uniformly continuous) defined on c_0 by real analytic functions remains open.

Question two. Given a C^k function f defined on a Banach space X with a separating polynomial, and a continuous $\varepsilon : X \to (0, +\infty)$, is it possible to find a real analytic function g on X such that $\|D^j f - D^j g\| \leq \varepsilon$ for $j = 1, ..., k$? Nothing is known about this problem, even in the case $k = 1$ and X being a Hilbert space. It is worth mentioning that if we replace real analytic with C^∞ and consider only $k = 1$, then the question has an affirmative answer, which was found by Moulis [M] in the cases $X = \ell_p, c_0$ and with range any Banach space (although her proof can be adapted to every Banach space with an unconditional basis and a C^∞ smooth bump function, see [AFGJL]), and by Hajek and Johanis [HJ] very recently for certain range spaces in the far more general case of a separable Banach space X with a C^∞ smooth bump function. And, interestingly enough, in both cases the solution came as a corollary of a theorem on approximation of Lipschitz functions by more regular Lipschitz functions. This leads us to the following natural question, which is the subject matter of the present paper.

Question three. Is it possible to approximate a Lipschitz function f defined on a Banach space X having a separating polynomial by Lipschitz, real analytic functions g? And is it possible to have the Lipschitz constants of the functions g be of the order of the Lipschitz constant of f (that is $\text{Lip}(g) \leq C\text{Lip}(g)$, where $C \geq 1$ is independent of f)? It is worth noting that every approximation method based on constructing real analytic approximations of partitions of unity in infinite dimensions (such as [K, F1, CH, J]) cannot work to give a solution to this problem, the reason being that in order to make the supports of the functions forming the partition locally finite one has to give up the idea of those functions having a common Lipschitz constant. That is, even if we consider C^p smooth functions instead of real analytic functions in this question, the standard approximation technique based on the use of C^p smooth partitions of unity does not provide a solution to the problem.

A partial answer to this problem can be obtained by combining the Deville-Fonf-Hajek results on real analytic approximation of convex functions with the following theorem of M. Cepedello-Boiso [Ce]: a Banach space is superreflexive if and only if every Lipschitz function can be approximated by differences of convex functions which are bounded on bounded sets, uniformly on bounded sets. It follows that, for $X = \ell_p$ or L_p, with p an even integer, and for every Lipschiz function $f : X \to \mathbb{R}$ there exists a sequence of real analytic functions $g_n : X \to \mathbb{R}$ which are Lipschitz on bounded sets, and such that $\lim_{n\to\infty} g_n = f$, uniformly on bounded sets. However, this method has two important disadvantages: 1) the approximation cannot be made to be uniform on X, and 2) even on a fixed bounded set B, we lose control on the Lipschitz constant of the approximations g_n: in fact, one has $\mathrm{Lip}(g_{n|_B}) \to \infty$ as $n \to \infty$.

A successful new approach to the question of approximating Lipschitz functions on infinite-dimensional Banach spaces was discovered by Fry in [F2], with the introduction of what one can call *sup- partitions of unity* (see the following section for a definition), a tool which has been thoroughly exploited in [HJ].

Very recently, Fry and Keener [FK] have constructed a real analytic approximation of a sup partition of unity and have employed it to show that, for every Banach space X having a separating polynomial, for every bounded open subset U of X, for every bounded Lipschitz function $f : U \to \mathbb{R}$, and for every $\varepsilon > 0$ there exists a Lipschitz, real analytic function $g : U \to \mathbb{R}$ such that $|f - g| \leq \varepsilon$.

A disadvantage of the sup partition approach to approximation is that it only works for bounded functions. In this paper we will simplify the construction in [FK], and we will combine this with some new tools (in particular we will introduce what we call a *gluing tube function*, see the next section) in order to eliminate these restrictions and obtain the following.

Theorem 1. *Let X be a separable Banach space which admits a separating polynomial. Then there exists a number $C \geq 1$ such that for every Lipschitz function $f : X \to \mathbb{R}$, and for every $\varepsilon > 0$ there exists a real analytic function $g : X \to \mathbb{R}$ such that*

(1) $|f(x) - g(x)| < \varepsilon$ *for all* $x \in X$
(2) g *is Lipschitz, with* $Lip(g) \leq C Lip(f)$.

As a matter of fact the proof of this result works for any Banach space X having a separating function with a Lipschitz holomorphic extension to a uniformly wide neighborhood of X in the complexification $\widetilde{X}$.

Definition 1. *A separating function Q on a Banach space X is a function $Q : X \to [0, +\infty)$ such that $Q(0) = 0$ and there exists $M, m > 0$ such that $Q(x) \geq m\|x\|$ whenever $\|x\| \geq M$.*

It is clear that X has a real analytic Lipschitz separating function (with a holomorphic extension to a uniformly wide neighborhood of X in $\widetilde{X}$) if and only if X has a real analytic Lipschitz function Q which satisfies the above definition with $M = m = 1$ (and with a holomorphic extension to a uniformly wide neighborhood of X in $\widetilde{X}$). So we also have the following.

Theorem 2. *Let X be a separable Banach space having a separating function $Q : X \to [0, \infty)$ with a Lipschitz holomorphic extension $\widetilde{Q}$ defined on a set of the form $\{x + iy \in \widetilde{X} : x, y \in X, \|y\| < \delta\}$ for some $\delta > 0$. Then there exists a number $C \geq 1$ such that for every Lipschitz function $f : X \to \mathbb{R}$, and for every $\varepsilon > 0$ there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X$, and $Lip(g) \leq C Lip(f)$.*

We will prove (see Lemma 2 below) that every Banach space with a separating polynomial (that is a polynomial $P : X \to \mathbb{R}$ such that $P(0) = 0 < \inf\{P(x) : \|x\| = 1\}$) also has a Lipschitz, real analytic separating function with a holomorphic extension to a uniformly wide neighborhood of X in $\widetilde{X}$. We do not know if the converse is true. A natural related question is: does the space c_0 have such a Lipschitz separating function?

2. A brief outline of the proof

The proof of our main result is quite long, and very technical at some points so, for the reader's convenience, we will next explain the main ideas of our construction (which we here intentionally oversimplify in order not to be burdened by important, but not very meaningful precision and notation).

As said in the introduction, we will show in Lemma 2 that every Banach space X with a separating polynomial p of degree n has a Lipschitz, real analytic separating function. This is done as follows: such a Banach space always has a $2n$-degree homogeneous polynomial q such that $\|x\|^{2n} \leq q(x) \leq K\|x\|^{2n}$ for all $x \in X$. Then the function $Q : X \to [0, +\infty)$ defined by

$$Q(x) = (1 + q(x))^{\frac{1}{2n}} - 1$$

is real analytic, Lipschitz, and separating.

The next step will be taking a dense sequence $\{x_n\}$ in X and constructing a equi-Lipschitz, real analytic analogue of a sup partition of unity $\{\varphi_n\}_{n \in \mathbb{N}}$ which is subordinated to the covering $X = \bigcup_{n=1}^{\infty} D_Q(x_n, 4)$, where $D_Q(x_n, 4) = \{x \in X : Q(x - x_n) < 4\}$. By this we mean a collection of real analytic functions $\varphi_n : X \to [0, 2]$, with holomorphic extensions $\widetilde{\varphi}_n$ defined on an open neighborhood $\widetilde{V}$ of X in the complexification $\widetilde{X}$ of X, such that:

(1) The collection $\{\varphi_n : X \to [0, 2] \,|\, n \in \mathbb{N}\}$ is equi-Lipschitz on X, with Lipschitz constant $L = 2\text{Lip}(Q)$.

(2) For each $x \in X$ there exists $m = m_x \in \mathbb{N}$ with $\varphi_m(x) > 1$.
(3) For every $x \in X$ the set $\{n \in \mathbb{N} : \varphi_n(x) > \varepsilon\}$ is finite.
(4) $0 \leq \varphi_n(x) \leq \varepsilon$ for all $x \in X \setminus D_Q(x_n, 4)$.
(5) The function $\widetilde{V} \ni z \mapsto \{\alpha_n \widetilde{\varphi}_n(z)\}_{n=1}^{\infty} \in \widetilde{c_0}$ is holomorphic for every sequence $\{\alpha_n\}$ such that $1 \leq \alpha_n \leq 1001$ for all n.

Next, there is a real analytic norm $\lambda : c_0 \to [0, \infty)$ which satisfies $\|y\|_\infty \leq \lambda(y) \leq 2\|y\|_\infty$ for every $y \in c_0$. Assuming $f : X \to [1, 1001]$ is 1-Lipschitz, define a function $g : X \to \mathbb{R}$ by

$$g(x) = \frac{\lambda(\{f(x_n)\varphi_n(x)\}_{n=1}^{\infty})}{\lambda(\{\varphi_n(x)\}_{n=1}^{\infty})}$$

(the sup-partition of unity approach to approximation consists in replacing the usual locally finite sum in a classical partition of unity by taking the norm λ. Since λ is equivalent to the supremum norm of c_0, the function g roughly behaves like $\sup_n\{f(x_n)\varphi_n(x)\}/\sup_n\{\varphi_n(x)\}$, which is a Lipschitz function approximating f). It can be checked that g is real analytic, 8518492Lip(Q)-Lipschitz, and $|f - g| \leq 8$. Moreover, given $\delta > 0$, there exists a neighborhood $\widetilde{U}_\delta$ of X in $\widetilde{X}$, *independent of f but dependent on the interval* $[1, 1001]$ *and the functions* φ_n, such that g has a holomorphic extension $\widetilde{g} : \widetilde{U}_\delta \to \mathbb{C}$ satisfying

$$|\widetilde{g}(z) - g(x)| \leq \delta \text{ for all } z = x + iy \in \widetilde{U}_\delta.$$

Proving such independence of $\widetilde{U}_\delta$ from f is a delicate matter that will be tackled with the help of a refinement of the main result of [CHP]: an estimation of the domain of existence of the holomorphic solutions to a family of complex implicit equations depending on a parameter (see Proposition 1 below).

If we were only interested in approximating bounded functions by Lipschitz, real analytic functions and we did not care about the Lipschitz constant of the approximations, a replacement of the interval $[1, 1001]$ with a suitable translation of the range of f in the above argument would finish our proof (up to scaling). However, as the size of the interval increases, so does the Lipschitz constant of f, and inversely, the size of $\widetilde{U}_\delta$ decreases. In order to prove our main result in full generality we have to work harder.

Up to scaling, the above argument shows the following: there exists $C \geq 1$ (depending only on X) such that, for every $\delta > 0$ there is an open neighborhood $\widetilde{U}_\delta$ of X in $\widetilde{X}$ such that, for every Lipschitz function $f : X \to [0, 1]$ with Lip$(f) \leq 1$, there exists a real analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\widetilde{g} : \widetilde{U}_\delta \to \mathbb{C}$, such that

(1) $|f(x) - g(x)| \leq 1/10$ for all $x \in X$.
(2) g is Lipschitz, with Lip$(g) \leq C$Lip(f).
(3) $|\widetilde{g}(x + iy) - g(x)| \leq \delta$ for all $z = x + iy \in \widetilde{U}_\delta$.

Now, given a 1-Lipschitz, bounded function $f : X \to [0, +\infty)$, we can cut its graph into pieces, namely we may define, for $n \in \mathbb{N}$, the functions

$$f_n(x) = \begin{cases} f(x) - n + 1 & \text{if } n - 1 \le f(x) \le n, \\ 0 & \text{if } f(x) \le n - 1, \\ 1 & \text{if } n \le f(x). \end{cases}$$

The functions f_n are clearly 1-Lipschitz and take values in the interval $[0, 1]$, so, for $\delta > 0$ (to be fixed later on) there exist a neighborhood $\widetilde{U}_\delta$ of X in $\widetilde{X}$ and C-Lipschitz, real analytic functions $g_n : X \to \mathbb{R}$, with holomorphic extensions $\widetilde{g}_n : \widetilde{U}_\delta \to \mathbb{C}$, such that for all $n \in \mathbb{N}$ we have that g_n is C-Lipschitz, $|g_n - f_n| \le 1/8$, and $|\widetilde{g}_n(x+iy) - g_n(x)| \le \delta$ for all $z = x+iy \in \widetilde{U}_\delta$.

Observe that the function $X \ni x \mapsto \{f_n(x)\}_{n=1}^\infty$ takes values in ℓ_∞, and more precisely in the image of the path $\beta : [0, +\infty) \to \ell_\infty$ defined by

$$\beta(t) = \left(1, \cdots, 1, \underbrace{t - n + 1}_{n^{\text{th}} \text{ place}}, 0, 0, \cdots \right) = \sum_{j=1}^{n-1} e_j + (t-n+1)e_n \text{ if } n-1 \le t \le n.$$

The path β is a 1-Lipschitz injection of $[0, +\infty)$ into ℓ_∞, with a uniformly continuous (but not Lipschitz) inverse $\beta^{-1} : \beta([0, +\infty)) \to [0, +\infty)$.

Define a uniformly continuous (not Lipschitz) function h on the path β by $h(\beta(t)) = t$ for all $t \ge 0$, that is $h(y) = \beta^{-1}(y)$ for $y \in \beta([0, +\infty))$. Then we have $f(x) = h(\{f_n(x)\}_{n=1}^\infty)$.

We will construct an open tube of radius $1/8$ (with respect to the supremum norm $\|\cdot\|_\infty$) around the path β in ℓ_∞, and a real-analytic approximate extension (with bounded derivative) H of the function h defined on this tube. Then, since $|g_n - f_n| \le 1/10$ and $\{f_n(x)\}_{n=1}^\infty$ takes values in the path β, the function $\{g_n(x)\}_{n=1}^\infty$ will take values in this tube, and therefore $g(x) := H(\{g_n(x)\}_{n=1}^\infty)$ will approximate $H(\{f_n(x)\}_{n=1}^\infty)$, which in turn approximates $h(\{f_n(x)\}_{n=1}^\infty) = f(x)$. Besides, since H has a bounded derivative on the tube and the functions g_n are C-Lipschitz then g will be CM-Lipschitz, where M is an upper bound of DH on the tube. Moreover, we will show that there exist $\delta > 0$ (this is the δ we had to fix) and a holomorphic extension $\widetilde{H}$ of H so that if $v \in \ell_\infty$ satisfies $\|v\|_\infty \le \delta$ then

$$|H(u + iv) - H(u)| \le 1.$$

We will call H a real analytic *gluing tube function.*

Thus, resetting C to CM, we will deduce that, there exists an open neighborhood $\widetilde{U} := \widetilde{U}_\delta$ of X in $\widetilde{X}$ such that, for every 1-Lipschitz, bounded function $f : X \to \mathbb{R}$, there exists a real analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\widetilde{g} : \widetilde{U} \to \mathbb{C}$, such that

(1) $|f(x) - g(x)| \le 1$ for all $x \in X$.
(2) g is C-Lipschitz.
(3) $|\widetilde{g}(x + iy) - g(x)| \le 1$ for all $z = x + iy \in \widetilde{U}$.

The last step of the proof consists in passing from bounded to unbounded functions. This can be achieved by constructing a real analytic approximation $\theta_n(Q(x))$ to a partition of unity subordinated to a covering of X consisting of crowns $C_n = \{x \in X : 2^{n-1} < Q(x) < 2^{n+1}\}$ of rapidly increasing diameter (so that $\sum_{n=1}^{\infty} \mathrm{Lip}(\theta_n) < 3$), and by approximating each restriction of f to C_n by a real analytic C-Lipschitz function g_n, in order to define

$$g(x) = \sum_{n=1}^{\infty} \theta_n(Q(x)) g_n(x),$$

which we will check is a $5C$-Lipschitz real analytic approximation of f. Up to scaling, our main theorem will then be proved.

3. Notation and basic definitions. Complexifications.

Our notation is standard, with X denoting a Banach space, and an open ball with centre x and radius r denoted $B_X(x, r)$ or $B(x, r)$ when the space is understood. If $\{f_j\}_j$ is a sequence of Lipschitz functions defined on X, then we will at times say this family is *equi-Lipschitz* if there is a common Lipschitz constant for all j. A *homogeneous polynomial of degree* n is a map, $P : X \to \mathbb{R}$, of the form $P(x) = A(x, x, ..., x)$, where $A : X^n \to \mathbb{R}$ is $n-$multilinear and continuous. For $n = 0$ we take P to be constant. A *polynomial of degree* n is a sum $\sum_{i=0}^{n} P_i(x)$, where the P_i are i-homogeneous polynomials.

Let X be a Banach space, and $G \subset X$ an open subset. A function $f : G \to \mathbb{R}$ is called *analytic* if for every $x \in G$, there are a neighbourhood N_x, and homogeneous polynomials $P_n^x : X \to \mathbb{R}$ of degree n, such that

$$f(x+h) = \sum_{n \geq 0} P_n^x(h) \text{ provided } x + h \in N_x.$$

Further information on polynomials may be found, for example, in [SS].

