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Nonlinear Analysis 67 (2007) 154–174
Nonlinear Analysis
www.elsevier.com/locate/na

Applications of proximal calculus to fixed point theory on Riemannian manifolds

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Received 15 December 2005; accepted 28 April 2006

Abstract

We prove a general form of a fixed point theorem for mappings from a Riemannian manifold into itself which are obtained as perturbations of a given mapping by means of general operations which in particular include the cases of sum (when a Lie group structure is given on the manifold) and composition. In order to prove our main result we develop a theory of proximal calculus in the setting of Riemannian manifolds.

MSC: 49J52; 58E30; 58C30; 47H10

Keywords: Proximal subdifferential; Riemannian manifolds; Fixed point theory

1. Introduction and tools

The proximal subdifferential of lower semicontinuous real-valued functions is a very powerful tool that has been extensively studied and used in problems of optimization, control theory, differential inclusions, Lyapunov Theory, stabilization, and Hamilton–Jacobi equations; see [5] and the references therein.

In this paper we introduce a notion of proximal subdifferential for functions defined on a Riemannian manifold M (either finite or infinite dimensional) and we develop the rudiments of a calculus for nonsmooth functions defined on M. We then establish a Decrease Principle from which we deduce Solvability and Implicit Function Theorems for nonsmooth functions on M. Our main results are applications of this Solvability Theorem: we provide several fixed point theorems for expansive and nonexpansive mappings and certain perturbations of such mappings

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0362-546X/$ - see front matter
doi:10.1016/j.na.2006.04.019

defined on M. Observe that, in general, very small perturbations of mappings having fixed points may lose them: consider for instance $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + \varepsilon$. Note also that most of the known fixed point theorems (such as Brouwer's, Lefschetz's, Schauder's and the Banach contraction mapping principle) rely either on compactness or on contractiveness, see [4,6] for instance. However, our results hold for (possibly expansive) mappings on (possibly noncompact) complete Riemannian manifolds.

Let us give a brief sample of the corollaries on fixed points that we will be deducing from our main theorems in the last section of this paper.

Corollary 1. *Let M be a complete Riemannian manifold with a positive injectivity radius $\rho = i(M)$. Let x_0 be a fixed point of a C^1 smooth mapping $G : M \to M$ such that G is C-Lipschitz on a ball $B(x_0, R)$. Let $H : M \to M$ be a differentiable mapping. Assume that $0 < 2R \max\{1, C\} < \rho$, that*

$$\langle L_{xH(G((x)))}h, L_{G(x)H(G(x))}dG(x)(h)\rangle_{F(x)} \le K < 1$$

for all $x \in B(x_0, R)$ and $h \in \mathrm{TM}_x$ with $\|h\|_x = 1$, and that $\|dH(y) - L_{yH(y)}\| < \varepsilon/C$ for every $y \in G(B(x_0, R))$, where $\varepsilon < 1 - K$, and $d(x_0, H(G(x_0))) < R(1 - K - \varepsilon)$. Then $F = H \circ G$ has a fixed point in $B(x_0, R)$.

This is a consequence of Theorem 36 below. Here L_{xy} stands for the parallel transport along the (unique in this setting) minimizing geodesic joining the points x and y. The hypotheses on H mean that H is relatively close to the identity, so the perturbation brought on G by its composition with H is relatively small.

Corollary 2. *Let $(M, +)$ be a complete Riemannian manifold with an abelian Lie group structure. Let x_0 be a fixed point of a C^1 function $G : M \to M$ satisfying the following condition:*

$$\langle h, dG(x_0)(h)\rangle_{x_0} \le K < 1 \quad \textit{for every } \|h\|_{x_0} = 1.$$

Then there exists a positive δ such that for every Lipschitz mapping $H : M \to M$ with Lipschitz constant smaller than δ, the mapping $G + H : M \to M$ has a fixed point provided that $d(x_0, x_0 + H(x_0)) < \delta$.

See Corollary 39 below.

The condition on the differential of G is satisfied, for instance, if G locally behaves like a multiple of a rotation round the point x_0, but notice that G may well be expansive. Consider for instance $G : \mathbb{R}^2 \to \mathbb{R}^2$, $G(x, y) = 23(y, -x)$; in this case we can take $K = 0$, but G is clearly expansive.

It should be stressed that these fixed point results are new even in the case when $M = \mathbb{R}^n$ or any Hilbert space. In particular we have the following.

Corollary 3. *Let X be a Hilbert space, and let x_0 be a fixed point of a differentiable mapping $G : X \to X$ satisfying the following condition:*

$$\langle h, DG(x)(h)\rangle \le K < 1 \quad \textit{for every} \quad x \in B(x_0, R) \quad \textit{and} \quad \|h\| = 1.$$

Then we have that:

(1) *If H is a differentiable L-Lipschitz mapping, with $L < 1 - K$, then $G + H$ has a fixed point in $B(x_0, R)$, provided that $\|H(x_0)\| < R(1 - K - L)$.*

(2) *If* $H : X \to X$ *is a differentiable mapping such that* $\|DH(G(x)) - I\| < \varepsilon$ *for every* $x \in B(x_0, R)$, *then* $F = H \circ G$ *has a fixed point in* $B(x_0, R)$, *provided that* $K + \varepsilon < 1$ *and* $\|H(x_0) - x_0\| < R(1 - K - \varepsilon)$.

All of these results and many other things will be proved in Section 2.

This paper should be compared with [3], where a theory of viscosity subdifferentials for functions defined on Riemannian manifolds is established and applied to show existence and uniqueness of viscosity solutions to Hamilton–Jacobi equations on such manifolds. See also [1] for proximal calculus on Riemannian manifolds applications, in particular, a Moreau–Yosida regularization for functions defined on Riemannian manifolds is presented.

On the other hand, proximal analysis is also a useful tool in the context of PDE's, see for instance [9,10,12], and we believe that the tools that we use here could also be employed in the study of PDE's on Riemannian manifolds.

Let us recall the definition of the proximal subdifferential for functions defined on a Hilbert space X. A vector $\zeta \in X$ is called a *proximal subgradient* of a lower semicontinuous function f at $x \in \operatorname{dom} f := \{y \in X : f(y) < +\infty\}$ provided there exist positive numbers σ and η such that

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2 \quad \text{for all } y \in B(x, \eta).$$

The set of all such ζ is denoted $\partial_P f(x)$, and is referred to as the *proximal subdifferential*, or P-subdifferential. A comprehensive study of this subdifferential and its numerous applications can be found in [5].

Before giving the definition of the proximal subdifferential for a function defined on a Riemannian manifold, we must establish a few preliminary results.

The following result is proved in [2, Corollary 2.4].

Proposition 4. *Let* X *be a real Hilbert space, and* $f : X \longrightarrow (-\infty, \infty]$ *be a proper, lower semicontinuous function. Then,*

$$\partial_P f(x) = \{\varphi'(x) : \varphi \in C^2(X, \mathbb{R}),\ f - \varphi \text{ attains a local minimum at } x\}.$$

In particular this implies that $\partial_P f(x) \subseteq D^- f(x)$, where $D^- f(x)$ is the viscosity subdifferential of f at x.

Lemma 5. *Let* X_1 *and* X_2 *be two real Hilbert spaces,* $\Phi : X_2 \to X_1$ *a* C^2 *diffeomorphism,* $f : X_1 \to (-\infty, +\infty]$ *a lower semicontinuous function. Then* $v \in \partial_P f(x_1)$ *if and only if* $D\Phi(x_2)^*(v) \in \partial_P(f \circ \Phi)(x_2)$, *where* $\Phi(x_2) = x_1$.

Proof. This is a trivial consequence of Proposition 4, bearing in mind that compositions with diffeomorphisms preserve local minima. □

Corollary 6. *Let* M *be a Riemannian manifold,* $p \in M$, (φ_i, U_i) $i = 1, 2$, *two charts with* $p \in U_1 \cap U_2$, *and* $\varphi_i(p) = x_i$. *Then* $\partial_P(f \circ \varphi_1^{-1})(x_1) \neq \emptyset$ *if and only if* $\partial_P(f \circ \varphi_2^{-1})(x_2) \neq \emptyset$. *Moreover,* $D(\varphi_1 \circ \varphi_2^{-1})(x_2)^*(\partial_P(f \circ \varphi_1^{-1})(x_1)) = \partial_P(f \circ \varphi_2^{-1})(x_2)$.

Now we can extend the notion of P-subdifferential to functions defined on a Riemannian manifold.

Notation 7. In the sequel, M will stand for a Riemannian manifold defined on a real Hilbert space X (either finite dimensional or infinite dimensional). As usual, for a point $p \in M$, TM_p

will denote the tangent space of M at p, and $\exp_p : \mathrm{TM}_p \to M$ will stand for the exponential function at p.

We will also make extensive use of the parallel transport of vectors along geodesics. Recall that, for a given curve $\gamma : I \to M$, numbers $t_0, t_1 \in I$, and a vector $V_0 \in \mathrm{TM}_{\gamma(t_0)}$, there exists a unique parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = V_0$. Moreover, the mapping defined by $V_0 \mapsto V(t_1)$ is a linear isometry between the tangent spaces $\mathrm{TM}_{\gamma(t_0)}$ and $\mathrm{TM}_{\gamma(t_1)}$, for each $t_1 \in I$. In the case when γ is a minimizing geodesic and $\gamma(t_0) = x$, $\gamma(t_1) = y$, we will denote this mapping by L_{xy}, and we call it the parallel transport from TM_x to TM_y along the curve γ. Note that the parallel transport L_{xy} is well defined when x and y are contained in a geodesic neighborhood, see Theorem 1.6.12 of [11].

