

# Smooth extensions of functions on separable Banach spaces

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ABSTRACT. Let  $X$  be a Banach space with a separable dual  $X^*$ . Let  $Y \subset X$  be a closed subspace, and  $f : Y \rightarrow \mathbb{R}$  a  $C^1$ -smooth function. Then we show there is a  $C^1$  extension of  $f$  to  $X$ .

## 1. Introduction

In this note we address the problem of the extension of smooth functions from subsets of Banach spaces to smooth functions on the whole space. For our results, smoothness is meant in the Fréchet sense, and we shall restrict our attention to real-valued functions. To state the problem more precisely, given a Banach space  $X$ , a closed subset  $Y$ , and a  $C^p$ -smooth function  $f : Y \rightarrow \mathbb{R}$ , when is it possible to find a  $C^p$ -smooth map  $F : X \rightarrow \mathbb{R}$  such that  $F|_Y = f$ ?

We should note that when  $Y$  is a complemented subspace of an arbitrary Banach space  $X$ , the extension problem can be easily solved. Indeed, let  $P : X \rightarrow Y$  be a continuous linear projection, and  $f : Y \rightarrow \mathbb{R}$  a  $C^p$ -smooth function. Then  $F(x) = f(Px)$  defines a  $C^1$  extension of  $f$  to  $X$ . Unfortunately, not every closed subspace  $Y$  of a separable Banach space  $X$  is complemented. In fact, a classic result of Lindenstrauss and Tzafriri [LT] states that the only Banach space all of whose closed subspaces are complemented is (up to renorming) a Hilbert space, so this trick only works when  $X$  is a Hilbert space.

When  $p = 0$ , this question is the problem of the continuous extension of functions from closed subsets. A complete characterization was given by the well known theorem of Tietze (see e.g., [Wi]) which we recall states that  $X$  is a normal space iff for every closed subset  $Y \subset X$  and continuous function  $f : Y \rightarrow \mathbb{R}$ , there exists a continuous extension  $F : X \rightarrow \mathbb{R}$  of  $f$ .

Such characterizations in the differentiable case, where  $p \geq 1$ , are more delicate. When  $X = \mathbb{R}$ ,  $Y \subset X$  is a subset, and  $p \geq 1$ , necessary and

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sufficient conditions (in terms of divided differences) for the existence of  $C^p$ -extensions to  $\mathbb{R}$  of  $C^p$  functions on  $Y$  were given by H. Whitney [W1], [W2]. Apparently, Whitney intended to find such a characterization in the case  $X = \mathbb{R}^n$  with  $n > 1$ , but a sequel to the paper [W2] never appeared.

Major advances in this area occurred some twenty years later with the fundamental work of Glaeser [G] who solved the problem when  $n \geq 1$  and  $p = 1$ . Subsequent work included that of Brudnyi and Shvartsman [BS1], [BS2], Bierstone, Milman and Pawlucka [BMP1], [BMP2], and in particular the striking results of C. Fefferman [Fe1], [Fe2], [Fe3]. For example, in [Fe2] a complete characterization is given of when a real-valued function defined on a compact subset of  $\mathbb{R}^n$  is the restriction of a  $C^m$ -smooth map on  $\mathbb{R}^n$ .

In this paper we consider the case when  $X$  is a separable Banach space which admits a  $C^1$ -smooth norm, a condition which is well known to be equivalent to  $X^*$  being separable [DGZ]. Then if  $Y \subset X$  is a closed subspace and  $f : Y \rightarrow \mathbb{R}$  is  $C^1$ -smooth, we show there exists a  $C^1$  extension  $F : X \rightarrow \mathbb{R}$ . If we require only that  $Y \subset X$  be closed and not necessarily a subspace, then a similar conclusion holds under the stronger assumption that  $f$  is defined on a neighbourhood  $U \supset Y$  and is  $C^1$ -smooth on  $Y$  as a function on  $X$  (i.e.,  $f'(y) \in X^*$  for  $y \in Y$  and  $y \rightarrow f'(y)$  is continuous). We observe, however, that in general the smooth extension problem has a negative solution. We give here three examples.

- (1) In [Z] (see also [DGZ, Theorem II.8.3, page 82]) an example is given of a separable Banach space  $Y \subset X = C[0, 1]$ , and a Gâteaux smooth norm on  $Y$  that cannot be extended to a Gâteaux smooth norm on  $X$ .
- (2) Also in [Z], it is shown that for  $1 < p < 2$ , there is a subspace  $Y \subset L_p$  isomorphic to Hilbert space, such that the Hilbertian norm of  $Y$  cannot be extended to a function  $\varphi$  on  $L_p$  which is Fréchet smooth on the unit sphere  $S_Y$  of  $Y$  as a function on  $X$  with  $y \rightarrow \varphi'(y)$  locally Lipschitz from  $S_Y$  to  $X^*$ . Since every  $C^2$  smooth function has a locally Lipschitz derivative, this immediately shows that, given a  $C^\infty$  smooth function  $f$  on  $Y \subset L_p$  (with  $1 < p < 2$ ), in general there is no  $C^2$  smooth extension of  $f$  to  $L_p$ .
- (3) As suggested to us by R. Aron [A] (see also Example 2.1 [AB]). Let  $X = C[0, 1]$  which has the Dunford-Pettis Property, and hence the polynomial Dunford-Pettis Property (i.e., if  $P : X \rightarrow \mathbb{R}$  is a polynomial and  $x_i \xrightarrow{w} 0$ , then  $P(x_i) \rightarrow P(0)$ ). Now  $Y = l_2 \subset X$  by the Banach-Mazur Theorem, and we consider  $f(x) = \|x\|_{l_2}^2$ . If  $f$  extended to a  $C^2$ -smooth function  $F$  on an open neighbourhood of  $l_2$  in  $X$ , then

$$P(h) = (1/2) F''(0)(h, h)$$

would be a polynomial on  $X$ . But  $e_i \xrightarrow{w} 0$  in  $l_2$  and so in  $X$ , and thus as noted above we would have  $1 = P(e_j) \rightarrow P(0) = 0$ , a contradiction.

One can compare the results of this note with the work of C.J. Atkin [At]. The programme of Atkin is to find smooth extension results for smooth functions  $f$  defined on finite unions of open, convex sets in separable Banach spaces  $X$  that do not admit  $C^p$ -smooth norms, or even  $C^p$ -smooth bump functions. In order to achieve this, however, it is assumed in [At] that the function  $f$  already possesses smooth extensions to all of  $X$  in a neighbourhood of every point in its domain. Finally we mention the result [DGZ, Proposition VIII.3.8], which states, in particular, that for weakly compactly generated  $X$  which admit  $C^p$ -smooth bump functions, for any closed subset  $Y \subset X$  and continuous function  $f : Y \rightarrow \mathbb{R}$ , there exists a continuous extension of  $f$  to  $X$  which is  $C^p$ -smooth on  $X \setminus Y$ .

We remark that the situation for analytic maps is quite different. Indeed, the paper by R. Aron and P. Berner [AB] characterizes the existence of analytic extensions from subspaces in terms of the existence of a linear extension operator. In particular, in the real case they prove, among many other equivalences, that if  $Y$  is a closed subspace of a Banach space  $X$ , the the following are equivalent (recall that  $Z$  is a  $\mathcal{C}$ -space if it is complemented in its second dual  $Z^{**}$ ):

- (1) For any  $\mathcal{C}$ -space  $Z$ , and (real) analytic map  $f : Y \rightarrow Z$  which is bounded on bounded sets, there exists an analytic extension  $F : X \rightarrow Z$  of  $f$ , also bounded on bounded sets.
- (2) There exists a continuous, linear extension operator  $T : Y^* \rightarrow X^*$ .

We have combined some recent work on smooth approximation of Lipschitz mappings [F] with some techniques of Moulis [M], the Bartle-Graves selector theorem, and the classical method of Tietze to deduce our principal results.