For a Banach space X, we define its (Taylor) complexification $\widetilde{X} = X \bigoplus iX$ with norm

$$\|x + iy\|_{\widetilde{X}} = \sup_{0 \leq \theta \leq 2\pi} \|\cos\theta\, x - \sin\theta\, y\|_X = \sup_{T \in X^*, \|T\| \leq 1} \sqrt{T(x)^2 + T(y)^2}.$$

We will sometimes denote, for $z = x + iy \in \widetilde{X}$, $x := \mathrm{Re}\, z$, $y := \mathrm{Im}\, z$. If $L : E \to F$ is a continuous linear mapping between two real Banach spaces then there is a unique continuous linear extension $\widetilde{L} : \widetilde{E} \to \widetilde{F}$ of L (defined by $\widetilde{L}(x + iy) = L(x) + iL(y)$) such that $\|\widetilde{L}\| = \|L\|$. For a continuous k-homogeneous polimomial $P : E \to \mathbb{R}$ there is also a unique continuous k-homogeneous polinomial $\widetilde{P} : \widetilde{E} \to \mathbb{C}$ such that $\widetilde{P} = P$ on $E \subset \widetilde{E}$, but the norm of P is not generally preserved: one has that $\|\widetilde{P}\| \leq 2^{k-1}\|P\|$. It follows that if $q(x)$ is a continuous polynomial on X, there is a unique continuous polynomial $\widetilde{q}(z) = \widetilde{q}(x + iy)$ on $\widetilde{X}$ where for $y = 0$ we have $\widetilde{q} = q$.

It also follows that if the Taylor series $f(x+h)=\sum_{n=0}^{\infty} D^n f(x)(h)/n!$ of a real analytic function $\underbrace{f:X}\to\mathbb{R}$ at a point x has radius of convergence r_x then the series $\sum_{n=0}^{\infty} \widetilde{D^n f(x)}(h)/n!$ has radius of convergence $r_x/2e$ (see for instance [CH, Lemma 0]). Consequently f has a holomorphic extension $\widetilde{f}$ to the neighborhood $\{x+iy : x,y\in X, \|x+iy\|_{\widetilde{X}} < r_x\}$ of X in $\widetilde{X}$.

We will also use the fact that for this complexification procedure the complexifications $\widetilde{c_0}, \widetilde{\ell_\infty}$ of the real Banach spaces c_0 and ℓ_∞ are just the usual complex versions of these spaces. That is, we have $\widetilde{c_0}=\{\{z_j\} : z_j\in\mathbb{C}, \lim_{j\to\infty} z_j = 0\}$, with norm $\|z\|_{\widetilde{c_0}} = \|\{z_j\}\|_{\widetilde{c_0}} = \max_j \{|z_j|\}$, and similarly for $\widetilde{l_\infty}$. For more information on complexifications (and polynomials) we recommend [MST].

In the sequel, all extensions of functions from X to $\widetilde{X}$, as well as subsets of $\widetilde{X}$, will be embellished with a tilde. All smoothness properties of a norm or Minkowski functional are assumed implicitly to not include the point 0.

4. Preliminary results: the Preiss norm and separating functions

4.1. **The Preiss norm.** As developed in [FPWZ], there is a real analytic norm on c_0 (hereafter referred to as the Preiss norm, $\|\cdot\|$) that is equivalent to the canonical supremum norm $\|\cdot\|_\infty$. Let us recall the construction. Let $C : c_0\to\mathbb{R}$ be given by $C(\{x_n\}) = \sum_{n=1}^{\infty} (x_n)^{2n}$. Let $W=\{x\in c_0 : C(x)\le 1\}$. Then $\|\cdot\|$ is the Minkowski functional of W; that is, $\|x\|$ is the solution for λ to $C\left(\lambda^{-1}x\right)=1$. The Preiss norm is analytic at all non-zero points in c_0. To see this, let us define the function $\widetilde{C} : V\to\mathbb{C}$ by $\widetilde{C}\left(\{z_n\}\right) = \sum_{n=1}^{\infty} (z_n)^{2n}$ where V is the subset of $\widetilde{l_\infty}$ for which the series converges. Then $\widetilde{C}$ is analytic at each $z\in\widetilde{c_0}$. Indeed, the partial sums are analytic as a consequence of the analyticity of the (complex linear) projection functions $p_j(\{z_i\})=z_j$. Since the series in the definition of $\widetilde{C}$ converges locally uniformly at each $z\in\widetilde{c_0}$ the analyticity of $\widetilde{C}$ on $\widetilde{c_0}$ follows. Also, for $z\in\widetilde{c_0}$ sufficiently close to c_0 and $\lambda\in\mathbb{C}\backslash\{0\}$ sufficiently close to $\mathbb{R}\backslash\{0\}$ we have $\frac{\partial\widetilde{C}(\lambda^{-1}z)}{\partial\lambda}\neq 0$, hence one can apply the complex Implicit Function Theorem (see e.g. [Ca], or [D] page 265, where the real result for Banach spaces is easily extended to the analytic case) to $F(z,\lambda)=\widetilde{C}\left(\lambda^{-1}z\right)-1$ to obtain a unique holomorphic solution $\widetilde{\lambda}(z)$ to $F(z,\lambda)=0$, with $\lambda := \widetilde{\lambda}\mid_{c_0} = \|\cdot\|$, defined on a neighborhood of $c_0\backslash\{0\}$ in $\widetilde{c_0}$. Now if $x=\{x_n\}$ satisfies $\|x\|_\infty = 1$, then $\sum_{n=1}^{\infty}\left(\|x\|^{-1} x_n\right)^{2n} = 1$ implies $\|x\|\ge 1$. On the other hand, if $\|x\|_\infty = 1/2$, then $C(x)\le\sum_{n=1}^{\infty}(1/2)^{2j} < 1$, implying $\|x\| < 1$. Hence, $(1/2)\|x\|\le\|x\|_\infty\le\|x\|$ for all x in c_0.We shall use the above notation throughout this article.

4.2. **Separating polynomials and separating Lipschitz analytic functions.** Let X be a Banach space. A *separating polynomial* on X is a polynomial q on X such that $0 = q(0) < \inf\{|q(x)| : x \in S_X\}$. It is known [FPWZ] that if X is superreflexive and admits a C^∞-smooth bump function then X admits a separating polynomial. The following lemma makes precise, observations of Kurzweil in [K].

Lemma 1. *Let X be a Banach space with a separating polynomial of degree n. Then there exist $K > 1$ and a $2n$ degree homogeneous separating polynomial $q : X \to [0, +\infty)$ such that $\|x\|^{2n} \leq q(x) \leq K\|x\|^{2n}$ for all $x \in X$.*

Proof. We may suppose that $p = \sum_{i=1}^n p_i$, where p_i is i-homogeneous for $1 \leq i \leq n$. Define $q = p_1^{2n} + p_2^{2n/2} + \cdots p_n^2$. Note that q is $2n$-homogeneous. As p is separating, there is some $\eta > 0$ such that $q(x) \geq \eta$ for all $x \in S_X$. By scaling, we may assume that $\eta = 1$. Then from the $2n$-homogeneity, $q\left(x\right) \geq \|x\|^{2n}$. Now, as q is continuous at 0, there is $\delta > 0$ with $\|x\| \leq \delta$ implies $q\left(x\right) \leq 1$, so again from $2n$-homogeneity, $1 \geq q\left(\frac{\delta x}{\|x\|}\right) = q\left(x\right) \frac{\delta^{2n}}{\|x\|^{2n}}$, and we are done. $\square$

Note that a separating polynomial, even though is Lipschitz on every bounded set, is never Lipschitz on all of X. In the proof of our main results we will require the existence of a globally Lipschitz, real analytic separating function on X. The following lemma provides us with such separating functions for every Banach space having a separating polynomial.

Lemma 2. *Let X be a Banach space with a separating polynomial q as in the previous Lemma. Then the function $Q : X \to [0, +\infty)$ defined by*

$$Q(x) = (1 + q(x))^{\frac{1}{2n}} - 1$$

is real analytic, satisfies $\inf_{\|x\| \geq 1} Q(x) > 0 = Q(0)$, and is Lipschitz on all of X. Moreover, there is some $\delta_Q > 0$ such that Q has an holomorphic extension $\widetilde{Q}$ to the open strip $W_{\delta_Q} := \{x + z : x \in X, z \in \widetilde{X}, \|z\|_{\widetilde{X}} < \delta_Q\}$ in $\widetilde{X}$ such that $\widetilde{Q}$ is still Lipschitz on W_{δ_Q}. Finally, we have that

$$Q(x) < 4r \implies \|x\| < 8r, \text{ for all } r \geq 1, x \in X.$$

In particular $Q(x) \geq \frac{1}{2}\|x\|$ for $\|x\| \geq 8$ and Q is separating.

Proof. Since the function $w \mapsto w^{1/2n}$ is well defined (taking the usual branch of log) and holomorphic on the half-plane $\operatorname{Re} w \geq 1/2$ in $\mathbb{C}$, it is clear that

$$\widetilde{Q}(x + z) = (1 + \widetilde{q}(x + z))^{1/2m}$$

will be well defined and holomorphic on such a strip W_{δ_Q} as soon as we find $\delta = \delta_Q > 0$ small enough so that $\operatorname{Re}(1 + \widetilde{q}(x + z)) \geq 1/2$ whenever $x \in X, z \in \widetilde{X}, \|z\| \leq \delta$.

Let A (resp. $\widetilde{A}$) denote a $2n$-multilinear form on X (resp. $\widetilde{X}$) such that $q(x) = A(x, ..., x)$ (resp. $\widetilde{q}(w) = \widetilde{A}(w, ..., w)$). We have, for $x \in X$ and $z \in \widetilde{X}$ with $\|z\| \leq \delta < 1$ ($\delta = \delta_Q$ is to be fixed later),

$$\begin{aligned}
&\operatorname{Re} \widetilde{q}(x+z) = \operatorname{Re} A(x+z, x+z, ..., x+z) \\
&= A(x, x, ..., x) + \text{ terms of the form } \operatorname{Re} A(y_1, ..., y_{2n}), \\
&\text{where } y_j \in \{x, z\} \text{ for all } j, \text{ and } y_k = z \text{ for at least one } k \\
&\geq A(x, ..., x) - \sum_{j=1}^{2n} \|\widetilde{A}\| \binom{2n}{j} \|x\|^{2n-j} \|z\|^j \\
&\geq \|x\|^{2n} - \|A\| \sum_{j=1}^{2n} \binom{2n}{j} \|x\|^{2n-j} \delta^j \\
&\geq \|x\|^{2n} - \delta \|A\| \sum_{j=1}^{2n} \binom{2n}{j} \|x\|^{2n-j} \\
&\geq \|x\|^{2n} - \delta \|A\| \left(\|x\| + 1\right)^{2n}.
\end{aligned}$$

Hence

$$\operatorname{Re}\left(1 + \widetilde{q}(x+z)\right) \geq \frac{1}{2} + \frac{1}{2} + \|x\|^{2n} - \delta \|A\| \left(\|x\| + 1\right)^{2n}.$$

Since the real function

$$f(t) = \frac{\frac{1}{2} + t^{2n}}{(|t| + 1)^{2n}}$$

is continuous, positive and satisfies $\lim_{|t| \to \infty} f(t) = 1$, there exists $\alpha > 0$ such that $f(t) \geq \alpha$, and therefore $1 + t^{2n} - \alpha(|t| + 1)^{2n} \geq 1/2$ for all $t \in \mathbb{R}$. If we set

$$\delta = \frac{1}{2} \min\{1, \alpha/\|A\|\},$$

this implies that $1 + \|x\|^{2n} - \delta \|A\| \left(\|x\| + 1\right)^{2n} \geq 1/2$ for all $x \in X$, and therefore

$$\operatorname{Re}\left(1 + \widetilde{q}(x+z)\right) \geq \frac{1}{2}, \text{ for all } x + z \in W_\delta,$$

and the function $\widetilde{Q}$ is holomorphic on W_δ.

Now let us check that $\widetilde{Q}$ is Lipschitz on W_δ. By the same estimation as above, for $x \in X$, $z \in \widetilde{X}$ with $\|z\| \leq \delta$, we have

$$|1 + \widetilde{q}(x+z)| \geq \operatorname{Re}\left(1 + \widetilde{q}(x+z)\right) \geq 1 + \|x\|^{2n} - \delta \|A\| \left(\|x\| + 1\right)^{2n} \geq 1/2,$$

and on the other hand, the derivative of $\widetilde{q}$ being a $(2n-1)$-homogeneous polynomial,

$$\left|\widetilde{q}'(x+z)\right| \leq \left\|\widetilde{q}'\right\| \left(\|x\| + \|z\|\right)^{2n-1} \leq \left\|\widetilde{q}'\right\| \left(\|x\| + \delta\right)^{2n-1}.$$

By combining these two inequalities we get, for $x+z \in W_\delta$,

$$\|\widetilde{Q}'(x+z)\| = \frac{\|\widetilde{q}'(x+z)\|}{\left|2n\,(1+\widetilde{q}(x+z))^{1-\frac{1}{2n}}\right|} \leq \frac{\|\widetilde{q}'\|\,(\|x\|+\delta)^{2n-1}}{\left(1+\|x\|^{2n}-\delta\|A\|\,(\|x\|+1)^{2n}\right)^{\frac{2n-1}{2n}}}.$$

Since the function

$$\mathbb{R} \ni t \mapsto \frac{\|\widetilde{q}'\|\,(|t|+\delta)^{2n-1}}{\left(1+|t|^{2n}-\delta\|A\|\,(|t|+1)^{2n}\right)^{\frac{2n-1}{2n}}} \in \mathbb{R}$$

is bounded on $\mathbb{R}$ when $\delta\|A\|<1$, it follows that the derivative $\widetilde{Q}'$ is bounded on the convex set W_δ, and therefore the function $\widetilde{Q}$ is Lipschitz on W_δ.

Finally, assume that $\|x\| \geq 8r$, $r \geq 1$. Since $q(x) \geq \|x\|^{2n}$ and the function $h(t) = (1+t^{2n})^{1/2n} - 1$ is increasing, we have

$$Q(x) \geq h(\|x\|) \geq h(8r).$$

Now, for $r \geq 1$ we have

$$(1+4r)^{2n} \leq 2^{2n}(4r)^{2n} \leq 1 + (8r)^{2n},$$

from which it follows that

$$h(8r) \geq 4r,$$

and therefore $Q(x) \geq 4r$. □

5. The proof of the main result

In the sequel we will be using the following notation: for the real analytic and Lipschitz separating function Q constructed in Lemma 2, and for every $x \in X$, $r > 0$, we define the Q-bodies

$$D_Q(x,r) = \{y \in X : Q(y-x) < r\}.$$

5.1. **Real analytic sup-partitions of unity.** The following Lemma provides us, for any given $r \geq 1, \varepsilon \in (0,1)$, with a real-analytic analogue of *sup-partitions of unity* (see [F2, HJ]) which, one could say, are *ε-subordinated* to a covering of the form $X = \bigcup_{j=1}^{\infty} D_Q(x_j, 4r)$.

Lemma 3. *Let $\widetilde{V} = W_{\delta_Q}$ be an open strip around X in $\widetilde{X}$ in which the function $\widetilde{Q}$ of Lemma 2 is defined. Given $r \geq 1, \varepsilon \in (0,1)$ and a covering of the form $X = \bigcup_{j=1}^{\infty} D_Q(x_j, r)$, there exists a sequence of holomorphic functions $\widetilde{\varphi}_n = \widetilde{\varphi}_{n,r,\varepsilon} : \widetilde{V} \to \mathbb{C}$, whose restrictions to X we denote by $\varphi_n = \varphi_{n,r,\varepsilon}$, with the following properties:*

1. *The collection $\{\varphi_{n,r,\varepsilon} : X \to [0,2] \,|\, n \in \mathbb{N}\}$ is equi-Lipschitz on X, with Lipschitz constant $L = 2Lip(Q)/r$ (independent of ε).*
2. *$0 \leq \varphi_{n,r,\varepsilon}(x) \leq 1+\varepsilon$ for all $x \in X$.*

(3) *For each* $x \in X$ *there exists* $m = m_{x,r} \in \mathbb{N}$ *(independent of* ε*) with* $\varphi_{m,r,\varepsilon}(x) > 1$*. Besides,* $m_{x,1} \geq m_{x,r} \geq m_{x,s}$ *whenever* $1 \leq r \leq s$*.*
(4) $0 \leq \varphi_{n,r,\varepsilon}(x) \leq \varepsilon$ *for all* $x \in X \setminus D_Q(x_n, 4r)$*.*
(5) *For each* $x \in X$ *there exist* $\delta_x > 0$*,* $a_x > 0$*, and* $n_x \in \mathbb{N}$ *(all of them independent of* r *or* ε*) such that*

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| < \frac{1}{n!a_x^n} \quad \text{for } n > n_x,\ r \geq 1, \varepsilon \in (0,1),\ z \in \widetilde{X} \text{ with } \|z\|_{\widetilde{X}} < \delta_x.$$

(6) *For each* $x \in X$ *there exists* $\delta_x > 0$ *(independent of* r, ε*) and* $n_{x,\varepsilon} \in \mathbb{N}$ *(independent of* r*) such that for* $\|z\|_{\widetilde{X}} < \delta_x$ *and* $n > n_{x,\varepsilon}$ *we have* $|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| < \varepsilon$*.*
(7) *For each* $x \in X$ *there exists* $\delta_{x,\varepsilon}$ *(independent of* r*) such that*

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| \leq 1 + 2\varepsilon \quad \text{for } n \in \mathbb{N}, r \geq 1,\ z \in \widetilde{X} \text{ with } \|z\|_{\widetilde{X}} \leq \delta_{x,\varepsilon}.$$

Proof. Define subsets $A_{1,r} = \{y_1 \in \mathbb{R} : -1 \leq y_1 \leq 4r\}$, and, for $n \geq 2$,

$$A_{n,r} = \{y = \{y_j\}_{j=1}^n \in \ell_\infty^n : -1-r \leq y_n \leq 4r,\ 2r \leq y_j \leq M_{n,r}+2r \text{ for } 1 \leq j \leq n-1\},$$

$$A'_{n,r} = \{y = \{y_j\}_{j=1}^n \in \ell_\infty^n : -1 \leq y_n \leq 3r,\ 3r \leq y_j \leq M_{n,r}+r \text{ for } 1 \leq j \leq n-1\},$$

$$\text{where } M_{n,r} = \sup\left\{Q\left(x - x_j\right) : x \in D_Q(x_n, 4r),\ 1 \leq j \leq n\right\}.$$

Let $b_n = b_{n,r,\varepsilon} : \ell_\infty^n \to [0,2]$ be the function defined by

$$b_n(y) = (1+\varepsilon)\max\{0, 1 - \frac{1}{r}\mathrm{dist}_\infty(y, A'_n)\},$$

where $\mathrm{dist}_\infty(y, A) = \inf\{\|y-a\|_\infty : a \in A\}$.

