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Approximate Morse-Sard type results for non-separable Banach spaces

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ABSTRACT

For the Banach spaces $E = c_0(\Gamma), \ell_p(\Gamma)$, where Γ is an arbitrary infinite set and $1 < p < \infty$, we show that for every (non-zero) quotient F of E, every continuous function $f : E \to F$ can be uniformly approximated by smooth functions with no critical points, that is for every continuous function $\varepsilon : E \to (0, \infty)$, there exists a C^k smooth function $g : E \to F$ such that $g'(x)$ is surjective and $\|f(x) - g(x)\| \leq \varepsilon(x)$ for all $x \in E$ ($k = \infty$ if $E = c_0(\Gamma)$ or p is an even integer, $k = p - 1$ if p is an odd integer and $k = [p]$ otherwise). Moreover, for a wide class of (not necessarily separable) infinite dimensional Banach spaces E and a suitable class of quotients F of E, every continuous function $f : E \to F$ can be uniformly approximated by C^k smooth functions with no critical points (k depending on the properties of the smoothness of the space E). In particular, for every Banach space E with C^k smooth partitions of unity ($k \in \mathbb{N} \cup \{\infty\}$) and an infinite dimensional separable complemented subspace with a C^k smooth and LUR norm, we show that every continuous function $f : E \to \mathbb{R}^n$

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($n \in \mathbb{N}$) can be uniformly approximated by C^1 smooth functions with no critical points.

1. Introduction

The main purpose of this paper is to extend, from the separable to the not necessarily separable situation, the results given in [1,25,7,5] concerning what one can regard as an approximate strong version of the Morse-Sard theorem for mappings between Banach spaces E and F, where E is infinite-dimensional. Namely, under some appropriate conditions on an infinite-dimensional Banach space E, for every (non-zero) quotient F of E (or a certain class of quotients F of E), every continuous function $f : E \to F$, and for every continuous function $\varepsilon : E \to (0, \infty)$, there exists a C^k smooth function $g : E \to F$ with no critical points (that is, $g'(x)$ is surjective for all $x \in E$) such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ for all $x \in E$. Here k depends on the smoothness of E.

Let us put our work in context. This kind of results is in connection with the classical Morse-Sard theorem [36,37] stating that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a C^k smooth function with $k > \max\{n - m, 0\}$ and C_f is the set of critical points of f, i.e. the set of points x with non surjective derivative $f'(x)$, then the set of critical values $f(C_f)$ is of Lebesgue measure zero in $\mathbb{R}^m$. This result has been very valuable in the areas of topology and analysis (see for instance the monographs of Hirsch [29] and Yomdi and Comte [43]). Subsequently, different versions of the Morse-Sard theorem have been obtained by Bates and Moreira [9,34].

There are also different versions of the Morse-Sard theorem in the infinite-dimensional setting. Smale [38] proved for X and Y separable connected smooth manifolds modelled on Banach spaces and $f : X \to Y$ a C^k smooth Fredholm map that $f(C_f)$ is of first category (and thus $f(C_f)$ has no interior points) provided $k > \max\{\text{index}(f'(x)), 0\}$ for all $x \in X$. The term index stands for the index of the Fredholm operator $f'(x)$, that is the difference between the dimension of the kernel of $f'(x)$ and the codimension of the image of $f'(x)$, which are both finite. The above assumptions impose that when X is infinite-dimensional then Y is infinite-dimensional. In contrast, Kupka constructed C^∞ smooth functions $f : \ell_2 \to \mathbb{R}$ such that their critical values $f(C_f)$ contain intervals [31]. Bates and Moreira proved that such kind of functions can even be taken to be polynomial of degree three [9,34].

For many applications of the Morse-Sard theorem it is enough to approximate any continuous function by a smooth function whose set of critical points has empty interior [29,43], so we can refer to this as *an approximate Morse-Sard theorem*. In connection with this, Eells and McAlpin proved that any continuous function $f : \ell_2 \to \mathbb{R}$ can be uniformly approximated by smooth functions g whose set of critical values $g(C_g)$ is of measure zero.

Azagra and Cepedello [1] showed that every continuous mapping $f : \ell_2 \to \mathbb{R}^m$ can be uniformly approximated by C^∞ smooth functions with no critical points. Recall that a bump function is a non-zero function with bounded support. Hajek and Johanis [25] established a similar result and proved that in any separable Banach space X containing c_0 and with a C^p smooth bump function ($p \in \mathbb{N} \cup \{\infty\}$) every continuous function $f : X \to \mathbb{R}$ can be uniformly approximated by a C^p smooth function g such that $g'(X)$ is of first category in X^* and $g'(X) \cap N = \emptyset$ for any pre-fixed countable set $N \subset X^*$. Later, Azagra and Jiménez-Sevilla [7] proved that in every separable Banach space X with a LUR and C^p smooth norm ($p \in \mathbb{N} \cup \{\infty\}$) every continuous function $f : X \to \mathbb{R}$ can be uniformly approximated by C^p smooth functions with no critical points. In particular for $p = 1$ the space X can be taken to be any separable Banach space with separable dual, thus providing a characterization of Banach spaces with separable dual. Also, the results in [7] yield the existence of approximations by C^p smooth functions with no critical points provided X is separable, has an unconditional basis and a C^p smooth Lipschitz bump function.

Recently, Azagra, Dobrowolski and García-Bravo [5] proved that every continuous function $f : E \to F$ can be uniformly approximated by C^k smooth functions with no critical points whenever $E = c_0, \ell_p, L_p$ with $p \in (1, \infty)$ and F is a (non-zero) quotient of E, being $k = \infty$ for $E = c_0, \ell_p$ with p even integer, $k = p - 1$ for p odd integer and k is the integer part of p otherwise. Similar approximation results are also establised in [5] under certain assumptions on a separable Banach space E and any (non-zero) quotient F of E.

Later, García-Bravo [21] established on a certain family of separable Banach spaces E (including c_0 and ℓ_p with $1 < p < \infty$) that for every C^1 smooth function $f : E \to \mathbb{R}^n$, any continuous function $\varepsilon : E \to (0, \infty)$ and any open set U containing the critical points of f there is a C^1 smooth function $g : E \to \mathbb{R}^n$ such that $||f(x) - g(x)|| < \varepsilon(x)$ for every $x \in E$, g has no critical points and $f = g$ outside U.