The parallel transport allows us to measure the length of the "difference" between vectors (or forms) which are in different tangent spaces (or in duals of tangent spaces, that is, fibers of the cotangent bundle), and do so in a natural way. Indeed, let γ be a minimizing geodesic connecting two points $x, y \in M$, say $\gamma(t_0) = x$, $\gamma(t_1) = y$. Take vectors $v \in \mathrm{TM}_x$, $w \in \mathrm{TM}_y$. Then we can define the distance between v and w as the number

$$\|v - L_{yx}(w)\|_x = \|w - L_{xy}(v)\|_y$$

(this equality holds because L_{xy} is a linear isometry between the two tangent spaces, with inverse L_{yx}). Since the spaces T^*M_x and TM_x are isometrically identified by the formula $v = \langle v, \cdot\rangle$, we can obviously use the same method to measure distances between forms $\zeta \in T^*M_x$ and $\eta \in T^*M_y$ lying on different fibers of the cotangent bundle.

For the sake of simplicity in the formulas, we will omit the norm and scalar product indices which indicate the tangent space where they are defined, whenever ambiguity does not appear.

Although most of the results that we will prove are valid for any Riemannian manifold, we will assume in the sequel that M is connected.

Definition 8. Let M be a Riemannian manifold, $p \in M$, $f : M \to (-\infty, +\infty]$ a lower semicontinuous function. We define the proximal subdifferential of f at p, denoted by $\partial_P f(p) \subset \mathrm{TM}_p$, as $\partial_P(f \circ \exp_p)(0)$ (it being understood that $\partial_P f(p) = \emptyset$ for all $p \notin \operatorname{dom} f$).

The following result is an immediate consequence of Lemma 5.

Proposition 9. *Let M be a Riemannian manifold, $p \in M$, (φ, U) a chart, with $p \in U$, and $f : M \to (-\infty, +\infty]$ a lower semicontinuous function. Then*

$$\partial_P f(p) = D\varphi(p)^*[\partial_P(f \circ \varphi^{-1})(\varphi(p))].$$

As a consequence of the definition of $\partial_P(f \circ \exp_p)(0)$ we get the following.

Corollary 10. *Let M be a Riemannian manifold, $p \in M$, $f : M \to (-\infty, +\infty]$ a lower semicontinuous function. Then $\zeta \in \partial_P f(p)$ if and only if there is a $\sigma > 0$ such that*

$$f(q) \geq f(p) + \langle \zeta, \exp_p^{-1}(q)\rangle - \sigma d(p, q)^2$$

for every q in a neighborhood of p.

We can also define the proximal superdifferential of a function f from a Hilbert space X into $[-\infty, \infty)$ as follows. A vector $\zeta \in X$ is called a *proximal supergradient* of an upper semicontinuous function f at $x \in \operatorname{dom} f$ if there are positive numbers σ and η such that

$$f(y) \leq f(x) + \langle \zeta, y - x\rangle + \sigma\|y - x\|^2 \quad \text{for all } y \in B(x, \eta).$$

and we denote the set of all such ζ by $\partial^P f(x)$, which we call the P-subdifferential of f at x.

Now, if M is a Riemannian manifold, $p \in M$, $f : M \to [-\infty, +\infty)$ an upper semicontinuous function. We define the proximal superdifferential of f at p, denoted by $\partial^P f(p) \subset TM_p$, as $\partial^P (f \circ \exp_p)(0)$. As before, we have that $\zeta \in \partial^P f(p)$ if and only if there is a $\sigma > 0$ such that

$$f(q) \leq f(p) + \langle \zeta, \exp_p^{-1}(q) \rangle + \sigma d(p,q)^2$$

for every q in a neighborhood of p. It is also clear that $\partial^P f(p) = -\partial_P(-f)(p)$.

Most of the following properties are easily translated from the corresponding ones for $M = X$ a Hilbert space (see [5]) through charts. Recall that a real-valued function f defined on a Riemannian manifold is said to be convex provided its composition $f \circ \alpha$ with any geodesic arc $\alpha : I \to M$ is convex as a function from $I \subset \mathbb{R}$ into $\mathbb{R}$.

Proposition 11. *Let M be a Riemannian manifold, $p \in M$, $f, g : M \to (-\infty, +\infty]$ lower semicontinuous functions. We have*

(i) *If f is C^2, then $\partial_P f(p) = \{df(p)\}$.*
(ii) *If f is convex, then $\zeta \in \partial_P f(p)$ if and only if $f(q) \geq f(p) + \langle \zeta, v \rangle$ for every $q \in M$ and $v \in \exp_p^{-1}(q)$.*
(iii) *If f has a local minimum at p, then $0 \in \partial_P f(p)$.*
(iv) *$\partial_P f(p) + \partial_P g(p) \subseteq \partial_P (f+g)(p)$, with equality if f or g is C^2.*
(v) *$\partial_P (cf)(p) = c\partial_P f(p)$, for $c > 0$.*
(vi) *If f is K-Lipschitz, then $\partial_P f(p) \subset \overline{B}(0, K)$.*
(vii) *$\partial_P f(p)$ is a convex subset of TM_p.*
(viii) *If $\zeta \in \partial_P f(p)$ and f is differentiable at p then $\zeta = df(p)$. Moreover, if M is connected and the exponential map $\exp_x : TM_x \to M$ is surjective for every $x \in M$, we also have*
(ix) *Every local minimum of a convex function f is global.*
(x) *If f is convex and $0 \in \partial_P f(p)$, then p is a global minimum of f.*

Proof. All the properties but perhaps (ii), (vi) and (viii) are easily shown to be true. Property (vi) follows from the fact that $\exp_p^{-1}(.)$ is almost 1-Lipschitz when restricted to balls of center 0_p and small radius.

Let us prove (ii). Let $q \in M$. Let $\gamma(t) = \exp_p(tv)$, $t \in [0, 1]$, which is a minimal geodesic joining p and q. The function $f \circ \gamma$ is convex and satisfies

$$\begin{aligned} f(\gamma(t)) &\geq f(\gamma(0)) + \langle \zeta, tv \rangle - \sigma d(\gamma(t)), (\gamma(0))^2 \\ &= f(\gamma(0)) + \langle (\zeta, t\gamma'(0)) \rangle - \sigma t^2 \end{aligned}$$

for some $\sigma > 0$ and $t > 0$ small. Hence $\langle (\zeta, \gamma'(0)) \rangle \in \partial_P (f \circ \gamma)(0)$, and consequently (bearing in mind that $f \circ \gamma$ is convex on a Hilbert space) $f(\gamma(t)) \geq f(\gamma(0)) + \langle (\zeta, t\gamma'(0)) \rangle$, which implies $f(q) \geq f(p) + \langle \zeta, v \rangle$.

To see (viii), note that Proposition 4 implies that $\zeta \in D^- f(p)$, that is, ζ is a viscosity subdifferential of f at p in the sense of [3]. Then, since f is differentiable, we have that $\zeta \in D^- f(p) = D^+ f(p) = \{df(p)\}$, so we conclude that $\zeta = df(p)$. □

The following important result is also local, it follows from [5, Theorem 1.3.1].

Theorem 12 (*Density Theorem*). *Let M be a Riemannian manifold, $p \in M$, $f : M \to (-\infty, +\infty]$ a lower semicontinuous function, $\varepsilon > 0$. Then there exists a point q such that $d(p,q) < \varepsilon$, $f(p) - \varepsilon \leq f(q) \leq f(p)$, and $\partial_P f(q) \neq \emptyset$.*

The following result can be deduced from [5, Theorem 1.8.3].

Theorem 13 (*Fuzzy Rule for the Sum*). *Let $f_1, f_2 : M \to (-\infty, \infty]$ be two lower semicontinuous functions such that at least one of them is Lipschitz near x_0. If $\zeta \in \partial_P(f_1 + f_2)(x_0)$ then, for every $\varepsilon > 0$, there exist x_1, x_2 and $\zeta_1 \in \partial_P f_1(x_1)$, $\zeta_2 \in \partial_P f_2(x_2)$ such that*

(a) $d(x_i, x_0) < \varepsilon$ *and* $|f_i(x_i) - f_i(x_0)| < \varepsilon$ *for* $i = 1, 2$.
(b) $\|\zeta - (L_{x_1 x_0}(\zeta)_1 + L_{x_2 x_0}(\zeta_2))\|_{x_0} < \varepsilon$.

The following theorem is also local, and is a consequence of the fuzzy chain rule known for functions defined on Hilbert spaces [5, Theorem 1.9.1, p. 59].

Theorem 14 (*Fuzzy Chain Rule*). *Let $g : N \to \mathbb{R}$ be lower semicontinuous, $F : M \to N$ be locally Lipschitz, and assume that g is Lipschitz near $F(x_0)$. Then, for every $\zeta \in \partial_P(g \circ F)(x_0)$ and $\varepsilon > 0$, there are $\tilde{x}$, $\tilde{y}$ and $\eta \in \partial_P g(\tilde{y})$ such that $d(\tilde{x}, x_0) < \varepsilon$, $d(\tilde{y}, F(x_0)) < \varepsilon$, $d(F(\tilde{x}), F(x_0)) < \varepsilon$, and*

$$L_{x\tilde{x}}\zeta \in \partial_P[\langle L_{\tilde{y}F(x_0)}(\eta), \exp^{-1}_{F(x_0)} \circ F(.)\rangle](\tilde{x}) + \varepsilon B_{\mathrm{TM}_{\tilde{x}}}.$$

The following result, which is local as well, relates the proximal subdifferential $\partial_P f(x)$ to the viscosity subdifferential $D^- f(x)$ of a function f defined on a Riemannian manifold M; see [3] for the definition of $D^- f(x)$ in the manifold setting.

Proposition 15. *Let $\xi_0 \in D^- f(x_0)$, $\epsilon > 0$. Then there exist $x \in B(x_0, \epsilon)$ and $\zeta \in \partial_P f(x)$ such that $|f(x) - f(x_0)| < \epsilon$ and $\|\xi_0 - L_{xx_0}(\zeta)\|_{x_0}$.*

Proof. This follows from [5, Proposition 3.4.5, p. 138]. □

The following result is the cornerstone in the proof of the Solvability Theorem stated below, which in turn will be the basis of the proofs of the applications we will present later on about fixed point theorems. This theorem is a version for manifolds of the classical Decrease Principle (see [5, Theorem 3.2.8, p. 122]). The proof given by Clarke et al. is based on the Mean Value Inequality, as far as we know this result is not known to be true for manifolds with the required generality. We present an alternative proof based on Ekeland's variational principle.