In the sequel, in order not to be too burdened by notation, will omit the subscripts r, ε whenever they do not play a relevant role locally in the argument.

It is clear that $\mathrm{support}(b_n) = A_n$, that $b_n = 1+\varepsilon$ on A'_n, and that b_n is $(2/r)$-Lipschitz (note in particular that the Lipschitz constant of b_n does not depend on $n \in \mathbb{N}$ or $\varepsilon \in (0,1)$). And anyway, since $r \geq 1$, $b_{n,r,\varepsilon}$ is 2-Lipschitz for all n, r, ε.

Since the function $b_n = b_{n,r,\varepsilon}$ is 2-Lipschitz and bounded by 2 on $\mathbb{R}^n$, it is a standard fact that the normalized integral convolutions of b_n with the Gaussian-like kernels $y \mapsto G_\kappa(y) := e^{-\kappa\sum_{j=1}^n 2^{-j}y_j^2}$,

$$x \mapsto \frac{1}{T_\kappa} b_n * G_\kappa(x) = \frac{1}{\int_{\mathbb{R}^n} e^{-\kappa\sum_{j=1}^n 2^{-j}y_j^2}dy}\int_{\mathbb{R}^n} b_n(y)e^{-\kappa\sum_{j=1}^n 2^{-j}(x_j-y_j)^2}dy,$$

$$\text{where } T_\kappa = T_{\kappa,n} = \int_{\mathbb{R}^n} e^{-\kappa\sum_{j=1}^n 2^{-j}y_j^2}dy,$$

converge to b_n uniformly on $\mathbb{R}^n$ as $\kappa \to +\infty$. Moreover, again using $0 \leq b_{n,r,\varepsilon} \leq 2$ and that all the functions $b_{n,r,\varepsilon}$ are 2-Lipschitz, it is easily seen that the limit

$$\lim_{\kappa\to+\infty}\frac{1}{T_{\kappa,n}} b_{n,r,\varepsilon} * G_\kappa(x) = b_{n,r,\varepsilon}(x)$$

is uniform with respect to $x \in \mathbb{R}^n, r \geq 1, \varepsilon \in (0,1)$. Therefore, for each $n \in \mathbb{N}$ we can find $\kappa_n = \kappa_{n,\varepsilon} > 0$ (not depending on r) large enough so that

$$|b_{n,r,\varepsilon}(x) - \frac{1}{T_{\kappa_{n,\varepsilon}}} b_{n,r,\varepsilon} * G_{\kappa_{n,\varepsilon}}(x)| \leq \varepsilon/2 \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{$*$}$$

For reasons that will become apparent later on, we shall also take each $\kappa_{n,\varepsilon}$ to be large enough so that

$$k_n^n \geq 2(\sqrt{2})^n (n!)^2. \tag{$**$}$$

Let us also define $\widetilde{T}_n = \int_{\mathbb{R}^n} e^{-\sum_{j=1}^n 2^{-j} y_j^2} dy$, and

$$T_n := T_{\kappa_n, n} = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy = \frac{1}{\kappa_n^{n/2}} \widetilde{T}_n,$$

and observe that by a change of variables

$$T_n = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} (A_j - y_j)^2} dy$$

for any $A_j \in \mathbb{R}$.

Now define $\nu_n : \ell_\infty^n \to \mathbb{R}$ by

$$\nu_n(x) := \frac{1}{T_{\kappa_n}} b_n * G_{\kappa_n}(x) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j}(x_j - y_j)^2} dy.$$

Let us note that

$$\nu_n(x) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j}(x_j - y_j)^2} dy = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(x - y) e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy,$$

and so

$$\begin{aligned}
\left|\nu_n(x) - \nu_n(x')\right| &= \left| \frac{1}{T_n} \int_{\mathbb{R}^n} \left(b_n(x-y) - b_n(x'-y)\right) e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \right| \\
&\leq \frac{1}{T_n} \int_{\mathbb{R}^n} \left|b_n(x-y) - b_n(x'-y)\right| e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \\
&\leq \frac{2}{r} \left\|x - x'\right\|_\infty \frac{1}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \\
&= \frac{2}{r} \left\|x - x'\right\|_\infty .
\end{aligned}$$

Hence, ν_n is $\frac{2}{r}$-Lipschitz. Note also that $\|\nu_n\|_\infty \leq \|b_n\|_\infty \leq 2$.

Next, consider the map $\lambda_n : X \to l_\infty^n$ given by

$$\lambda_n(x) = \left(Q(x - x_1), ..., Q(x - x_n)\right),$$

where Q is the real analytic and Lipschitz separating function constructed in Lemma 2.

Then for $n \geq 1$ we define (real) analytic maps $\varphi_n = \varphi_{n,r,\varepsilon} : X \to \mathbb{R}$ by

$$\begin{aligned}\varphi_n(x) &= \nu_n(\lambda_n(x)) = \nu_n\left(\{Q(x-x_j)\}_{j=1}^n\right) \\ &= \frac{1}{T_n}\int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j}(Q(x-x_j)-y_j)^2} dy.\end{aligned}$$

Since Q is Lipschitz on X, we have that

$$\begin{aligned}\left|\varphi_n(x) - \varphi_n(x')\right| &= \left|\nu_n(\lambda_n(x)) - \nu_n(\lambda_n(x'))\right| \\ &\leq \frac{2}{r}\left\|\lambda_n(x) - \lambda_n(x')\right\|_\infty \\ &= \frac{2}{r}\left\|\left\{Q(x-x_j) - Q(x'-x_j)\right\}_{j=1}^n\right\|_\infty \\ &\leq \frac{2}{r}\mathrm{Lip}(Q)\left\|x-x'\right\|_X,\end{aligned}$$

hence the collection $\{\varphi_{n,r,\varepsilon} : n \in \mathbb{N}\}$ is uniformly Lipschitz on X, with constant $\frac{2}{r}\mathrm{Lip}(Q)$.
We can extend the maps $\varphi_{n,r,\varepsilon}$ to complex valued maps defined on W_{δ_Q} (see Lemma 2), calling them $\widetilde{\varphi}_{n,r,\varepsilon}$. Namely (where $x \in X$, $z \in \widetilde{X}$),

$$\widetilde{\varphi}_n(x+z) = \frac{1}{T_n}\int_{\mathbb{R}^n} b_n(y)\, e^{-\kappa_n \sum_{j=1}^n 2^{-j}\left(\widetilde{Q}(x-x_j+z)-y_j\right)^2} dy$$

Note that the $\widetilde{\varphi}_n$ are well defined (as the b_n have compact supports) and are holomorphic where $\widetilde{Q}$ is (namely on $\widetilde{V}$). Hence the above calculation establishes (1) as $\widetilde{\varphi}_n\mid_X = \varphi_n$. Bearing in mind that $\|\nu_n\|_\infty \leq \|b_n\|_\infty$, it is also clear that $0 \leq \varphi_n(x) \leq 1+\varepsilon$ for all $x \in X, n \in \mathbb{N}$, which proves (2).

Let us show (3). For each fixed $x \in X$, there exists $m = m_{x,r}$ with $x \in D_Q(x_{n_x}, 3r)$ but with $x \notin D_Q(x_i, 3r)$ for $i < m$. Observe that, as $r \geq 1$, $x \notin D_Q(x_i, 3)$ for $i < m_{x,r}$, so we necessarily have $m_{x,1} \geq m_{x,r}$ for all $r \geq 1$ (in general, it is also clear that $m_{x,r} \geq m_{x,s}$ if $r \leq s$).

This implies that the point $\left(Q(x-x_1), Q(x-x_2), ..., Q\left(x-x_{m_{x,r}}\right)\right)$ belongs to $A'_{m,r}$, where the function $b_{m_{x,r}}$ takes the value $1+\varepsilon$. According to $(*)$, we have

$$|1+\varepsilon-\varphi_m(x)| =$$

$$|b_m(x) - (\frac{1}{T_{\kappa_{n_x}}} b_m * G_{\kappa_m})(Q(x-x_1), Q(x-x_2), ..., Q(x-x_m))| \leq \varepsilon/2,$$

which yields $\varphi_m(x) \geq 1+\varepsilon-\varepsilon/2 > 1$.

Property (4) is shown similarly: if $Q(x-x_n) \geq 4r$ then the point $(Q(x-x_1), ..., Q(x-x_n))$ lies in a region of $\mathbb{R}^n$ where the function b_n takes the value 0, and $(*)$ immediately gives us $\varphi_n(x) \leq \varepsilon/2$.

We finally show the more delicate properties (5), (6) and (7). For $x \in X$ and $z \in \widetilde{X}$ with $\|z\| < \delta_Q$, according to Lemma 2, we have

$$\widetilde{Q}\,(x - x_j + z) = Q\,(x - x_j) + Z_j,$$

$$\text{where } Z_j \in \mathbb{C} \text{ with } \ |Z_j| \le C\,\|z\|_{\widetilde{X}}\,,$$

C being the Lipschitz constant of $\widetilde{Q}$ on the strip W_{δ_Q}.

Now

$$\begin{aligned}\left(\widetilde{Q}\,(x - x_j + z) - y_j\right)^2 &= \left(Q\,(x - x_j) - y_j + Z_j\right)^2 \\ &= \left(Q\,(x - x_j) - y_j\right)^2 + 2\left(Q\,(x - x_j) - y_j\right) Z_j + Z_j^2.\end{aligned}$$

Hence, for $\|z\|_{\widetilde{X}} < \delta_Q$ we have

$$\begin{aligned}&\operatorname{Re}\left(\widetilde{Q}\,(x - x_j + z) - y_j\right)^2 \\ &= \left(Q\,(x - x_j) - y_j\right)^2 + 2\left(Q\,(x - x_j) - y_j\right)\operatorname{Re} Z_j + \operatorname{Re}(Z_j^2) \\ &= \left(Q(x - x_j) - y_j + \operatorname{Re} Z_j\right)^2 - \left(\operatorname{Re} Z_j\right)^2 + \operatorname{Re}(Z_j^2) \\ &\ge \left(Q(x - x_j) - y_j + \operatorname{Re} Z_j\right)^2 - 2C^2\|z\|_{\widetilde{X}}^2\,.\end{aligned}$$

Next, for each $x \in X$ there exists n_x so that $Q\,(x - x_{n_x}) < 1 \le r$ (note in particular that n_x is independent of r, ε), and also for $n > n_x$ and $y \in A_n = \operatorname{support}(b_n)$ we have $y_{n_x} \ge 2r$. Hence, $y_{n_x} - Q\,(x - x_{n_x}) > 2r - r = r \ge 1$. Thus, for $\|z\|_{\widetilde{X}} < \min\{1/2C, \delta_Q\}$, $n > n_x$ and $y \in A_n$ we have

$$\begin{aligned}&\left(Q\,(x - x_{n_x}) - y_{n_x} + \operatorname{Re}(Z_{n_x})\right)^2 = \left(y_{n_x} - Q\,(x - x_{n_x}) - \operatorname{Re}(Z_{n_x})\right)^2 \\ &\ge \left(y_{n_x} - Q\,(x - x_{n_x}) - C\,\|z\|_{\widetilde{X}}\right)^2 \ge \left(1 - C\,\|z\|_{\widetilde{X}}\right)^2 \\ &\ge (1 - \frac{1}{2})^2 = \frac{1}{4}.\end{aligned}$$

It follows that for $\|z\|_{\widetilde{X}} < \min\{1/2C, \delta_Q\}$, $n > n_x$, and $y \in A_n$,

$$\sum_{j=1}^{n} 2^{-j} \operatorname{Re}\left(\widetilde{Q}\,(x - x_j + z) - y_j\right)^2$$

$$\geq \sum_{j=1}^{n} 2^{-j}\,(Q(x-x_j) - y_j + \operatorname{Re} Z_j)^2 - 2C^2 \sum_{j=1}^{n} 2^{-j} \|z\|_{\widetilde{X}}^2$$

$$= \frac{1}{2}\sum_{j=1}^{n} 2^{-j}\,(Q(x-x_j) - y_j + \operatorname{Re} Z_j)^2 + \frac{1}{2}\sum_{j=1}^{n} 2^{-j}\,(Q(x-x_j) - y_j + \operatorname{Re} Z_j)^2 - 2C^2 \sum_{j=1}^{n} 2^{-j} \|z\|_{\widetilde{X}}^2$$

$$\geq \frac{1}{2}\sum_{j=1}^{n} 2^{-j}\,(Q(x-x_j) - y_j + \operatorname{Re} Z_j)^2 + \frac{1}{2} 2^{-n_x} \frac{1}{4} - 2C^2 \sum_{j=1}^{n} 2^{-j}\, \|z\|_{\widetilde{X}}^2$$

$$\geq \frac{1}{2}\sum_{j=1}^{n} 2^{-j}\,(Q(x-x_j) - y_j + \operatorname{Re} Z_j)^2 + 2^{-n_x - 3} - 2C^2\, \|z\|_{\widetilde{X}}^2 .$$

Define

$$\delta_x = \sqrt{\frac{2^{-n_x-4}}{C^2}}, \;\; a_x = 2^{-n_x-4},$$

and note that these numbers are independent of r, ε. For every $n > n_x$, $y \in A_n$, and $z \in \widetilde{X}$ with $\|z\|_{\widetilde{X}} \leq \delta_x$, we have

$$\begin{aligned}
&\sum_{j=1}^{n} 2^{-j} \operatorname{Re}\left(\widetilde{Q}\,(x - x_j + z) - y_j\right)^2 \\
&\geq \frac{1}{2}\sum_{j=1}^{n} 2^{-j}\,(Q(x-x_j) - y_j + \operatorname{Re} Z_j)^2 + 2^{-n_x-3} - 2C^2 \delta_x^2 \\
&= \frac{1}{2}\sum_{j=1}^{n} 2^{-j}\,(Q(x-x_j) - y_j + \operatorname{Re} Z_j)^2 + a_x.
\end{aligned}$$

Therefore, for every $n > n_x$ and $z \in \widetilde{X}$ with $\|z\|_{\widetilde{X}} \leq \delta_x$ we can estimate

$$\begin{aligned}
|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| &= \left| \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y)\, e^{-\kappa_n \sum_{j=1}^n 2^{-j}\left(\widetilde{Q}(x_0 - x_j + z) - y_j\right)^2} dy \right| \\
&= \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y)\, e^{-\kappa_n \operatorname{Re} \sum_{j=1}^n 2^{-j}\left(\widetilde{Q}(x_0 - x_j + z) - y_j\right)^2} dy \\
&\leq \frac{2}{T_n} \int_{A_{n,r}} e^{-\kappa_n a_x - \frac{1}{2}\kappa_n \sum_{j=1}^n 2^{-j}(Q(x-x_j) + \operatorname{Re}(Z_j) - y_j)^2} dy \\
&\leq \frac{2e^{-\kappa_n a_x}}{T_n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\kappa_n \sum_{j=1}^n 2^{-j}(Q(x-x_j) + \operatorname{Re}(Z_j) - y_j)^2} dy \\
&= \frac{2e^{-\kappa_n a_x}}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} u_j^2} \left(\sqrt{2}\right)^n du \\
&= 2(\sqrt{2})^n e^{-\kappa_n a_x}.
\end{aligned}$$

Now, by choice of $\kappa_n = \kappa_{n,\varepsilon}$ (see $(**)$ above), we have

$$2(\sqrt{2})^n e^{-\kappa_n a_x} \leq \frac{2(\sqrt{2})^n n!}{\kappa_n^n a_x^n} \leq \frac{1}{a^n n!}.$$

Hence

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| \leq \frac{1}{n! a_x^n}$$

for all $n > n_x$, all $r \geq 1, \varepsilon \in (0,1)$, and all $z \in \widetilde{X}$ with $\|z\| \leq \delta_x$. This establishes (5).

In particular by taking $n_{x,\varepsilon} > n_x$ sufficiently large, we can guarantee that for $n > n_{x,\varepsilon}$ and $\|z\|_{\widetilde{X}} < \delta_x$ we have $|\widetilde{\varphi}_n(x_0 + z)| < \varepsilon$, which proves (6).