We refer to the introduction of the paper [5] for more background about Morse-Sard results. Here, we want to emphasize that in the literature there are no approximate Morse-Sard results for nonseparable spaces, even in the fundamental case where E is a nonseparable Hilbert space and $F = \mathbb{R}$.

The results stated in this paper are also connected with the area of ranges of derivatives of a C^k smooth function $f : E \to \mathbb{R}$ ($k \in \mathbb{N} \cup \{\infty\}$), and more generally for functions $f : E \to F$, for E and F Banach spaces. In particular, the size and the shape of ranges of derivatives has been extensively studied by many authors. Azagra and Deville [2] constructed a C^1 smooth bump function f such that $f'(X) = X^*$ on every Banach space admitting a C^1 smooth Lipschitz bump function, in contrast with James' characterization of reflexive spaces as those Banach spaces E where the subdifferential of the norm $\partial|| \cdot ||$ verifies $\partial|| \cdot ||(S_E) = S_{E^*}$, where S_E and S_{E^*} are the unit spheres of E and E^*, respectively. Also, Azagra, Deville and Jiménez-Sevilla in [3] and Azagra, Fabian and Jiménez-Sevilla in [4] established conditions on the Banach spaces X and Y for the existence of Fréchet (Gâteaux, C^k) smooth functions $f : X \to Y$ with bounded support so that the range of the derivatives (successive derivatives for $k > 1$) fill the space of continuous linear (symmetric k-linear for $k > 1$) mappings from X to Y. Subsequent results have been obtained by Borwein, Fabian and Loewen in [12], Borwein, Fabian, Kortezov and Loewen in [11] and Azagra, Fabian and Jiménez-Sevilla in [4] to establish sufficient conditions on the Banach spaces X, Y and on an open (or closed) set A of continuous linear (symmetric k-linear for $k > 1$) mappings from X to Y to get the existence of a Fréchet (Gâteaux, C^k) smooth function with bounded support such that the range of the derivatives (successive derivatives for $k > 1$) is the set A. In contrast, Hajek [24] proves that Fréchet differentiable real-valued functions on c_0 with locally uniformly continuous derivative have locally compact derivative. Hajek proves that the same is true for Fréchet differentiable mappings from c_0 into any Banach space F with non trivial type. Also, the results of Hajek and Johanis in [25] mentioned above provide C^p smooth functions whose range of first derivatives is of first category. We refer the reader to [25,30,16] and references therein for more information on this area. Also related with this subject is the existence of smooth bump functions not satisfying Rolle's theorem on every infinite dimensional Banach space E with a smooth bump function (see [6] and references therein).

In this paper, we deal with a stronger version of an approximate Morse-Sard theorem, namely the uniform approximation of any continuous function $f : E \to F$ by C^k smooth functions with no critical points under suitable assumptions on the Banach spaces E and F (being F a non-zero quotient of E) and $k \in \mathbb{N} \cup \{\infty\}$).

Our notation is standard. E, F, X, Y will be real Banach spaces with norm denoted by $\|\cdot\|$, dual norm denoted by $\|\cdot\|^*$, closed unit ball $B_{\|\cdot\|}$, unit sphere $S_{\|\cdot\|}$, etc. If a mapping $f : E \to F$ is (Fréchet) differentiable, its derivative at a point $x \in E$ will be denoted indistinctly by $Df(x)$ or $f'(x)$. The critical set of f (that is, the set of all points $x \in E$ such that the linear mapping $Df(x) : E \to F$ is not surjective) will be denoted by C_f. So $C_f = \emptyset$ if and only if f has no critical points. For a pair of Banach spaces E, F, we denote by $C^k(E, F)$ the class of C^k smooth functions defined on E with values in F, where $k \in \mathbb{N} \cup \{\infty\}$. If $F = \mathbb{R}$, we shall write $C^k(E, \mathbb{R})$ or $C^k(E)$. Other necessary definitions will be provided as they are needed below. For any undefined terms in Banach space theory we refer to the monographs [17,18,26].

This paper is structured as follows. In Section 2 we will recall some tools that will be essential to our proofs, such as M-bases, homeomorphic embeddings into $c_0(\Gamma)$ with smooth coordinate functions, smooth partitions of unity, and diffeomorphisms extracting certain sets.

In Section 3 we will prove the following two results concerning C^k smooth approximations with no critical points to continuous functions defined on the classical infinite dimensional Banach spaces $c_0(\Gamma)$ and $\ell_p(\Gamma)$, with $p \in (1, \infty)$ for any infinite set Γ. Let us recall that, if Γ is an infinite set, $c_0(\Gamma)$ is defined as the space

$$c_0(\Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \subset \mathbb{R}^\Gamma : \text{ for every } \varepsilon > 0 \text{ the set } \{\gamma \in \Gamma : |x_\gamma| \geq \varepsilon\} \text{ is finite}\}.$$

We endow this space with the supremum norm $\|(x_\gamma)_{\gamma\in\Gamma}\|_\infty = \sup_{\gamma\in\Gamma}|x_\gamma|$.

Similarly, for every $1 < p < \infty$ we define $\ell_p(\Gamma)$ as the space

$$\ell_p(\Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \subset \mathbb{R}^\Gamma : \sum_{\gamma\in\Gamma} |x_\gamma|^p < \infty\},$$

endowed with the norm $\|(x_\gamma)_{\gamma\in\Gamma}\|_p = \left(\sum_{\gamma\in\Gamma}|x_\gamma|^p\right)^{1/p}$.

Both $(c_0(\Gamma), \|\cdot\|_\infty)$ and $(\ell_p(\Gamma), \|\cdot\|_p)$ are Banach spaces, and when Γ is uncountable they become non-separable.

In order to simplify the notation, we will write $\|f - g\| < \varepsilon$ to refer to the inequality $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in E$.

