PERTURBED SMOOTH LIPSCHITZ EXTENSIONS OF UNIFORMLY CONTINUOUS FUNCTIONS ON BANACH SPACES

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ABSTRACT. We show that if Y is a separable subspace of a Banach space X such that both X and the quotient X/Y have C^p-smooth Lipschitz bump functions, and U is a bounded open subset of X, then, for every uniformly continuous function $f : Y \cap U \to \mathbb{R}$ and every $\varepsilon > 0$, there exists a C^p-smooth Lipschitz function $F : X \to \mathbb{R}$ such that $|F(y) - f(y)| \leq \varepsilon$ for every $y \in Y \cap U$.

If we are given a separable subspace Y of a Banach space X and a continuous (resp. Lipschitz) function $f : Y \to \mathbb{R}$, under what conditions can we ensure the existence of a C^p-smooth (Lipschitz) perturbed extension of f? That is, for a given $\varepsilon > 0$, does there exist a C^p-smooth (Lipschitz) function $F : X \to \mathbb{R}$ so that $|F(y) - f(y)| \leq \varepsilon$ for all $y \in Y$? Such an F will be called a smooth perturbed extension of f.

Of course there are several conditions under which the answer is "yes" in a trivial way (for the non-Lipschitz case). For instance, when X has C^p-smooth partitions of unity, and also when Y has a C^p-smooth bump function and the subspace Y is complemented in X, such perturbed extensions F are easily proved to exist. However, in the Lipschitz case, or when the space X does not have smooth partitions of unity and Y is not complemented in X, it is not quite clear what the answer is.

These questions are interesting in the theory of Banach spaces because, for instance, while trying to prove a theorem, one might be able to construct a continuous (resp. Lipschitz) function f with some nice properties, but defined only on a certain separable subspace Y of X, and then one might want to obtain a smooth (resp. and Lipschitz) function F defined on the whole of X that behaves on Y almost the same way as f does.

In this paper we will try to give a solution to the above question in the Lipschitz case. This problem is clearly related to the question concerning uniform approximation of Lipschitz functions by smooth Lipschitz functions on infinite-dimensional spaces, which has remained unasked and open until recent times: in [7] R. Fry has shown that, on any separable Banach space with a C^1-smooth bump function, uniformly continuous functions can be approximated by C^1-smooth Lipschitz functions, uniformly on bounded sets.

We will use some of the ideas of the proof of [7], combined with some techniques for starlike bodies, to prove the following theorem, which provides a solution to the above "smooth perturbed extension" problem.

Theorem 1. *Let $(X, \| \cdot \|_X)$ be a Banach space and Y a separable subspace of X such that both X and X/Y admit a Lipschitz C^p-smooth bump function, where $p \in \mathbb{N} \cup \{\infty\}$. Let us consider U, an open subset of X, and a bounded uniformly continuous function $f : Y \cap U \to \mathbb{R}$. Then, for every $\varepsilon > 0$ there exists a C^p-smooth Lipschitz function $K : X \to \mathbb{R}$ such that $|K(x) - f(x)| \leq \varepsilon$ for all $x \in Y \cap U$.*

Here $p \in \mathbb{N} \cup \{\infty\}$. Note that, if we take $X = Y$, we obtain a generalization of the main result of [7] (which was proved only for $p = 1$). Note also that if the open set U is assumed to be bounded, then it is not necessary to demand that f be bounded, because every uniformly continuous function is bounded on bounded sets.

As we have already said, starlike bodies and their Minkowski functionals will play an important role in our proof. Let us recall the definition and some of the elementary properties of starlike bodies. A closed subset A of a Banach space X is said to be a starlike body if there exists a point a_0 in the interior of A such that every ray emanating from a_0 meets ∂A, the boundary of A, at most once. We will say that a_0 is a center of A. There can obviously exist many centers for a given starlike body. Up to a suitable translation, we can always assume that $a_0 = 0$ is the origin of X, and we will often do so, unless otherwise stated. For a starlike body A with center a_0, we define the characteristic cone of A as

$$ccA = \{x \in X | a_0 + r(x - a_0) \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A with respect to the center a_0 as

$$\mu_{A,a_0}(x) = \mu_A(x) = \inf\{t > 0 \,|\, x - a_0 \in t(-a_0 + A)\} \text{ for all } x \in X.$$

Note that $\mu_A(x) = \mu_{-a_0+A}(x - a_0)$ for all $x \in X$. It is easily seen that μ_A is a continuous function that satisfies $\mu_A(a_0 + rx) = r\mu_A(a_0 + x)$ for every $r \geq 0$ and $x \in X$, and $\mu_A^{-1}(0) = ccA$. Moreover, $A = \{x \in X | \mu_A(x) \leq 1\}$, and $\partial A = \{x \in X \mid \mu_A(x) = 1\}$. Conversely, if $\psi : X \to [0, \infty)$ is continuous and satisfies $\psi(a_0 + \lambda x) = \lambda\psi(a_0 + x)$ for all $\lambda \geq 0$, then $A_\psi = \{x \in X \mid \psi(x) \leq 1\}$ is a starlike body.