As for (7), we can write, as above, $\widetilde{Q}(x - x_j + z) = Q(x - x_j) + Z_j$, where $Z_j \in \mathbb{C}$ with $|Z_j| \leq C \|z\|_{\widetilde{X}} \leq C\delta_x$, and we have

$$\begin{aligned}
&\operatorname{Re}\left(\widetilde{Q}(x - x_j + z) - y_j\right)^2 \\
&= (Q(x - x_j) - y_j)^2 + 2(Q(x - x_j) - y_j)\operatorname{Re} Z_j + \operatorname{Re}(Z_j^2) \\
&= (Q(x - x_j) - y_j + \operatorname{Re} Z_j)^2 - (\operatorname{Re} Z_j)^2 + \operatorname{Re}(Z_j^2) \\
&\geq (Q(x - x_j) - y_j + \operatorname{Re} Z_j)^2 - 2C^2\delta_x^2,
\end{aligned}$$

hence

$$
\begin{aligned}
|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| &= \left| \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y)\, e^{-\kappa_n \sum_{j=1}^n 2^{-j} \left(\widetilde{Q}(x_0 - x_j + z) - y_j \right)^2} dy \right| \\
&= \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y)\, e^{-\kappa_n \operatorname{Re} \sum_{j=1}^n 2^{-j} \left(\widetilde{Q}(x_0 - x_j + z) - y_j \right)^2} dy \\
&\leq \frac{1+\varepsilon}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} \left[(Q(x-x_j) + \operatorname{Re} Z_j - y_j)^2 - 2C^2 \delta_x^2 \right]} dy \\
&= \frac{(1+\varepsilon) e^{2\kappa_n C^2 \delta_x^2}}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} (Q(x-x_j) + \operatorname{Re} Z_j - y_j)^2} dy \\
&= (1+\varepsilon) e^{2\kappa_n C^2 \delta_x^2}.
\end{aligned}
$$

This is true for every $n \in \mathbb{N}$, $z \in \widetilde{X}$ with $\|z\|_{\widetilde{X}} \leq \delta_x$. On the other hand, for $n > n_x$ we already know that

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| \leq \frac{1}{a_x n!}$$

if $\|z\|_{\widetilde{X}} \leq \delta_x$, and since this sequence converges to 0 we can find $m_{x,\varepsilon}$ such that

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| \leq \frac{1}{a_x n!} < 1 + 2\varepsilon \text{ for } n \geq m_{x,\varepsilon}, \|z\|_{\widetilde{W}} \leq \delta_x.$$

Then, since $\kappa_{n,\varepsilon}$ does not depend on r, we can define

$$\delta_{x,\varepsilon} = \min \left\{ \delta_x, \sqrt{\frac{1}{2\kappa_{1,\varepsilon} C^2} \log \left(\frac{1+2\varepsilon}{1+\varepsilon} \right)}, \ldots, \sqrt{\frac{1}{2\kappa_{m_{x,\varepsilon},\varepsilon} C^2} \log \left(\frac{1+2\varepsilon}{1+\varepsilon} \right)} \right\},$$

so we get

$$|\widetilde{\varphi}_{n,r,\varepsilon}(x+z)| \leq 1 + 2\varepsilon$$

if $\|z\|_{\widetilde{X}} \leq \delta_{x,\varepsilon}$, for all $n \in \mathbb{N}$ and $r \geq 1$. □

5.2. **A refinement of the implicit function theorem for holomorphic mappings on complex Banach spaces.** We shall use the following refinement of [CHP, Theorem 1.1], which provides a lower bound on the size of the domains of existence of the solutions to a family of complex analytic implicit equations, in a uniform manner with respect to the equation, whenever appropriate bounds are available for the functions defining the equations and its first derivatives with respect to the dependent variable.

Proposition 1. *Let $\psi : E \times \mathbb{C} \times P \to \mathbb{C}$ be a function, where E is a complex Banach space and P is a set of parameters. Let $(x_0, \mu^0) \in E \times \mathbb{C}$, $R > 0$, and suppose that for each $p \in P$ the function*

$$(x, \mu) \mapsto \psi_p(x, \mu) = \psi(x, \mu, p)$$

is (complex) analytic on $B = \{(x, \mu) : \|x - x_0\| \leq R,\ |\mu - \mu^0| \leq R\}$*, and that there exist* $M, A > 0$ *such that*

$$\left|\frac{\partial \psi_p}{\partial \mu}(x_0, \mu_0)\right| \geq A, \quad \text{and} \ \ |\psi_p(x, \mu)| \leq M \ \ \text{for all } (x, \mu) \in B, p \in P.$$

Then there exist $0 < r = r(A, R, M) < R$ *and* $s = s(A, R, M) > 0$ *so that for each* $p \in P$ *there exists a unique analytic function* $\mu = \mu_p(x)$*, defined on the ball* $B(x_0, s)$*, such that* $\psi_p(x, \mu) = 0$ *for* $(x, \mu) \in B(x_0, s) \times B(\mu^0, r)$ *if and only if* $\mu = \mu_p(x)$ *for* $x \in B(x_0, s)$*.*

Proof. This is essentially a restatement of Theorem 1.1 in [CHP], where the result is proved (with precise expressions for r and s) in the case when P is a singleton and $E = \mathbb{C}^n$. The part of the proof in [CHP] that does not follow directly from the implicit function theorem for holomorphic mappings combines an estimation of the Taylor series of ψ_p at (x_0, μ_0) with respect to the variable $\mu \in \mathbb{C}$, and an application of Rouche's theorem to the functions $\mathbb{C} \ni \mu \mapsto \psi_p(x_0, \mu) \in \mathbb{C}$ and $\mathbb{C} \ni \mu \mapsto \psi_p(x, \mu) \in \mathbb{C}$. In these two steps the variable $x \in E$ can be regarded as a parameter, so the same argument applies, with obvious changes, to the present situation. $\square$

5.3. **Proof of the main result in the case of a 1-Lipschitz function taking values in** $[0, 1]$**.** Let us define

$$A = \{\alpha = \{\alpha_n\}_{n=1}^{\infty} \in \ell_\infty \ : \ 1 \leq \alpha_n \leq 1001 \ \text{ for all } n \in \mathbb{N}\}.$$

For every $\alpha \in A$ and $z = \{z_n\}_{n=1}^{\infty} \in \widetilde{c_0} \setminus \{0\}$, let us denote $\alpha z := \{\alpha_n z_n\}_{n=1}^{\infty}$, and observe that $\alpha z \in \widetilde{c_0} \setminus \{0\}$. In fact the mapping $\alpha : \widetilde{c_0} \to \widetilde{c_0}$, defined by $\alpha(z) = \alpha z$ is a linear isomorphism satisfying $\|z\|_\infty \leq \|\alpha z\|_\infty \leq 1001\|z\|_\infty$.

Lemma 4. *Let* $\varphi_n = \varphi_{n,r,\varepsilon}, \widetilde{\varphi}_n = \widetilde{\varphi}_{n,r,\varepsilon}$ *be the collections of functions defined in Lemma 3. We have that:*

(1) *There exists an open neighborhood* $\widetilde{V}_1 \subset \widetilde{V}$ *of* X *in* $\widetilde{X}$ *such that the mapping* $\widetilde{\Phi}_{\alpha,r,\varepsilon} = \widetilde{\Phi} : \widetilde{V}_1 \to \widetilde{c_0}$ *defined by*

$$\widetilde{\Phi}(z) = \{\alpha_n \widetilde{\varphi}_n(z)\}_{n=1}^{\infty}$$

is holomorphic, for any $\alpha \in A$*,* $r \geq 1$*,* $\varepsilon \in (0, 1)$*.*

(2) *Fix* $\varepsilon \in (0, 1)$*. Then there exists an open neighborhood* $\widetilde{W_\varepsilon}$ *of* X *in* $\widetilde{X}$ *such that for every* $\alpha \in A$*,* $r \geq 1$*, there is a holomorphic extension* $\widetilde{F}_{\alpha,r,\varepsilon}$ *of* $\lambda \circ \Phi_{\alpha,r,\varepsilon}$*, defined on* $\widetilde{W_\varepsilon}$*, where* λ *is the Preiss norm from section 4.1, and such that*

$$|\widetilde{F}_{\alpha,r,\varepsilon}(x + iy) - \lambda \circ \Phi_{\alpha,r,\varepsilon}(x)| \leq \varepsilon$$

for every $x, y \in X$ *with* $x + iy \in \widetilde{W_\varepsilon}$*.*

Proof. Because the numerical series $\sum_{n=1}^{\infty} 1/n!a_x^n$ is convergent, and bearing in mind property (5) of Lemma 3, we have that the series of functions

$$\sum_{n=1}^{\infty} |\alpha_n \widetilde{\varphi}_n(z)|$$

is uniformly convergent on the ball $B_{\widetilde{X}}(x, \delta_x)$, which clearly implies that the series

$$\sum_{n=1}^{\infty} \alpha_n \widetilde{\varphi}_n(z) e_n$$

is uniformly convergent for $z \in B_{\widetilde{X}}(x, \delta_x)$, for any $\{a_n\}_{n=1}^{\infty} \in \ell_\infty$. If we set $\widetilde{V}_1 = \bigcup_{x \in X} B_{\widetilde{X}}(x, \delta_x)$ and note that for each $n \in \mathbb{N}$ the mapping

$$\widetilde{V} \ni z \mapsto \varphi_n(z) e_n \in \widetilde{c_0}$$

is holomorphic, it is clear that $\widetilde{\Phi}_{\alpha,r,\varepsilon} = \sum_{n=1}^{\infty} \widetilde{\varphi}_{n,r,\varepsilon} e_n$, being a series of holomorphic mappings which converges locally uniformly in $\widetilde{V}_1$, defines a holomorphic mapping from $\widetilde{V}_1$ into $\widetilde{c_0}$. Note that $\widetilde{V}_1$ is independent of α, r, ε because so is δ_x.

Let us now prove (2). Define the function $\psi_{\alpha,r,\varepsilon} : \widetilde{V}_1 \times \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by

$$\widetilde{\psi}_{\alpha,r,\varepsilon}(w, \mu) = \sum_{n=0}^{\infty} \left(\frac{\alpha_n \widetilde{\varphi}_{n,r,\varepsilon}(w)}{\mu} \right)^{2n} - 1.$$

The function $\widetilde{\psi}_{\alpha,r,\varepsilon}$ is holomorphic on $\widetilde{V}_1$, because it is the composition of $\widetilde{\Phi}_{\alpha,r,\varepsilon} : \widetilde{V}_1 \to \widetilde{c_0}$ with the function $\widetilde{C} : \widetilde{c_0} \times \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $\widetilde{C}(\{z_n\}) = \sum_{n=1}^{\infty} z_n^{2n} - 1$, which is also holomorphic on $\widetilde{c_0} \times \mathbb{C} \setminus \{0\}$.

For $\varepsilon > 0$ fixed and $x \in X$ given, we are going to apply the preceding Proposition to the family of functions $\widetilde{\psi}_{\alpha,r,\varepsilon}$ indexed by $P = \{(\alpha, r) : \alpha \in A, r \geq 1\}$. So we have to find appropriate bounds for $\widetilde{\psi}_{\alpha,r,\varepsilon}$ and $\partial \psi_{\alpha,r,\varepsilon} / \partial \mu$.

For our given $x \in X$, we can apply properties (5) and (7) of Lemma 3 to find $n_x, a_x, \delta_x > 0$ (independent of r, ε) and $\delta_{x,\varepsilon} > 0$ (independent of r) such that

$$|\widetilde{\varphi}_{n,r,\varepsilon}(w)| < \frac{1}{n! a_x^n} \quad \text{for } n > n_x,\ r \geq 1,\ w \in \widetilde{X} \text{ with } \|w - x\|_{\widetilde{X}} < \delta_x.$$

and

$$|\widetilde{\varphi}_{n,r,\varepsilon}(w)| \leq 1 + 2\varepsilon \quad \text{for } n \in \mathbb{N}, r \geq 1,\ w \in \widetilde{X} \text{ with } \|w - x\|_{\widetilde{X}} \leq \delta_{x,\varepsilon}.$$

Let us fix $0 < R_{x,\varepsilon} < \min\{\varepsilon, 1/6, \delta_x, \delta_{x,\varepsilon}\}$, and observe that $\widetilde{V}_1 \supset B_{\widetilde{X}}(x, R_{x,\varepsilon})$.

Note that $\lambda(\Phi_{\alpha,r,\varepsilon}(x)) \geq \|\{\alpha_n \varphi_n(x)\}_{n=1}^{\infty}\|_\infty > 1$ by property (3) of Lemma 3, so in particular $|\mu| \geq \lambda(\Phi_{\alpha,r,\varepsilon}(x)) - R_{x,\varepsilon} \geq 1 - 1/6 = 5/6$ whenever $|\mu - \lambda(\Phi_{\alpha,r,\varepsilon}(x))| \leq R_{x,\varepsilon}$.

Then we have, for every $(\alpha, r) \in P$, for $\|w - x\|_{\widetilde{X}} \leq R_{x,\varepsilon}$, and for $|\mu - \lambda(\Phi_{\alpha,r,\varepsilon}(x))| \leq R_{x,\varepsilon}$,

$$|\widetilde{\psi}_{\alpha,r,\varepsilon}(w,\mu)| \leq \left| \sum_{n=1}^{\infty} \left(\frac{\alpha_n \widetilde{\varphi}_{n,r,\varepsilon}(w)}{\mu} \right)^{2n} - 1 \right|$$
$$\leq 1 + \sum_{n=1}^{n_x} \left(\frac{1001(1+2\varepsilon)}{5/6} \right)^{2n} + \sum_{n=n_x+1}^{\infty} \left(\frac{1}{n! a_x^n} \right)^{2n} := M_{x,\varepsilon}$$

(observe that $M_{x,\varepsilon}$ is independent of $(r, \alpha) \in P$).

On the other hand we can apply property (3) of Lemma 3 to find, for every $r \geq 1$, numbers

$$m_{x,r} \leq m_{x,1} \text{ such that } \varphi_{m_{x,r},r,\varepsilon}(x) > 1.$$

Since $\lambda \leq 2\| \cdot \|_\infty$ on c_0, and using property (2) of Lemma 3, we have

$$\lambda(\Phi_{\alpha,r,\varepsilon}(x)) \leq 2\|\Phi_{\alpha,r,\varepsilon}(x)\|_\infty \leq 2002\|\Phi_{1,r,\varepsilon}(x)\|_\infty \leq 4004.$$

So, if we set

$$A_x := \frac{1}{4004^{2m_{x,1}+1}}$$

(clearly independent of $(r, \alpha) \in P$), and we use the fact that $m_{x,r} \leq m_{x,1}$ for all $r \geq 1$, we can estimate

$$\left| \frac{\partial \psi_{\alpha,r,\varepsilon}}{\partial \mu}(x, \lambda(\Phi_{\alpha,r,\varepsilon}(x))) \right| = \left| - \sum_{n=1}^{\infty} 2n \left(\frac{\alpha_n \varphi_n(x)}{\lambda(\Phi_{\alpha,r,\varepsilon}(x))} \right)^{2n} \cdot \frac{1}{\lambda(\Phi_{\alpha,r,\varepsilon}(x))} \right|$$
$$\geq \sum_{n=1}^{\infty} 2n \left(\frac{\varphi_n(x)}{4004} \right)^{2n} \cdot \frac{1}{4004} \geq \frac{2m_{x,r}}{4004} \left(\frac{\varphi_{m_{x,r}}(x)}{4004} \right)^{2m_{x,r}}$$
$$\geq \frac{m_{x,r}}{4004} \left(\frac{1}{4004} \right)^{2m_{x,r}} \geq \frac{1}{4004} \left(\frac{1}{4004} \right)^{2m_{x,r}} \geq \frac{1}{4004} \left(\frac{1}{4004} \right)^{2m_{x,1}} = A_x.$$

Therefore, according to the preceding Proposition, we can find a number $s_x = s_{x,\varepsilon} > 0$ (independent of α, r) such that there is a holomorphic solution $\widetilde{\mu} = \widetilde{F}_{\alpha,r,\varepsilon}(w)$ to the implicit equation $\widetilde{\psi}_{\alpha,r,\varepsilon}(w,\mu) = 0$, defined on the ball $B_{\widetilde{X}}(x, s_x)$, for every $\alpha \in A, r \geq 1$. Since the solution is locally unique, two holomorphic functions which coincide on a neighborhood of a point are equal to one another in the connected component of that point, and the function $w \mapsto \widetilde{\lambda}(\widetilde{\Phi}_{\alpha,r,\varepsilon}(w))$ also solves the implicit equation $\widetilde{\psi}_{\alpha,r,\varepsilon}(w,\mu) = 0$ for μ in terms of w in a neighborhood of x, one easily deduces that $\widetilde{F}_{\alpha,r,\varepsilon}$ can be defined on all of

$$\widetilde{W}_\varepsilon := \bigcup_{x \in X} B_{\widetilde{X}}(x, s_{x,\varepsilon}),$$

and that $\widetilde{F}_{\alpha,r,\varepsilon}$ is a holomorphic extension of $\lambda \circ \Phi_{\alpha,r,\varepsilon}$.

Besides, by the same Proposition, we also have that for every $\alpha \in A, r \geq 1$ the function $\widetilde{F}_{\alpha,r,\varepsilon}$ maps the ball $B_{\widetilde{X}}(x, s_x)$ into a disc $B_{\mathbb{C}}(\lambda(\Phi_{\alpha,r,\varepsilon}(x)), r_x)$, where $0 < r_{x,\varepsilon} < R_{x,\varepsilon} \leq \varepsilon$. Hence we have that

$$|\widetilde{F}_{\alpha,r,\varepsilon}(x+iy) - \lambda \circ \Phi_{\alpha,r,\varepsilon}(x)| \leq \varepsilon$$

for every $x, y \in X$ with $x + iy \in \widetilde{W}_\varepsilon$. $\square$

Now we turn to the proof of Theorem 1. We shall first consider the case when $f : X \to [1, 1001]$ is L-Lipschitz with $0 < L \leq 1$. Define $r = 1/L$, so $r \geq 1$, rename $\varepsilon = \delta \in (0, 10^{-4})$, and apply Lemma 3 to find a corresponding collection of functions $\varphi_n = \varphi_{n,\frac{1}{L},\delta}$, $n \in \mathbb{N}$. The functions φ_n are thus $2C/r$-Lipschitz, where we denote $C = \mathrm{Lip}(Q)$.