Theorem 1.1. *Let $E = c_0(\Gamma)$, where Γ is any infinite set, and let F be a (non-zero) quotient of E. Then for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists $g \in C^\infty(E, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.*

Theorem 1.2. *Let $E = \ell_p(\Gamma)$, where Γ is any infinite set and $1 < p < \infty$, and let F be a (non-zero) quotient of E. Then for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists $g \in C^k(E, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points. Here $k = \infty$ if p is even, $k = p-1$ if p is odd and $k = [p]$, where $[p]$ is the integer part of p, if $p \notin \mathbb{N}$.*

We will establish much more general results in other sections of the paper, but we think it is important to provide a separate proof of these theorems for classical Banach spaces, for the following two reasons. One is that some readers will only be interested in understanding why this kind of result is true for these spaces (especially for the Hilbert space), and the proofs can be written in a much more simple and transparent way in these cases. The other reason is that some of the ideas and techniques that we will be using in the proofs of the rest of our theorems are already present in these simple cases, so once the readers are acquainted with them they will be able to focus on the specific difficulties of the more general cases.

The proof of Theorem 1.2 requires two key tools: one is the existence of certain deleting diffeomorphisms on $\ell_p(\Gamma)$ given in [5] by Azagra, Dobrowolski and García-Bravo, and the other one is the existence of certain homeomorphic embeddings from $\ell_p(\Gamma)$ into $c_0(\Gamma')$ (for some infinite set Γ') whose coordinate functions are C^k smooth, given by Toruńczyk in [40]. On the other hand, the proof of Theorem 1.1 only requires the existence of certain C^∞ smooth partitions of unity of $c_0(\Gamma)$ with functions that locally depend on a finite number of coordinates. As said above, these essential tools will be explained in the next section.

Since every Hilbert space E is isometric to $\ell_2(\Gamma)$, where Γ is any set with the same cardinal as a complete orthonormal system in E, Theorem 1.2 immediately yields the following fundamental corollary.

Corollary 1.3. *Let E be an infinite dimensional Hilbert space, and let F be a (non-zero) quotient of E. Then for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists $g \in C^\infty(E, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.*

We now proceed to describe some of our more general and technical results in this paper.

Definition 1.4. We say that a Banach space E has a decomposition of the form $E = \bigoplus_{n\in\mathbb{N}} E_n$ for a sequence $\{E_n\}_{n=1}^\infty$ of subspaces of E, whenever the following holds:

(i) For every $n \in \mathbb{N}$, E_n is a closed subspace of E;
(ii) $E_n \cap E_m = \{0\}$ whenever $n \neq m$;
(iii) Every $x \in E$ can be written in a unique way as a sum $x = \sum_{n=1}^\infty x_n$ with $x_n \in E_n$ for all n;
(iv) The canonical projections $P_n : E \to E_n$ given by $P_n(x) = x_n$ are continuous; in particular each E_n is complemented in E.

In Section 4 we prove the following two main results for a target space F (finite or infinite dimensional).

Theorem 1.5. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let X, Y and F be Banach spaces such that:*

(1) *X has C^k smooth partitions of unity.*
(2) *Y is infinite-dimensional and has a C^k smooth and LUR norm.*
(3) *Y is reflexive, $Y = \bigoplus_{n\in\mathbb{N}} Y_n$.*
(4) *F is a (non-zero) quotient of Y_n for every $n \in \mathbb{N}$.*
(5) *The canonical projection $Q : Y \to \bigoplus_{j \text{ odd}} Y_j$ given by $Q(y) = \sum_{j \text{ odd}} y_j$, for every $y = \sum_{j=1}^\infty y_j \in Y$ with $y_j \in Y_j$, is well defined and continuous.*

Then for every continuous mapping $f : X \oplus Y \to F$ and every continuous function $\varepsilon : X \oplus Y \to (0, \infty)$ there exists $g \in C^k(X \oplus Y, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.

Regarding condition (5) in Theorem 1.5, it is worth mentioning that this condition holds in the case that the decomposition $Y = \oplus_{n\in\mathbb{N}} Y_n$ is “unconditional”, that is, there is a constant $K \geq 1$ such that $\| \sum_{j\in\sigma} y_j \| \leq K \| \sum_{j\in\tau} y_j \|$ whenever σ and τ are finite subsets of integers with $\sigma \subset \tau$ and $y_j \in Y_j$ for all $j \in \tau$. Also, if all the Y_n are isomorphic to Y, then condition (4) in Theorem 1.5 (and Theorem 1.6 below) can be replaced by "F is a non-zero quotient of Y".

Notice that, in Theorem 1.5, if $k \geq 2$ then the assumption on the existence of a C^k smooth and LUR norm on Y yields in particular the superreflexivity of the space Y (see [19] or [17, Chapter V, Proposition I.3]). So Theorem 1.5 for $k \geq 2$ only concerns certain superreflexive Banach spaces Y. Let us also note that not every superreflexive Banach space can be equivalently renormed with a C^2 smooth and LUR norm (for example, ℓ_p, for $1 < p < 2$, cannot be renormed in such a way). Now, in Theorem 1.5, if we do not assume Y to be reflexive (so this only involves case $k = 1$), we have to require additional conditions on Y to get the same conclusion, as we do in the following theorem.

Theorem 1.6. *Let X, Y and F be Banach spaces such that:*

(1) *X has C^1 smooth partitions of unity.*
(2) *Y is infinite-dimensional and has a C^1 smooth and LUR norm.*
(3) *$Y = \bigoplus_{n\in\mathbb{N}} Y_n$, each Y_n has a shrinking M-basis and the union of all these M-bases, which we denote by $\{u_i, u_i^*\}_{i\in I} \subset \bigcup_{n\in\mathbb{N}} (Y_n \times Y_n^*)$ is a shrinking M-basis of Y.*
(4) *F is a (non-zero) quotient of Y_n for every $n \in \mathbb{N}$.*
(5) *The canonical projections*

$$Q_m : Y \to (\oplus_{j=1}^m Y_j) \oplus (\bigoplus_{\substack{j \text{ odd} \\ j>m}} Y_j)$$

given by

$$Q_m(y) = \sum_{j=1}^m y_j + \sum_{\substack{j \text{ odd} \\ j>m}} y_j,$$

for every $y = \sum_{j=1}^\infty y_j \in Y$ with $y_j \in Y_j$, are well defined and have norm one for all $m \in \mathbb{N}$. That is, $\|Q_m(y)\| \leq \|y\|$ for all $y \in Y$ and all $m \in \mathbb{N}$.