Our notation is standard, with  $X$  typically denoting a (real) Banach space. We shall denote an open ball with centre  $x \in X$  and radius  $r > 0$  either by  $B_r(x)$ ,  $B(x; r)$ , or  $B_r$  if the centre is understood. We write the closed unit ball of a Banach space  $X$  as  $B_X$ . If  $Y \subset X$ , we denote the restriction of a function  $f : X \rightarrow \mathbb{R}$  to  $Y$  by  $f|_Y$ , and we say that a map  $K : X \rightarrow \mathbb{R}$  is an extension of  $f : Y \rightarrow \mathbb{R}$  if  $K(y) = f(y)$  for all  $y \in Y$ . We denote the Fréchet derivative of a function  $g$  at  $x$  in the direction  $h$  by  $g'(x)(h)$ . As noted above, if  $Y \subset X$  and  $U \supset Y$  is open, we say that  $f : U \rightarrow \mathbb{R}$  is  $C^1$ -smooth on  $Y$  as a function on  $X$  if  $f'(y) \in X^*$  and  $y \rightarrow f'(y)$  is continuous for  $y \in Y$ . For any undefined terms we refer the reader to [FHHMPZ], [DGZ].

## 2. Main Results

An essential tool shall be the following consequence of the main theorem in [F], see also [HJ].

**Lemma 1.** *There exists a constant  $C_0 \geq 1$  such that, for every separable Banach  $X$  space with a  $C^1$ -smooth norm, for every subspace  $Y \subseteq X$ , every Lipschitz function  $f : X \rightarrow \mathbb{R}$ , and every  $\varepsilon > 0$ , there exists a  $C^1$ -smooth function  $K : X \rightarrow \mathbb{R}$  such that*

- (1)  $|f(x) - K(x)| < \varepsilon$  for all  $x \in X$ ,
- (2)  $Lip(K) \leq C_0 Lip(f)$ , and
- (3)  $\|K'(y)\|_{X^*} \leq C_0 Lip(f|_Y)$  for all  $y \in Y$  (in particular the Lipschitz constant of the restriction of  $K$  to  $Y$  is of the order of the Lipschitz constant of the restriction of  $f$  to  $Y$ ).

This lemma can be deduced with some work from the results in either [F, HJ] but here, for the sake of completeness, we shall give a self-contained proof which moreover provides a simple method of constructing *sup-partitions of unity*.

**Proof of the Lemma.** Let us first assume that  $f : X \rightarrow [1, 1001]$ . Define  $\eta = Lip(f|_Y)$ ,  $L = Lip(f)$ ,  $R = 1/\eta$  and  $r = 1/L$  (in the event that  $Lip(f|_Y) = 0$ , take any  $\eta \in (0, L)$ , and observe that when  $Lip(f) = 0$  the result is trivial, so we may assume  $L > 0$ ). Obviously,  $\eta \leq L$  and  $r \leq R$ .

Since  $f$  is  $\eta$ -Lipschitz on  $Y$  and  $Y$  is separable we can cover  $Y$  by a countable family of balls  $B(y_n, R)$  of radius  $R$ , where  $\{y_n\}$  is a dense subset in  $Y$ , in such a way that if  $y, y' \in B(y_n, 4R) \cap Y$  then  $|f(y) - f(y')| \leq 8$ . Similarly, since  $f$  is  $L$ -Lipschitz on  $X$ , we can cover the set  $\{x \in X : \text{dist}(x, Y) \geq r/4\}$  by a countable family of balls  $B(x_n, r/32)$  of radius  $r/32$ , where  $\{x_n\}$  is dense in  $\{x \in X : \text{dist}(x, Y) \geq r/4\}$ , with the properties that the balls  $B(x_n, r/8)$  of radius  $4r/32$  do not touch the set  $\{x \in X : \text{dist}(x, Y) < r/8\}$ , and that if  $x, x' \in B(x_n, r/8)$  then  $|f(x) - f(x')| \leq 1/4$ .

Also note that the open slabs  $D_{y_n} := \{x \in X : \text{dist}(x, Y) < r, \|x - y_n\| < R\}$  cover  $Y$  and if we denote

$$D_{y_n}^4 := \{x \in X : \text{dist}(x, Y) < r, \|x - y_n\| < 4R\}$$

then these sets have the property that if  $x, x' \in D_{y_n}^4$  then  $|f(x) - f(x')| \leq 10$ .

**Claim 1.** *There exists a sequence of  $C^1$  functions  $\varphi_n : X \rightarrow \mathbb{R}$  with the following properties:*

- (1) *The collection  $\{\varphi_n : X \rightarrow [0, 1] \mid n \in \mathbb{N}\}$  is uniformly Lipschitz on  $X$ , with Lipschitz constant  $8/r = 8L$ .*
- (2)  $\|\varphi_n'(y)\|_{X^*} \leq 2/R = 2\eta$  for all  $y \in Y$ . In fact,

$$Lip(\varphi_n|_{\{x \in X : \text{dist}(x, Y) < r/4\}}) \leq 2\eta.$$

- (3) *For each  $x$  with  $\text{dist}(x, Y) < r/4$  there exists  $n \in \mathbb{N}$  with  $\varphi_n(x) = 1$ .*

- (4) For each  $x \in X$  there exists  $\delta > 0$  and  $n_x \in \mathbb{N}$  such that for  $z \in B(x, \delta)$  and  $n > n_x$  we have  $\varphi_n(z) = 0$ .  
(5)  $\varphi_n(x) = 0$  for all  $x \notin D_{y_n}^4$ .

The existence of a family of functions  $\varphi_n$  satisfying properties (1), (3), (4) and (5) on all of  $X$  is known from [F, HJ]. What is new about this claim is that, in the present situation, one can also require (property (2)) that the derivatives of  $\varphi_n$  are bounded on  $Y$  by a constant of the order of  $\text{Lip}(f|_Y)$ , which could be very small compared to the global Lipschitz constant of  $f$ . We say that the collection of functions  $\varphi_n$  forms a *sup-partition of unity* on  $\{x \in X : \text{dist}(x, Y) < r\}$ , subordinated to the covering  $\{D_{y_n} : n \in \mathbb{N}\}$ .

**Proof of the Claim.** Define subsets  $A_1 = \{u_1 \in \mathbb{R} : -1 \leq u_1 \leq 4r\}$ , and, for  $n \geq 2$ ,

$$A_n = \{\{u_j\}_{j=1}^n \in \ell_\infty^n : -1-R \leq u_n \leq 4R, 2R \leq u_j \leq M_n+2 \text{ for } 1 \leq j \leq n-1\},$$

$$A'_n = \{\{u_j\}_{j=1}^n \in \ell_\infty^n : -1 \leq u_n \leq 3R, 3R \leq u_j \leq M_n+2-R \text{ for } 1 \leq j \leq n-1\},$$

$$\text{where } M_n = \sup\{\|x - y_j\| : x \in B(y_n, 4R), 1 \leq j \leq n\}.$$

Let  $b_n : \ell_\infty^n \rightarrow [0, 2]$  be the function defined by

$$b_n(y) = \max\{0, 1 - \frac{1}{R} \text{dist}_\infty(y, A'_n)\},$$

where  $\text{dist}_\infty(y, A) = \inf\{\|y - a\|_\infty : a \in A\}$ . It is clear that  $\text{support}(b_n) = A_n$ , that  $b_n = 1$  on  $A'_n$ , and that  $b_n$  is  $(1/R)$ -Lipschitz (note in particular that the Lipschitz constant of  $b_n$  does not depend on  $n$ ).