Define a function $g : X \to \mathbb{R}$ by

$$g(x) = \frac{\lambda(\{f(x_n)\varphi_n(x)\}_{n=1}^\infty)}{\lambda(\{\varphi_n(x)\}_{n=1}^\infty)} = \frac{\lambda \circ \Phi_{\alpha,r,\delta}(x)}{\lambda \circ \Phi_{1,r,\delta}(x)}$$

where here we denote $\alpha = \{f(x_n)\}_{n=1}^\infty \in A$, and also $1 = (1, 1, 1, ...) \in A \subset \ell_\infty$.

The function g is well defined, is real analytic, and has a holomorphic extension $\widetilde{g}$ defined on the neighborhood $\widetilde{U} := \widetilde{W}_\delta$ of X in $\widetilde{X}$ provided by Lemma 4, given by

$$\widetilde{g}(w) = \frac{\widetilde{F}_{\alpha,r,\delta}(w)}{\widetilde{F}_{1,r,\delta}(w)}.$$

Since the functions φ_n are $2C/r$-Lipschitz, the norm $\lambda : c_0 \to \mathbb{R}$ is 2-Lipschitz (with respect to the usual norm $\|\cdot\|_\infty$ of c_0), and $1 \leq f \leq 1001$, we have

$$\begin{aligned}
&|\lambda(\{f(x_n)\varphi_n(x)\}_{n=1}^\infty) - \lambda(\{f(x_n)\varphi_n(y)\}_{n=1}^\infty)| \leq \\
&2\|\{f(x_n)(\varphi_n(x) - \varphi_n(y))\}_{n=1}^\infty)\|_\infty \leq 2002\|\{\varphi_n(x) - \varphi_n(y)\}_{n=1}^\infty)\|_\infty \leq \\
&\frac{4004C}{r}\|x - y\|,
\end{aligned}$$

that is the function $\lambda \circ \Phi_{\alpha,r,\delta}$ is $4004C/r$-Lipschitz on X, and is bounded by 2002. Similarly, since the function $t \mapsto 1/t$ is 1-Lipschitz on $[1, \infty)$ and $\lambda \circ \Phi_{1,r,\delta}$ is bounded below by 1, we have that the function $1/\lambda \circ \Phi_{1,r,\delta}$ is also $4004C/r$-Lipschitz on X, and bounded above by 1. Therefore the product satisfies

$$\mathrm{Lip}(g) \leq 2002 \times \frac{4004C}{r} + 1 \times \frac{4004C}{r} = \frac{8020012C}{r} = 8020012C\mathrm{Lip}(f).$$

On the other hand, we have

$$
\begin{aligned}
|g(x)-f(x)| &= \left|\frac{\lambda\left(\{f\left(x_j\right)\varphi_j\left(x\right)\}\right)}{\lambda\left(\{\varphi_j\left(x\right)\}\right)} - f\left(x\right)\right| \\
&= \left|\frac{\lambda\left(\{f\left(x_j\right)\varphi_j\left(x\right)\}\right)}{\lambda\left(\{\varphi_j\left(x\right)\}\right)} - \frac{f\left(x\right)\lambda\left(\{\varphi_j\left(x\right)\}\right)}{\lambda\left(\{\varphi_j\left(x\right)\}\right)}\right| \\
&= \frac{1}{\lambda\left(\{\varphi_j\left(x\right)\}\right)}\left|\lambda\left(\{f\left(x_j\right)\varphi_j\left(x\right)\}\right) - \lambda\left(\{f\left(x\right)\varphi_j\left(x\right)\}\right)\right| \\
&\leq \frac{1}{\lambda\left(\{\varphi_j\left(x\right)\}\right)}\lambda\left(\{f(x_j)\varphi_j\left(x\right)\} - \{f\left(x\right)\varphi_j\left(x\right)\}\right) \\
&= \frac{1}{\lambda\left(\{\varphi_j\left(x\right)\}\right)}\lambda\left(\{\varphi_j\left(x\right)\left(f\left(x_j\right)-f\left(x\right)\right)\}\right).
\end{aligned}
$$

Now recall that

$$
\begin{aligned}
\lambda\left(\{\varphi_j\left(x\right)\left(f\left(x_j\right)-f\left(x\right)\right)\}\right) &\leq 2\left\|\{\varphi_j\left(x\right)\left(f\left(x_j\right)-f\left(x\right)\right)\}\right\|_\infty \\
&= 2\max_j\left\{\varphi_j\left(x\right)\left|f\left(x_j\right)-f\left(x\right)\right|\right\}.
\end{aligned}
$$

Set $J=\{j : x\in D_Q(x_j,4r)\}$. For $j\in J$, according to Lemma 2 we have $\|x-x_j\|_X<8r$ and so, because $\mathrm{Lip}(f)=1/r$,

$$\varphi_j\left(x\right)\left|f\left(x_j\right)-f\left(x\right)\right| \leq \varphi_j\left(x\right)8.$$

It follows that for $j\in J$

$$\frac{\varphi_j\left(x\right)\left|f\left(x_j\right)-f\left(x\right)\right|}{\lambda\left(\{\varphi_j\left(x\right)\}\right)} \leq \frac{\varphi_j\left(x\right)8}{\left\|\{\varphi_j\left(x\right)\}\right\|_\infty} \leq 8.$$

On the other hand, for $j\notin J$ we have by part (4) of Lemma 3, that

$$\varphi_j\left(x\right)\left|f\left(x_j\right)-f\left(x\right)\right| \leq 2002\varphi_j\left(x\right) \leq 2002\delta < 1.$$

Hence, given that $\lambda\left(\{\varphi_j\left(x\right)\}\right)\geq 1$, we have for $j\notin J$

$$\frac{\varphi_j\left(x\right)\left|f\left(x_j\right)-f\left(x\right)\right|}{\lambda\left(\{\varphi_j\left(x\right)\}\right)} \leq 1.$$

It follows that

$$|g(x)-f(x)|\leq 8.$$

If we reset C to $8020012C$, this argument proves Theorem 1 in the case when $\varepsilon=8$, $f:X\to[1,1001]$, $\mathrm{Lip}(f)\leq 1$.

Moreover, according to property (2) of Lemma 4, we have

$$|\widetilde{F}_{\alpha,r,\delta}(x+iy)-\lambda(\Phi_{\alpha,r,\delta}(x))|\leq\delta$$

for every $x, y \in X$ with $x+iy \in \widetilde{W}_\delta$, $\alpha \in A$. Therefore, taking into account that $\lambda(\Phi_{\alpha,r,\delta}(x)) \leq 2002$ and $1 \leq \lambda(\Phi_{1,r,\delta}(x)) \leq 2$, we have

$$|\widetilde{g}(x+iy) - g(x)| = \left| \frac{\widetilde{F}_{\alpha,r,\delta}(x+iy)}{\widetilde{F}_{1,r,\delta}(x+iy)} - \frac{\lambda(\Phi_{\alpha,r,\delta}(x))}{\lambda(\Phi_{1,r,\delta}(x))} \right| =$$

$$\left| \frac{1}{\widetilde{F}_{1,r,\delta}(x+iy)\lambda(\Phi_{1,r,\delta}(x))} \right| \cdot |\lambda(\Phi_{1,r,\delta}(x)) \left(\widetilde{F}_{\alpha,r,\delta}(x+iy) - \lambda(\Phi_{\alpha,r,\delta}(x)) \right)$$

$$+\lambda(\Phi_{\alpha,r,\delta}(x)) \left(\lambda(\Phi_{1,r,\delta}(x)) - \widetilde{F}_{1,r,\delta}(x+iy) \right) | \leq$$

$$\frac{1}{1-\delta}(2\delta + 2002\delta) = \frac{\delta}{1-\delta} 2004$$

for every $x, y \in X$ with $x+iy \in \widetilde{W}_\delta$, $\alpha \in A$, $r \geq 1$.

Up to scaling and subtracting appropriate constants we have thus proved the following intermediate result.

Proposition 2. *Let X be a Banach space having a separating polynomial. Then there exists $C \geq 1$ (depending only on X) such that, for every $\delta > 0$ there is an open neighborhood $\widetilde{U}_\delta$ of X in $\widetilde{X}$ such that, for every Lipschitz function $f : X \to [0,1]$ with $Lip(f) \leq 1$, there exists a real analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\widetilde{g} : \widetilde{U}_\delta \to \mathbb{C}$, such that*

1. *$|f(x) - g(x)| \leq 1/10$ for all $x \in X$.*
2. *g is Lipschitz, with $Lip(g) \leq C Lip(f)$.*
3. *$|\widetilde{g}(x+iy) - g(x)| \leq \delta$ for all $z = x+iy \in \widetilde{U}_\delta$.*

Next we are going to see that this result remains true for all bounded, nonnegative functions f with $\mathrm{Lip}(f) \leq 1$ if we only allow C to be slightly larger and we replace $1/10$ by 1.

For a 1-Lipschitz, bounded function $f : X \to [0, +\infty)$ we define, for $n \in \mathbb{N}$, the functions

$$f_n(x) = \begin{cases} f(x) - n + 1 & \text{if } n-1 \leq f(x) \leq n, \\ 0 & \text{if } f(x) \leq n-1, \\ 1 & \text{if } n \leq f(x). \end{cases}$$

Note that since f is bounded there exists $N \in \mathbb{N}$ such that $f_n = 0$ for all $n \geq N$. The functions f_n are clearly 1-Lipschitz and take values in the interval $[0,1]$, so, for a given $\delta > 0$ (to be fixed later on), by what has been proved immediately above, there exist a neighborhood $\widetilde{U}_\delta$ of X in $\widetilde{X}$ and C-Lipschitz, real analytic functions $g_n : X \to \mathbb{R}$, with holomorphic extensions $\widetilde{g}_n : \widetilde{U}_\delta \to \mathbb{C}$, such that for all $n \in \mathbb{N}$ we have that g_n is C-Lipschitz, $|g_n - f_n| \leq 1/10$, and $|\widetilde{g}(x+iy) - g(x)| \leq \delta$ for all $z = x+iy \in \widetilde{U}_\delta$. Since $f_n = 0$ for all $n \geq N$, we may obviously assume that $g_n = 0$ for all $n \geq N$, as well.

Now define a path $\beta : [0, +\infty) \to \ell_\infty$ by,

$$\beta(t) = \left(1, \cdots, 1, \underbrace{t - n + 1}_{n^{\text{th}} \text{ place}}, 0, 0, \cdots \right) = \sum_{j=1}^{n-1} e_j + (t-n+1)e_n \text{ if } n-1 \le t \le n.$$

Clearly the path β is a 1-Lipschitz injection of $[0, +\infty)$ into ℓ_∞, with a uniformly continuous (but not Lipschitz) inverse $\beta^{-1} : \beta([0, +\infty)) \to [0, +\infty)$.

Define a uniformly continuous (not Lipschitz) function h on the path β by $h(\beta(t)) = t$ for all $t \ge 0$, that is $h(y) = \beta^{-1}(y)$ for $y \in \beta([0, +\infty))$. Then we have $f(x) = h(\{f_n(x)\}_{n=1}^\infty)$.

5.4. **The gluing tube function.** Now we are going to construct an open tube of radius $1/8$ (with respect to the supremum norm $\|\cdot\|_\infty$) around the path β in ℓ_∞, and a real-analytic approximate extension (with bounded derivative) H of the function h defined on this tube. This construction is meant to be used as follows: since $|g_n - f_n| \le 1/10$ and $\{f_n(x)\}_{n=1}^\infty$ takes values in the path β, then $\{g_n(x)\}_{n=1}^\infty$ will take values in this tube, and therefore $g(x) := H(\{g_n(x)\}_{n=1}^\infty)$ will approximate $H(\{f_n(x)\}_{n=1}^\infty)$, which in turn approximates $h(\{f_n(x)\}_{n=1}^\infty) = f(x)$. Besides, since H has a bounded derivative on the tube and the functions g_n are C-Lipschitz then g will be CM-Lipschitz, where M is an upper bound of DH on the tube.

Lemma 5. *There exists a real-analytic mapping $G : \ell_\infty \to \ell_\infty$ with holomorphic extension $\widetilde{G} : \widetilde{\ell_\infty} \to \widetilde{\ell_\infty}$ such that:*

(1) *G diffeomorphically maps the* straight tube *$\mathcal{S}$ defined by $\{x \in \widetilde{\ell_\infty} : |x_n| < 1/7$ for all $n \ge 2, x_1 > -1/7\}$ onto a* twisted tube *$\mathcal{T}$ around the path β in such a way that*

$$\{x \in \ell_\infty : dist(x, \beta([0,\infty))) < \frac{1}{8}\} \subseteq G(\mathcal{S}) \subseteq \{x \in \ell_\infty : dist(x, \beta([0,\infty))) < \frac{1}{6}\}.$$

(2) *The function $H : \mathcal{T} \to \mathbb{R}$ defined by $H = e_1^* \circ G^{-1}$ satisfies*

$$\|x - \beta(t)\|_\infty < \frac{1}{8} \implies |H(x) - t| < \frac{3}{4}.$$

(3) *The maps $G_{|\mathcal{S}}$ and $(G_{|\mathcal{S}})^{-1}$ have bounded derivatives on the sets $\mathcal{S}$ and $\mathcal{T}$ respectively.*

(4) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|\widetilde{G}(x + iy) - G(x)\|_\infty \le \varepsilon$$

for all $x, y \in \mathcal{S}$ with $\|y\|_\infty \le \delta$.

(5) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that a holomorphic extension $\widetilde{G^{-1}}$ of G^{-1} is defined from $\widetilde{\mathcal{T}_\delta} := \{u + iv : u \in \mathcal{T}, v \in \ell_\infty, \|v\|_\infty < \delta\}$ into $\widetilde{\ell_\infty}$, with the property that*

$$\|\widetilde{G^{-1}}(u + iv) - G^{-1}(u)\|_\infty \le \varepsilon$$

for all $x \in \mathcal{S}, u \in \mathcal{T}$ and $v \in \ell_\infty$ such that $u + iv \in \widetilde{\mathcal{T}_\delta}$.

Proof. Let $\overline{\Phi}:\mathbb{R}^2\to\mathbb{R}^2$ be a C^∞ mapping such that

(i) $\overline{\Phi}$ is the identity on the rectangle $[-1/6,3/4]\times[-1/6,1/6]$;

(ii) $\overline{\Phi}$ maps isometrically the rectangle $[5/4,2]\times[-1/6,1/6]$ onto the rectangle $[1-1/6,1+1/6]\times[1/4,1]$, and $D\Phi(s,t)(1,0)=(0,1)$ for all $(s,t)\in[3/2,2]\times[-1/6,1/6]$;

(iii) $\overline{\Phi}$ is a diffeomorphism from $[0,2]\times[-\frac{1}{6},\frac{1}{6}]$ onto $\overline{\Phi}\left([0,2]\times[-\frac{1}{6},\frac{1}{6}]\right)$;

(iv) considering, by abusing notation, that $\beta:[0,2]\to\mathbb{R}^2\subset\ell_\infty$, we have

$$\{(s,t)\in\mathbb{R}^2:\operatorname{dist}((s,t),\beta([0,2]))<\frac{1}{8}+\frac{1}{64}\}\subseteq$$
$$\overline{\Phi}([0,2]\times[-1/7,1/7])\subseteq\{(s,t)\in\mathbb{R}^2:\operatorname{dist}((s,t),\beta([0,2]))<\frac{1}{6}-\frac{1}{64}\},$$
$$\|\overline{\Phi}(t,0)-\beta(t)\|\leq\frac{1}{64}\text{ for all }t\in[0,2];$$

(v) $\overline{\Phi}(s,t)=(0,0)$ if $|s|,|t|\geq 10$.

Notice that the local isometry property in (ii) is provided by a $\pi/2$-rotation followed by a translation, which is an affine isometry with respect to the norm $\|\cdot\|_\infty$ as well as to the Euclidean norm in $\mathbb{R}^2$. In fact this local isometry is given by $(x,y)\mapsto(1-y,x-1)$.

Observe that such a mapping $\overline{\Phi}$ also satisfies

(vi) $|e_1^*\circ\overline{\Phi}^{-1}(x)-t|<\frac{1}{2}+\frac{1}{8}$ whenever $\|x-\beta(t)\|_\infty<\frac{1}{8}, t\in[0,2]$.

Indeed, assume $\|x-\beta\|_\infty<1/8$. If $x\in[0,\frac{3}{4}]\times[-\frac{1}{6},\frac{1}{6}]\cup[1-\frac{1}{6},1+\frac{1}{6}]\times[\frac{1}{4},1]$ it is clear, by (ii), that $|e_1^*\circ\overline{\Phi}^{-1}(x)-e_1^*\circ\overline{\Phi}^{-1}(\beta(t))|<1/8$, and also using (iv),

$$\begin{aligned}&|e_1^*\circ\overline{\Phi}^{-1}(x)-t|=|e_1^*\circ\overline{\Phi}^{-1}(x)-e_1^*\circ\overline{\Phi}^{-1}(\overline{\Phi}(t,0))|\leq\\&|e_1^*\circ\overline{\Phi}^{-1}(x)-e_1^*\circ\overline{\Phi}^{-1}(\beta(t))|+|e_1^*\circ\overline{\Phi}^{-1}(\beta(t))-e_1^*\circ\overline{\Phi}^{-1}(\overline{\Phi}(t,0))|=\\&|e_1^*(x)-e_1^*(\beta(t))|+|e_1^*(\beta(t))-e_1^*(\overline{\Phi}(t,0))|<\\&\frac{1}{8}+\frac{1}{64}.\end{aligned}$$

On the other hand, if $x\notin[0,\frac{3}{4}]\times[-\frac{1}{6},\frac{1}{6}]\cup[1-\frac{1}{6},1+\frac{1}{6}]\times[\frac{1}{4},1]$ then, by $(ii),(iii)$ and (iv), we have $\overline{\Phi}^{-1}(x)\in[\frac{3}{4},\frac{5}{4}]\times[-\frac{1}{6},\frac{1}{6}]$ and in particular $e_1^*\circ\overline{\Phi}^{-1}(x)\in[\frac{3}{4},\frac{5}{4}]$, which together with $\|x-\beta(t)\|_\infty<1/8$ implies $t\in[\frac{3}{4}-\frac{1}{8},\frac{5}{4}+\frac{1}{8}]$. Therefore

$$\frac{3}{4}-\frac{5}{4}-\frac{1}{8}<e_1^*\circ\overline{\Phi}^{-1}(x)-t<\frac{5}{4}-\frac{3}{4}+\frac{1}{8},$$

that is $|e_1^*\circ\overline{\Phi}^{-1}(x)-t|<\frac{1}{2}+\frac{1}{8}$. In any case (vi) is satisfied.