Then for every continuous mapping $f : X \oplus Y \to F$ and every continuous function $\varepsilon : X \oplus Y \to (0, \infty)$ there exists $g \in C^1(X \oplus Y, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.

Regarding condition (5) in Theorem 1.6, it is worth mentioning that this condition holds in the case that the decomposition $Y = \oplus_{n\in\mathbb{N}} Y_n$ is "unconditional" with associated constant $K = 1$, i.e. $\| \sum_{j\in\sigma} y_j \| \leq \| \sum_{j\in\tau} y_j \|$ whenever σ and τ are finite subsets of integers with $\sigma \subset \tau$ and $y_j \in Y_j$ for all $j \in \tau$.

As in the preceding section, a key tool in the proofs of Theorem 1.5 and Theorem 1.6 is the existence of certain homeomorphic embeddings from $X \oplus Y$ into $c_0(\Gamma')$ (for a certain set Γ') whose coordinate functions are C^k smooth. Also, the proofs require the existence of certain deleting diffeomorphisms on $X \oplus Y$ except for the case $Y = c_0(\Gamma)$ (for any infinite set Γ). In the latter case we have that in fact, the partial derivative with respect to the second coordinate y of the approximating function g is a surjective operator at every point $(x, y) \in X \oplus Y$. Let us record this fact in the following proposition.

Proposition 1.7. *Let $k \in \mathbb{N} \cup \{\infty\}$, let Γ be an infinite set and let X and F be Banach spaces such that:*

(1) *X has C^k smooth partitions of unity.*
(2) *F is a (non-zero) quotient of $c_0(\Gamma)$.*

Then for every continuous mapping $f : X \oplus c_0(\Gamma) \to F$ and every continuous function $\varepsilon : X \oplus c_0(\Gamma) \to (0, \infty)$ there exists $g \in C^k(X \oplus c_0(\Gamma), F)$ such that $\|f - g\| < \varepsilon$ and $\frac{\partial g}{\partial y}(x, y)$ is surjective for every $(x, y) \in X \oplus c_0(\Gamma)$. In particular, g has no critical points.

Note that all reflexive Banach spaces admit a C^1 smooth LUR norm and a shrinking M-basis (see [41, Corollary 6] or [27, Theorem 6.1]).

In Section 5, we deal with the particular case of a finite dimensional target space F. The fact that F is finite dimensional allows us to dispense with some of the assumptions made in Theorem 1.5 and Theorem 1.6. In this case, our main result reads as follows.

Theorem 1.8. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let X be a Banach space with C^k smooth partitions of unity, let Y be a separable infinite dimensional Banach space with a C^k smooth and LUR norm and let F be a (non-zero) finite dimensional space. Then for every continuous function $f : X \oplus Y \to F$ and every continuous function $\varepsilon : X \oplus Y \to (0, \infty)$ there exists $g \in C^k(X \oplus Y, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.*

We note that for the case of $E = \{0\} \oplus Y = Y$, where Y is separable and $k = 1$, this result yields a characterization of the approximate strong Morse-Sard property for finite dimensional targets F: it is necessary and sufficient that E has C^1 partitions of unity. This was previously known for $F = \mathbb{R}$ (see [7]) but the result is new for higher dimensions of F. In fact we have the following.

Corollary 1.9. *Let E be a Banach space with C^1 smooth partitions of unity. Assume there exists an infinite dimensional separable and complemented subspace $Y \subset E$. Let F be a (non-zero) finite dimensional space. Then, for every continuous function $f : E \to F$ and every continuous function $\varepsilon : X \to (0, \infty)$ there exists $g \in C^1(E, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.*

Recall that a Banach space Y is Weakly Compactly Generated (WCG for short) whenever Y is the closed linear span of a weakly compact set of Y. In particular, separable Banach spaces are WCG. Reformulating Theorem 1.8 in a different way we get the following corollary which we will prove in Section 5 as well.

Corollary 1.10. *Let $k \in \mathbb{N} \cup \{\infty\}$, let X be a Banach space with C^k smooth partitions of unity, let Y be an infinite dimensional Banach space with a LUR and C^k smooth norm and let F be a (non-zero) finite dimensional space. For $k = 1$, let us assume in addition that Y is WCG. Then for every continuous function $f : X \oplus Y \to F$ and every continuous function $\varepsilon : X \oplus Y \to (0, \infty)$ there exists $g \in C^k(X \oplus Y, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.*

Remark 1.11. The role of the space X in our main results Theorems 1.5, 1.6, 1.8 and Corollary 1.10, can be taken by somehow *bad* spaces in terms of smoothness. For instance, Haydon gives in [28] examples of Banach spaces $C_0(T)$, where T is a tree, that have C^∞ smooth partitions of unity but no C^1 smooth equivalent norm (note that this never occurs for separable spaces).
We note as well that there exist Banach spaces with C^1 smooth and LUR norm, as the Johnson-Lindenstrauss space JL_0 (see [44] for the definition and more properties about this space), which do not lie under the assumptions given for the space Y in any of the main results of this paper. The reason is that JL_0 does not have a M-basis and cannot have a separable complemented subspace, because since it is c_0-saturated this would imply by Sobczyk's theorem [39] that c_0 is complemented in JL_0, which is not true (see [44, page 1764]). It is indeed an open question for us if one may have approximated Morse-Sard type results for this kind of spaces.

The proof of Theorem 1.8 relies on three key tools. We are already familiar with two of them: the existence of certain deleting diffeomorphisms on $X \oplus Y$ and the existence of certain homeomorphic embeddings from $X \oplus Y$ into $c_0(\Gamma')$ (for a suitable infinite set Γ') with C^k smooth coordinate functions. When Y is not reflexive, we need a third key tool: the construction of a residual set of C^1 smooth and LUR (equivalent) norms on Y whose dual norms are Fréchet differentiable at the points of a pre-fixed subspace of countable Hamel dimension of Y^* (except at 0). Here the residuality is considered within the Baire space $(\mathcal{N}_Y, \rho)$ of all equivalent norms on Y with the Hausdorff metric (see Section 5 for the definitions). Let us describe it in what follows. For a given Banach space Y with norm p, let us denote by NA_p the set of elements $x^* \in Y^*$ such that x^* attains its p^*-norm (being p^* the dual norm of p), that is, there is $x \in S_p := \{x \in Y : p(x) = 1\}$ satisfying $p^*(x^*) = x^*(x)$. Also, we denote by $\mathrm{cone}(W)$ the cone generated by a set W of a Banach space Y (see Section 5.1 for the definitions), and we say that a set is K_σ whenever it is a countable union of compact sets. Specifically we will prove the following renorming results.