We will say that A is a C^p-smooth starlike body provided its Minkowski functional μ_A is C^p-smooth on the set $X \setminus ccA = X \setminus \mu_A^{-1}(0)$. This is equivalent to saying that ∂A is a C^p-smooth one-codimensional submanifold of X such that no affine hyperplane tangent to ∂A contains a ray emanating from the center a_0. Throughout this paper, $p = 1, 2, \ldots, \infty$. Similarly, we will say that a starlike body A is Lipschitz if μ_A is Lipschitz on X.

The following result is implicitly proved in [6, Proposition II.5.1]; see also [8].

Proposition 2. *Let X be a Banach space and $p \in \mathbb{N} \cup \{\infty\}$. The following assertions are equivalent.*

(1) *X admits a C^p-smooth Lipschitz bump function.*
(2) *There exist numbers $a, b > 0$ and a Lipschitz function $\psi : X \to [0, \infty)$ which is C^p-smooth on $X \setminus \{0\}$, homogeneous (that is $\psi(tx) = |t|\psi(x)$ for all $t \in \mathbb{R}, x \in X$), and such that $a\| \cdot \| \leq \psi \leq b\| \cdot \|$.*
(3) *There exists a symmetric bounded C^p-smooth Lipschitz starlike body E with center $0 \in X$.*

As a consequence we can deduce the following result, which plays a technical but fundamental role in our proof.

Corollary 3. *Let X be a Banach space, Y a closed subspace of X, and $p \in \mathbb{N} \cup \{\infty\}$. Consider the following assertions.*

(1) *X/Y admits a C^p-smooth Lipschitz bump function.*

(2) *There exist numbers $a, b > 0$ and a Lipschitz function $\psi : X \to [0, \infty)$ that is homogeneous, C^p-smooth on $X \setminus Y$, and such that $a \operatorname{dist}(x, Y) \leq \psi(x) \leq b \operatorname{dist}(x, Y)$ for all $x \in X$.*

(3) *There exists a symmetric C^p-smooth Lipschitz starlike body E with center $0 \in X$ and whose characteristic cone is Y .*

Then $(1) \implies (2) \implies (3)$.

Proof. We denote by $| \cdot |$ the canonical norm on X/Y, and let π be the canonical quotient map of X onto X/Y. If (1) is true, then, according to Proposition 2, there exist positive real numbers a, b and a Lipschitz function $\Psi : X/Y \to [0, \infty)$, that is homogeneous, C^p-smooth on $X/Y \setminus \{0\}$, and such that $a| \cdot | \leq \Psi \leq b| \cdot |$. Then the function $\psi = \Psi \circ \pi$ makes (2) hold true. To see that $(2) \implies (3)$ it is enough to define E as the set $\{x \in X : \psi(x) \leq 1\}$. $\square$

Proof of Theorem 1. Without loss of generality, and taking into account the well-known fact that every uniformly continuous function can be uniformly approximated by Lipschitz functions (see [2] for instance), we may assume that f is Lipschitz.

Moreover, since f is bounded and Lipschitz on $Y \cap U$, there exists a bounded Lipschitz function $F : Y \to \mathbb{R}$ that is an extension of f. Indeed, let L be the Lipschitz constant of f on $Y \cap U$, and let $C > 0$ be such that $|f(x)| + 1 \leq C$. The function $g : Y \to \mathbb{R}$ defined as

$$g(y) = \inf\{f(u) + L\|y - u\| : u \in Y \cap U\}$$

is easily seen to be a Lipschitz extension of f to Y. Then, the function $F : Y \to \mathbb{R}$ defined by

$$F(y) = \begin{cases} -C & \text{if} & g(y) \leq -C, \\ g(y) & \text{if} & -C \leq g(y) \leq C, \\ C & \text{if} & C \leq g(y) \end{cases}$$

is a bounded Lipschitz extension of f to Y.