Since the function  $b_n$  is uniformly continuous and bounded on  $\mathbb{R}^n$ , it is a standard fact that the normalized integral convolutions of  $b_n$  with the Gaussian-like kernels  $y \mapsto G_\kappa(y) := e^{-\kappa \sum_{j=1}^n 2^{-j} y_j^2}$ ,

$$x \mapsto \frac{1}{T_\kappa} b_n * G_\kappa(x) = \frac{1}{\int_{\mathbb{R}^n} e^{-\kappa \sum_{j=1}^n 2^{-j} y_j^2} dy} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa \sum_{j=1}^n 2^{-j} (x_j - y_j)^2} dy,$$

$$\text{where } T_\kappa = \int_{\mathbb{R}^n} e^{-\kappa \sum_{j=1}^n 2^{-j} y_j^2} dy,$$

converge to  $b_n$  uniformly on  $\mathbb{R}^n$  as  $\kappa \rightarrow +\infty$ . Therefore, for each  $n \in \mathbb{N}$  we can find  $\kappa_n > 0$  large enough so that

$$|b_n(x) - \frac{1}{T_{\kappa_n}} b_n * G_{\kappa_n}(x)| \leq 1/10 \text{ for all } x \in \mathbb{R}^n. \quad (*)$$

Define  $\nu_n : \ell_\infty^n \rightarrow \mathbb{R}$  by

$$\nu_n(x) := \frac{1}{T_{\kappa_n}} b_n * G_{\kappa_n}(x) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j} (x_j - y_j)^2} dy.$$

Let us note that

$$\frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j} (x_j - y_j)^2} dy = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(x - y) e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy,$$

and so

$$\begin{aligned}
|\nu_n(x) - \nu_n(x')| &= \left| \frac{1}{T_n} \int_{\mathbb{R}^n} (b_n(x-y) - b_n(x'-y)) e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \right| \\
&\leq \frac{1}{T_n} \int_{\mathbb{R}^n} |b_n(x-y) - b_n(x'-y)| e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \\
&\leq \frac{1}{R} \|x - x'\|_\infty \frac{1}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \\
&= \frac{1}{R} \|x - x'\|_\infty.
\end{aligned}$$

Hence,  $\nu_n$  is  $\frac{1}{R}$ -Lipschitz. Note also that  $0 \leq \nu_n(x) \leq \|b_n\|_\infty = 1$  for all  $x \in X$ .

Now take a  $C^\infty$  function  $\alpha : \mathbb{R} \rightarrow [0, 1]$  such that:

$$\begin{aligned}
&\alpha \text{ is } 2\text{-Lipschitz;} \\
&\alpha(t) = 0 \text{ if } t \leq 1/10; \\
&\alpha(t) = 1 \text{ if } t \geq 9/10.
\end{aligned}$$

Then the composition  $\alpha \circ \nu_n$  is a  $C^\infty$  function so that

$$\begin{aligned}
&\alpha \circ \nu_n \text{ is } 2/R\text{-Lipschitz;} \\
&\alpha \circ \nu_n(x) = 0 \text{ if } x \notin A_n; \\
&\alpha \circ \nu_n = 1 \text{ if } x \in A_n'.
\end{aligned}$$

Consider the quotient space  $X/Y$ , with its quotient map  $q : X \rightarrow X/Y$ . The mapping  $T \ni (X/Y)^* \rightarrow T \circ q \in X^*$  defines a continuous linear injection from  $(X/Y)^*$  into  $X^*$ , and since  $X^*$  is separable so is  $(X/Y)^*$ . Hence  $X/Y$  has an equivalent  $C^1$  smooth norm (which we will also denote  $\|\cdot\|$ ) with the property that

$$\text{dist}(x, Y) \leq \|q(x)\| \leq 2\text{dist}(x, Y) \text{ for all } x \in X.$$

In particular the function  $x \mapsto \|q(x)\|$  is 2-Lipschitz on  $X$ , as is easily checked.

Take also a  $C^\infty$  function  $\beta : \mathbb{R} \rightarrow [0, 1]$  such that

$$\begin{aligned}
&\beta \text{ is } 3/r\text{-Lipschitz;} \\
&\beta(t) = 0 \text{ if } t \geq r; \\
&\beta(t) = 1 \text{ if } t \leq r/2.
\end{aligned}$$

Next, consider the map  $\lambda_n : X \rightarrow l_\infty^n$  given by

$$\lambda_n(x) = (\|x - y_1\|, \dots, \|x - y_n\|).$$

Then for  $n \geq 1$  we define the maps  $\varphi_n : X \rightarrow \mathbb{R}$  by

$$\varphi_n(x) = \beta(\|q(x)\|) \alpha(\nu_n(\lambda_n(x))) = \beta(\|q(x)\|) \alpha\left(\nu_n\left(\{\|x - y_j\|\}_{j=1}^n\right)\right).$$

Since  $\nu_n$  is constant in a neighborhood of each point  $v = (v_1, \dots, v_n) \in \mathbb{R}^n$  with  $v_i = 0$  for some  $i$ , it is immediately seen that  $\varphi_n$  is of class  $C^1$  on  $X$ .

Now, if  $\text{dist}(x, Y) < r/4 < \text{dist}(x', Y)$  we have that  $\beta(\|q(x)\|) = 1 = \beta(\|q(x')\|)$  and

$$\begin{aligned} |\varphi_n(x) - \varphi_n(x')| &= |\alpha \circ \nu_n(\lambda_n(x)) - \alpha \circ \nu_n(\lambda_n(x'))| \\ &\leq \frac{2}{R} \|\lambda_n(x) - \lambda_n(x')\|_\infty = \frac{2}{R} \left\| \left\{ \|x - x_j\| - \|x' - x_j\| \right\}_{j=1}^n \right\|_\infty \\ &\leq \frac{2}{R} \|x - x'\|_X, \end{aligned}$$

hence the collection  $\{\varphi_n\}$  is uniformly Lipschitz on the open neighborhood  $\{x \in X : \text{dist}(x, Y) < r/4\}$  of the subspace  $Y$ , with constant  $\frac{2}{R} = 2\eta$ . In particular we have that

$$\|\varphi_n'(y)\|_{X^*} \leq 2\eta \text{ for all } y \in Y,$$

which shows (2). On the other hand, from the definition of the  $\varphi_n$ , it is immediately checked that these functions are uniformly Lipschitz on all of  $X$ , with constant  $2/R + 6/r \leq 8/r$ . This shows (1).

Let us show (3). For each fixed  $x \in X$  with  $\text{dist}(x, Y) < r/4$  there exists  $n_x$  with  $x \in B(y_{n_x}, 3R)$  but with  $x \notin B(y_i, 3R)$  for  $i < n_x$ . This implies that the point  $(\|x - y_1\|, \|x - y_2\|, \dots, \|x - y_{n_x}\|)$  belongs to  $A'_{n_x}$ , where the function  $\alpha \circ \nu_{n_x}$  takes the value 1. Besides  $\beta(\|q(x)\|) = 1$ . Hence by the definition of  $\varphi_n$ , we have  $\varphi_{n_x}(x) = 1$ .

Property (5) is shown similarly: if  $\|x - y_n\| \geq 4R$  then the point  $(\|x - y_1\|, \dots, \|x - y_n\|)$  lies in a region of  $\mathbb{R}^n$  where the function  $\alpha \circ \nu_n$  takes the value 0, hence  $\varphi_n(x) = 0$ . Or, if  $\text{dist}(x, Y) \geq r$  then  $\|q(x)\| \geq r$  and  $\beta(\|q(x)\|) = 0$ , hence  $\varphi_n(x) = 0$ .