Denote $\overline{\Phi}=(\overline{\varphi},\overline{\psi})$. Since the coordinate functions $\overline{\varphi},\overline{\psi}$ have bounded support, their derivatives (of all orders) are bounded, and in particular Φ is a Lipschitz mapping with a Lipschitz derivative.

Now set

$$\overline{\Phi}_{(1,2)}(x) = \overline{\varphi}(x_1, x_2)e_1 + \overline{\psi}(x_1, x_2)e_2 + \sum_{k\geq 3} x_k e_k,$$

and, for $n \geq 1$, define $\overline{\Phi}_{(n+1,n+2)} : \ell_\infty \to \ell_\infty$ by

$$\overline{\Phi}_{(n+1,n+2)}(x) = \sum_{j=1}^{n}(1 - x_{j+1})e_j + \overline{\varphi}(x_1 - n, x_{n+2})e_{n+1} + \overline{\psi}(x_1 - n, x_{n+2})e_{n+2} + \sum_{k\geq n+3} x_k e_k.$$

One can check that the mappings $\overline{\Phi}_{(n,n+1)}$ are C^∞ smooth on ℓ_∞, satisfy $\overline{\Phi}_{(n,n+1)}(x) = \overline{\Phi}_{(n+1,n+2)}(x)$ whenever $n+1/4 \leq x_1 \leq n+3/4$ and $|x_j| < 1/6$ for all $j \geq 2$. Moreover, there exists $N > 0$ such that

$$\|\overline{\Phi}_{(n,n+1)}(x)\|_\infty \leq N \text{ for all } x \in \mathcal{S}, n \in \mathbb{N}$$

and

$$\|D^k\overline{\Phi}_{(n,n+1)}(x)\| \leq N \text{ for all } x \in \ell_\infty, n \in \mathbb{N}, k \in \{1, 2\}.$$

Then define $\overline{G} : \mathcal{S} \subset \ell_\infty \to \ell_\infty$ by

$$\overline{G}(x) = \sum_{n=0}^{\infty} \overline{\theta}(x_1 - n)\overline{\Phi}_{(n+1,n+2)}(x),$$

where $\overline{\theta} : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that

$\overline{\theta}(t) = 0$ if and only if $t \in (-\infty, -1/2] \cup [3/4, +\infty)$;
$\overline{\theta}(t) = 1$ if and only if $t \in [-1/4, 1/2]$;
$\overline{\theta}'(t) > 0$ if and only if $t(-1/2, -1/4)$;
$\overline{\theta}(t) = 1 - \overline{\theta}(t - 1)$ if $t \in (1/2, 3/4)$.

Note in particular that the collection of functions $t \mapsto \overline{\theta}(t - n)$, $n \in \mathbb{N}$, form a C^∞ partition of unity on $\mathbb{R}$.

It is not difficult to check that the mapping $\overline{G} : \mathcal{S} \subset \ell_\infty \to \ell_\infty$ has properties (1), (2) and (3) of the statement (with the slightly sharper bound $\frac{1}{2} + \frac{1}{8}$, instead of $\frac{3}{4}$, in (2); this is shown as in (vi) above). In particular $\overline{G}^{-1}$ is uniformly continuous and there exists $m > 0$ such that

$$\|D\overline{G}(x)(h)\| \geq m\|h\| \text{ for all } x, h \in \ell_\infty.$$

But of course $\overline{G}$ is not real analytic. In order to obtain a required real-analytic function G we shall substitute $\overline{\varphi}, \overline{\psi}$ and $\overline{\theta}$ with real-analytic approximations of these functions defined by appropriate integral convolutions with Gaussian kernels in $\mathbb{R}^2$ and $\mathbb{R}$. Namely, let us define

$\varphi(s,t) = a_\kappa \int_{\mathbb{R}^2} \overline{\varphi}(u,v) \exp\left(-\kappa\left[(s-u)^2 + (t-v)^2\right]\right) dudv,$
$\psi(s,t) = a_\kappa \int_{\mathbb{R}^2} \overline{\psi}(u,v) \exp\left(-\kappa\left[(s-u)^2 + (t-v)^2\right]\right) dudv,$
$\theta(t) = b_\kappa \int_{-\infty}^{+\infty} \overline{\theta}(u)e^{-\kappa(t-u)^2}du,$
$\Phi_{(n+1,n+2)}(x) = \sum_{j=1}^{n}(1 - x_{j+1})e_j + \varphi(x_1 - n, x_{n+2})e_{n+1} +$
$+ \psi(x_1 - n, x_{n+2})e_{n+2} + \sum_{k\geq n+3} x_k e_k.$

where

$a_\kappa = 1/\int_{\mathbb{R}^2} \exp\left(-\kappa\left[u^2+v^2\right]\right) dudv,$ and
$b_\kappa = 1/\int_{-\infty}^{+\infty} e^{-\kappa v^2} dv.$

By taking κ large enough we may assume that the functions φ, ψ, θ and their first and second derivatives are as close to $\overline{\varphi}, \overline{\psi}, \overline{\theta}$ and their first and second derivatives, respectively, as we want. Therefore the mappings $\overline{\Phi}_{(n,n+1)}$ and their first and second derivatives can be taken as close as needed to $\Phi_{(n,n+1)}$ and their first and second derivatives, say

$$\|D^{(k)}\Phi_{(n,n+1)}(x) - D^{(k)}\overline{\Phi}_{(n,n+1)}(x)\|_\infty \leq \varepsilon/2$$

for all $x \in \ell_\infty$, $k = 0, 1, 2$. Moreover, it is easily checked that the holomorphic extensions $\widetilde{\varphi}, \widetilde{\psi}$ of φ, ψ to $\mathbb{C}^2$, as well as the holomorphic extension of θ to $\mathbb{C}$ (defined by the same formulae by letting $s, t \in \mathbb{C}$) satisfy

$$\begin{aligned}
&|\varphi(s+ir, t+iw)| \leq e^{\kappa(r^2+w^2)}|\varphi(s,t)|,\\
&|\psi(s+ir, t+iw)| \leq e^{\kappa(r^2+w^2)}|\psi(s,t)|,\\
&|\theta(a+ib)| \leq e^{\kappa b^2}|\theta(a)|
\end{aligned}$$

and in particular we have

$$\begin{aligned}
&|\varphi(s+ir, t+iw)| \leq (1+\varepsilon)|\varphi(s,t)|,\\
&|\psi(s+ir, t+iw)| \leq (1+\varepsilon)|\psi(s,t)|,\\
&|\theta(a+ib)| \leq (1+\varepsilon)|\theta(a)|
\end{aligned}$$

provided that $\max\{|r|^2+|w|^2, |b|^2\} < \frac{\log(1+\varepsilon)}{\kappa}$. It follows that the corresponding holomorphic extension $\widetilde{\Phi}_{(n,n+1)}$ of $\Phi_{(n,n+1)}$ to $\widetilde{\ell}_\infty$ (defined by a similar formula just replacing φ, ψ with $\widetilde{\varphi}, \widetilde{\psi}$) satisfies

$$\|\widetilde{\Phi}_{(n,n+1)}(x+iz)\|_\infty \leq (1+\varepsilon)\|\Phi_{(n,n+1)}(x)\|_\infty$$

for all $x, z \in \ell_\infty$ such that $\|z\|_\infty \leq \sqrt{\frac{\log(1+\varepsilon)}{2\kappa}}$.

Now define

$$G(x) = \sum_{n=0}^{\infty} \theta(x_1 - n)\Phi_{(n+1,n+2)}(x).$$

Let us see that G is real analytic and has a holomorphic extension $\widetilde{G}$ to $\widetilde{\ell}_\infty$ defined by

$$\widetilde{G}(x+iz) = \sum_{n=0}^{\infty} \widetilde{\theta}(x_1 - n + iz_1)\widetilde{\Phi}_{(n+1,n+2)}(x+iz).$$

It is enough to check that the series of holomorphic functions defining $\widetilde{G}$ is locally uniformly absolutely convergent. Taking into account that the mappings $\widetilde{\Phi}_{(n,n+1)}$ are clearly uniformly bounded on bounded sets (meaning that for every $R > 0$ there exists $K > 0$ such that $\|\widetilde{\Phi}_{(n,n+1)}(x + iz)\| \leq K$ for all $x, z \in B(0, R)$ and all $n \in \mathbb{N}$), this amounts to showing that

$$\sum_{n=0}^{\infty} |\widetilde{\theta}(t - n + is)| < +\infty$$

locally uniformly for $t + is \in \mathbb{C}$. Assume $|t|, |s| < R \leq n$, $u \in [-1/2, 3/4]$, then we have $-\kappa|t - n|^2 + 2\kappa(t - n)u \leq -\kappa(n - R)^2 + \frac{3}{2}\kappa(n + R)$, hence

$$0 \leq \theta(t - n) = \frac{1}{\int_{-\infty}^{\infty} e^{-\kappa v^2} dv} \int_{-\infty}^{\infty} \overline{\theta}(u) e^{-\kappa(t-n-u)^2} du \leq$$
$$\frac{1}{\int_{-\infty}^{\infty} e^{-\kappa v^2} dv} e^{-\kappa\left[(n-R)^2 - \frac{3}{2}(n+R)\right]} \int_{-1/2}^{3/4} e^{-\kappa u^2} du \leq e^{-\kappa\left[(n-R)^2 - \frac{3}{2}(n+R)\right]},$$

and consequently

$$|\widetilde{\theta}(t - n + is)| \leq \theta(t - n) e^{\kappa s^2} \leq e^{-\kappa\left[(n-R)^2 - \frac{3}{2}(n+R) - R^2\right]}.$$

Since

$$\sum_{n \geq R}^{\infty} e^{-\kappa\left[(n-R)^2 - \frac{3}{2}(n+R) - R^2\right]} < +\infty,$$

it is then clear that the series $\sum_{n=0}^{\infty} |\widetilde{\theta}(t - n + is)| <$ is bounded on $\{t + is : |t|, |s| < R\}$ by an absolutely convergent numerical series, and therefore it is locally uniformly convergent.

Now let us show that, given $\varepsilon > 0$, G satisfies $\|G(x) - \overline{G}(x)\|_\infty \leq \varepsilon$ for all $x \in \mathcal{S}$, provided $\kappa > 0$ is large enough. Let us first observe that we can take $\kappa > 0$ sufficiently large so that

$$\sum_{n=0}^{\infty} |\theta(t - n) - \overline{\theta}(t - n)| \leq \frac{\varepsilon}{2(N + 1)}.$$

Indeed, on the one hand, if $|t - n| \geq 2$ we have $\overline{\theta}(t - n) = 0$, and $|t - n|^2 - \frac{3}{2}|t - n| = |t - n|\left(|t - n| - \frac{3}{2}\right) \geq \frac{1}{2}|t - n|$, hence

$$|\theta(t - n) - \overline{\theta}(t - n)| = \theta(t - n) = \frac{1}{\int_{-\infty}^{\infty} e^{-\kappa v^2} dv} \int_{-\infty}^{\infty} \overline{\theta}(u) e^{-\kappa(t-n-u)^2} du \leq$$
$$\frac{1}{\int_{-\infty}^{\infty} e^{-\kappa v^2} dv} e^{-\kappa\left(|n-t|^2 - \frac{3}{2}|t-n|\right)} \int_{-1/2}^{3/4} e^{-\kappa u^2} du \leq e^{-\kappa|t-n|/2},$$

consequently

$$\sum_{|t-n| \geq 2} |\theta(t - n) - \overline{\theta}(t - n)| \leq \sum_{|t-n| \geq 2} e^{-\kappa|t-n|/2} \leq 2 \sum_{n=2}^{\infty} e^{-\kappa n/2},$$

and because $2\sum_{n=2}^{\infty} e^{-\kappa n/2} \to 0$ as $\kappa \to +\infty$, we may assume κ is large enough so that

$$\sum_{|t-n|\geq 2} |\theta(t-n) - \overline{\theta}(t-n)| \leq \frac{\varepsilon}{4(N+1)}.$$

On the other hand, we may also assume $\kappa > 0$ is large enough so that $|\theta(u) - \overline{\theta}(u)| \leq \varepsilon/16(N+1)$ for all $u \in \mathbb{R}$, and clearly there are at most four integers n with $|t-n| < 2$, so we have

$$\sum_{|t-n|<2} |\theta(t-n) - \overline{\theta}(t-n)| \leq 4\frac{\varepsilon}{16(N+1)} = \frac{\varepsilon}{4(N+1)}.$$

By combining the two last inequalities we get

$$\sum_{n=0}^{\infty} |\theta(t-n) - \overline{\theta}(t-n)| \leq \frac{\varepsilon}{2(N+1)}$$

as required.

Now, for every $x \in \mathcal{S}$ we can estimate

$$\begin{aligned}
\|G(x) - \overline{G}(x)\|_\infty &\leq \|\sum_{n=0}^{\infty} \big(\theta(x_1-n) - \overline{\theta}(x_1-n)\big)\, \Phi_{n+1,n+2}(x)\| + \\
&+\|\sum_{n=0}^{\infty} \overline{\theta}(x_1-n)\left(\Phi_{n+1,n+2}(x) - \overline{\Phi}_{n+1,n+2}(x)\right)\| \\
&\leq \sum_{n=0}^{\infty} \|\Phi_{n+1,n+2}(x)\|\, |\theta(x_1-n) - \overline{\theta}(x_1-n)| + \\
&+ \sum_{n=0}^{\infty} \overline{\theta}(x_1-n) \|\Phi_{n+1,n+2}(x) - \overline{\Phi}_{n+1,n+2}(x)\| \\
&\leq (N+1)\sum_{n=0}^{\infty} |\theta(x_1-n) - \overline{\theta}(x_1-n)| + \frac{\varepsilon}{2}\sum_{n=0}^{\infty} \overline{\theta}(x_1-n) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

By using the facts that the derivatives θ', θ'' of θ decrease exponentially at infinity and approximate $\overline{\theta}', \overline{\theta}''$, and by performing similar calculations, one can also show that

$$\|DG(x) - D\overline{G}(x)\| \leq \varepsilon, \text{ and } \|D^2G(x) - D^2\overline{G}(x)\| \leq \varepsilon, \text{ for all } x \in \mathcal{S}$$

provided $\kappa > 0$ is large enough.

Let us now see that G is a diffeomorphism from $\mathcal{S}$ onto $\mathcal{T} := G(\mathcal{S})$. We know that $\overline{G}$ is of class C^∞ with bounded derivatives of all orders, in particular $\overline{G}$ is Lipschitz and has a Lipschitz derivative. Moreover $D\overline{G}(x)$ is a linear isomorphism for all $x \in \mathcal{S}$ and the mapping $x \mapsto \|D\overline{G}(x)^{-1}\|$ is bounded above on $\mathcal{S}$ by a number $M > 0$. This implies that if $L : \ell_\infty \to \ell_\infty$ is a linear mapping such that $\|L\| < 1/M$ then $D\overline{G}(x) + L$ is a linear isomorphism as well, for every $x \in \mathcal{S}$. Since we may assume that

$\|D\overline{G}(x) - DG(x)\| < 1/M$ for all $x \in \mathcal{S}$, we have that $DG(x)$ is a linear isomorphism for all $x \in \mathcal{S}$

Now we recall that a standard proof of the inverse mapping theorem (the one that uses the fixed point theorem for contractive mappings) shows that if a function F (defined on an open set U of a Banach space X and taking values in X) has the property that $DF(x)$ is a linear isomorphism for all x, the mappings $x \mapsto DF(x)$ and $x \mapsto DF(x)^{-1}$ are bounded, and the mapping $x \mapsto DF(x)$ is Lipschitz, then there exist uniform lower bounds $r, s > 0$ (depending only on $\mathrm{Lip}(DF)$ and the bounds for $DF, (DF)^{-1}$) such that F maps diffeomorphically each open ball $B(x,r)$ contained in U onto an open subset of X containing the ball $B(F(x), s)$; see for instance [AMR, Proposition 2.5.6]. Since $D\overline{G}, D^2\overline{G}$ are bounded, and DG, D^2G are being assumed to be close enough to these functions for large κ, it follows that DG, D^2G are uniformly bounded as well for all $\kappa > 0$ large enough, and we can apply the mentioned fact with $F = G$ to conclude that there exists $r > 0$ such that, for all $\kappa > 0$ large enough, G is a diffeomorphism from each open ball $B(x,r) \subset \mathcal{S}$ onto an open subset of ℓ_∞, and in particular G is uniformly locally injective (at distances less that r). The same property is of course true of $\overline{G}$.