Since f and F are bounded, we may obviously assume that $F, f \geq 1$.

From the assumption that X and X/Y have C^p-smooth Lipschitz bumps we infer two facts. First, the existence of a symmetric C^p-smooth Lipschitz starlike body $B_X \subset E$ with center the origin (see Proposition 2). Therefore the Minkowski functional $\mu_E : X \to [0, \infty)$ is Lipschitz, homogeneous, and C^p-smooth on $X \setminus \{0\}$. Second (see Corollary 3), there are a Lipschitz function $\rho : X \to [0, \infty)$ and a real number $M > 1$ such that

(1) $\rho(tx) = |t|\rho(x)$ for all $t \in \mathbb{R}$, $x \in X$;

(2) $\rho^{-1}(0) = Y$;

(3) ρ is C^p-smooth on $X \setminus Y$;

(4) $\operatorname{dist}(x, Y) \leq \rho(x) \leq M\operatorname{dist}(x, Y)$, for all $x \in X$.

Let $(e_i)_{i=1}^{\infty}$ be the usual basis of c_0. Let us consider the function $\Phi : c_0 \to \mathbb{R}$ defined by $\Phi(x) = \sum_{n=1}^{\infty} x_n^{2n}$ for each $x = (x_n)_{n=1}^{\infty} \in c_0$. The set $A = \{x \in c_0 : \Phi(x) \leq 1\}$

is closed, convex, bounded and symmetric with respect to the origin (which is contained as an interior point). Hence $\|\cdot\| := \mu_A(\cdot)$ is a C^∞-smooth (in fact real-analytic) norm equivalent to the canonical norm $\|\cdot\|_\infty$ (see [6, Theorem V.1.5 and Remark V.1.6]). It is easy to verify that $\frac{1}{2}\|\cdot\| \leq \|\cdot\|_\infty \leq \|\cdot\|$. Moreover, $\|\cdot\|$ has the following nice property. $\square$

Fact 4. *If $x = (x_n)_{n=1}^\infty, y = (y_n)_{n=1}^\infty \in c_0$ satisfy $|x_n| \leq |y_n|$ for every $n \in \mathbb{N}$, then $\|x\| \leq \|y\|$.*

Proof. For every $\lambda > 0$ we have that

$$\sum_{n=1}^\infty \left(\frac{x_n}{\lambda}\right)^{2n} \leq \sum_{n=1}^\infty \left(\frac{y_n}{\lambda}\right)^{2n}.$$

It follows that $\inf\{\lambda > 0 : \Phi(y/\lambda) \leq 1\} \leq \inf\{\lambda > 0 : \Phi(x/\lambda) \leq 1\}$, that is, $\|x\| \leq \|y\|$. $\square$

Fix $\varepsilon \in (0,1)$. Since $F : Y \to \mathbb{R}$ is Lipschitz and, in particular, uniformly continuous, there exists $\delta > 0$ such that, for all $x, y \in Y$, we have

$$\mu_E(x - y) \leq \delta \Rightarrow |F(x) - F(y)| \leq \varepsilon.$$

For every $x \in X$ and $r > 0$, the set

$$B(x, r; \mu_E) = \{y \in X : \mu_E(y - x) < r\}$$

is open. By the separability of Y, we can take a dense sequence $(x_j)_{j=1}^\infty$ in Y. We have that

$$Y \subset \bigcup_{j=1}^\infty B\left(x_j, \frac{\delta}{8}; \mu_E\right).$$

Let us define the open subset $V := \bigcup_{j=1}^\infty B(x_j, \frac{\delta}{8}; \mu_E)$. In the sequel, we will say that a family of functions $\{h_i\}_{i \in I}$ is uniformly Lipschitz if there is a number $C > 0$ so that h_i is C-Lipschitz for every $i \in I$.

Claim 5. There exists a family $\{\Psi_j : j \in \mathbb{N}\}$ of non-negative C^p-smooth functions defined on X such that:

(i) $\{\Psi_j : j \in \mathbb{N}\}$ is uniformly Lipschitz and uniformly bounded.

(ii) For every $x \in V$, there exist W_x, an open neighborhood of $x \in V$, and $j_x \in \mathbb{N}$, such that $\Psi_j|_{W_x} = 0$ if $j > j_x$.

(iii) For every $x \in V$, there exists $j_0 \in \mathbb{N}$ such that $\Psi_{j_0}(x) = 1$.

(iv) For every $x \in V$ and $j \in \mathbb{N}$, if $\mu_E(x - x_j) \geq \delta$, then $\Psi_j(x) = 0$.