Theorem 16 (*Decrease Principle*). *Let M be a complete Riemannian manifold. Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function, and $x_0 \in \mathrm{dom} f$. Assume that there exist $\delta > 0$ and $\rho > 0$ such that $\|\zeta\|_x \geq \delta$ for any x with $d(x, x_0) < \rho$ and any $\zeta \in \partial_P f(x)$. Then $\inf\{f(x) : d(x, x_0) \leq \rho\} \leq f(x_0) - \rho\delta$.*

Proof. We will use the following restatement of Ekeland's variational principle (Theorem 1 in [8]).

Theorem 17 (*Ekeland's Variational Principle*). *Let V be a complete metric space and $F : V \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function such that $F \neq +\infty$ and F is bounded from below. Let $\varepsilon > 0$ be given, and a point $u \in V$ such that*

$$F(u) \leq \inf_V F + \varepsilon.$$

Then for any $\lambda > 0$ there exists some point $v \in V$ such that

$$F(v) \leq F(u), \qquad d(u, v) \leq \lambda,$$

and the function

$$w \mapsto F(w) + \frac{\varepsilon}{\lambda} d(v, w)$$

has a strict minimum at v, that is,

$$F(w) + \frac{\varepsilon}{\lambda} d(v, w) > F(v) \quad \forall w \in V, w \neq v.$$

In order to prove Theorem 16 we can obviously assume that $f(x_0) = 0$. Define V to be the closed ball $\{x : d(x, x_0) \leq \rho\}$, so (V, d) is a complete metric space. Define $\varepsilon := -\inf_V f$; since $f(x_0) = 0$, then $\varepsilon \geq 0$. We immediately see that $\varepsilon > 0$, otherwise x_0 would be a minimum of f, and then $0 \in \partial_P f(x_0)$, so that the hypothesis would be contradicted. To prove the theorem, we want to see that $\inf_V f \leq -\rho\delta$, that is $\varepsilon \geq \rho\delta$.

Suppose by contradiction that $\varepsilon < \rho\delta$: then $\varepsilon/\delta < \rho$ so it is possible to choose λ such that

$$\frac{\varepsilon}{\delta} < \lambda < \rho.$$

We know that $0 = f(x_0) \leq \inf_V f + \varepsilon = 0$. We can then apply Ekeland's principle with the above choice of ε, λ, to obtain a point v such that $d(x_0, v) \leq \lambda$, and the function

$$w \mapsto f(w) + \frac{\varepsilon}{\lambda} d(v, w)$$

attains a strict minimum at v. Since $\lambda < \rho$, then v is in the interior of V, so

$$0 \in \partial_P \left(f(\cdot) + \frac{\varepsilon}{\lambda} d(v, \cdot) \right)(v).$$

Now fix $\varepsilon' > 0$ such that $\varepsilon' < \rho - \lambda$, and $B(v, \varepsilon')$ is a geodesic ball (so that parallel transport is well defined). By applying the fuzzy sum rule Theorem 13, we can find points x_1, x_2 such that $d(x_i, x_0) < \varepsilon'$ for $i = 1, 2$ and we can find

$$\zeta_1 \in \partial_P f(x_1), \qquad \zeta_2 \in \partial_P \left(\frac{\varepsilon}{\lambda} d(v, \cdot) \right)(x_2)$$

such that $\|L_{x_1 v}\zeta_1 - L_{x_2 v}\zeta_2\|_v < \varepsilon'$.

Note that

$$d(x_0, x_1) \leq d(x_0, v) + d(v, x_1) \leq \lambda + \varepsilon' < \rho.$$

Since $d(v, \cdot)$ is 1-Lipschitz, $\|\zeta_2\|_{x_2} \leq \varepsilon/\lambda$. By parallel transport and the triangular inequality we get

$$\|\zeta_1\|_{x_1} = \|L_{x_1 v}\zeta_1\|_v \leq \varepsilon' + \|L_{x_2 v}\zeta_2\|_v \leq \varepsilon' + \frac{\varepsilon}{\lambda} < \delta,$$

achieving contradiction. □

Under the same conditions we have the following corollary.

Corollary 18. *Let $\varepsilon > 0$ and x_0 satisfy $f(x_0) < \inf f + \varepsilon$. For every $\lambda > 0$, there exist $z \in B(x_0, \lambda)$ and $\zeta \in \partial_P f(z)$ such that $f(z) < \inf f + \varepsilon$ and $\|\zeta\| < \varepsilon/\lambda$.*

Proof. Otherwise, there is $\lambda > 0$ so that for every $z \in B(x_0, \lambda)$ and every $\zeta \in \partial_P f(z)$ we have $\|\zeta\| \geq \frac{\varepsilon}{\lambda}$ (we may assume that $f(z) < \inf f + \varepsilon$ by lower semicontinuity). Then, by the Decrease Principle, we have

$$\inf_{B(x_0, \lambda)} f(x) \leq f(x_0) - \lambda\frac{\varepsilon}{\lambda} < \inf f,$$

a contradiction. □

Now, from the Decrease Principle, we are going to obtain important information about solvability of equations on any complete Riemannian manifold M.

Let $U \subset M$, A be an arbitrary set of parameters α. Let $F : M \times A \to [0, +\infty]$ be a function satisfying that for every $\alpha \in A$ the function $F_\alpha : M \to [0, +\infty]$ defined by $F_\alpha(x) = F(x, \alpha)$ is lower semicontinuous and proper (not everywhere ∞). We denote the set $\{x \in U : F(x, \alpha) = 0\}$ by $\phi(\alpha)$. Then we have the following version of the Solvability Theorem (see [5, Theorem 3.3.1, p. 126]).

Theorem 19 (*Solvability Theorem*). *Let V and U be open subsets of M, and $\delta > 0$. Assume that*

$$\alpha \in A, \qquad x \in V, \qquad F(x, \alpha) > 0, \qquad \zeta \in \partial_P F_\alpha(x) \Rightarrow \|\zeta\| \geq \delta.$$

Then for every $x \in M$ and $\alpha \in A$, we have

$$\min\{d(x, V^c), d(x, U^c), d(x, \phi(\alpha))\} \leq \frac{F(x, \alpha)}{\delta}.$$

Proof. Otherwise there exist x_0, α_0 and $\rho > 0$ such that

$$\min\{d(x_0, V^c), d(x_0, U^c), d(x_0, \phi(\alpha_0))\} > \rho > \frac{F(x_0, \alpha_0)}{\delta},$$

and in particular $B(x_0, \rho) \subset U \cap V$ and $d(x_0, \phi(\alpha_0)) > \rho$, which implies that $F(x, \alpha_0) > 0$ for every $x \in B(x_0, \rho)$. Hence we have $\|\zeta\| \geq \delta$ for every $\zeta \in \partial_P F_{\alpha_0}(x)$ with $x \in B(x_0, \rho)$. Therefore, by the Decrease Principle, $0 \leq \inf_{x \in B(x_0, \rho)} F_{\alpha_0} \leq F(x_0, \alpha_0) - \rho\delta < 0$, which is a contradiction. □

Of course, the most interesting fact about the above inequality is that, in many situations (such of that of the following corollary) we can deduce $d(x, \phi(\alpha)) \leq F(x, \alpha)/\delta$, which implies that $\phi(\alpha)$ is nonempty. The situation in which the above theorem is most often applied is when we have identified a point (x_0, α_0) at which $F = 0$, with V and Ω being neighborhoods of x. This is especially interesting in the cases when the functions involved are not known to be smooth or the derivatives do not satisfy the conditions of the Implicit Function Theorem. For instance, we can deduce the following result.

Corollary 20. *Let $x_0 \in M$, $\varepsilon > 0$, and $\delta > 0$. Assume that*

$$\alpha \in A, \qquad d(x, x_0) < 2\varepsilon, \qquad F(x, \alpha) > 0, \qquad \zeta \in \partial_P F_\alpha(x) \Rightarrow \|\zeta\| \geq \delta.$$

Then we have that the equation $F(z, \alpha) = 0$ has a solution for z in $B(x_0, 2\varepsilon)$ provided that there is an $\tilde{x} \in B(x_0, \varepsilon)$ satisfying $F(\tilde{x}, \alpha) < \varepsilon\delta$.

Proof. It is enough to apply the Solvability Theorem with $U = V = B(x_0, 2\varepsilon)$. We have that $\min\{d(\tilde{x}, V^c), d(\tilde{x}, U^c), d(\tilde{x}, \phi(\alpha))\} < \varepsilon$, and consequently $d(\tilde{x}, \phi(\alpha)) < \varepsilon$, because both $d(\tilde{x}, U^c)$ and $d(\tilde{x}, V^c)$ are greater than ε. Hence $\phi(\alpha) \neq \emptyset$. □

2. Main results: Applications to fixed point theory

Now we are going to show how the Solvability Theorem allows us to deduce some interesting fixed point theorems for possibly expansive mappings and certain perturbations of such mappings defined on Riemannian manifolds M.

We first need to establish a couple of auxiliary results.