We finally show (4). If  $\text{dist}(x, Y) \leq r \leq R$  then, since the sequence  $\{y_n\}$  is dense in  $Y$ , there exists  $n_x \in \mathbb{N}$  such that  $\|x - y_{n_x}\| < 2R$ . Take  $\delta = 2R - \|x - y_{n_x}\| > 0$ . Then for all  $z \in B(x, \delta)$  we also have  $\|z - y_{n_x}\| < 2R$  and, by the definition of  $A_n$ ,

$$\lambda_n(z) = (\|z - y_1\|, \dots, \|z - y_{n_x}\|, \dots, \|z - y_n\|) \notin A_n \text{ for } n > n_x,$$

hence, bearing in mind that  $\alpha \circ \nu_x = 0$  outside  $A_n$ , we get

$$\varphi_n(z) = 0 \text{ for all } n > n_x, z \in B(x, \delta).$$

On the other hand, if  $\text{dist}(x, Y) > r$  then  $\beta(\|q(z)\|) = 0$  for all  $z \in B(x, \delta')$ , where  $\delta' = \text{dist}(x, Y) - r > 0$ , and therefore  $\varphi_n(z) = 0$  for all  $n \in \mathbb{N}, z \in B(x, \delta')$ . ■

**Remark 1.** Note that if  $Y = X$  then  $q = 0$  and  $\beta(\|q(x)\|) = 1$  for all  $x$  (hence there is no need to use this term in the definition of  $\varphi_n$ ). In this case the above proof gives a simple method of constructing  $2/R$ -Lipschitz sup-partitions of unity subordinated to any covering by balls of radius  $R$  of  $X$ .

Moreover these sup-partitions of unity are of the same order of smoothness as the norm of  $X$ .

By replacing  $y_n$  with  $x_n$ ,  $R$  with  $r/32$ , and  $\beta$  with a different  $C^1$  function  $\beta : \mathbb{R} \rightarrow [0, 1]$  such that

$$\begin{aligned} \beta(t) &= 0 \text{ for } t \leq r/4 - r/16 \\ \beta(t) &= 1 \text{ for } t \geq r/4 - r/32 \\ \text{Lip}(\beta) &\leq 36/r, \end{aligned}$$

one can similarly show:

**Claim 2.** *There exists a sequence of  $C^1$  functions  $\psi_n : X \rightarrow \mathbb{R}$  with the following properties:*

- (1) *The collection  $\{\psi_n : X \rightarrow [0, 1] \mid n \in \mathbb{N}\}$  is uniformly Lipschitz on  $X$ , with Lipschitz constant  $136/r = 136L$ .*
- (2) *For each  $x$  with  $\text{dist}(x, Y) \geq r/4$  there exists  $n \in \mathbb{N}$  with  $\psi_n(x) = 1$ .*
- (3) *For each  $x \in X$  there exists  $\delta > 0$  and  $n_x \in \mathbb{N}$  such that for  $z \in B(x, \delta)$  and  $n > n_x$  we have  $\varphi_n(z) = 0$ .*
- (4)  *$\psi_n(x) = 0$  for all  $x \notin B(x_n, r/8)$ .*

*In particular all of the functions  $\psi_n$  vanish on the set  $\{x \in X : \text{dist}(x, Y) < r/8\}$ .*

Now let  $\|\cdot\|_{c_0}$  be a  $C^\infty$  smooth equivalent norm to the usual norm  $\|\cdot\|_\infty$  of  $c_0$  and such that

$$\|x\|_\infty \leq \|x\|_{c_0} \leq 2\|x\|_\infty \text{ for all } x \in c_0.$$

Let us define a collection of  $C^1$  functions  $\Phi_n : X \rightarrow [0, 1]$  by

$$\Phi_n(x) = \begin{cases} \varphi_k(x) & \text{if } n = 2k - 1 \text{ is odd,} \\ \psi_k(x) & \text{if } n = 2k \text{ is even.} \end{cases}$$

Notice that, according to properties (4) of Claim 1 and (3) of Claim 2, the mapping  $X \ni x \rightarrow \{\Phi_n(x)\}_{n=1}^\infty \in c_0$  is well defined and  $C^1$  smooth (as the tails of the sequence eventually vanish locally).

Define a function  $g : X \rightarrow \mathbb{R}$  by

$$g(x) = \frac{\|\{a_n \Phi_n(x)\}_{n=1}^\infty\|_{c_0}}{\|\{\Phi_n(x)\}_{n=1}^\infty\|_{c_0}},$$

where

$$a_n = \begin{cases} f(y_k) & \text{if } n = 2k - 1 \text{ is odd,} \\ f(x_k) & \text{if } n = 2k \text{ is even.} \end{cases}$$

The function  $g$  is well defined because  $\|\{\Phi_n(x)\}_{n=1}^\infty\|_{c_0} \geq \|\{\Phi_n(x)\}_{n=1}^\infty\|_\infty = 1$  by properties (3) of Claim 1 and (2) of Claim 2, and is  $C^1$  smooth on  $X$  by the previous observation and because  $a_n \geq 1$ .

Since the functions  $\Phi_n$  are  $136 \times L$ -Lipschitz and  $|a_n| \leq 1001$  we have

$$\begin{aligned} \|\{a_n \Phi_n(x)\}_{n=1}^\infty - \{a_n \Phi_n(z)\}_{n=1}^\infty\|_{c_0} &\leq 2\|\{a_n (\Phi_n(x) - \Phi_n(z))\}_{n=1}^\infty\|_\infty \leq \\ &2002 \times 136 \times L \|x - z\|, \end{aligned}$$

that is the function  $\|\{a_n \Phi_n(\cdot)\}_{n=1}^\infty\|_{c_0}$  is  $2002 \times 136 \times L$ -Lipschitz on  $X$ , and is bounded by 2002. Similarly, since the function  $t \mapsto 1/t$  is 1-Lipschitz on  $[1, \infty)$  and  $\{\Phi_n(\cdot)\}_{n=1}^\infty$  is bounded below by 1, we have that the function  $1/\|\{\Phi_n(\cdot)\}_{n=1}^\infty\|_{c_0}$  is  $1 \times 2 \times 136 \times L$ -Lipschitz on  $X$  and bounded above by 1. Therefore the product satisfies

$$\text{Lip}(g) \leq 2002 \times (1 \times 2 \times 136 \times L) + 1 \times (2002 \times 136 \times L) = 816816 \times L.$$

When we restrict  $g$  to the set  $\{x \in X : \text{dist}(x, Y) < r/8\}$ , all the even terms of the sequence  $\{\Phi_n(x)\}_{n=1}^\infty$  vanish, so the only functions that matter are the  $\varphi_k$ , which are  $2\eta$ -Lipschitz on this set, and the above calculation can be performed replacing  $L$  with  $\eta$  to show that

$$\text{Lip}(g|_{\{x \in X : \text{dist}(x, Y) < r/4\}}) \leq 816816 \times \eta,$$

which implies

$$\|g'(y)\|_{X^*} \leq 816816\eta \text{ for all } y \in Y.$$

Finally, bearing in mind that the supports of the  $\varphi_n$  are contained in the slabs  $D_{y_n}^4$ , that the supports of the  $\psi_n$  are contained in the balls  $B(x_n, r/8)$ , and that on each of these sets the oscillation of  $f$  is bounded by 10, it is easy to check that

$$|f(x) - g(x)| \leq 20 \text{ for all } x \in X.$$

This argument proves the Lemma in the case when  $\varepsilon = 20$  and  $f : X \rightarrow [0, 1000]$ .

We next see that this result remains true for functions  $f$  taking values in  $\mathbb{R}$  if we replace 20 with 50 and we allow  $C_0$  to be slightly larger than 816816. Indeed, by considering the function  $h = \theta \circ g$ , where  $\theta$  is a  $C^\infty$  smooth function  $\theta : \mathbb{R} \rightarrow [0, 1000]$  such that  $|t - \theta(t)| \leq 30$  if  $t \in [0, 1000]$ ,  $\theta(t) = 0$  for  $t \leq 21$ , and  $\theta(t) = 1000$  for  $t \geq 979$ , we get the following result: there exists  $C_0 := 816816 \times \text{Lip}(\theta)$  such that for every  $L$ -Lipschitz function  $f : X \rightarrow [0, 1000]$  whose restriction to  $Y$  is  $\eta$ -Lipschitz there exists a  $C^1$  function  $h : X \rightarrow [0, 1000]$  such that

- (1)  $|f(x) - h(x)| \leq 50$  for all  $x \in X$
- (2)  $h$  is  $C_0 L$ -Lipschitz
- (3)  $\|h'(y)\|_{X^*} \leq C_0 \eta$
- (4)  $f(x) = 0 \implies h(x) = 0$ , and  $f(y) = 1000 \implies h(y) = 1000$ .