Thus, in order to prove that G maps $\mathcal{S}$ diffeomorphically onto its image it only remains to be seen that G is globally injective as well (at distances greater than r, at least for $\kappa > 0$ large enough). Clearly we have $0 < \inf\{\|\overline{G}(v) - \overline{G}(w)\|_\infty : v, w \in \mathcal{S}, \|v - w\|_\infty \geq r\}$, so we can take ε with

$$0 < \varepsilon < \frac{1}{4} \inf\{\|\overline{G}(v) - \overline{G}(w)\|_\infty : v, w \in \mathcal{S}, \|v - w\|_\infty \geq r\},$$

and of course we may assume $\kappa > 0$ is large enough so that $\|G(x) - \overline{G}(x)\|_\infty < \varepsilon$ for all $x \in \mathcal{S}$. Now take $x, y \in \mathcal{S}$. If $G(x) = G(y)$ then necessarily $\|x - y\|_\infty \geq r$, and we have

$$\begin{aligned}
&\|\overline{G}(x) - \overline{G}(y)\|_\infty = \\
&\|\overline{G}(x) - G(x) + G(y) - \overline{G}(y)\|_\infty \leq \|\overline{G}(x) - G(x)\|_\infty + \|G(y) - \overline{G}(y)\|_\infty \\
&\leq 2\varepsilon < \inf\{\|\overline{G}(v) - \overline{G}(w)\|_\infty : v, w \in \mathcal{S}, \|v - w\| \geq r\},
\end{aligned}$$

a contradiction.

This proves the first part of (1). The second part of (1) follows easily from the definitions of $\overline{\Phi}$, $\overline{G}$ and the fact that G approximates $\overline{G}$ (we may assume $0 < \varepsilon < 1/64$ and $\kappa > 0$ large enough so that $\|G(x) - \overline{G}(x)\| \leq \varepsilon$ for all $x \in \mathcal{S}$). Property (2) can be obtained from a similar property for $\overline{H} := e_1^* \circ \overline{G}^{-1}$ (namely that $|t - \overline{H}(x)| < \frac{1}{2} + \frac{1}{8}$ whenever $\|x - \beta(t)\|_\infty < 1/8$), and from the fact that G^{-1} approximates $\overline{G}^{-1}$, for $\kappa > 0$ large enough.[1]

[1] This is a consequence of the following exercise: if f_k is a sequence of diffeomorphisms which uniformly converges to a diffeomorphism f with a uniformly continuous inverse f^{-1}, then f_k^{-1} uniformly converges to f^{-1}.

The bounds in (3) can be obtained from similar bounds for $D\overline{G}, (D\overline{G})^{-1}$, and the facts that these derivatives are uniformly approximated by DG, $(DG)^{-1}$, respectively, for $\kappa > 0$ large enough.[2]

As for property (4), we shall first check that $D\widetilde{G}$ is bounded on a neighborhood of $\mathcal{S}$ in $\widetilde{\ell}_\infty$ of the form $\{x + iz : x \in \mathcal{S}, z \in \ell_\infty, \|z\| \leq 1\}$, so $\widetilde{G}$ is Lipschitz on this set. Indeed, on the one hand, if $t, s \in \mathbb{R}, |s| \leq 1$, we have

$$
\begin{aligned}
&|(\widetilde{\theta})'(t+is-n)| = \\
&\left| \frac{1}{\int_{-\infty}^{\infty} e^{-\kappa v^2} dv} \int_{-\infty}^{\infty} \overline{\theta}(u) 2(t+is-n-u) e^{-\kappa(t+is-n-u)^2} du \right| \leq \\
&\frac{1}{\int_{-\infty}^{\infty} e^{-\kappa v^2} dv} 2e^{\kappa} \left(|t-n| + \frac{3}{4} + 1 \right) e^{-\kappa\left(|t-n|^2 - \frac{3}{2}|t-n|\right)} \int_{-1/2}^{3/4} e^{-\kappa u^2} du \leq \\
&2e^{\kappa} \left(|t-n| + 2 \right) e^{-\kappa\left(|t-n|^2 - \frac{3}{2}|t-n|\right)}.
\end{aligned}
$$

For $|t-n| < 2$ this expression is bounded by $8e^{4\kappa}$, and since there are at most four integers n with $|t-n| < 2$ we have

$$
\sum_{|t-n|<2} |(\widetilde{\theta})'(t+is-n)| \leq 32e^{4\kappa}.
$$

And for $|t-n| \geq 2$ we can estimate

$$
\begin{aligned}
&\sum_{|t-n|\geq 2} |(\widetilde{\theta})'(t+is-n)| \leq \sum_{|t-n|\geq 2} 2e^{\kappa} \left(|t-n| + 2 \right) e^{-\kappa\left(|t-n|^2 - \frac{3}{2}|t-n|\right)} \leq \\
&\sum_{|t-n|\geq 2} 2e^{\kappa} \left(|t-n| + 2 \right) e^{-\kappa|t-n|/2} \leq 2e^{\kappa} \sum_{m=1}^{\infty} (m+2) e^{-\kappa m/2} < +\infty.
\end{aligned}
$$

Therefore there exists $0 < C := 32e^{4\kappa} + 2e^{\kappa} \sum_{m=1}^{\infty} (m+2) e^{-\kappa m/2} < +\infty$ such that

$$
\sum_{n=0}^{\infty} |(\widetilde{\theta})'(t+is-n)| \leq C
$$

for all $t, s \in \mathbb{R}$ with $|s| \leq 1$. We also have

$$
\sum_{n=0}^{\infty} |\widetilde{\theta}(t+is-n)| \leq C' \quad \text{for all } \; t, s \in \mathbb{R}, \; |s| \leq 1
$$

for some $C' > 0$.

On the other hand, it is easy to see that there exists $C'' > 0$ such that

$$
\|D\widetilde{\Phi}(x+iz)\| \leq C'' \quad \text{and} \quad \|\widetilde{\Phi}(x+iz)\| \leq C''
$$

[2]This is a consequence of the following (easy to prove) fact: if f_k is a sequence of diffeomorphisms which uniformly converges to a diffeomorphism f such that there exists $m > 0$ with $\|Df(x)(h)\| \geq m\|h\|$ for all x, h, then the sequence of derivatives $D(f_k^{-1})$ converges to $D(f^{-1})$.

for all $x \in \mathcal{S}$, $z \in \ell_\infty$, $\|z\| \leq 1$. Therefore we get

$$\|D\widetilde{G}(x+iz)\| \leq \sum_{n=0}^{\infty} |(\widetilde{\theta})'(x_1+iz_1-n)| \, \|\widetilde{\Phi}_{(n+1,n+2)}(x+iz)\| +$$

$$+\sum_{n=0}^{\infty} |\widetilde{\theta}(x_1+iz_1-n)| \, \|D\widetilde{\Phi}_{(n+1,n+2)}(x+iz)\| \leq$$

$$\sum_{n=0}^{\infty} |(\widetilde{\theta})'(x_1+iz_1-n)| C'' + \sum_{n=0}^{\infty} |\widetilde{\theta}(x_1+iz_1-n)| C'' \leq C''(C+C')$$

for all $x \in \mathcal{S}, z \in \ell_\infty, \|z\| \leq 1$.

Since $D\widetilde{G}$ is bounded on the convex set $\{x+iz : x \in \mathcal{S}, z \in \ell_\infty, \|z\| \leq 1\}$, it immediately follows from the mean vale theorem that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathcal{S}, z \in \ell_\infty$ and $\|z\| \leq \delta$ then $\|\widetilde{G}(x+iz) - G(x)\| \leq \varepsilon$. This shows (4).

Similar calculations show that $D^2\widetilde{G}$ is also bounded on $\{x + iz : x \in \mathcal{S}, z \in \ell_\infty, \|z\| \leq 1\}$. And we already know that DG and $(DG)^{-1}$ are bounded on $\mathcal{S}$. Then, according to Proposition 2.5.6 of [AMR] (or rather, by an analogous statement for holomorphic mappings, which can be proved in the same fashion by refining a standard proof of the inverse mapping theorem for holomorphic mappings), there exist $r, s > 0$ (depending only on the bounds for DG and $(DG)^{-1}$ on $\mathcal{S}$ and of the bound of $D^2\widetilde{G}$ on $\{x+iz : x \in \mathcal{S}, z \in \ell_\infty, \|z\| \leq 1\}$) such that, for every $x \in \mathcal{S}$, the mapping $\widetilde{G}$ is a holomorphic diffeomorphism from the ball $B_{\widetilde{\ell_\infty}}(x, r)$ onto an open subset of $\widetilde{\ell_\infty}$ which contains the ball $B_{\widetilde{\ell_\infty}}(G(x), s)$. In particular, for every $y = G(x) \in \mathcal{T}$ there exists a holomorphic extension $(\widetilde{G^{-1}})_y$ of G^{-1} defined on the ball $B_{\widetilde{\ell_\infty}}(G(x), s)$ and which maps this ball diffeomorphically within the ball $B_{\widetilde{\ell_\infty}}(x, r)$.

Now define $\widetilde{G^{-1}}(u+iv) = (\widetilde{G^{-1}})_y(u+iv)$ if $u + iv \in B_{\widetilde{\ell_\infty}}(y, s)$ for some $y \in \mathcal{T}$. This mapping is well defined. **For**, if $w = u + iv \in B_{\widetilde{\ell_\infty}}(y_1, s) \cap B_{\widetilde{\ell_\infty}}(y_2, s)$ with $y_1, y_2 \in \mathcal{T}, y_1 \neq y_2$, then there exist $y_3 = G(x_3) \in \mathcal{T}$ and $t \in (0, s)$ such that $B_{\widetilde{\ell_\infty}}(y_3, t) \subset B_{\widetilde{\ell_\infty}}(y_1, s) \cap B_{\widetilde{\ell_\infty}}(y_2, s)$, and since $\widetilde{G}$ maps diffeomorphically the ball $B_{\widetilde{\ell_\infty}}(x_3, r)$ onto an open set containing $B_{\widetilde{\ell_\infty}}(y_3, t)$, and the mappings $(\widetilde{G^{-1}})_{y_1}, (\widetilde{G^{-1}})_{y_2}, (\widetilde{G^{-1}})_{y_3}$ are local inverses of $\widetilde{G}$ defined on $B_{\widetilde{\ell_\infty}}(y_3, t)$ and taking values in $B_{\widetilde{\ell_\infty}}(x_3, r)$, we necessarily have $(\widetilde{G^{-1}})_{y_1} = (\widetilde{G^{-1}})_{y_2} = (\widetilde{G^{-1}})_{y_3}$ on the open ball $B_{\widetilde{\ell_\infty}}(y_3, t)$, by uniqueness of the inverse. But then, by the identity theorem for holomorphic mappings, we must have $(\widetilde{G^{-1}})_{y_1} = (\widetilde{G^{-1}})_{y_2}$ on the open connected set $B_{\widetilde{\ell_\infty}}(y_1, s) \cap B_{\widetilde{\ell_\infty}}(y_2, s)$, and in particular $(\widetilde{G^{-1}})_{y_1}(w) = (\widetilde{G^{-1}})_{y_2}(w)$. Therefore $\widetilde{G^{-1}}$ is a holomorphic extension of G^{-1} to the open neighborhood $\widetilde{\mathcal{T}}_s := \{u + iv : y \in \mathcal{T}, v \in$

$\ell_\infty, \|v\| < s\}$ of $\mathcal{T}$ which maps $\widetilde{\mathcal{T}_s}$ into $\widetilde{\mathcal{S}_r} := \{x+iz : x \in \mathcal{S}, z \in \ell_\infty, \|z\| < r\}$. Hence

$$\|\widetilde{G^{-1}}(u+iv) - G^{-1}(u)\| < r,$$

for all $u \in \mathcal{T}$, $v \in \ell_\infty$ with $\|v\| < s$. Obviously we can assume $r < \varepsilon$, so (5) is proved. ⊠

5.5. **Proof of the main result in the case of a 1-Lipschitz, bounded function.** Let us continue with the proof of our main result in the case of a nonnegative, bounded, 1-Lipschitz function. Fixing $\varepsilon = 1$, there exists $\delta > 0$ so that properties (4) and (5) of the preceding Lemma hold. Now, given a bounded 1-Lipschitz function $f : X \to \mathbb{R}$, we may assume (up to the addition of a constant) that f takes values in the interval $[0, +\infty)$. Let $N \in \mathbb{N}$ be such that $N \geq f(x) \geq 0$ for all $x \in X$. By Proposition 2 we can find real analytic functions $g_1, ..., g_N$, with holomorphic extensions $\widetilde{g}_1, ...\widetilde{g}_N$ defined on $\widetilde{U} := \widetilde{U}_\delta \supset X$, such that

(1) $|f_i(x) - g_i(x)| \leq 1/10$ for all $x \in X$.
(2) g_i is C-Lipschitz.
(3) $|\widetilde{g}_i(x+iy) - g_i(x)| \leq \delta$ for all $z = x+iy \in \widetilde{U}$.

Now let us define g by

$$X \ni x \mapsto (g_1(x), ..., g_N(x), 0, 0, ...) \in \mathcal{T} \ni y \mapsto G^{-1}(y) \in \mathcal{S} \subset \ell_\infty \ni z \mapsto z_1,$$

that is $g = e_1^* \circ G^{-1} \circ \{g_i\}_{i=1}^\infty = H \circ \{g_i\}_{i=1}^\infty$, where we understand $g_i = 0$ for all $i > N$.

Note that, in spite of G^{-1} having a bounded derivative, because $\mathcal{T}$ is not convex we cannot deduce that G^{-1} is Lipschitz. As a matter of fact this mapping is not Lipschitz, though it is uniformly continuous. Nevertheless, since X is convex and the function H has a bounded derivative (because so does G^{-1}), we do have that g is Lipschitz. And since the mapping

$$X \ni x \mapsto (g_1(x), ..., g_N(x), 0, 0, ...) \in \ell_\infty$$

is obviously C-Lipschitz, it follows that $g = H \circ \{g_i\}_{i=1}^\infty$ is CM-Lipschitz, where M is an upper bound of DH on the tube $\mathcal{T}$.

Because $(f_1(x), \cdots, f_N(x), 0, 0, \cdots) \in \beta([0, N])$, and also $|f_i - g_i| < 1/8$, we have $(g_1(x), \cdots, g_N(x), 0, 0, \cdots) \in \mathcal{T}$ and, by property (2) of the preceding Lemma,

$$|g(x) - f(x)| = |H(\{g_n(x)\}_{n=1}^\infty) - h(\{f_n(x)\}_{n=1}^\infty)| \leq 3/4.$$

The function g is clearly real analytic, with holomorphic extension $\widetilde{g}(z) = (\widetilde{e}_1^* \circ \widetilde{G^{-1}})(\widetilde{g}_1(z), \widetilde{g}_2(z), ..., \widetilde{g}_N(z), 0, 0, ...)$ defined on $\widetilde{U}$. And, because $|\widetilde{g}_i(x+iy) - g_i(x)| \leq \delta$ for all $z = x+iy \in \widetilde{U}$, we have, using property (5) of the preceding Lemma, that $|\widetilde{g}(x+iy) - g(x)| \leq 1$ for all $x+iy \in \widetilde{U}$.

By resetting C to CM, we have thus proved the following version of our main result for bounded functions.

Proposition 3. *Let X be a Banach space having a separating polynomial. Then there exist $C \geq 1$ (depending only on X) and an open neighborhood $\widetilde{U}$ of X in $\widetilde{X}$ such that, for every 1-Lipschitz, bounded function $f : X \to \mathbb{R}$, there exists a real analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\widetilde{g} : \widetilde{U} \to \mathbb{C}$, such that*

(1) $|f(x) - g(x)| \leq 1$ *for all* $x \in X$.
(2) g *is* C*-Lipschitz.*
(3) $|\widetilde{g}(x + iy) - g(x)| \leq 1$ *for all* $z = x + iy \in \widetilde{U}$.

5.6. **Proof of the main result in the case of a (possibly unbounded) Lipschitz function.** Let $f : X \to \mathbb{R}$ be a (possibly unbounded) 1-Lipschitz function. For $n \in \mathbb{N}, n \geq 2$, let us define the *crowns*

$$C_n := \{x \in X \,:\, 2^{n-1} \leq Q(x) \leq 2^{n+1}\},$$

and for $n = 1$ set

$$C_1 = \{x \in X \,:\, Q(x) \leq 4\}.$$

Let f_n denote a bounded, 1-Lipschitz extension of $f_{|C_n}$ to X (defined for instance by $x \mapsto \max\{-\|f_{|C_n}\|_\infty, \min\{\|f_{|C_n}\|_\infty, \inf_{y \in C_n}\{f(y) + \|x - y\|\}\}\})$.

According to the preceding Proposition there exist $C \geq 1$, an open neighborhood $\widetilde{U}$ of X in $\widetilde{X}$, and real analytic functions $g_n : X \to \mathbb{R}$, with holomorphic extensions $\widetilde{g_n} : \widetilde{U} \to \mathbb{C}$, such that

(1) $|f_n(x) - g_n(x)| \leq 1$ for all $x \in X$.
(2) g_n is C-Lipschitz.
(3) $|\widetilde{g_n}(x + iy) - g_n(x)| \leq 1$ for all $z = x + iy \in \widetilde{U}$.

For $n = 1$, let $\overline{\theta}_1 : \mathbb{R} \to [0, 1]$ be a C^∞ function such that

(1) $\overline{\theta}_1(t) = 1 \iff t \in [0, 2]$;
(2) $\overline{\theta}_1(t) > 0 \iff t \in (-1, 4)$;
(3) $\overline{\theta}_1'(t) < 0 \iff t \in (2, 4)$;
(4) $\mathrm{Lip}(\overline{\theta}_1) \leq 1$.