Lemma 21. *Let X, Y be Hilbert spaces, $F : X \to Y$ Lipschitz, and $g : Y \to \mathbb{R}$ of class C^2 near $F(x_0)$. Define $f = g \circ F : X \to \mathbb{R}$. Then*

$$\partial_P f(x_0) \subseteq \partial_P \left(\langle dg(F(x_0)), F(\cdot)\rangle\right)(x_0).$$

Proof. Take $\zeta \in \partial_P f(x_0)$, that is,

$$f(x) - \langle \zeta, x\rangle + \sigma\|x - x_0\|^2 \geq f(x_0) - \langle \zeta, x_0\rangle$$

for x near x_0, with $\sigma > 0$. Let S be the graph of the mapping F, a subset of $X \times Y$. Another way of writing the previous inequality is the following:

$$g(y) - \langle \zeta, x\rangle + \sigma\|x - x_0\|^2 + I_S(x, y) \geq g(F(x_0)) - \langle \zeta, x_0\rangle$$

for x near x_0, where I_S is the indicator function of S, that is, $I_S(x, y) = 0$ if $(x, y) \in S$, and $I_S(x, y) = +\infty$ otherwise. This means that the function

$$H(x, y) := g(y) - \langle \zeta, x\rangle + \sigma\|x - x_0\|^2 + I_S(x, y) := h(x, y) + I_S(x, y)$$

attains a local minimum at $(x_0, F(x_0))$. Therefore

$$\begin{aligned}(0, 0) \in \partial_P H(x_0, F(x_0)) &= dh(x_0, F(x_0)) + \partial_P I_S(x_0, F(x_0))\\ &= (-\zeta, dg(F(x_0))) + \partial_P I_S(x_0, F(x_0)) = (-\zeta, dg(F(x_0))) + N_S^P(x_0, F(x_0)),\end{aligned}$$

where $N_S^P(x_0, F(x_0))$ denotes the proximal normal cone of S at $(x_0, F(x_0))$, see [5, p. 22–30]. This means that $(\zeta, -dg(F(x_0))) \in N_S^P(x_0, F(x_0))$, that is (according to [5, Proposition 1.1.5]), for some $\sigma > 0$ we have

$$\begin{aligned}\langle(\zeta, -dg(F(x_0))), (x, F(x)) - (x_0, F(x_0))\rangle &\leq \sigma\|(x, F(x)) - (x_0, F(x_0))\|^2\\ = \|(x - x_0)\|^2 + \|F(x) - F(x_0)\|^2 &\leq \sigma(1 + K)\|x - x_0\|^2,\end{aligned}$$

where K is the Lipschitz constant of F. Therefore,

$$\langle \zeta, x - x_0\rangle - \sigma(1 + K)\|x - x_0\|^2 \leq \langle dg(F(x_0)), F(x)\rangle - \langle dg(F(x_0)), F(x_0)\rangle,$$

which means that $\zeta \in \partial_P(\langle dg(F(x_0)), F(\cdot)\rangle)(x_0)$. □

Lemma 22. *Let M be a Riemannian manifold, $F : M \to M$ Lipschitz, $x_0 \in M$ satisfying that $x_0 \neq F(x_0)$ and that $d(x, y)$ is C^2 near $(x_0, F(x_0))$, $f(x) = d(x, F(x))$. Then*

$$\partial_P f(x_0) \subset v + \partial_P\langle -L_{x_0 F(x_0)} v, (\exp^{-1}_{F(x_0)} \circ F)(.)\rangle(x_0)$$

where $v = \frac{\partial d(x_0, F(x_0))}{\partial x}$.

Proof. By using the preceding lemma, we deduce that

$$\begin{aligned}\partial_P f(x_0) &\subset \partial_P\left(\langle v, \exp^{-1}_{x_0}(.)\rangle(x_0) + \left\langle \frac{\partial d(x_0, F(x_0))}{\partial y}, (\exp^{-1}_{F(x_0)} \circ F)(.)\right\rangle\right)(x_0)\\ &= \partial_P(\langle v, \exp^{-1}_{x_0}(.)\rangle(x_0) + \langle -L_{x_0 F(x_0)} v, (\exp^{-1}_{F(x_0)} \circ F)(.)\rangle)(x_0)\\ &= D(\langle v, \exp^{-1}_{x_0}(.)\rangle)(x_0) + \partial_P\langle -L_{x_0 F(x_0)} v, (\exp^{-1}_{F(x_0)} \circ F)(.)\rangle(x_0)\\ &= v + \partial_P\langle -L_{x_0 F(x_0)} v, (\exp^{-1}_{F(x_0)} \circ F)(.)\rangle(x_0). \quad \square\end{aligned}$$

Now we consider a family of Lipschitz mappings F_α, $\alpha \in A$. Let P_α denote the set of fixed points of F_α. We are going to apply the Solvability Theorem to the function $f(x,\alpha) = d(x, F_\alpha(x))$, with $U = M$. Under the hypotheses of Lemma 22 we have the following.

Theorem 23. *Let M be a complete Riemannian manifold, and $F_\alpha : M \to M$, $\alpha \in A$, be a family of Lipschitz mappings satisfying the hypotheses of Lemma 22 at every point $x \in V \subset M$ with $F_\alpha(x) \neq x$. Assume that there is a positive δ such that $\|v_\alpha + \zeta\| \geq \delta$ for every*

$$\zeta \in \partial_P \langle -v_\alpha, (L_{F_\alpha(x)x} \circ \exp^{-1}_{F_\alpha(x)} \circ F_\alpha)(.)\rangle(x),$$

where $x \in V$, $x \notin P_\alpha$, and $v_\alpha = \frac{\partial d(x, F_\alpha(x))}{\partial x}$. Then we have that

$$\min\{d(x, V^c), d(x, P_\alpha)\} \leq \frac{d(x, F_\alpha(x))}{\delta}$$

for every $x \in V$ and $\alpha \in A$.

Proof. This follows immediately from Lemma 22 and Theorem 19. □

Remark 24. The condition that $d(x, y)$ is C^2 near $(x, F(x))$ whenever $F(x) \neq x$ is satisfied, for instance, if $M = X$ is a Hilbert space, or if M is finite dimensional and $F(x) \notin \mathrm{cut}(x)$, where $\mathrm{cut}(x)$ denotes the cut locus of the point x.

The statement of Theorem 23 might seem rather artificial at first glance but, as the rest of the section will show, it has lots of interesting consequences.

Corollary 25. *Let M be a complete finite dimensional manifold or $M = X$ (a Hilbert space). Assume that $F : M \to M$ is Lipschitz and $F(x) \notin \mathrm{cut}(x)$ for every $x \in M$. Assume also that there is a constant $0 < K < 1$ such that the functions*

$$y \mapsto \left\langle \frac{\partial d(x, F(x))}{\partial x}, (L_{F(x)x} \circ \exp^{-1}_{F(x)})(F(y)) \right\rangle$$

are K-Lipschitz near x. Then F has a fixed point.

Proof. We have that $\partial_P \langle -\frac{\partial d(x,F(x))}{\partial x}, (L_{F(x)x} \circ \exp^{-1}_{F(x)} \circ F)(.)\rangle(x) \subset \overline{B}(0, K)$, hence $\delta = 1 - K$ does the work (with $V = M$). □

The following corollary is of course well known, but still it is very interesting that it can be proved just by using the above results on proximal subgradients (and without requiring any smoothness of the distance function in M).

Corollary 26. *Let M be a complete Riemannian manifold, and suppose that $F : M \to M$ is K-Lipschitz, with $K < 1$. Then F has a unique fixed point.*

Proof. Uniqueness follows from the fact that $d(F(x), F(y)) < d(x, y)$ whenever $x \neq y$. In order to get existence, let us observe that, in the situation of Lemma 22, if smoothness of the distance function fails, we may use the following fact:

$$\partial_P f(x_0) \subset \partial_L f(x_0) \subset \bigcup_{\|v\|=1} [v + \partial_L \langle -v, (L_{F(x_0)x_0} \circ \exp^{-1}_{F(x_0)} \circ F)(.)\rangle(x_0)],$$

where $\partial_L f(x_0)$, the limiting subdifferential, is defined locally in the natural way, see [5, p. 61] for the definition of $\partial_L f(x_0)$ in the Hilbert space. Next we observe that the function

$$x \mapsto \langle -v, (L_{F(x_0)x_0} \circ \exp^{-1}_{F(x_0)} \circ F)(x)\rangle$$

is $(K+\varepsilon)$-Lipschitz near x_0, with $(K+\varepsilon<1)$, hence

$$\partial_L\langle -v, (L_{F(x_0)x_0}\circ \exp^{-1}_{F(x_0)}\circ F)(.)\rangle(x_0)\subset \overline{B}(0, K+\varepsilon).$$

Therefore $\bigcup_{\|v\|=1}[v+\partial_L\langle -v, (L_{F(x_0)x_0}\circ \exp^{-1}_{F(x_0)}\circ F)(.)\rangle(x_0)]$ does not meet the ball $B(0, 1-K-\varepsilon)$, and consequently neither does $\partial_P f(x_0)$, so we can apply the Solvability Theorem as well. □

The following results, which are also consequences of Theorem 23, allow us to explore the behavior of small Lipschitz perturbations of certain mappings with fixed points. Let us first observe that very small Lipschitz perturbations of mappings having fixed points may lose them: consider for instance $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x+\varepsilon$. The proofs of these results in their most general (and powerful) form are rather technical. In order that the main ideas of the proofs become apparent to the reader, we will first establish the main theorem and its corollaries in the case of C^1 smooth mappings of a Hilbert space, and then we will proceed to study more general versions for nonsmooth perturbations on Riemannian manifolds.

Theorem 27. *Let X be a Hilbert space, $G:X\to X$ a C^1 smooth mapping, and $J:X\times X\to X$ satisfying that*

(i) *G is C-Lipschitz on $B(x_0, R)$;*
(ii) *$\langle h, DG(x)(h)\rangle\leq K<1$ for every $x\in B(x_0,R)$ and $\|h\|=1$;*
(iii) *the mapping $J_y:X\to X$, $J_y(x)=J(x,y)$ is L-Lipschitz for all $y\in X$;*
(iv) *the mapping $J_x:X\to X$, $J_x(y)=J(x,y)$ is differentiable, and $\|\frac{\partial J}{\partial y}(x,y)-I\|\leq\varepsilon/C$ for every $x\in B(x_0,R)$ and $y\in G(B(x_0,R))$;*
(v) *$L+K+\varepsilon<1$, and*
(vi) *$\|J(x_0, G(x_0))-x_0\|<R(1-(L+K+\varepsilon))$.*

Then the mapping $F:M\to M$, defined by $F(x)=J(x,G(x))$, has a fixed point in the ball $B(x_0,R)$.