Let us assume for a while that Claim 5 is proved and let us see how the proof of Theorem 1 is finished.

By (ii) of Claim 5, for every $x \in V$ there exist W_x, an open neighborhood of $x \in V$, and $j_x \in \mathbb{N}$ such that

$$y \in W_x \ , \ j > j_x \Rightarrow \Psi_j(y) = 0.$$

Then the mapping

$$V \ni x \longrightarrow \left\{\Psi_j(x)\right\}_{j=1}^\infty \in c_{00} \subset c_0$$

locally takes values on finite-dimensional subspaces of c_{00}. Clearly it is also Lipschitz and C^p-smooth (note (i) of Claim 5). Let us define $\Psi : V \to \mathbb{R}$ by

$$\Psi(x) = \left\|\left\{\Psi_j(x)\right\}_{j=1}^\infty\right\|,$$

which is a Lipschitz function. To prove that Ψ is C^p-smooth, it suffices to check that Ψ is bounded below by a strictly positive real number. Indeed, by (iii) of Claim 5, for each $x \in V$, there exists $j_0 \in \mathbb{N}$ such that $\Psi_{j_0}(x) = 1$, and thus

$$\Psi(x) = \left\| \left\{ \Psi_j(x) \right\}_{j=1}^\infty \right\| \geq \left\| \left\{ \Psi_j(x) \right\}_{j=1}^\infty \right\|_\infty \geq \Psi_{j_0}(x) = 1.$$

It follows that Ψ is a C^p-smooth function and $\Psi \geq 1$.

Define then $K_0 : V \to [0, \infty)$ by

$$K_0(x) = \frac{\left\| \left\{ F(x_j) \Psi_j(x) \right\}_{j=1}^\infty \right\|}{\Psi(x)}.$$

Taking into account that $F \geq 1$, it is obvious that

$$\left\| \left\{ F(x_j) \Psi_j(x) \right\}_{j=1}^\infty \right\| \geq \left\| \left\{ F(x_j) \Psi_j(x) \right\}_{j=1}^\infty \right\|_\infty \geq \left\| \left(\Psi_j(x) \right)_{j=1}^\infty \right\|_\infty \geq 1.$$

Therefore K_0 is C^p-smooth. Let us see that K_0 is a Lipschitz function. Since F is bounded, and the family $\{\Psi_j : j \in \mathbb{N}\}$ is uniformly bounded, it is obvious that the collection $\{F(x_j)\Psi_j : j \in \mathbb{N}\} \subset C^p(V)$ is uniformly Lipschitz and uniformly bounded, which implies that the numerator of K_0 is bounded and Lipschitz. Besides, since Ψ is Lipschitz and $\Psi \geq 1$, it follows that K_0 is Lipschitz (note that the inverse of a Lipschitz function bounded below by a strictly positive real number is Lipschitz).

Finally let us see that $|K_0(x) - F(x)| \leq \varepsilon$ for every $x \in Y$. Fix $x \in Y$. Then,

$$|K(x) - F(x)| = \left| \frac{\left\| \left\{ F(x_j) \Psi_j(x) \right\}_{j=1}^\infty \right\|}{\Psi(x)} - F(x) \right|$$

$$= \left| \frac{\left\| \left\{ F(x_j) \Psi_j(x) \right\}_{j=1}^\infty \right\|}{\Psi(x)} - \frac{F(x)\Psi(x)}{\Psi(x)} \right| = \frac{1}{\Psi(x)} \left\| \left\{ \Psi_j(x) \left(F(x_j) - F(x) \right) \right\}_{j=1}^\infty \right\|.$$

Let us consider the set $J := \{ j \in \mathbb{N} : \mu_E(x - x_j) < \delta \}$. We can distinguish two situations.

Case 1. If $j \in J$, then $\mu_E(x - x_j) < \delta$, and thus $|F(x) - F(x_j)| \leq \varepsilon$.

Case 2. Assume that $j \notin J$. Then $\mu_E(x - x_j) \geq \delta$ and, by (iv) of Claim 5, we have $\Psi_j(x) = 0$.