Now, for a  $L$ -Lipschitz function  $f : X \rightarrow [0, +\infty)$  so that  $\text{Lip}(f|_Y) = \eta$ , we can write  $g(x) = \sum_{n=0}^\infty f_n(x)$ , where

$$f_n(x) = \begin{cases} f(x) - 1000n & \text{if } 1000n \leq f(x) \leq 1000(n+1), \\ 0 & \text{if } f(x) \leq 1000n, \\ 1000 & \text{if } 1000(n+1) \leq f(x) \end{cases}$$

and the sum is locally finite. The functions  $g_n$  are clearly  $L$ -Lipschitz, satisfy  $\text{Lip}((g_n)|_Y) \leq \eta$  and take values in the interval  $[0, 1000]$ , so there are  $C^1$  functions  $h_n : X \rightarrow [0, 1000]$  such that for all  $n \in \mathbb{N}$  we have that  $h_n$  is  $C_0 L$ -Lipschitz,  $\|h'_n(y)\|_{X^*} \leq C_0 \eta$  for all  $y \in Y$ ,  $|f_n - h_n| \leq 50$ , and  $h_n$  is

0 or 1000 wherever  $f_n$  is 0 or 1000. It is easy to check that the function  $h : X \rightarrow [0, +\infty)$  defined by  $h = \sum_{n=0}^{\infty} h_n$  is  $C^1$  smooth,  $C_0$ -Lipschitz, and satisfies  $|f - h| \leq 50$  and  $\|h'(y)\|_{X^*} \leq C_0\eta$ . This argument shows that there is  $C_0 \geq 1$  such that for any  $L$ -Lipschitz function  $f : X \rightarrow [0, +\infty)$  with  $\text{Lip}(f) = \eta$ , there exists a  $C^1$  function  $h : X \rightarrow [0, +\infty)$  such that

- (1)  $|f(x) - h(x)| \leq 50$  for all  $x \in X$
- (2)  $h$  is  $C_0$ -Lipschitz
- (3)  $\|h'(y)\|_{X^*} \leq C_0\eta$  for all  $y \in Y$
- (4)  $f(x) = 0 \implies h(x) = 0$ .

Finally, for an arbitrary  $L$ -Lipschitz function  $f : X \rightarrow \mathbb{R}$ , we can write  $f = f^+ - f^-$  and apply this result to find  $C^1$  smooth,  $C_0$ -Lipschitz functions  $h^+, h^- : X \rightarrow [0, +\infty)$  so that  $h := h^+ - h^-$  is  $C^1$  smooth,  $C_0L$ -Lipschitz,  $\|h'\|_{X^*} \leq C_0\eta$  on  $Y$ ,  $|f - h| \leq 50$ . This proves the Lemma for  $\varepsilon = 50$ .

For an arbitrary  $\varepsilon \in (0, 50)$ , let us consider the function  $g : X \rightarrow \mathbb{R}$  defined by  $g(x) = \frac{50}{\varepsilon}f(\frac{\varepsilon}{50}x)$ . It is immediately checked that  $\text{Lip}(g) = \text{Lip}(f) = L$  and  $\text{Lip}(g|_Y) = \text{Lip}(f|_Y) = \eta$ , so by the result above there exists a  $C^1$  smooth,  $C_0L$ -Lipschitz function  $h$  with  $\|h'\|_{X^*}$  bounded by  $C_0\eta$  on  $Y$  and such that  $|g(x) - h(x)| \leq 50$  for all  $x$ , which implies that the function  $K(z) := \frac{\varepsilon}{50}h(\frac{50}{\varepsilon}z)$  is  $C_0\eta$ -Lipschitz and satisfies  $|f(z) - K(z)| \leq \varepsilon$  for all  $z \in X$ . ■

We next establish the existence of a continuous and bounded selection of the Hahn-Banach extension operator  $y^* \in Y^* \rightarrow G(y^*) \in 2^{X^*}$ , where

$$G(y^*) = \{x^* \in X^* : x^*(y) = y^*(y) \text{ for all } y \in Y\}.$$

**Lemma 2.** *For every Banach space  $X$  and every closed subspace  $Y \subset X$  there exist a continuous mapping  $H : Y^* \rightarrow X^*$  and a number  $M \geq 1$  such that*

- (1)  $H(y^*)(y) = y^*(y)$  for every  $y^* \in Y^*$ ,  $y \in Y$ ;
- (2)  $\|H(y^*)\|_{X^*} \leq M\|y^*\|_{Y^*}$  for every  $y^* \in Y^*$ .

**Proof.** This is a consequence of the Bartle-Graves selector theorem (see [DGZ, page 299]) which states: *Let  $W$  and  $Z$  be Banach spaces and let  $T$  be a bounded linear mapping of  $W$  onto  $Z$ . Then there exists a continuous (nonlinear in general) mapping  $B$  of  $Z$  into  $W$  such that  $(T \circ B)w = w$  for every  $w \in W$ .* Moreover, it follows from the proof of this result that there exists an  $M > 1$  such that  $\|B(w)\| \leq M\|w\|$ . If we apply this theorem with  $W = X^*$ ,  $Z = Y^*$ , to the mapping  $T : X^* \rightarrow Y^*$  defined by  $T(x^*) = x^*|_Y$ , which is a continuous linear surjection with  $\|T\| = 1$  (by the Hahn-Banach theorem), we obtain our continuous map  $H = B : Y^* \rightarrow X^*$  with the property that the restriction of  $H(y^*)$  to  $Y$  is  $y^*$ , for every  $y^* \in Y^*$ , and such that  $\|H(y^*)\|_{X^*} \leq M\|y^*\|_{Y^*}$ . ■

Now we are in a situation to deduce an approximation result which is of independent interest and which, combined with some ideas of the Tietze proof, will yield our main results on smooth extension.

**Theorem 1.** *Let  $X$  be a separable Banach space which admits a  $C^1$ -smooth norm, and  $Y \subset X$  a closed subspace. Let  $f : Y \rightarrow \mathbb{R}$  be a  $C^1$ -smooth function, and  $F$  a continuous extension of  $f$  to  $X$ . Let  $H : Y^* \rightarrow X^*$  be any extension operator as in Lemma 2. Then, for every  $\varepsilon > 0$ , there exists a  $C^1$ -smooth map  $g : X \rightarrow \mathbb{R}$  such that*

- (1)  $|F(x) - g(x)| < \varepsilon$  on  $X$ , and
- (2)  $\|H(f'(y)) - g'(y)\|_{X^*} < \varepsilon$  on  $Y$ .

Furthermore, if the given  $C^1$  function  $f$  is Lipschitz on  $Y$  and  $F$  is a Lipschitz extension of  $f$  to  $X$  with  $\text{Lip}(F) = \text{Lip}(f)$  (for instance  $F(x) = \inf_{y \in Y} \{f(y) + \text{Lip}(f)\|x - y\|\}$ ), then the function  $g$  can be chosen to be Lipschitz on  $X$  and with the additional property that

- (3)  $\text{Lip}(g) \leq C \text{Lip}(f)$ ,

where  $C > 1$  is a constant only depending on  $X$ .

**Proof.** First note that by the Tietze Theorem, the continuous extension  $F$  always exists. We modify the proof of Theorem 4 in [AFGJL] employing Lemma 1. It will be convenient to use the following notation: given a point  $y_k \in Y$ , we define  $T_k$  to be the natural  $H$ -extension of the first order Taylor Polynomial of  $f$  at  $y_k$ ; namely,  $T_k(x) = f(y_k) + H(f'(y_k))(x - y_k)$ . Note in particular that  $T_k \in C^\infty(X, \mathbb{R})$ , with  $T'_k(x) = H(f'(y_k))$  for every  $x \in X$ , and  $T'_k(y) \upharpoonright_Y = f'(y_k)$  for all  $y \in Y$ .