Proof. This theorem, as it is stated (that is, without assuming J is differentiable), is a consequence of Theorem 35 below. We will give an easy proof of this statement for the case when J is differentiable and we are in a Hilbert space setting. Let us take a $\zeta\in\partial_P(\langle -v, F(\cdot)\rangle)(x)=\langle -v, dF(x)(\cdot)\rangle$, that is,

$$\begin{aligned}\zeta &= \langle -v, dF(x)(\cdot)\rangle=\left\langle -v, \frac{\partial J}{\partial x}(x,G(x))(\cdot)+\frac{\partial J}{\partial y}(x,G(x))(DG(x)(\cdot))\right\rangle\\ &=\left\langle -v, \frac{\partial J}{\partial x}(x,G(x))(\cdot)\right\rangle+\left\langle -v, \frac{\partial J}{\partial y}(x,G(x))(DG(x)(\cdot))\right\rangle.\end{aligned}$$

Then, bearing in mind that $x\mapsto J_y(x)$ is L-Lipschitz and $\partial J/\partial y$ is ε/C-close to the identity, we have

$$\begin{aligned}\langle\zeta, -v\rangle &=\left\langle -v, \frac{\partial J}{\partial x}(x,G(x))(-v)\right\rangle+\left\langle -v, \frac{\partial J}{\partial y}(x,G(x))(DG(x)(-v))\right\rangle\\ &\leq L+\langle -v, DG(x)(-v)\rangle+\frac{\varepsilon}{C}\|DG(x)\|\leq L+K+\varepsilon<1.\end{aligned}$$

Therefore, $\|v+\zeta\|\geq\langle v, v+\zeta\rangle=\langle v,v\rangle+\langle v,\zeta\rangle=1-\langle\zeta,-v\rangle\geq 1-(L+K+\varepsilon):=\delta>0$ and, according to Theorem 23 (here we take A to be a singleton), we get that

$$\min\{R, d(x_0, P)\} \le \frac{\|F(x_0) - x_0\|}{\delta} = \frac{\|J(x_0, G(x_0)) - x_0\|}{\delta} < R,$$

and consequently $P \ne \emptyset$ (that is, F has a fixed point in $V = U := B(x_0, R)$). □

Remark 28. The above proof shows that Theorem 27 remains true if we only require G to be differentiable (not necessarily C^1) but in turn we demand that J is differentiable as well.

It is also worth noting that condition (ii) can be replaced with the (formally weaker) following one:

(ii)′ $\langle x - F(x), DG(x)(x - F(x))\rangle \le K\|x - F(x)\|^2$ for every $x \in B(x_0, R)$.

When x_0 is a fixed point of G condition (vi) means that $J(x_0, x_0)$ is close to x_0. The mapping J can be viewed as a general means of perturbation of the mapping G. When we take a function J of the form $J(x, y) = y + H(x)$ we obtain the following corollary, which ensures the existence of fixed points of the mapping $G + H$ when H is L-Lipschitz and relatively small near x_0 (a fixed point of G).

Corollary 29. *Let X be a Hilbert space, and let x_0 be a fixed point of a differentiable mapping $G : X \to X$ satisfying the following condition:*

$$\langle h, DG(x)(h)\rangle \le K < 1 \quad \textit{for every } x \in B(x_0, R) \textit{ and } \|h\| = 1.$$

Let H be a differentiable L-Lipschitz mapping, with $L < 1 - K$. Then $G + H$ has a fixed point, provided that $\|H(x_0)\| < R(1 - K - L)$.

Proof. Define $J(x, y) = y + H(x)$. Note that the above proof of Theorem 27 works for any differentiable mappings G and J (not necessarily C^1). In order to deduce the corollary it is enough to observe that $\partial H/\partial y = I$, so condition (iv) of Theorem 27 is satisfied for $\varepsilon = 0$. □

Let us observe that, when $R = +\infty$, we do not need to require that x_0 be a fixed point of G, and no restriction on the size of $H(x_0)$ is necessary either. As a consequence we have the following.

Corollary 30. *Every mapping $F : X \to X$ of the form $F = T + H$, where T is linear and satisfies $\langle h, T(h)\rangle \le K < 1$ for every $\|h\| = 1$, and H is L-Lipschitz, with $L < 1 - K$, has a fixed point.*

Remark 31. If X is finite dimensional, the conditions on T are satisfied but requiring that $\mathrm{Re}\lambda < 1$ for every eigenvalue λ. On the other hand, let us observe that the function F may be expansive, that is $\|F(x) - F(y)\| > \|x - y\|$ for some, or even all, $x \ne y$. Consider for instance the mapping $T : \ell_2 \to \ell_2$ defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = 5(x_2, -x_1, x_4, -x_3, \ldots);$$

in this case T is clearly expansive but we can take $K = 0$. This result should be compared with [7, Corollary 1.6, p. 24].

As a consequence of Corollary 29 we can also deduce the following local version of the result.

Corollary 32. *Let x_0 be a fixed point of a differentiable function $G : X \to X$ satisfying the following condition:*

$$\langle h, DG(x_0)(h)\rangle \le K < 1 \quad \textit{for every } \|h\| = 1.$$

Let H be a differentiable L-Lipschitz function. Then there exists a positive constant α_0 such that the function $G + \alpha H$ has a fixed point, for every $\alpha \in (0, \alpha_0)$.

Another way to perturb a mapping G with a fixed point x_0 is to compose it with a function H which is close to the identity. When we take J of the form $J(x, y) = H(y)$ in Theorem 27 we obtain the following.

Corollary 33. *Let X be a Hilbert space, and x_0 be a fixed point of a differentiable mapping $G : X \to X$ such that*

$$\langle h, DG(x)(h)\rangle \leq K < 1$$

for every $x \in B(x_0, R)$ and $h \in X$ with $\|h\| = 1$. Let $H : X \to X$ be a differentiable mapping such that $\|DH(G(x)) - I\| < \varepsilon$ for every $x \in B(x_0, R)$. Then $F = H \circ G$ has a fixed point in $B(x_0, R)$, provided that $K + \varepsilon < 1$ and $\|H(x_0) - x_0\| < R(1 - K - \varepsilon)$.

Proof. For $J(x, y) = H(y)$ we have that $x \mapsto J_y(x)$ is constant, hence 0-Lipschitz for every y, and we can apply Theorem 27 with $L = 0$ (and bearing in mind Remark 28). □

Note that the third corollary stated in the introduction is the sum of Theorem 35, Corollary 39.

Finally we will consider an extension of Theorem 27 and the above corollaries to the setting of nonsmooth mappings on Riemannian manifolds. We will make use of the following fact about *partial proximal subdifferentials*.

Lemma 34. *Let M be a Riemannian manifold, and $f : M \times M \to \mathbb{R}$. For each $x \in M$ let us define the partial function $f_x : M \to \mathbb{R}$ by $f_x(y) = f(x, y)$, and define also $f_y : M \to \mathbb{R}$ by $f_y(x) = f(x, y)$. Assume that $\zeta = (\zeta_1, \zeta_2) \in \partial_P f(x_0, y_0)$. Then $\zeta_1 \in \partial_P f_{y_0}(x_0)$, and $\zeta_2 \in \partial_P f_{x_0}(y_0)$.*

Proof. Since $\zeta \in \partial_P f(x_0, y_0)$ there exists a C^2 function $\varphi : M \times M \to \mathbb{R}$ such that $f - \varphi$ attains a local minimum at (x_0, y_0), and

$$\left(\frac{\partial \varphi}{\partial x}(x_0, y_0), \frac{\partial \varphi}{\partial y}(x_0, y_0)\right) = d\varphi(x_0, y_0) = \zeta = (\zeta_1, \zeta_2).$$

Then it is obvious that $x \mapsto f_{y_0}(x) - \varphi(x, y_0)$ attains a local minimum at x_0 as well, so

$$\zeta_1 = \frac{\partial \varphi}{\partial x}(x_0, y_0) \in \partial_P f_{y_0}(x_0).$$

In the same way we see that $\zeta_2 = \frac{\partial \varphi}{\partial y}(x_0, y_0) \in \partial_P f_{x_0}(y_0)$. □

Now we can prove our main result about perturbation of mappings with fixed points. As said before, the mapping J should be regarded as a general form of perturbation of G. We use the following notation:

$$\mathrm{sing}(x) := \{y \in M : d(\cdot, x)^2 \text{ is not differentiable at } y\}.$$

When M is finite dimensional it is well known that $\mathrm{sing}(x) \subseteq \mathrm{cut}(x)$, and both $\mathrm{sing}(x)$ and $\mathrm{cut}(x)$ are sets of measure zero.

Theorem 35. *Let M be a complete Riemannian manifold, $G : M \to M$, $J : M \times M \to M$ and $F : M \to M$ defined by $F(x) = J(x, G(x))$ be mappings such that:*

(i) $F(x) \notin \text{sing}(x) \cup \text{sing}(G(x))$ *for every* $x \in B(x_0, R)$*, and moreover there are (unique) minimizing geodesics joining* $F(x)$ *to* x *and* $F(x)$ *to* $G(x)$*;*
(ii) G *is* C^1 *smooth and* $\langle L_{xF(x)}h, L_{G(x)F(x)}dG(x)(h)\rangle_{F(x)} \le K < 1$ *for all* $x \in B(x_0, R)$ *and* $h \in \text{TM}_x$ *with* $\|h\|_x = 1$*;*
(iii) G *is* C*-Lipschitz on* $B(x_0, R)$*;*
(iv) J *is locally Lipschitz;*
(v) *the mapping* $x \mapsto J_y(x) := J(x, y)$ *is* L*-Lipschitz for every* $y \in M$*;*
(vi) *the mapping* $y \mapsto J_x(y) := J(x, y)$ *is differentiable for every* $x \in B(x_0, R)$*, there is a unique minimizing geodesic joining* $J(x, y)$ *to* y*,* $J(x, y) \notin \text{sing}(y)$ *and*

$$\left\| \frac{\partial J}{\partial y}(x, y) - L_{yJ(x,y)} \right\| \le \frac{\varepsilon}{C}$$

for every $x \in B(x_0, R)$ *and* $y \in G(B(x_0, R))$*;*
(vii) $L + K + \varepsilon < 1$*; and*
(viii) $d(x_0, J(x_0, G(x_0))) < R(1 - (L + K + \varepsilon))$.