By collecting the conclusions of the distinguished cases, we deduce that

$$\Psi_j(x) |F(x_j) - F(x)| \leq \varepsilon \Psi_j(x)$$

for every $j \geq 1$. Since $\left\{ \Psi_j(x) \left(F(x_j) - F(x) \right) \right\}_{j=1}^\infty \in c_{00}$, taking Fact 4 into account we may estimate

$$|K_0(x) - F(x)| = \frac{1}{\Psi(x)} \left\| \left\{ \Psi_j(x) \left(F(x_j) - F(x) \right) \right\}_{j=1}^\infty \right\|$$

$$\leq \frac{1}{\Psi(x)} \left\| \varepsilon \left\{ \Psi_j(x) \right\}_{j=1}^\infty \right\| = \varepsilon \frac{\Psi(x)}{\Psi(x)} = \varepsilon.$$

Hence $K_0 : V \to \mathbb{R}$ is a Lipschitz C^p-smooth function such that $|K_0(x) - F(x)| \leq \varepsilon$ for every $x \in Y$.

Finally, let us see how we can extend the function K_0 from the open set V to the whole of X. Since $B_X \subset E$, there exists a real number $m \geq 1$ that satisfies $m\| \cdot \|_X \leq \mu_E \leq \| \cdot \|_X$. By property (4) of ρ, it is easy to see that

$$Y \subset \{x \in X : \rho(x) < \frac{\delta}{8}\} \subset V \subset \{x \in X : \rho(x) < \frac{M\delta}{8m}\}.$$

Let us pick two positive real numbers r_1 and r_2 such that $0 < r_1 < r_2 < \frac{\delta}{8}$. Take a C^∞-smooth function $\theta : \mathbb{R} \to [0,1]$ such that $\theta^{-1}(1) = (-\infty, r_1]$ and $\theta^{-1}(0) = [r_2, \infty)$. Then, the function $K : X \to \mathbb{R}$ defined by

$$K(x) = \begin{cases} \theta(\rho(x)) K_0(x) & \text{if } x \in V, \\ 0 & \text{if } x \notin V \end{cases}$$

is Lipschitz and C^p-smooth. Moreover, K and K_0 are equal on the open set

$$\{x \in X : \rho(x) < r_1\},$$

which contains the subspace Y. Therefore $|K(x) - F(x)| = |K_0(x) - F(x)| \leq \varepsilon$ for every $x \in Y$. In particular, we have that

$$|K(x) - f(x)| = |K(x) - F(x)| \leq \varepsilon,$$

for each $x \in U \cap Y$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To finish the proof of Theorem 1, it only remains to show Claim 5.

Proof of Claim 5. Let us take two Lipschitz functions $\xi_1, \xi_2 \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\xi_1' \geq 0, \xi_1^{-1}(0) = (-\infty, \delta/2], \; \xi_1^{-1}(1) = [\delta, \infty);$$
$$\xi_2' \leq 0, \; \xi_2^{-1}(1) = (-\infty, \delta/4], \; \xi_2^{-1}(0) = [\delta/2, \infty).$$

For every $j \in \mathbb{N}$ and every $x \in X$, we define

$$f_j(x) = \xi_1(\mu_E(x - x_j)), \; g_j(x) = \xi_2(\mu_E(x - x_j)).$$

Since ξ_1, ξ_2 and μ_E are Lipschitz, it follows that $\{f_j : j \in \mathbb{N}\} \cup \{g_j : j \in \mathbb{N}\}$ is a uniformly Lipschitz family. Moreover, all the functions f_j and g_j are C^p-smooth (notice that ξ_1 and ξ_2 are constant on a neighborhood of the origin). Let $\Phi_1 : X \to c_{00} \subset c_0$ be the zero function. For every $j \geq 2$, let us define $\Phi_j : X \to c_{00}$ by

$$\Phi_j(x) = (g_1(x), ..., g_{j-1}(x), 0, 0, ...).$$

It is clear that $\{\Phi_j : j \in \mathbb{N}\}$ is uniformly Lipschitz, and each Φ_j is C^p-smooth. Take a C^∞-smooth function $\eta : \mathbb{R} \to [0, \infty)$ such that η is non-decreasing, $\eta(t) = t$ if $t \geq 1$, and $\eta = 1/2$ on a neighborhood of the origin.

For each $j \in \mathbb{N}$, we define $S_j : X \to [0, \infty)$ by

$$S_j(x) = f_j(x) + \eta(\|\Phi_j(x)\|).$$

Bearing in mind that f_j and Φ_j are C^p-smooth, η is C^∞-smooth, $\|\cdot\|$ is a C^∞-smooth norm, and η is constant on a neighborhood of the origin, it follows that S_j is a C^p-smooth function. Moreover, taking into account that $\{f_j : j \in \mathbb{N}\} \cup \{\Phi_j : j \in \mathbb{N}\}$ is a uniformly Lipschitz family, and that η and ξ are Lipschitz, it follows that $\{S_j : j \in \mathbb{N}\}$ is a uniformly Lipschitz family of non-negative C^p-smooth functions defined on X.