Proposition 1.12. *Let Y be a Banach space and let $W \subset S_{\|\cdot\|^*}$ be a K_σ subset. Then the set of norms $p \in \mathcal{N}_Y$ such that its dual norm p^* is Fréchet differentiable at the points of* $\operatorname{cone}(W) \setminus \{0\}$ *is residual in $(\mathcal{N}_Y, \rho)$. In particular, the set of norms $p \in \mathcal{N}_Y$ such that* $\operatorname{cone}(W) \subset \mathrm{NA}_p$ *is residual in $(\mathcal{N}_Y, \rho)$.*

Remark 1.13. Notice that, in Proposition 1.12 as well as in Corollary 1.14, we can take as W the set $W = H \cap S_{||\cdot||^*}$, being H a subspace of Y^* of countable Hamel dimension (and thus $\operatorname{cone}(W) = H$).

Corollary 1.14. *Let Y be a Banach space with a LUR norm $\|\cdot\|$ whose dual norm $\|\cdot\|^*$ is LUR. Let $W \subset S_{\|\cdot\|^*}$ be a K_σ subset. Then the set of norms $p \in \mathcal{N}_Y$ such that both p and its dual norm p^* are LUR and p^* is Fréchet differentiable at the points of* $\operatorname{cone}(W) \setminus \{0\}$ *is residual in $(\mathcal{N}_Y, \rho)$. In particular, the set of norms $p \in \mathcal{N}_Y$ such that both p and its dual norm p^* are LUR and* $\operatorname{cone}(W) \subset \mathrm{NA}_p$ *is residual in $(\mathcal{N}_Y, \rho)$.*

We refer to Vanderwerff [42], Hajek [23], Dantas, Hajek and Russo [13–15] for more information on C^1 smooth norms (or more generally C^k smooth norms and analytic norms) on dense subspaces of Banach spaces.

Finally, in the last section, we will show that, when stating the existence of C^k smooth approximations with no critical points to continuous functions from an (infinite-dimensional) Banach space X to any (non-zero) quotient space F of X, it is enough to consider $F = X$ as explained in the following fact.

Fact 1.15. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let X be a Banach space. Then the following are equivalent:*

(1) *For every (non-zero) quotient F of X, every continuous mapping $f : X \to F$ and every continuous function $\varepsilon : X \to (0, \infty)$ there exists $g \in C^k(X, F)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.*
(2) *For every continuous mapping $f : X \to X$ and every continuous function $\varepsilon : X \to (0, \infty)$ there exists $g \in C^k(X, X)$ such that $\|f - g\| < \varepsilon$ and g has no critical points.*

As in the results given in [7], the existence of C^k smooth approximations with no critical points to real-valued continuous functions defined on a Banach space X yields two important corollaries: (1) *a nonlinear C^k smooth Hahn-Banach separation theorem* (Corollary 6.1) and (2) *C^k smooth approximations of closed sets* (Corollary 6.2), which shall be stated in the last section.

2. Preliminaries

Let us gather several tools that we will need in our proofs.

2.1. M-bases and embeddings into $c_0(\Gamma)$ with C^1 smooth coordinate functions

Let us denote by $\operatorname{span}\{A\}$ (or $\operatorname{span}(A)$) the span of a set A of a normed space E, by $\overline{\operatorname{span}}\{A\}$ (or $\overline{\operatorname{span}}(A)$) the closed linear span of A and by $\overline{\operatorname{span}}^{w^*}\{A\}$ (or $\overline{\operatorname{span}}^{w^*}(A)$) the *weak** closed linear span of A.

Definition 2.1. Let E be a Banach space and Γ an infinite set. A family $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma} \subset E\times E^*$ is called a Markushevich basis for E, henceforth called an M-basis, if $x_\gamma^*(x_\beta) = \delta_{\gamma\beta}$ for all $\gamma, \beta \in \Gamma$, $E = \overline{\operatorname{span}}\{x_\gamma : \gamma \in \Gamma\}$ and $E^* = \overline{\operatorname{span}}^{w^*}\{x_\gamma^* : \gamma \in \Gamma\}$. It is moreover called shrinking provided that $E^* = \overline{\operatorname{span}}\{x_\gamma^* : \gamma \in \Gamma\}$.

Observe that if $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma} \subset E \times E^*$ is an M-basis, the fact that $E^* = \overline{\operatorname{span}}^{w^*}\{x_\gamma^* : \gamma \in \Gamma\}$ implies that $\{x_\gamma^*\}_{\gamma\in\Gamma}$ is a total set over E, meaning that if $x_\gamma^*(x) = 0$ for all γ then $x = 0$.

It is nowadays well-known that reflexive spaces admit shrinking M-bases (see [27, Theorem 6.1]). Moreover we will use that reflexive spaces E have renormings that are locally uniformly rotund (LUR) and C^1 smooth on $E\setminus\{0\}$ (see [41, Corollary 6]). Joining these two facts we can give a simple proof of the following result about the existence of homeomorphic embeddings into $c_0(\Gamma)$ with C^1 smooth coordinate functions. We note that this result was first proved by Lindenstrauss [33] using strongly the results from [32]. However he did not use the existence of shrinking M-basis for reflexive spaces. Lemma 2.2, together with Theorem 2.8 was shown in [40] to derive the existence of C^1 smooth partitions of unity for reflexive spaces.