Then the mapping $F : M \to M$ *has a fixed point in the ball* $B(x_0, R)$*.*

Moreover, when M *is finite dimensional, conditions* (i) *and* (vi) *on the singular sets can be replaced by:*

(i)′ $F(x) \notin \text{cut}(x) \cup \text{cut}(G(x))$ *for every* $x \in B(x_0, R)$*, and*
(vi)′ *the mapping* $y \mapsto J_x(y) := J(x, y)$ *is differentiable for every* $x \in B(x_0, R)$*, and* $J(x, y) \notin \text{cut}(y)$ *and*

$$\left\| \frac{\partial J}{\partial y}(x, y) - L_{yJ(x,y)} \right\| \le \frac{\varepsilon}{C}$$

for every $x \in B(x_0, R)$ *and* $y \in G(B(x_0, R))$*.*

At first glance this statement may seem to be overburdened with assumptions, but it turns out that all of them are either useful or necessary, as we will see from its corollaries and in the next remarks. Before giving the proof of the theorem, let us make some comments on these assumptions.

1. The condition in (i)′ that $F(x) \notin \text{cut}(x)$ for all $x \in B(x_0, R)$ is necessary. Indeed, in the simplest case when there is no perturbation at all, that is, $J(x, y) = y$, if we take G to be a continuous mapping from the sphere S^2 into itself and $G(x) \in \text{sing}(x) = \text{cut}(x) = \{-x\}$ for all $x \in S^2$, then G is the antipodal map and has no fixed point. Therefore, in order that $G : S^2 \to S^2$ has a fixed point, there must exist some x_0 with $G(x_0) \notin \text{cut}(x_0)$ and, therefore, by continuity, $G(x) \notin \text{cut}(x)$ for every x in a neighborhood of x_0.

On the other hand, not only is this a necessary condition, but also very natural in these kinds of problems. For instance, one can deduce from the Hairy Ball Theorem that if $G : S^2 \to S^2$ is a continuous mapping such that $G(x) \notin \text{cut}(x)$ for every $x \in S^2$ then G has a fixed point. Indeed, for every $x \in S^2$, the condition $G(x) \notin \text{cut}(x)$ implies the existence of a unique $v_x \in TS^2_x$ with $\|v_x\| < \pi$ such that $\exp_x(v_x) = G(x)$. The mapping $S^2 \ni x \mapsto v_x \in TS^2$ defines a continuous field of tangent vectors to S^2. If G did not have any fixed point then we would have $v_x \ne 0$ for all x, which contradicts the Hairy Ball Theorem.

2. The other condition in (i)′ that $F(x) \notin \text{cut}G(x)$ is also natural in this setting and very easily satisfied if we mean F to be a relatively small perturbation of G. For instance, if M has a positive injectivity radius $\rho = i(M) > 0$ and F is ρ-close to G, that is, $d(F(x), G(x)) < \rho$ for $x \in B(x_0, R)$, then $F(x) \notin \text{cut}(G(x))$ for $x \in B(x_0, R)$.

3. Since the main aim of the present theorem is to establish corollaries in which we have a mapping $G : M \to M$ with a fixed point x_0 and we perturb G by composing with or summing a mapping H with certain properties, thus obtaining a mapping F which is relatively close to G, and then we want to be able to guarantee that this perturbation of G still has a fixed point, it turns out that condition (i)′ of the theorem is not really restrictive. Indeed, since $G(x_0) = x_0$, $J(x_0, x_0)$ is relatively close to x_0 (see property (viii)), the mappings G and $F = J \circ G$ are continuous and there always exists a convex neighborhood of x_0 in M, it is clear that there must be some $R > 0$ such that $F(x) \notin \mathrm{cut}(x)$ for every $x \in B(x_0, R)$.

4. The second part of condition (vi) means that, in its second variable, J is relatively close to the identity, a natural condition to apply if we mean the function $F(x) = J(x, G(x))$ to be a relatively small perturbation of G.

5. The requirement that G is Lipschitz on $B(x_0, R)$ is not a strong one. On the one hand, since G is C^1 it is locally Lipschitz, condition (iii) is met provided R is small enough. On the other hand, when M is finite dimensional, by local compactness of M and continuity of dG, condition (iii) is always true for any R.

6. Condition (ii) is met in many interesting situations: for example, when the behavior of G in a neighborhood of x_0 is similar to a multiple of a rotation. Consider for instance M, the surface $z = x^2 + y^2$ in $\mathbb{R}^3$, and $G(x, y, z) = (5y, -5x, 25z)$. Then $dG(0)$ is the linear mapping $T(x, y) = 5(y, -x)$, and it is clear that for every $K \in (0, 1)$ there is some $R > 0$ such that (ii) is satisfied. Of course the origin is a fixed point of G. Theorem 35 tells us that any relatively small perturbation of G still has a fixed point (relatively close to the origin).

7. Notice also that Theorem 35 gives, in the case of $M = X$ a Hilbert space, the statement of Theorem 27, which is stronger than the version already proved, because here the mapping J is not necessarily differentiable, only $\frac{\partial J}{\partial y}$ needs to exist. This is one of the reasons why the proof of Theorem 35 is much more complicated than the one already given for the Hilbert space. This seemingly small difference is worth the effort of the proof, because, for instance, in Theorem 38, we only have to ask that the perturbing function H is Lipschitz, not necessarily everywhere differentiable with a bounded derivative.

Proof of Theorem 35. Let us define $\tilde{G}(x) = (x, G(x))$. Let us fix a point $x \in B(x_0, R)$ with a subgradient

$$\begin{aligned}\zeta \in\ & \partial_P \left(\langle -v, L_{F(x)x} \circ \exp_{F(x)}^{-1} \circ F(\cdot)\rangle\right)(x)\\ &= \partial_P \left(\langle -v, L_{F(x)x} \circ \exp_{F(x)}^{-1} \circ J(\cdot)\rangle \circ \tilde{G}(\cdot)\right)(x),\end{aligned}$$

where $v = \partial d(x, F(x))/\partial x$. We want to see that $\|v + \zeta\| \geq \lambda > 0$ for some $\lambda > 0$ independent of x, ζ, and such that $d(J(x_0, G(x_0)), x_0) < R\lambda$. Then, by using Theorem 23, we will get that

$$\min\{R, d(x_0, P)\} \leq \frac{d(F(x_0), x_0)}{\lambda} = \frac{d(J(x_0, G(x_0)), x_0)}{\lambda} < R,$$

hence $P \neq \emptyset$, that is, F has a fixed point in $V = U = B(x_0, R)$. So let us prove that there exists such a number λ.

Since J is locally Lipschitz we can find positive numbers C', δ_0 such that J is C'-Lipschitz on the ball $B(\tilde{G}(x), \delta_0)$.

By continuity of $\tilde{G}$ and $d\tilde{G}$ and the properties of exp, for any given $\varepsilon' > 0$ we can find $\delta_1 > 0$ such that

$$\|d(\exp^{-1}_{\tilde{G}(x)})(\tilde{G}(\tilde{x})) - L_{\tilde{G}(\tilde{x})\tilde{G}(x)}\| < \frac{\varepsilon'}{1+\|d\tilde{G}(x)\|}, \quad \text{and} \quad \|d\tilde{G}(\tilde{x})\| \le 1 + \|d\tilde{G}(x)\|$$

whenever $\tilde{x} \in B(x, \delta_1)$, and therefore, by the chain rule,

$$\|d(\exp^{-1}_{\tilde{G}(x)} \circ \tilde{G})(\tilde{x}) - L_{\tilde{G}(\tilde{x})\tilde{G}(x)} d\tilde{G}(\tilde{x})\| < \frac{\varepsilon'}{1+\|d\tilde{G}(x)\|} \|d\tilde{G}(\tilde{x})\| \le \varepsilon' \tag{1}$$

for every $\tilde{x} \in B(x_0, \delta_1)$. On the other hand, since $\exp^{-1}_{F(x)}$ is an almost isometry near $F(x)$ and the mapping J is continuous, we can find $\delta_2 > 0$ such that if $\tilde{y}, \tilde{z} \in B(\tilde{G}(x), \delta_2)$ then

$$\|\exp^{-1}_{F(x)}(J(\tilde{y})) - \exp^{-1}_{F(x)}(J(\tilde{z}))\| \le (1+\varepsilon') d(J(\tilde{y}), J(\tilde{z})).$$

In particular, bearing in mind the fact that the mapping J_y is L-Lipschitz for all y, we deduce that

$$\|\exp^{-1}_{F(x)}(J(z, y)) - \exp^{-1}_{F(x)}(J(z', y))\| \le (1+\varepsilon')\, L\, d(z, z') \tag{2}$$

for all $(z, y), (z', y) \in B(\tilde{G}(x), \delta_2)$.

In a similar manner, because $d\exp^{-1}_{F(x)}(\tilde{z})$ is arbitrarily close to $L_{\tilde{z}F(x)}$ provided $\tilde{z}$ is close enough to $F(x)$, and $J(\tilde{y})$ is close to $F(x) = J(\tilde{G}(x))$ when $\tilde{y}$ is close to $\tilde{G}(x)$, we can find a number $\delta_3 > 0$ such that

$$\|d\exp^{-1}_{F(x)}(J(\tilde{y})) - L_{J(\tilde{y})F(x)}\| \le \varepsilon' \tag{3}$$

provided that $\tilde{y} \in B(\tilde{G}(x), \delta_3)$.