Now, using the separability of  $X$ , the closedness of  $Y \subset X$ , and the continuity of  $F$ , we can construct a covering  $\mathcal{C} = \{B_{r_j}\}_{j=1}^\infty \cup \{B_{s_k}\}_{k=1}^\infty$  of  $X$ , by open balls with centres  $x_j$  and  $y_k$  respectively, with the following properties:

- (i). We have  $B_{2r_j} \subset X \setminus Y$ , and  $|F(x) - F(x_j)| < \varepsilon/2C_0$  on  $B_{2r_j}$ ,
- (ii). The collection  $\{B_{s_k}\}_k \subset X$  covers  $Y$  with centres  $y_k \in Y$  and radii  $s_k$  chosen using the smoothness of  $f$  on  $Y$  and the norm-norm continuity of the extension operator  $H$ , so that  $\|T'_k(y) - f'(y)\|_{Y^*} < \varepsilon/8C_0$  and  $\|T'_k(y) - H(f'(y))\|_{X^*} < \varepsilon/8C_0$  on  $B_{2s_k} \cap Y$ .

It will be useful in the sequel, to employ an alternate notation for the open balls  $B_{r_j}$  and  $B_{s_k}$ . We let  $\beta : \mathbb{N} \rightarrow \mathcal{C}$  be a bijection where for each  $i$ ,  $\beta(i) = B(\beta_1(i); \beta_2(i))$ . Let  $\varphi_j \in C^1(X, [0, 1])$  with bounded derivative so that  $\varphi_j = 1$  on  $B(\beta_1(j); \beta_2(j))$  and  $\varphi_j = 0$  outside of  $B(\beta_1(j); 2\beta_2(j))$ .

By Lemma 1 applied to  $T_k(y) - f(y)$  on  $B_{2s_k} \cap Y$ , we may choose  $C^1$ -smooth maps  $\delta_k : X \rightarrow \mathbb{R}$  so that on each  $B_{2s_k} \cap Y$  we have both

$$|T_k(y) - f(y) - \delta_k(y)| < 2^{-k-2} \varepsilon M_k^{-1},$$

and  $\|\delta'_k(y)\|_{X^*} < \varepsilon/8$ , where  $M_k = \sum_{i=1}^k \widetilde{M}_i$  and  $\widetilde{M}_i = \sup_{x \in Y \cap B_{2s_i}} \|\varphi'_i(x)\|_{X^*}$ .

Then we also have, for  $y \in B_{2s_k} \cap Y$  using our estimate above,

$$\begin{aligned} \|T'_k(y) - H(f'(y)) - \delta'_k(y)\|_{X^*} &\leq \|T'_k(y) - H(f'(y))\|_{X^*} + \|\delta'_k(y)\|_{X^*} \\ &< \varepsilon/8C_0 + \varepsilon/8 \leq \varepsilon/4. \end{aligned}$$

Set  $\Delta_i(x) = T_k(x) - \delta_k(x)$  if  $\beta(i) = B_{s_k}$  is a ball from the subcollection  $\{B_{s_l}\}_{l=1}^\infty$  covering  $Y$ , and  $\Delta_i = F(x_j)$  if  $\beta(i) = B_{r_j}$  belongs to the subcollection  $\{B_{r_l}\}_{l=1}^\infty$  covering  $X \setminus Y$ .

Next, we define

$$h_i = \varphi_i \prod_{k < i} (1 - \varphi_k),$$

and

$$g(x) = \sum_i h_i(x) \Delta_i(x)$$

Note that for each  $x$ , if  $n := n(x) := \min\{m : x \in \beta(m)\}$ , then because  $1 - \varphi_n(x) = 0$  and  $\beta(n)$  is open, it follows from the definition of the  $h_j$  that there is a neighbourhood  $N \subset \beta(n)$  of  $x$  so that for  $z \in N$ ,  $g(z) = \sum_{j \leq n} h_j(z) \Delta_j(z)$ , and  $\sum_j h_j(z) = \sum_{j \leq n} h_j(z)$ . Also, by a straightforward calculation, again using the fact that  $\varphi_n = 1$  on  $\beta(n)$ , we have that  $\sum_j h_j(z) = 1$  for  $z \in \beta(n)$ , and so for all  $z \in X$ .

Now, fix any  $x_0 \in X$ , and let  $n_0 = n(x_0)$  and a neighborhood  $N_0$  of  $x_0$  be as above. For each  $j \leq n_0$  define the functions  $V_j : N_0 \rightarrow \mathbb{R}$  and  $W_j : N_0 \rightarrow \mathbb{R}$  by

$$V_j(x) = \begin{cases} 0 & \text{if } \beta_1(j) \notin Y \\ |T_k(x) - F(x) - \delta_k(x)| & \text{if } \beta_1(j) = y_k \end{cases}$$

and

$$W_j(x) = \begin{cases} 0 & \text{if } \beta_1(j) \in Y \\ |F(x_i) - F(x)| & \text{if } \beta_1(j) = x_i \end{cases}$$

Then for any  $x \in N_0$  we have that

$$\begin{aligned} |g(x) - F(x)| &\leq \sum_{j \leq n_0} h_j(x) \max\{V_j(x), W_j(x)\} \\ &\leq \sum_{j \leq n_0} h_j(x) \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Now define the function  $\alpha$  so that when  $\beta_1(j) \in Y$ ,  $\beta_1(j) = y_{\alpha(j)}$ . Recall that  $B_{2r_j} \cap Y = \emptyset$  for every  $j$ , and that  $\varphi_j = 0$  off of  $B_{2r_j}$ . Hence, if  $y \in Y$ , then in the sum  $g(y)$  only those indices  $j$  such that  $\beta_1(j) \in Y$  are non-zero.

Recall also that  $\sum_j h_j(x) = 1$  for all  $x \in X$  (and hence  $\sum_j h'_j(x) = 0$ ), and so for  $y \in Y$  we have,  $H(f'(y)) = \sum_j h'_j(y) f(y) + \sum_j h_j(y) H(f'(y))$ .

And,  $g'(y) = \sum_j h'_j(y) (T_{\alpha(j)}(y) - \delta_{\alpha(j)}(y)) + \sum_j h_j(y) (T'_{\alpha(j)}(y) - \delta'_{\alpha(j)}(y))$ .

Finally, a straightforward calculation shows that  $\|h'_j(x)\| \leq M_{\alpha(j)}$  for  $\beta_1(j) \in Y$ . With these observations in mind, we have,

$$\begin{aligned}
& \|g'(y) - H(f'(y))\|_{X^*} \\
& \leq \sum_{\substack{j \leq n(y) \\ \beta_1(j) \in Y}} (\|h'_j(y)\| |T_{\alpha(j)}(y) - f(y) - \delta_{\alpha(j)}(y)| \\
& \quad + h_j(y) \|T'_{\alpha(j)}(y) - H(f'(y)) - \delta'_{\alpha(j)}(y)\|_{X^*}) \\
& < \sum_{\substack{j \leq n(y) \\ \beta_1(j) \in Y}} \|h'_j(y)\| \left(2^{-\alpha(j)-2} \varepsilon M_{\alpha(j)}^{-1}\right) + \sum_{\substack{j \leq n(y) \\ \beta_1(j) \in Y}} h_j(y) \frac{\varepsilon}{4} \\
& < \sum_{\substack{j \leq n(y) \\ \beta_1(j) \in Y}} M_{\alpha(j)} \left(2^{-\alpha(j)-2} \varepsilon M_{\alpha(j)}^{-1}\right) + \frac{\varepsilon}{4} < \varepsilon.
\end{aligned}$$

As  $H(f'(y))|_Y = f'(y)$ , we also have the estimate  $\|g'(y) - f'(y)\|_{Y^*} < \varepsilon$ .