Lemma 2.2. *([33]) Let E be a Banach space with an (equivalent) C^1 smooth and LUR norm $\|\cdot\|$ in E and a shrinking M-basis $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma}$ in E with $\|x_\gamma^*\| = 1$ for all $\gamma \in \Gamma$. Assuming $0 \notin \Gamma$, the mapping $u : E \to c_0(\{0\} \cup \Gamma)$ defined by*

$$u(x)_\gamma = \begin{cases} \|x\|^2 & \text{if } \gamma = 0 \\ x_\gamma^*(x) & \text{if } \gamma \in \Gamma, \end{cases}$$

($u(x)_\gamma$ being the γ-coordinate of $u(x)$) is a homeomorphic embedding and $x \to u(x)_\gamma$ is C^1 smooth for every $\gamma \in \{0\} \cup \Gamma$.

Proof. It is clear that the mapping $L : E \to \ell_\infty(\Gamma)$ defined by $L(x)_\gamma = x_\gamma^*(x)$ if $\gamma \in \Gamma$ is linear and continuous (where $L(x)_\gamma$ is the γ-coordinate of $L(x)$). In fact, since $\|x_\gamma^*\| = 1$ for all $\gamma \in \Gamma$, we have $\|L(x)\| = \sup\{|x_\gamma^*(x)| : \gamma \in \Gamma\} \le \|x\|$ for all $x \in E$. Besides, since $L : E \to \ell_\infty(\Gamma)$ is bounded, $L(\operatorname{span}\{x_\gamma : \gamma \in \Gamma\}) \subset c_0(\Gamma)$, and $c_0(\Gamma)$ is closed in $\ell_\infty(\Gamma)$, we have that $L : E \to c_0(\Gamma)$. Therefore u is well-defined and continuous, and the fact that $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma}$ is an M-basis in E implies that L is injective, hence u is injective too.

Let us see that $u^{-1} : u(E) \to E$ is continuous as well. If $(y_k)_{k\in\mathbb{N}} \subset u(E)$ converges to some $y \in u(E)$, say $y_k = u(x_k) \xrightarrow{k\to\infty} u(x) = y$. For $\gamma = 0$, $u(x_k)_\gamma = \|x_k\|^2 \xrightarrow{k\to\infty}$

$u(x)_\gamma = \|x\|^2$. Therefore $(\|x_k\|)_k$ is bounded. Moreover we have that $x^*_\gamma(x_k) \xrightarrow{k\to\infty} x^*_\gamma(x)$, for all γ, which together with the boundedness of $(\|x_k\|)_k$ and the fact that $\{x_\gamma, x^*_\gamma\}_{\gamma\in\Gamma}$ is a shrinking M-basis in E imply that x_k weakly converges to x. Finally, the weak convergence and the convergence of norms $\lim_{k\to\infty}\|x_k\| = \|x\|$ yield that $(x_k)_k$ converges to x in the norm topology, as the norm $\|\cdot\|$, being LUR, has the Kadec-Klee property [17]. □

2.2. Smooth partitions of unity and LFC functions. Toruńczyk result

Let Γ be a set. For every $\gamma \in \Gamma$ let us introduce the linear functionals $e^*_\gamma : c_0(\Gamma) \to \mathbb{R}$ that map any vector $x = (x_\gamma)_{\gamma\in\Gamma}$ to its γ-coordinate, that is $e^*_\gamma(x) = x_\gamma$.

Definition 2.3. For $k \in \mathbb{N}\cup\{\infty\}$, We will say that a C^k smooth function $f : c_0(\Gamma) \to \mathbb{R}$ locally depends on finitely many coordinates (LFC or LFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$) if for every $x \in c_0(\Gamma)$ there is an open neighbourhood U_x of x, a finite set $\{\gamma_1, \dots, \gamma_n\} \subset \Gamma$ and a C^k smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ so that for every $y \in U_x$ we have

$$f(y) = \varphi(e^*_{\gamma_1}(y), \dots, e^*_{\gamma_n}(y)) = \varphi(y_{\gamma_1}, \dots, y_{\gamma_n}).$$

Observe that for all $y \in U_x$ we have $f'(y) \in \operatorname{span}\{e^*_{\gamma_k} : k = 1, \dots, n\}$ and therefore

$$\bigcap_{k=1}^{n} \operatorname{Ker} e^*_{\gamma_k} \subset \operatorname{Ker} f'(y).$$

The following result can be found in [26, Page 284] (for the separable case the original proof is due to N. Kuiper and can be found in [10]).

Theorem 2.4. *For any set Γ the space $c_0(\Gamma)$ admits an equivalent C^∞ smooth LFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ norm.*

In order to construct our smooth approximation in the proofs of the main theorems it will be important for us to have smooth and LFC partitions of unity at our disposal. Recall that the (open) support of a function $\psi : X \to \mathbb{R}$ is defined as $\operatorname{supp}_0 \psi = \{x \in X : \psi(x) \neq 0\}$.

Definition 2.5. Let X be a Banach space and a family $\{\psi_\alpha\}_{\alpha\in\Delta}$ of continuous functions $\psi_\alpha : X \to [0,\infty)$ for all $\alpha \in \Delta$. The family $\{\psi_\alpha\}_{\alpha\in\Delta}$ is called a partition of unity on X if for every $x \in X$ there is a neighbourhood U_x in X which intersect only finitely many of $\{\operatorname{supp}_0 \psi_\alpha\}_{\alpha\in\Delta}$ and $\sum_{\alpha\in\Delta} \psi_\alpha(x) = 1$ for all $x \in X$.

Definition 2.6. We say that a Banach space X admits $\mathcal{C}$-partitions of unity for a certain class $\mathcal{C}$ of continuous functions defined on X if for any open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of X there is a partition of unity $\{\phi_\beta\}_{\beta\in\Omega}$ on X such that $\{\phi_\beta\}_{\beta\in\Omega} \subset \mathcal{C}$ and for each β there is α

such that $\operatorname{supp}_0 \phi_\beta \subset U_\alpha$ (we will say that $\{\psi_\beta\}_{\beta\in\Omega}$ is subordinated to $\{U_\alpha\}_{\alpha\in\Delta}$). The type of families $\mathcal{C}$ that we consider throughout the paper allows us to have for a Banach space X with $\mathcal{C}$-partitions of unity and for every open covering $\{U_\alpha\}_{\alpha\in\Omega}$ a partition of unity $\{\psi_\alpha\}_{\alpha\in\Omega} \subset \mathcal{C}$ such that $\operatorname{supp}_0 \psi_\alpha \subset U_\alpha$ for all α (and say with this meaning that $\{\psi_\alpha\}_{\alpha\in\Omega}$ is subordinated to $\{U_\alpha\}_{\alpha\in\Omega}$). So we will use this last property in a Banach space X admitting $\mathcal{C}$-partitions of unity.