Because of the continuity properties of the parallel transport and the geodesics (a consequence of their being solutions of differential equations which exhibit continuous dependence with respect to the initial data), we may find numbers $\delta_4, \delta_5, \delta_6 > 0$ such that:

$$\|L_{J(\tilde{y})F(x)} L_{\tilde{y}_2 J(\tilde{y})} - L_{\tilde{y}_2 F(x)}\| \le \varepsilon' \text{ provided that } d(\tilde{y}, \tilde{G}(x)) < \delta_4; \tag{4}$$

$$\|dG(\tilde{x}) L_{x\tilde{x}} - L_{G(x)G(\tilde{x})} dG(x)\| \le \varepsilon' \text{ provided that } d(x, \tilde{x}) < \delta_5, \qquad \text{and} \tag{5}$$

$$\|L_{\tilde{y}_2 F(x)} L_{G(x)\tilde{y}_2} - L_{G(x)F(x)}\| \le \varepsilon' \text{ provided that } d(\tilde{y}_2, G(x)) < \delta_6. \tag{6}$$

Let us take any $\delta < \min\{\delta_0, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$. By the fuzzy chain rule Theorem 14, we have that there are points $\tilde{y} = (\tilde{y}_1, \tilde{y}_2) \in M \times M$, $\tilde{x} \in M$, and a subgradient

$$\eta \in \partial_P \left(\langle -v, L_{F(x)x} \circ \exp^{-1}_{F(x)} \circ J(\cdot) \rangle \right)(\tilde{y}) \tag{7}$$

such that $d(\tilde{y}, \tilde{G}(x)) < \delta$, $d(\tilde{x}, x) < \delta$, $d(\tilde{G}(\tilde{x}), \tilde{G}(x)) < \delta$, and

$$L_{x\tilde{x}}\zeta \in \partial \left(\langle L_{\tilde{y}\tilde{G}(x)}(\eta), \exp^{-1}_{\tilde{G}(x)} \circ \tilde{G} \rangle \right)(\tilde{x}) + \delta B_{\mathrm{TM}_{\tilde{x}}}.$$

Since the mapping $\tilde{G}$ is differentiable this means, according to property (x) of Proposition 11, that

$$L_{x\tilde{x}}\zeta \in \langle L_{\tilde{y}\tilde{G}(x)}(\eta), d\left(\exp^{-1}_{\tilde{G}(x)} \circ \tilde{G}\right)(\tilde{x})(\cdot) \rangle + \delta B_{\mathrm{TM}_{\tilde{x}}}.$$

But, by Eq. (1) above, and taking into account that $\|\eta\| \leq C'$ (because J is C'-Lipschitz on $B(\tilde{G}(x), \delta) \ni \tilde{y}$), we have that

$$\begin{aligned}
&\langle L_{\tilde{y}\tilde{G}(x)}(\eta), d\left(\exp^{-1}_{\tilde{G}(x)} \circ \tilde{G}(\cdot)\right)(\tilde{x})(\cdot)\rangle + \delta B_{\mathrm{TM}_{\tilde{x}}} \\
&\quad \subseteq \langle L_{\tilde{y}\tilde{G}(x)}(\eta), L_{\tilde{G}(\tilde{x})\tilde{G}(x)} \circ d\tilde{G}(\tilde{x})(\cdot)\rangle + (\delta + \varepsilon'\|\eta\|) B_{\mathrm{TM}_{\tilde{x}}} \\
&\quad \subseteq \langle L_{\tilde{y}\tilde{G}(x)}(\eta), L_{\tilde{G}(\tilde{x})\tilde{G}(x)} \circ d\tilde{G}(\tilde{x})(\cdot)\rangle + (\delta + \varepsilon' C') B_{\mathrm{TM}_{\tilde{x}}},
\end{aligned}$$

so we get that, defining $\eta = (\eta_1, \eta_2)$,

$$\begin{aligned}
L_{x\tilde{x}}\zeta &\in \langle L_{\tilde{y}\tilde{G}(x)}(\eta), L_{\tilde{G}(\tilde{x})\tilde{G}(x)} \circ d\tilde{G}(\tilde{x})(\cdot)\rangle + (\delta + \varepsilon' C') B_{\mathrm{TM}_{\tilde{x}}} \\
&= \langle (\eta_1, \eta_2), L_{\tilde{G}(x)\tilde{y}} L_{\tilde{G}(\tilde{x})\tilde{G}(x)} \circ d\tilde{G}(\tilde{x})(\cdot)\rangle + (\delta + \varepsilon' C') B_{\mathrm{TM}_{\tilde{x}}} \\
&= \langle \eta_1, L_{x\tilde{y}_1} L_{\tilde{x}x}(\cdot)\rangle + \langle \eta_2, L_{G(x)\tilde{y}_2} L_{G(\tilde{x})G(x)} \circ dG(\tilde{x})(\cdot)\rangle + (\delta + \varepsilon' C') B_{\mathrm{TM}_{\tilde{x}}}. \qquad (8)
\end{aligned}$$

Now, from inequality (2) above and taking into account that $L_{F(x)x}$ is an isometry, we get that the functions

$$z \mapsto \langle -v, L_{F(x)x} \circ \exp^{-1}_{F(x)} \circ J(z, y)\rangle$$

are $L(1+\varepsilon')$-Lipschitz on $B(x, \delta_2)$ for every $y \in B(G(x), \delta_2)$. Then, since

$$(\eta_1, \eta_2) = \eta \in \partial_P \left(\langle -v, L_{F(x)x} \circ \exp^{-1}_{F(x)} \circ J(\cdot)\rangle\right)(\tilde{y}), \qquad (9)$$

and by using Lemma 34, we deduce that

$$\|\eta_1\| \leq (1+\varepsilon')L. \qquad (10)$$

On the other hand, since the mapping $y \mapsto J(x, y)$ is differentiable, by looking at Eq. (9) above, and again using Proposition 11(x) and Lemma 34, we have that

$$\begin{aligned}
\eta_2 &= \frac{\partial\left(\langle -v, L_{F(x)x} \circ \exp^{-1}_{F(x)} \circ J(y_1, y_2)\rangle\right)}{\partial y_2}(\tilde{y}) \\
&= \frac{\partial\left(\langle -L_{xF(x)}v, \circ \exp^{-1}_{F(x)} \circ J(y_1, y_2)\rangle\right)}{\partial y_2}(\tilde{y}) \\
&= \left\langle -L_{xF(x)}(v), d\exp^{-1}_{F(x)}(J(\tilde{y}))\left(\frac{\partial J}{\partial y_2}(\tilde{y})\right)(\cdot)\right\rangle. \qquad (11)
\end{aligned}$$

Besides, bearing in mind Eqs. (3) and (4) and the assumption (vi) of the statement, we have

$$\begin{aligned}
&\left\| d\exp^{-1}_{F(x)}(J(\tilde{y})) \circ \frac{\partial J}{\partial y_2}(\tilde{y}) - L_{\tilde{y}_2F(x)} \right\| \\
&\quad \leq \left\| d\exp^{-1}_{F(x)}(J(\tilde{y})) \circ \frac{\partial J}{\partial y_2}(\tilde{y}) - L_{J(\tilde{y})F(x)} \circ L_{\tilde{y}_2J(\tilde{y})} \right\| + \varepsilon' \\
&\quad = \left\| d\exp^{-1}_{F(x)}(J(\tilde{y})) \circ \left(\frac{\partial J}{\partial y_2}(\tilde{y}) - L_{\tilde{y}_2J(\tilde{y})}\right) \right. \\
&\qquad \left. + \left(d\exp^{-1}_{F(x)}(J(\tilde{y})) - L_{J(\tilde{y})F(x)}\right) \circ L_{\tilde{y}_2J(\tilde{y})} \right\| + \varepsilon' \\
&\quad \leq \|d\exp^{-1}_{F(x)}(J(\tilde{y}))\| \left\| \frac{\partial J}{\partial y_2}(\tilde{y}) - L_{\tilde{y}_2J(\tilde{y})} \right\|
\end{aligned}$$

$$+ \|d\exp^{-1}_{F(x)}(J(\tilde y)) - L_{J(\tilde y)F(x)}\| \, \|L_{\tilde y_2 J(\tilde y)}\| + \varepsilon'$$
$$\le (1+\varepsilon')\cdot\frac{\varepsilon}{C} + \varepsilon'\cdot 1 + \varepsilon' = (1+\varepsilon')\frac{\varepsilon}{C} + 2\varepsilon',$$

which, combined with (11), yields

$$\begin{aligned}\langle \eta_2, h\rangle &= \left\langle -L_{xF(x)}(v), d\exp^{-1}_{F(x)}(J(\tilde y))\left(\frac{\partial J}{\partial y_2}(\tilde y)\right)(h)\right\rangle \\ &\le \langle -L_{xF(x)}(v), L_{\tilde y_2 F(x)} h\rangle + \left((1+\varepsilon')\frac{\varepsilon}{C} + 2\varepsilon'\right)\|h\|\end{aligned}$$

for all $h \in \mathrm{TM}_{\tilde y_2}$. By taking $h = L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} dG(\tilde x)(-L_{x\tilde x} v)$ in this expression we get

$$\begin{aligned}&\langle \eta_2, L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} dG(\tilde x)(-L_{x\tilde x} v)\rangle \\ &\quad\le \langle -L_{xF(x)} v, L_{\tilde y_2 F(x)} L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} dG(\tilde x)(-L_{x\tilde x} v)\rangle \\ &\qquad + \left((1+\varepsilon')\frac{\varepsilon}{C} + 2\varepsilon'\right)\|dG(\tilde x)\| \\ &\quad\le \langle -L_{xF(x)} v, L_{\tilde y_2 F(x)} L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} dG(\tilde x)(-L_{x\tilde x} v)\rangle + \left((1+\varepsilon')\frac{\varepsilon}{C} + 2\varepsilon'\right) C \\ &\quad= \langle -L_{xF(x)} v, L_{\tilde y_2 F(x)} L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} dG(\tilde x)(-L_{x\tilde x} v)\rangle + (1+\varepsilon')\varepsilon + 2\varepsilon' C. \qquad (12)\end{aligned}$$