Let $h \in C^\infty(\mathbb{R}, [0,1])$ be a Lipschitz function such that h is non-increasing and $h^{-1}(1) = (-\infty, 1/2]$ and $h^{-1}(0) = [1, \infty)$. We define

$$\Psi_j = h \circ S_j,$$

for each $j \in \mathbb{N}$. It is patent that $\{\Psi_j : j \in \mathbb{N}\} = \{h \circ S_j : j \in \mathbb{N}\} \subset C^p(X, [0,1])$ is a uniformly Lipschitz family. Therefore, the collection $\{\Psi_j : j \in \mathbb{N}\}$ satisfies (i) of Claim 5. Let us prove that $\{\Psi_j : j \in \mathbb{N}\}$ also satisfies (ii), (iii) and (iv).

(ii) Pick $x \in V$. Choose $j_x \in \mathbb{N}$ and $r > 0$ such that $W_x := B(x, r; \mu_E) \subset B(x_{j_x}, \frac{\delta}{8}; \mu_E)$. If $y \in W_x$, then $\mu_E(y - x_{j_x}) < \frac{\delta}{8}$ and $g_{j_x}(y) = 1$. Therefore, for each $j > j_x$, we have that $\|\Phi_j(y)\| \geq \|\Phi_j(y)\|_\infty \geq |g_{j_x}(y)| = 1$. Hence $\eta(\|\Phi_j(y)\|) = \|\Phi_j(y)\| \geq 1$ and $S_j(y) \geq 1$. It follows from the definitions of the function h that $\Psi_j(y) = h(S_j(y)) = 0$ for each $y \in W_x$ and $j > j_x$.

(iii) Let us fix $x \in V$. Notice that $V \subset \bigcup_{j \in \mathbb{N}} B(x_j, \frac{\delta}{2}; \mu_E)$. If $x \in B(x_1, \frac{\delta}{2}; \mu_E)$, then $f_1(x) = 0 = \|\Phi_1(x)\|$ and $\eta(\|\Phi_1(x)\|) = 1/2$. If $x \notin B(x_1, \frac{\delta}{2}; \mu_E)$, there exists $j_0 > 1$ such that $x \in B(x_{j_0}, \frac{\delta}{2}; \mu_E) \setminus \bigcup_{i < j_0} B(x_i, \frac{\delta}{2}; \mu_E)$. From the definition of the functions f_j, g_j and η, it follows that $f_{j_0}(x) = 0$ and $\eta((\|\Phi_{j_0}(x)\|)) = \frac{1}{2}$. Therefore, for each $x \in V$, there exists $j_0 \in \mathbb{N}$ such that $\Psi_{j_0}(x) = h(S_{j_0}(x)) = h(1/2) = 1$.

(iv) Take $x \in V$ and $j \in \mathbb{N}$. Assume that $\mu_E(x - x_j) \geq \delta$. From the definition of f_j, it follows that $f_j(x) = \xi_1(\mu_E(x - x_j)) = 1$. Hence $S_j(x) = f_j(x) + \eta(\|\Phi_j(x)\|) \geq f_j(x) = 1$ and $\Psi_j(x) = h(S_j(x)) = 0$. $\qquad \square$

An application. As an illustration of the usefulness of our main result, let us mention that D. Azagra and A. Montesinos have used Theorem 1 to show that, for every infinite-dimensional Banach space X with a separable subspace Y so that X and X/Y have C^p-smooth Lipschitz bumps, there exist C^p-diffeomorphisms deleting points with bounded supports, that is, there exist C^p-diffeomorphisms $h : X \to X \setminus \{0\}$ such that $h(x) = x$ if $\|x\| \geq 1$; see [1] for details.

Some related open questions. (1) Is it always true that if X has a C^p-smooth Lipschitz bump and $Y \subset X$ is separable, then the quotient X/Y has a C^p-smooth bump as well?

(2) Is "X/Y has a C^p-smooth Lipschitz bump" necessary in Theorem 1?

(3) Does Theorem 1 hold true when f is not bounded?

(4) Is Theorem 1 true if we drop the Lipschitz condition everywhere?

(5) If X has a separating polynomial, can we obtain real-analytic perturbed extensions F?

(6) What happens when Y is not separable?

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