For the next result see [40] or also [26, p. 422].

Lemma 2.7. *([40]) For any set Γ the space $c_0(\Gamma)$ admits C^∞-smooth and LFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ partitions of unity.*

We need to state for future use Toruńczyk's result [40, Theorem 1], about the existence of certain type of partitions of unity for Banach spaces which admit homeomorphic embeddings into $c_0(\Gamma)$ with C^1 smooth coordinate functions.

Theorem 2.8. *([40]) Let E be a Banach space and let $\mathcal{S}$ be a family of real valued continuous functions defined on E satisfying*

(1) *For every function $g : E \to \mathbb{R}$, every open cover $\mathcal{U} = \{U_i\}_{i\in\Delta}$ of E and every family $\{\psi_i : i \in \Delta\} \subset \mathcal{S}$, if $\psi_i|_{U_i} = g|_{U_i}$ for all $i \in \Delta$, then $g \in \mathcal{S}$.*
(2) *If $n \in \mathbb{N}$, $\Psi \in C^\infty(\mathbb{R}^n)$ and $\psi_1, \dots, \psi_n \in \mathcal{S}$, then $\Psi(\psi_1, \dots, \psi_n) \in \mathcal{S}$.*

Then E admits $\mathcal{S}$-partitions of unity if and only if there is a set Γ and a homeomorphic embedding $u : E \to c_0(\Gamma)$ so that $e^*_\gamma \circ u \in \mathcal{S}$ for every $\gamma \in \Gamma$.

2.3. Diffeomorphic extraction of closed sets

Finally we need a result which guarantees that we can diffeomorphically extract certain closed sets. This is basically a restatement of [5, Theorem 1.4].[1]

Theorem 2.9. *([5]) Let E be a Banach space, $k \in \mathbb{N} \cup \{\infty\}$, and $\mathcal{C} \subset E$ be a closed set with the property that, for each $x \in \mathcal{C}$, there exist a neighbourhood U_x of x in E, Banach spaces $E_{(1,x)}$ and $E_{(2,x)}$, and a continuous mapping $f_x : \mathcal{C}_x \to E_{(2,x)}$, where $\mathcal{C}_x$ is a closed subset of $E_{(1,x)}$, such that:*

(1) *$E = E_{(1,x)} \oplus E_{(2,x)}$;*
(2) *$E_{(1,x)}$ has C^k smooth partitions of unity;*
(3) *$E_{(2,x)}$ is infinite-dimensional and has an equivalent C^k smooth norm;*

[1] We note that there is an omission in the assumptions of [5, Theorem 1.4]: in item (3) it is important that the C^p smooth norm in $E_{(2,x)}$ is equivalent to the restriction of the original one in E.

(4) *$\mathcal{C} \cap U_x \subset G(f_x)$, where*

$$G(f_x) = \{y = (y_1, y_2) \in E_{(1,x)} \oplus E_{(2,x)} \,:\, y_2 = f_x(y_1),\, y_1 \in C_x\}.$$

Then, for every open cover $\mathcal{G}$ of E there exists a C^k diffeomorphism $d : E \to E \setminus \mathcal{C}$ which is limited by $\mathcal{G}$ (that is, the family of sets $\{\{x, d(x)\} : x \in E\}$ refines $\mathcal{G}$, meaning that for every $x \in E$ there is a $G_x \in \mathcal{G}$ such that both x and $d(x)$ are in G_x).

In some cases we will use the above theorem in the following form.

Corollary 2.10. *([5]) Let E be a Banach space, $k \in \mathbb{N} \cup \{\infty\}$, and $\mathcal{C} \subset E$ be a closed set with the property that, for each $x \in \mathcal{C}$, there exist a neighbourhood U_x of x in E and Banach spaces $E_{(1,x)}$ and $E_{(2,x)}$ such that:*

(1) *$E = E_{(1,x)} \oplus E_{(2,x)}$;*
(2) *$E_{(1,x)}$ has C^k smooth partitions of unity;*
(3) *$E_{(2,x)}$ is infinite-dimensional and has an equivalent C^k smooth norm;*
(4) *$\mathcal{C} \cap U_x \subset E_{(1,x)} \oplus \{0\}$.*

Then, for every open cover $\mathcal{G}$ of E there exists a C^k diffeomorphism $d : E \to E \setminus \mathcal{C}$ which is limited by $\mathcal{G}$.

3. Proofs for the classical spaces $c_0(\Gamma)$ and $\ell_p(\Gamma)$

3.1. Proof of Theorem 1.1

The proof for the space $c_0(\Gamma)$ is simpler than that for $\ell_p(\Gamma)$, since in this case we do not have to extract any critical set. Let us write $E = c_0(\Gamma)$. By Lemma 2.7, the space E admits C^∞ smooth partitions of unity $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ where the functions ψ_α locally are of the form

$$x \mapsto \psi_\alpha(x) = \varphi_\alpha\left(e^*_{\gamma_1}(x), ..., e^*_{\gamma_n}(x)\right), \tag{3.1}$$

where the functionals $e^*_\gamma : E \to \mathbb{R}$ are defined by $e^*_\gamma(x) = x_\gamma$ (the γ-th coordinate of $x \in c_0(\Gamma)$) for certain indexes $\gamma_1, ..., \gamma_n \in \Gamma$ with $n \in \mathbb{N}$ (depending on the neighbourhood), and certain C^∞ smooth function $\varphi_\alpha : \mathbb{R}^n \mapsto \mathbb{R}$ (also depending on the neighbourhood).

Thus, given f and ε as in the statement, we may find an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of E and a partition of unity $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ subordinated to $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ such that each ψ_α locally is of the form (3.1) and

$$\|f(x) - f(y)\| < \varepsilon(y)/4 \quad \text{and } \varepsilon(x) < 2\varepsilon(y) \quad \text{for all } x, y \in U_\alpha, \text{ and for each } \alpha \in \mathcal{A}.$$

Without loss of generality we may assume that for every $\alpha \in \mathcal{A}$ the diameter of the set U_α is less than $\varepsilon(x_\alpha)/4$, where x_α is a point in U_α that we fix from now on.