Now, by combining Eqs. (8), (10), (12), (5), (6) and assumption (ii), we obtain

$$\begin{aligned}\langle \zeta, -v\rangle &= \langle L_{x\tilde x}\zeta, -L_{x\tilde x} v\rangle \\ &\le (\delta + \varepsilon' C') + \langle \eta_1, L_{x\tilde y_1} L_{\tilde x x}(-L_{x\tilde x} v)\rangle \\ &\quad + \langle \eta_2, L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} \circ dG(\tilde x)(-L_{x\tilde x} v)\rangle \\ &\le (\delta + \varepsilon' C') + (1+\varepsilon')L \\ &\quad + \langle -L_{xF(x)} v, L_{\tilde y_2 F(x)} L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} \circ dG(\tilde x)(-L_{x\tilde x} v)\rangle \\ &\quad + (1+\varepsilon')\varepsilon + 2\varepsilon' C \\ &= \langle -L_{xF(x)} v, L_{\tilde y_2 F(x)} L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} \circ dG(\tilde x)(-L_{x\tilde x} v)\rangle \\ &\quad + \delta + \varepsilon' C' + (1+\varepsilon')L + (1+\varepsilon')\varepsilon + 2\varepsilon' C \\ &\le \delta + \varepsilon' C' + (1+\varepsilon')L + (1+\varepsilon')\varepsilon + 2\varepsilon' C \\ &\quad + \langle -L_{xF(x)} v, L_{\tilde y_2 F(x)} L_{G(x)\tilde y_2} L_{G(\tilde x)G(x)} L_{G(x)G(\tilde x)} \circ dG(x)(-v)\rangle + \varepsilon' \\ &= \delta + \varepsilon' C' + (1+\varepsilon')L + (1+\varepsilon')\varepsilon + 2\varepsilon' C + \varepsilon' \\ &\quad + \langle -L_{xF(x)} v, L_{\tilde y_2 F(x)} L_{G(x)\tilde y_2} \circ dG(x)(-v)\rangle \\ &\le \delta + \varepsilon' C' + (1+\varepsilon')L + (1+\varepsilon')\varepsilon + 2\varepsilon' C + \varepsilon' \\ &\quad + \langle -L_{xF(x)} v, L_{F(x)G(x)} \circ dG(x)(-v)\rangle + \varepsilon' C \\ &\le K + \delta + \varepsilon' C' + (1+\varepsilon')L + (1+\varepsilon')\varepsilon + 2\varepsilon' C + \varepsilon' + \varepsilon' C,\end{aligned}$$

that is,

$$\langle \zeta, -v\rangle \le \mu(\delta, \varepsilon') := K + \delta + \varepsilon'(1 + C') + (1+\varepsilon')L + (1+\varepsilon')\varepsilon + 3\varepsilon' C. \qquad (13)$$

Since δ and ε' can be chosen to be arbitrarily small and

$$\lim_{(\delta,\varepsilon')\to(0,0)} \mu(\delta, \varepsilon') = K + L + \varepsilon,$$

this argument shows that

$$\langle \zeta, -v \rangle \leq K + L + \varepsilon. \tag{14}$$

Finally, this implies that

$$\begin{aligned} \|v + \zeta\| &\geq \langle v, v + \zeta \rangle = \langle v, v \rangle + \langle v, \zeta \rangle \\ &= 1 - \langle \zeta, -v \rangle \geq 1 - (K + L + \varepsilon) := \lambda > 0, \end{aligned}$$

and λ is clearly independent of x, ζ. Moreover, according to assumption (vi), we have that $d(x_0, J(x_0, G(x_0))) < R(1 - (L + K + \varepsilon)) = R\lambda$, so we got all we needed. □

Finally let us see what Theorem 35 means when we consider some special cases of the perturbing mapping J. In the general case of a complete Riemannian manifold, if we have a mapping $G : M \to M$ having an almost fixed point x_0 and satisfying certain conditions, and we compose G with a mapping H which is relatively close to the identity, we get that $F = H \circ G$ has a fixed point. More precisely, we have the following.

Theorem 36. *Let M be a complete Riemannian manifold, and $G : M \to M$ a C^1 smooth function such that G is C-Lipschitz on a ball $B(x_0, R)$. Let $H : M \to M$ be a differentiable mapping. Assume that $H(G(x)) \notin \operatorname{sing}(x) \cup \operatorname{sing}(G(x))$ for every $x \in B(x_0, R)$, that*

$$\langle L_{xH(G((x)))}h, L_{G(x)H(G(x))}dG(x)(h)\rangle_{F(x)} \leq K < 1$$

for all $x \in B(x_0, R)$ and $h \in \mathrm{TM}_x$ with $\|h\|_x = 1$, and that $\|dH(y) - L_{yH(y)}\| < \varepsilon/C$ for every $y \in G(B(x_0, R))$, where $\varepsilon < 1 - K$, and $d(x_0, H(G(x_0))) < R(1 - K - \varepsilon)$. Then $F = H \circ G$ has a fixed point in $B(x_0, R)$.

If M is finite dimensional one can replace $\operatorname{sing}(z)$ with $\operatorname{cut}(z)$ everywhere.

Proof. It is enough to consider the mapping $J(x, y) = H(y)$. Since $x \mapsto J_y(x)$ is constant for every y, we can apply Theorem 35 with $L = 0$. □

Notice that when we take $0 < R < \rho = i(M)$, the global injectivity radius of M, we obtain the first corollary mentioned in the general introduction.

As another consequence we also have a local version of the result, whose statement becomes simpler.

Theorem 37. *Let M be a complete Riemannian manifold. Let x_0 be a fixed point of a C^1 function $G : M \to M$ satisfying the following condition:*

$$\langle h, dG(x_0)(h) \rangle \leq K < 1 \quad \text{for every } \|h\| = 1.$$

Then there exists a positive δ such that for every differentiable mapping $H : M \to M$ such that $\|dH(y) - L_{yH(y)}\| < \delta$ for every y near x_0, the composition $H \circ G : M \to M$ has a fixed point provided that $d(x_0, H(x_0)) < \delta$.

If M is endowed with a Lie group structure a natural extension of Corollary 29 holds: we can perturb the function G by summing a small function H with a small Lipschitz constant, and we get that $G + H$ has a fixed point.

Theorem 38. *Let $(M, +)$ be a complete Riemannian manifold with an abelian Lie group structure. Let $G : M \to M$ be a C^1 smooth function which is C-Lipschitz on a ball $B(x_0, R)$.*

Let $H : M \to M$ be an L-Lipschitz function. Assume that $G(x) + H(x) \notin \mathrm{sing}(x) \cup \mathrm{sing}(G(x))$ for every $x \in B(x_0, R)$, and that

$$\langle L_{x(H(x)+G(x))}h, L_{G(x)((H(x)+G(x)))}dG(x)(h)\rangle_{F(x)} \leq K < 1$$

for all $x \in B(x_0, R)$ and $h \in \mathrm{TM}_x$ with $\|h\|_x = 1$. Then $G + H$ has a fixed point, provided that $L < 1 - K$ and $d(x_0, x_0 + H(x_0)) < R(1 - K - L)$.

Again, if M is finite dimensional one can replace $\mathrm{sing}(z)$ with $\mathrm{cut}(z)$ everywhere.

Proof. Define $J(x, y) = y + H(x)$. We have that

$$\frac{\partial J}{\partial y}(x, y) = L_{yJ(x,y)},$$

so we can apply Theorem 35 with $\varepsilon = 0$. □

Let us conclude with an analogue of Corollary 32, which can be immediately deduced from Theorem 38.

Corollary 39. *Let $(M, +)$ be a complete Riemannian manifold with an abelian Lie group structure. Let x_0 be a fixed point of a C^1 function $G : M \to M$ satisfying the following condition:*

$$\langle h, dG(x_0)(h)\rangle \leq K < 1 \quad \text{for every } \|h\| = 1.$$

Then there exists a positive δ such that for every Lipschitz mapping $H : M \to M$ with Lipschitz constant smaller than δ, the mapping $G + H : M \to M$ has a fixed point provided that $d(x_0, x_0 + H(x_0)) < \delta$.

This is the second corollary mentioned in the introduction.

Let us show an easy example of a situation in which the above results are applicable. Let M be the cylinder defined by $x^2 + y^2 = 1$ in $\mathbb{R}^3$, and let $G : M \to M$ be the mapping defined by $G(x, y, z) = (x, -y, -z)$. Take p_0 to be either $(1, 0, 0)$ or $(-1, 0, 0)$ (the only two fixed points of G). We have that G is 1-Lipschitz and $\langle L_{pq}h, L_{G(p)q}dG(p)(h)\rangle = -1 := K$ whenever $q \notin \mathrm{cut}(p) \cup \mathrm{cut}(G(p))$. Then we can apply Theorem 36 with $R = \pi/2$ to obtain that, if we take any differentiable mapping $H : M \to M$ such that $H(G(p)) \notin \mathrm{cut}(p) \cup \mathrm{cut}(G(p))$ for every $p \in B(p_0, \pi/2)$ and $\|dH(p) - L_{pH(p)}\| < \varepsilon$ for every $p \in G(B(p_0, R))$ and $d(p_0, H(G(p_0))) < R(1 - K - \varepsilon)$, where $0 < \varepsilon < 2$, then the composition $F = H \circ G$ has a fixed point in $B(p_0, \pi/2)$.

In a similar way one can also apply Theorem 38 to obtain that, when M is endowed with the natural Lie group structure of $S^1 \times \mathbb{R}$, the mapping $G + H$ has a fixed point near p_0 provided $H : M \to M$ is a relatively small Lipschitz function.

Acknowledgements

We thank Francisco R. Ruiz del Portal for several valuable conversations concerning the results of this paper.

The first-named author was supported by a Marie Curie Intra-European Fellowship of the European Community, Human Resources and Mobility Programme under contract number MEIF CT2003-500927. The second-named author was supported by BFM2003-06420.

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