Let us now consider the case when  $f$  is  $C^1$  and Lipschitz on  $Y$  and  $F$  is any Lipschitz extension of  $f$  to  $X$  with  $\text{Lip}(F) = \text{Lip}(f)$ . In this case we have to modify the definition of the functions  $\Delta_i$  as follows.

We let  $\Delta_i(x) = T_k(x) - \delta_k(x)$  if  $\beta(i) = B_{s_k}$  is a ball from the subcollection  $\{B_{s_l}\}_{l=1}^{\infty}$  covering  $Y$  where  $\delta_k$  is chosen (by using Lemma 1) so that  $|T_k(y) - f(y) - \delta_k(y)| < 2^{-i-2} \varepsilon M_k^{-1}$ ,  $\text{Lip}(\delta_k) \leq C_0 \text{Lip}(T_k - F) \leq 2C_0 \text{Lip}(F)$ , and  $\|\delta'_k(y)\|_{X^*} < \varepsilon/8$ , where now the  $M_i$  are defined by  $M_i = \sum_{j=1}^i \widetilde{M}_j$  and  $\widetilde{M}_j = \sup_{x \in B(\beta_1(j); 2\beta_2(j))} \|\varphi'_j(x)\|_{X^*}$ . We also let  $\Delta_i(x) = F_\ell(x)$  if  $\beta(i) = B_{r_\ell}$  belongs to the subcollection  $\{B_{r_j}\}_{j=1}^{\infty}$  covering  $X \setminus Y$ , where the function  $F_\ell$  is again chosen by using Lemma 1 so that  $|F_\ell(x) - F(x)| < 2^{-i-2} \varepsilon M_l^{-1}$  on  $B_{2r_l}$  and  $\text{Lip}(F_\ell) \leq C_0 \text{Lip}(F) = C_0 \text{Lip}(f)$ . Note that, with these choices, we have

$$|\Delta_i(x) - F(x)| < 2^{-i-2} \varepsilon M_i^{-1} \text{ on } B(\beta_1(i), \beta_2(i)), \text{ and}$$

$$\text{Lip}(\Delta_i) \leq C_1 \text{Lip}(f),$$

where  $C_1 := M + 3C_0$  (with  $M$  as in Lemma 2(2)) is a constant depending only on  $X$ .

Now define the  $C^1$  function  $g : X \rightarrow \mathbb{R}$  by

$$g(x) = \sum_i \Delta_i(x) h_i(x).$$

As above, one can check that  $|g(x) - F(x)| < \varepsilon$  for all  $x \in X$ , and also  $\|g'(y) - H(f'(y))\|_{X^*} < \varepsilon$  for all  $y \in Y$ ; that is  $g$  satisfies properties (1) and (2) of the statement. Let us see that  $g$  satisfies (3) as well. Noting that  $\text{Lip}(h_j) \leq M_j$ , we can estimate, for every  $x, z \in X$ ,

$$\begin{aligned}
& g(x) - g(z) \\
&= \sum_j \Delta_j(x)h_j(x) - \sum_j \Delta_j(z)h_j(z) \\
&= \sum_j (\Delta_j(x) - F(x))(h_j(x) - h_j(z)) + \sum_j (\Delta_j(x) - \Delta_j(z))h_j(z) \\
&\leq \sum_j \frac{\varepsilon}{2^{j+2}M_j} \text{Lip}(h_j)\|x - z\| + \sum_j \text{Lip}(\Delta_j)\|x - z\|h_j(z) \\
&\leq \left(\frac{\varepsilon}{4} + C_1\text{Lip}(f)\right)\|x - z\| \leq C\text{Lip}(f)\|x - z\|,
\end{aligned}$$

provided that  $\varepsilon > 0$  is chosen small enough (recall that we are assuming  $\text{Lip}(f) > 0$ ), and where  $C = 2C_1 > 1$ , a constant only depending on  $X$ . This shows that  $\text{Lip}(g) \leq C\text{Lip}(f)$ . ■

**Theorem 2.** *Let  $X$  be a separable Banach space which admits a  $C^1$ -smooth norm. Let  $Y \subset X$  be a closed subspace, and  $f : Y \rightarrow \mathbb{R}$  a  $C^1$ -smooth and Lipschitz function. Then there is a  $C^1$  and Lipschitz extension  $g : X \rightarrow \mathbb{R}$  of  $f$  such that  $\text{Lip}(g) \leq C\text{Lip}(f)$ , where  $C$  is a constant depending only on  $X$ .*

**Proof.** First note that if  $h$  is a bounded, Lipschitz function defined on  $Y$ , there always exists a bounded, Lipschitz extension of  $h$  to  $X$ , with the same Lipschitz constant, and bounded by the same constant (defined for instance by  $x \mapsto \max\{-\|h\|_\infty, \min\{\|h\|_\infty, \inf_{y \in Y}\{h(y) + \text{Lip}(h)\|x - y\|\}\}$ ). For the purposes of the proof, we denote such an extension by  $\bar{h}$ .

We are going to define our function  $g$  by means of a series constructed by induction. By Theorem 1 there exists a  $C^1$  function  $g_1 : X \rightarrow \mathbb{R}$  such that

- $|\bar{f} - g_1| < 2^{-1}\varepsilon$  on  $X$ ,
- $\|f'(y) - g_1'(y)\|_{Y^*} < 2^{-1}\varepsilon/C$  for  $y \in Y$  (note in particular that this implies  $\text{Lip}(f - g_1|_Y) \leq 2^{-1}\varepsilon/C$ ), and
- $\text{Lip}(g_1) \leq C\text{Lip}(f)$

Now, for  $n \geq 2$ , suppose that we have chosen  $g_1, \dots, g_n$ , real-valued and  $C^1$ -smooth on  $X$  such that for all  $x \in X$  and  $y \in Y$ ,

$$\left| \overline{\left( f - \sum_{i=1}^n g_i \right)}(x) \right| < 2^{-n}\varepsilon,$$

$$\left\| f'(y) - \sum_{i=1}^n g'_i(y) \right\|_{Y^*} < 2^{-n}\varepsilon/C,$$

and

$$\text{Lip}(g_n) \leq C \text{Lip} \left( f - \left( \sum_{j=1}^{n-1} g_j \right) |_Y \right).$$

It is clear that an application of Theorem 1 to the function  $f - (g_1)|_Y$  provides us with a function  $g_2$  which, together with  $g_1$ , makes the above properties true for  $n = 2$ . Hence we can proceed to the general step of our inductive construction.

Consider the function  $l = f - \sum_{i=1}^n g_i$ , which is  $C^1$ -smooth on  $Y$ .

By Theorem 1, we can find a  $C^1$ -smooth map  $g_{n+1}$  on  $X$  such that we have,

$$(2.1) \quad |\bar{l}(x) - g_{n+1}(x)| < 2^{-n-1}\varepsilon \text{ on } X,$$

and for  $y \in Y$ ,

$$(2.2) \quad \|l'(y) - g'_{n+1}(y)\|_{Y^*} < 2^{-n-1}\varepsilon/C,$$

and also,

$$(2.3) \quad \text{Lip}(g_{n+1}) \leq C \text{Lip}(l).$$

From (2.5), we have in particular,  $|f(y) - \sum_{i=1}^{n+1} g_i(y)| = |l(y) - g_{n+1}(y)| < 2^{-n-1}\varepsilon$  on  $Y$ , and so  $\left| \overline{\left( f - \sum_{i=1}^{n+1} g_i \right)}(x) \right| < 2^{-n-1}\varepsilon$  on  $X$ . This together with (2.6) and (2.7) completes the inductive step.