Now let $\overline{\theta}_n : \mathbb{R} \to [0, 1]$, $n \in \mathbb{N}$, $n \geq 2$, be a sequence of C^∞ functions with the following properties:

(1) $\overline{\theta}_n(t) > 0 \iff t \in (2^{n-1}, 2^{n+1})$;
(2) $\overline{\theta}_n'(t) > 0 \iff t \in (2^{n-1}, 2^n)$;
(3) $\overline{\theta}_n'(t) < 0 \iff t \in (2^n, 2^{n+1})$;
(4) $\overline{\theta}_n(2^n) = 1$;
(5) $\overline{\theta}_n(t) = 1 - \overline{\theta}_{n-1}(t)$ whenever $t \in (2^{n-1}, 2^n)$;
(6) $\mathrm{Lip}(\overline{\theta}_n) \leq 1/2^{n-2}$.

Note that the functions $\overline{\theta}_n$ form a partition of unity on $\mathbb{R}$, and

$$\sum_{n=1}^{\infty} \mathrm{Lip}(\overline{\theta}_n) \leq 3.$$

Then we also have that the functions $x \mapsto \overline{\theta}_n(Q(x))$, $n \in \mathbb{N}$, form a partition of unity subordinated to the covering $\bigcup_{n\in\mathbb{N}} C_n = X$, and the sum of the Lipschitz constants of these functions is bounded by $3\mathrm{Lip}(Q)$.

Define real analytic functions $\theta_n : \mathbb{R} \to [0,1]$ by

$$\theta(t) = a_n \int_{\mathbb{R}} \overline{\theta}_n(s) e^{-\kappa_n (t-s)^2} ds,$$

where

$$a_n := \int_{\mathbb{R}} e^{-\kappa_n s^2} ds,$$

and κ_n is large enough so that

(1) $e^{-\kappa_n} \leq 1/2^n(1 + \|g_n\|_\infty)$;
(2) $|\theta_n(t) - \overline{\theta}_n(t)| \leq 1/2^{n+1}(C + \|g_n\|_\infty)$ for all $t \in \mathbb{R}$;
(3) $|\theta_n'(t) - \overline{\theta}_n'(t)| \leq 1/2^{n+1}\mathrm{Lip}(Q)(1 + \|g_n\|_\infty)$ for all $t \in \mathbb{R}$.

Note also that

$$\mathrm{Lip}(\theta_n) = \mathrm{Lip}(\overline{\theta}_n),$$

and in particular $\sum_{n=1}^\infty \mathrm{Lip}(\theta_n) \leq 3$.

Let us define $g : X \to \mathbb{R}$ and $\widetilde{g} : \widetilde{U} \to \mathbb{C}$ by

$$g(x) = \sum_{n=1}^\infty \theta_n(Q(x)) g_n(x), \quad \text{and} \quad \widetilde{g}(z) = \sum_{n=1}^\infty \widetilde{\theta}_n(\widetilde{Q}(z)) \widetilde{g}_n(z),$$

where

$$\widetilde{\theta}(u + iv) = a_n \int_{\mathbb{R}} \overline{\theta}_n(s) e^{-\kappa_n (u+iv-s)^2} ds.$$

In order to see that g is well defined and real-analytic, with holomorphic extension $\widetilde{g}$ defined on $\widetilde{U}$, it is enough to show that the series of holomorphic mappings defining $\widetilde{g}$ is locally uniformly and absolutely convergent on $\widetilde{U}$. And, since $|\widetilde{g_n}(z)| \leq \|g_n\|_\infty + 1$ for all $z \in \widetilde{U}$, it is sufficient to check that

$$\sum_{n=1}^\infty |\widetilde{\theta}_n(\widetilde{Q}(z))| \, (1 + \|g_n\|_\infty) < +\infty$$

locally uniformly on $\widetilde{U}$.

We may assume that $\mathrm{Lip}(\widetilde{Q}) \leq C$. Then, for each $x \in X$, according to Lemma 2, we can write

$$\widetilde{Q}(x + z) = Q(x) + Z_x, \text{ where } Z_x \in \mathbb{C}, \, |Z_x| \leq C\|z\|_{\widetilde{X}}.$$

Fix $x \in X$. There exists $n_x \in N$ so that $x \in C_{n_x}$ and in particular $Q(x) \leq 2^{n_x+1}$.

Now, if $n \geq n_x + 3$ and $s \in \text{support}(\overline{\theta}_n) = [2^{n-1}, 2^{n+1}]$ we have $s \geq 2^{n_x+2}$ and therefore, for $\|z\|_{\widetilde{X}} \leq 1/2C$,

$$\begin{aligned}
&\operatorname{Re}\left(\widetilde{Q}\left(x+z\right)-s\right)^{2} \\
&= \left(Q\left(x\right)-s\right)^{2} + 2\left(Q\left(x\right)-s\right)\operatorname{Re} Z_x + \operatorname{Re}(Z_x^2) \\
&= (Q(x)-s+\operatorname{Re} Z_x)^2 - (\operatorname{Re} Z_x)^2 + \operatorname{Re}(Z_x^2) \\
&\geq (Q(x)-s+\operatorname{Re} Z_x)^2 - \frac{1}{2} \\
&\geq \frac{1}{2}(s-Q(x)-\operatorname{Re} Z_x)^2 + \frac{1}{2}(2^{n_x+2}-2^{n_x+1}-\frac{1}{2})^2 - \frac{1}{2} \\
&\geq \frac{1}{2}(s-Q(x)-\operatorname{Re} Z_x)^2 + 1.
\end{aligned}$$

Hence, for $n \geq n_x + 3$ and $\|z\|_{\widetilde{X}} \leq 1/2C$, we have, according to the choice of κ_n, that

$$\begin{aligned}
&|\widetilde{\theta}_n(\widetilde{Q}(x+z))| \leq a_n \int_{\mathbb{R}} \overline{\theta}_n(s) e^{-\kappa_n \operatorname{Re}(\widetilde{Q}(x+z)-s)^2} ds \\
&\leq a_n \int_{\mathbb{R}} e^{-\kappa_n\left(\frac{1}{2}(s-Q(x)-\operatorname{Re} Z_x)^2+1\right)} ds \\
&= e^{-\kappa_n} a_n \int_{\mathbb{R}} e^{-\kappa_n \frac{1}{2}(s-Q(x)-\operatorname{Re} Z_x)^2} ds = e^{-\kappa_n} a_n \int_{\mathbb{R}} e^{-\kappa_n u^2} \sqrt{2} du \\
&= \sqrt{2} e^{-\kappa_n} \leq \sqrt{2} \frac{1}{2^n(1+\|g_n\|_\infty)}.
\end{aligned}$$

Then it is clear that the series

$$\sum_{n=1}^{\infty} |\widetilde{\theta}_n(\widetilde{Q}(x+z))| \left(1+\|g_n\|_\infty\right)$$

is uniformly convergent for $\|z\|_{\widetilde{X}} \leq 1/2C$. Since we can obviously assume $\widetilde{U} \subset \{x+iy : x \in X, \|y\| \leq 1/2C\}$ (and in fact this is always the case, see the proof of Lemma 3), this argument shows that the series

$$\sum_{n=1}^{\infty} \widetilde{\theta}_n(\widetilde{Q}(z)) \widetilde{g}_n(z)$$

defines a holomorphic function $\widetilde{g}$ on $\widetilde{U}$.

Next let us show that the real analytic function g approximates f on X and is $4C$-Lipschitz. Let us define an auxiliary C^∞ function $\overline{g}$ by

$$\overline{g}(x) = \sum_{n=1}^{\infty} \overline{\theta}_n(Q(x)) g_n(x).$$

Since the functions $\overline{\theta}_n(Q(x))$ form a partition of unity subordinated to the covering $\bigcup_{n\in\mathbb{N}} C_n = X$, $|g_n - f_n| \leq 1$, and $f_n = f$ on C_n, it is immediately checked that

$$|\overline{g}(x) - f(x)| \leq 1 \text{ for all } x \in X.$$

On the other hand we have

$$\begin{aligned}
|g(x) - \overline{g}(x)| &= |\sum_{n=1}^{\infty}(\theta_n(Q(x)) - \overline{\theta}_n(Q(x)))g_n(x)| \\
&\leq \sum_{n=1}^{\infty} \|g_n\|_\infty |\theta_n(Q(x)) - \overline{\theta}_n(Q(x))| \\
&\leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.
\end{aligned}$$

By combining the last two estimations we get

$$|g(x) - f(x)| \leq 2 \text{ for all } x \in X.$$

As for the Lipschitz constant of g, observe first that, because support$(\overline{\theta}_n) = C_n$ and $f_n = f$ on C_n, we can write, for every $x, y \in X$,

$$f(x) = \sum_{n=1}^{\infty} f(x)\overline{\theta}_n(Q(y)) = \sum_{n=1}^{\infty} f_n(x)\overline{\theta}_n(Q(y)).$$

Then we have, on the one hand, for every $x, y \in X$,

$$\begin{aligned}
&\overline{g}(x) - \overline{g}(y) = \sum_{n=1}^{\infty} \overline{\theta}_n(Q(x))g_n(x) - \sum_{n=1}^{\infty} \overline{\theta}_n(Q(y))g_n(y) = \\
&\sum_{n=1}^{\infty} (g_n(x) - f_n(x)) \left(\overline{\theta}_n(Q(x)) - \overline{\theta}_n(Q(y))\right) + \sum_{n=1}^{\infty}(g_n(x) - g_n(y))\overline{\theta}_n(Q(y)) \\
&\leq \sum_{n=1}^{\infty} \mathrm{Lip}(\overline{\theta}_n)\mathrm{Lip}(Q)\|x - y\| + \sum_{n=1}^{\infty} C\|x - y\|\overline{\theta}_n(Q(y)) \\
&\leq (3\mathrm{Lip}(Q) + C)\|x - y\| \leq 4C,
\end{aligned}$$

so $\overline{g}$ is $4C$-Lipschitz. And on the other hand,

$$\begin{aligned}
&\mathrm{Lip}(g - \overline{g}) \leq \sum_{n=1}^{\infty} \mathrm{Lip}(Q)\mathrm{Lip}(\theta_n - \overline{\theta}_n)\|g_n\|_\infty + \sum_{n=1}^{\infty} \|\theta_n - \overline{\theta}_n\|_\infty C \\
&\leq \frac{1}{2} + \frac{1}{2} = 1,
\end{aligned}$$

which implies

$$\mathrm{Lip}(g) \leq \mathrm{Lip}(\overline{g}) + 1 \leq 4C + 1 \leq 5C.$$

If we reset C to $5C$, we have thus shown that there exist $C \geq 1$ and an open neighborhood of $\widetilde{U}$ of X in $\widetilde{X}$ so that for every 1-Lipschitz function $f : X \to \mathbb{R}$ there exists a C-Lipschitz, real analytic function $g : X \to \mathbb{R}$, with holomorphic extension $\widetilde{g}:\widetilde{U}\to\mathbb{C}$, such that $|f-g|\leq 2$. This proves our main theorem in the case $\varepsilon\geq 2$ and $\mathrm{Lip}(f)\leq 1$.

Finally, for arbitrary $\varepsilon\in(0,2)$ and $\mathrm{Lip}(f):=L\in(0,\infty)$, we consider the function $F:X\to\mathbb{R}$ defined by $F(x)=\frac{2}{\varepsilon}f(\frac{\varepsilon}{2L}x)$. It is immediately checked that $\mathrm{Lip}(F)=1$, so by the result above there exists a C-Lipschitz, real analytic function $G:X\to\mathbb{R}$ (with holomorphic extension $\widetilde{G}:\widetilde{U}\to\mathbb{C}$) such that $|F-G|\leq 2$. If we define $g(x)=\frac{\varepsilon}{2}G(\frac{2L}{\varepsilon}x)$, we get a real analytic function $g:X\to\mathbb{R}$ with $\mathrm{Lip}(g)=C\mathrm{Lip}(f)$, and such that $|g-f|\leq\varepsilon$. This proves Theorem 1 in full generality. $\square$

Remark 1. However, note that the neighborhood of X in $\widetilde{X}$ on which a holomorphic extension $\widetilde{g}$ of g is defined is $\widetilde{U}_{L,\varepsilon}:=\frac{\varepsilon}{2L}\widetilde{U}$. In particular, as $L=\mathrm{Lip}(f)$ increases to ∞ (or as ε decreases to 0), the neighborhood $\widetilde{U}_{L,\varepsilon}$ of X where a holomorphic extension of the approximation g of f is defined shrinks to X.

This unfortunate dependence of $\widetilde{U}_{L,\varepsilon}$ on L,ε (which, according to the Cauchy-Riemann equations, is an inherent disadvantage of any conceivable scaling procedure, not just the one we have used) prevents our going further and combining our main result with a real analytic refinement of Moulis' techniques [M] in order to prove the following conjecture: that every C^1 function defined on a Banach space having a separating polynomial can be C^1-finely approximated by real analytic functions.

It is no use, either, trying to avoid scaling by choosing r of the order of ε/L and reworking the statements and the proofs of Lemma 3 and the subsequent results in order to allow $r\in(0,1)$, because in such case the neighborhood $\widetilde{V}$ where the holomorphic extensions $\widetilde{\varphi}_n$ of the functions φ_n are defined and conveniently bounded will be contained in $\widetilde{V}_r=\{x+iy: x,y\in X, \|y\|<r\}$, which also shrinks to X as r decreases to 0.

References

[AMR] R. Abraham, J.E. Marsden, T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Second Edition, Applied Mathematical Sciences vol. 75, Springer-Verlag, New York, 1988.

[AFGJL] D. Azagra, R. Fry, J. Gómez Gil, J.A. Jaramillo, and M. Lovo, *C^1-fine approximation of functions on Banach spaces with unconditional basis,* Oxford Quarterly J. Math., **56** (2005), 13-20.

[AFM] D. Azagra, R. Fry, and A. Montesinos, Perturbed Smooth Lipschitz Extensions of Uniformly Continuous Functions on Banach Spaces, *Proc. Amer. Math. Soc.*, 133 (2005), 727-734.

[Ca] H. Cartan, *Calcul différentiel.* Hermann, Paris 1967.

[Ce] M. Cepedello, Approximation of Lipschitz functions by Δ-convex functions in Banach spaces, *Isr. J. Math.* 106 (1998), 269-284.

[CH] M. Cepedello, and P. Hajek, *Analytic approximations of uniformly continuous functions in real Banach spaces.* J. Math. Anal. Appl. 256 (2001), no. 1, 80–98.

[CHP] H.C. Chang, W. He, N. Prabhu, *The analytic domain in the implicit function theorem*, Journal of Inequalities in Pure and Applied Mathematics volume 4 (2003), issue 1, article 12.

[DGZ] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, vol. 64, Pitman Monographs and Surveys in Pure and Applied Mathematics, 1993.

[DFH1] R. Deville, V. Fonf and P. Hajek, Analytic and C^k approximation of norms in separable Banach spaces, *Studia Math.* 120 (1996), 61-74.

[DFH1] R. Deville, V. Fonf and P. Hajek, Analytic and polyhedral approximation of convex bodies in separable polyhedral banach spaces, *Israel J. Math.* 105 (1998), 139-154.

[D] J. Dieudonné, Foundations of Modern Analysis, *Academic Press*, (New York, 1960).

[FHHMPZ] M. Fabian, P. Habala, P. Hájek, V.M. Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry,* CMS books in mathematics **8**, Springer-Verlag, 2001.

[FPWZ] M. Fabian, D. Preiss, J.H.M. Whitfield and V.E. Zizler, Separating polynomials on Banach spaces, *Quart. J. Math. Oxford* (2) 40 **(**1989**),** 409-422.

[F1] R. Fry, *Analytic approximation on* c_0. J. Funct. Anal. 158 (1998), no. 2, 509–520.

[F2] R. Fry, Approximation by functions with bounded derivative on Banach spaces, *Bull. Aust. Math. Soc.* 69 (2004), 125-131.

[FK] R. Fry and L. Keener, *Approximation by Lipschitz, analytic maps on certain Banach spaces*, preprint, arXiv:0810.5600v2 [math.FA]

[HJ] P. Hájek and M. Johanis, *Smooth approximations*, preprint.

[J] M. Jiménez-Sevilla, *A note on the range of the derivatives of analytic approximations of uniformly continuous functions on* c_0. J. Math. Anal. Appl. 348 (2008), no. 2, 573–580.

[K] J. Kurzweil, *On approximation in real Banach spaces*, Studia Math. 14 (1953), 214-231.

[M] N. Moulis, *Approximation de fonctions différentiables sur certains espaces de Banach,* Ann. Inst. Fourier, Grenoble **21,** 4 (1971), 293-345.

[MST] G. Munoz, Y. Sarantopulos and A. Tonge, Complexifications of real Banach spaces, polynomials and multilinear maps, *Studia Math.* 134 (1999), 1-33.

[NS] A.M. Nemirovski and S.M. Semenov, On polynomial approximation in function spaces, *Mat. Sb.* 21 (1973), 255-277.

[SS] K. Sundaresan and S. Swaminathan, Geometry and Nonlinear Analysis in Banach Spaces, *Lecture Notes in Mathematics* 1131, (Springer-Verlag, 1985).

[W] H. Whitney, Analytic extensions of differential functions in closed sets, *Trans. Amer. Math. Soc.* 36 (1934), 63-89.

ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain

E-mail address: daniel_azagra@mat.ucm.es

Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, B.C., Canada

E-mail address: Rfry@tru.ca

Department of Mathematics and Statistics, University of Northern British Columbia, Prince George, B.C., Canada

E-mail address: keener@unbc.ca