Now, from (2.6) we have  $\text{Lip} \left( f - \left( \sum_{i=1}^{n+1} g_i \right) |_Y \right) \leq 2^{-n}\varepsilon/C$ , and so from (2.7) we obtain,

$$\|g'_{n+1}(x)\| \leq \text{Lip}(g_{n+1}) \leq C \text{Lip} \left( f - \left( \sum_{j=1}^n g_j \right) |_Y \right) \leq C 2^{-n}\varepsilon/C = 2^{-n}\varepsilon.$$

Hence the series  $\sum_j g'_j(x)$  is absolutely and uniformly convergent on  $X$ . Similarly, we have the estimate  $|g_{n+1}(x)| \leq 2^{-n+1}\varepsilon$ . Therefore the series

$$g(x) = \sum_{n=1}^{\infty} g_n(x)$$

defines a  $C^1$  function on  $X$ , which coincides with  $f$  on  $Y$  because of the first inequality in the inductive assumptions. Finally, we have

$$\text{Lip}(g) \leq \text{Lip}(g_1) + \sum_{n=2}^{\infty} \text{Lip}(g_n) \leq C \text{Lip}(f) + \sum_{n=2}^{\infty} 2^{-(n-1)}\varepsilon \leq 2C \text{Lip}(f),$$

provided that  $\text{Lip}(f) > 0$  (which we can always assume) and  $\varepsilon$  is small enough. ■

**Remark 2.** With some more work in the proofs of the preceding theorems one could show that the constant  $C$  can be taken to be any number  $C > M$ , where  $M$  is as in Lemma 2(2). Unfortunately the proof of the Bartle-Graves extension theorem does not give us any useful estimation about the size of  $M$ , and in general  $M$  is going to be quite large, so we cannot hope that any refinement of the above proofs will yield a statement of Theorem 2 in which  $C$  can be chosen to be any number bigger than 1.

**Theorem 3.** *Let  $X$  be a separable Banach space which admits a  $C^1$ -smooth norm. Let  $Y \subset X$  be a closed subspace, and  $f : Y \rightarrow \mathbb{R}$  a  $C^1$ -smooth function. Then there is a  $C^1$  extension of  $f$  to  $X$ .*

**Proof.** Since  $f$  is  $C^1$  on  $Y$ , there exists  $\{B_j\} := \{B(y_j; r_j)\}_{j=1}^{\infty}$ , a countable covering of  $Y$  by open balls in  $X$  such that  $f$  is Lipschitz on  $B_j \cap Y$  for each  $j \in \mathbb{N}$ . Let  $U = \bigcup_{j=1}^{\infty} B_j$ , and  $W = X \setminus Y$ .

Consider the mapping  $h : X \rightarrow X$  defined by

$$h(x) = \frac{1}{1 + \|x\|} x.$$

It is easily checked that  $h$  is a  $C^1$  diffeomorphism from  $X$  onto its open unit ball  $\text{int}B_X$  (with inverse  $h^{-1}(y) = (1/(1 - \|y\|))y$ ), that  $h$  has a bounded derivative, and that  $h$  preserves lines and in particular leaves the subspace  $Y$  invariant. By composing  $h$  with suitable dilations and translations we get  $C^1$  diffeomorphisms  $h_j : X \rightarrow B_j$  such that  $h_j$  is Lipschitz for each  $j$ . And, by composing the restrictions to  $Y$  of these  $h_j$  with our function  $f$ , we get  $C^1$  and Lipschitz functions  $f_j := f \circ (h_j)|_Y : Y \rightarrow \mathbb{R}$ . According to the preceding result there exist  $C^1$  (and Lipschitz) extensions  $G_j : X \rightarrow \mathbb{R}$  of  $f_j$ . Then the composition

$$g_j = G_j \circ h_j^{-1}$$

defines a  $C^1$  extension of  $f|_{B_j \cap Y}$  to  $B_j$ . Put  $g_0 \equiv 1$ .

Now let  $\{\varphi_0\} \cup \{\varphi_j\}_{j=1}^{\infty}$  be a  $C^1$  partition of unity subordinated to the open covering  $\{W\} \cup \{B_j\}_{j=1}^{\infty}$  of  $X$  (such partitions of unity always exist for separable spaces with  $C^1$  norms, see [DGZ, Theorem VIII.3.2, page 351]). Define

$$g(x) = \sum_{j=0}^{\infty} \varphi_j(x) g_j(x).$$

Then it is clear that  $g$  is a  $C^1$  extension of  $f$  to  $X$ . ■

**Corollary 1.** *Let  $M$  be a separable Banach manifold modelled on a Banach space  $X$  which admits a  $C^1$  norm, and let  $N$  be a closed  $C^1$  submanifold of  $M$ . Then every  $C^1$  function  $f : N \rightarrow \mathbb{R}$  has a  $C^1$  extension to  $M$ .*

**Proof.** Let  $\{V_j\}_{j=1}^\infty$  be a covering of  $N$  by open sets in  $M$  so that there are  $C^1$  diffeomorphisms  $\psi_j : V_j \rightarrow X$  such that  $\psi_j(N \cap V_j) = Y$ , where  $Y$  is a closed subspace of  $X$ .

The functions  $f \circ \psi_j^{-1} : Y \rightarrow \mathbb{R}$  are  $C^1$  and (by the preceding theorem) there are  $C^1$  extensions  $G_j : X \rightarrow \mathbb{R}$ , which in turn give, by composition,  $C^1$  extensions  $g_j := G_j \circ \psi_j$  of  $f|_{V_j \cap N}$  to  $V_j$ .

Then, if  $\{\theta\} \cup \{\theta_j\}_{j=1}^\infty$  is a  $C^1$  partition of unity subordinated to the open covering  $\{M \setminus N\} \cup \{V_j\}_{j=1}^\infty$  of  $M$  (note that a separable Banach manifold modelled on a Banach space  $X$  admits  $C^1$  partitions of unity if and only if  $X$  does), the function

$$g(x) = \sum_j \theta_j(x) g_j(x)$$

is a  $C^1$  extension of  $f$  to  $M$ . ■

If  $Y$  is not required to be a closed subspace of  $X$  but is merely closed, results similar to Theorem 1 and Theorem 2 can be obtained. However, the differentiability requirements on  $f$  must be strengthened. The proofs, which we omit, closely parallel those for Theorem 1 and Theorem 2, where the essential difference is that  $f'(y)$  is extended to directions off of  $Y$ , not by the Bartle-Graves generated  $H(f'(y))$ , but by explicit hypothesis. Of course one must also verify that Lemma 1 still holds in the case when  $Y$  is a closed subset of  $X$ . It is easy to establish such a version of Lemma 1 by replacing the function  $\|q(x)\|$  in its proof with a Lipschitz  $C^1$  approximation of the distance function to  $Y$  (which in turn can be constructed with the help of a sup-partition of unity provided by Claim 1, see also [F, HJ]).

**Theorem 4.** *Let  $X$  be a separable Banach space which admits a  $C^1$ -smooth norm,  $Y \subset X$  a closed subset, and  $U \supset Y$  a neighbourhood of  $Y$ . Let  $\varepsilon > 0$ , and  $f : U \rightarrow \mathbb{R}$  be a map which is  $C^1$ -smooth on  $Y$  as a function on  $X$ . Then there exists a  $C^1$ -smooth map  $g : X \rightarrow \mathbb{R}$  such that,*

- (1)  $|f(y) - g(y)| < \varepsilon$  on  $Y$ ,
- (2)  $\|f'(y) - g'(y)\|_{X^*} < \varepsilon$  on  $Y$ .

**Theorem 5.** *Let  $X$  be a separable Banach space which admits a  $C^1$ -smooth norm. Let  $Y \subset X$  be a closed subset, and  $U$  an open set containing  $Y$ . Let  $f : U \rightarrow \mathbb{R}$  be a map  $C^1$ -smooth on  $Y$  as a function on  $X$ . Then there is a  $C^1$  extension of  $f|_Y$  to  $X$ .*

We have the following easy corollary.

**Corollary 2.** *Let  $X$  be a separable Banach space which admits a  $C^1$ -smooth norm. Let  $U \subset X$  be open and  $f : U \rightarrow \mathbb{R}$  a  $C^1$ -smooth function. Then for any open set  $V \subset U$  with  $\overline{V} \subset U$ , there is a  $C^1$  extension of  $f|_V$  to  $X$ .*

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