Let us define $g : E \to F$ by

$$g(x) = \sum_{\alpha \in \mathcal{A}} \psi_\alpha(x) \left(f(x_\alpha) + T(x - x_\alpha) \right), \tag{3.2}$$

where $T : E \to F$ is a continuous linear operator that we define as follows. Note that $\sharp(\Gamma \times \mathbb{N}) = \sharp\Gamma$, so we may write

$$\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n,$$

where $\sharp\Gamma_n = \sharp\Gamma$ and $\Gamma_n \cap \Gamma_m = \emptyset$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Hence we have

$$E = \bigoplus_{n \in \mathbb{N}} E_n,$$

where $E_n = c_0(\Gamma_n)$ for all $n \in \mathbb{N}$, and now the direct sum is understood in the c_0 sense. For every $n \in \mathbb{N}$, we let $P_n : E \to E_n$ denote the canonical projection. Since $c_0(\Gamma_n)$ is isometric to $c_0(\Gamma)$ and F is a linear quotient of the latter, for every $n \in \mathbb{N}$ there exists a continuous linear surjection $T_n : E_n \to F$; by dividing by the norm of T_n, if necessary, we may assume that $\|T_n\| = 1$. We define $T : E \to F$ by

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n \circ P_n.$$

A routine calculation yields

$$\|f(x) - g(x)\| < \varepsilon(x) \text{ for all } x \in E. \tag{3.3}$$

Let us end the proof by showing that g has no critical points.

Since the sum defining g is locally finite and each ψ_α is locally of the form (3.1), given $z \in E$ there exists an open neighbourhood V_z of z, there exist $\alpha_1, ..., \alpha_m \in \mathcal{A}$, $\gamma_{1,j}, ..., \gamma_{n_j,j} \in \Gamma$ for $j = 1, ..., m$, and there exist C^∞ smooth functions $\varphi_{\alpha_j} : \mathbb{R}^{n_j} \mapsto \mathbb{R}$ for $j = 1, ..., m$ such that

$$g(x) = \sum_{j=1}^{m} \psi_{\alpha_j}(x) \left(f(x_{\alpha_j}) + T(x - x_{\alpha_j}) \right) \quad \text{ for all } x \in V_z$$

where

$$\psi_{\alpha_j}(x) = \varphi_{\alpha_j}\left(e^*_{\gamma_{1,j}}(x), ..., e^*_{\gamma_{n_j,j}}(x) \right) \text{ for all } x \in V_z.$$

A straightforward calculation shows that

$$Dg(x) = T + \sum_{j=1}^{m} \sum_{i=1}^{n_j} \mu_{i,j}(x) e^*_{\gamma_{i,j}}, \tag{3.4}$$

where

$$\mu_{i,j}(x) := \frac{\partial \varphi_{\alpha_j}}{\partial y_i}(e^*_{\gamma_{1,j}}(x), ..., e^*_{\gamma_{n_j,j}}(x)) \left(f(x_{\alpha_j}) + T(x - x_{\alpha_j})\right)$$

for all $x \in V_z$ (and $\frac{\partial \varphi_{\alpha_j}}{\partial y_i}$ is the partial derivative of φ_{α_j} with respect to the i-th variable y_i). We may choose $n \in \mathbb{N}$ large enough so that

$$\{\gamma_{i,j} : 1 \leq i \leq n_j, 1 \leq j \leq m\} \cap \Gamma_n = \emptyset,$$

which implies that

$$E_n \subset \bigcap_{j=1}^{m} \bigcap_{i=1}^{n_j} \mathrm{Ker}(e^*_{\gamma_{i,j}}). \tag{3.5}$$

Now, given $w \in F$, since $T_n : E_n \to F$ is surjective, we may find $v \in E_n$ with $T_n(v) = 2^n w$, hence $T(v) = w$, and, according to (3.5),

$$e^*_{\gamma_{i,j}}(v) = 0$$

for all $1 \leq i \leq n_j, 1 \leq j \leq m$. Thus, by substituting in (3.4) we obtain

$$Dg(x)(v) = T(v) = w,$$

showing that $Dg(x) : E \to F$ is surjective for all $x \in V_z$. □

3.2. Proof of Theorem 1.2

Recall that the norm $|| \cdot ||_p$ in $E = \ell_p(\Gamma)$ is LUR (locally uniformly rotund) for every $p \in (1, \infty)$ and the function $\| \cdot \|^p$ is C^k smooth, where $k = \infty$ if p is an even integer, $k = p - 1$ if p is an odd integer and $k = [p]$ otherwise (for the proof of the smoothness result, see for instance [17, Chapter V. Theorem 1.1]).

According to the result of Toruńczyk [40] the mapping $u : E \to c_0(\{0\} \cup \Gamma)$ defined by the coordinate functions

$$u(x)_\gamma = \begin{cases} \|x\|^p & \text{if } \gamma = 0 \\ x_\gamma & \text{if } \gamma \in \Gamma \end{cases} \tag{3.6}$$

is a homeomorphic embedding[2] (here we are assuming $0 \notin \Gamma$). Define now $\mathcal{S}_k$ to be the family of functions $\psi : E \to \mathbb{R}$ so that $\psi \in C^k(E)$ and ψ is locally of the form

$$\psi(x) = \varphi\left(\|x\|^p, e^*_{\gamma_1}(x), ..., e^*_{\gamma_n}(x)\right), \tag{3.7}$$

where the functionals $e^*_\gamma : E \to \mathbb{R}$ are defined by $e^*_\gamma(x) = x_\gamma$ (the γ-th coordinate of $x \in E$), for certain indexes $\gamma_1, ..., \gamma_n \in \Gamma$ with $n \in \mathbb{N}$ (depending on the neighbourhood), and certain C^k smooth function $\varphi : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ (also depending on the neighbourhood). Observe that these properties imply that if $\Psi \in C^\infty(\mathbb{R}^n)$ and $\psi_1, \ldots, \psi_n \in \mathcal{S}_k$, then $\Psi(\psi_1, \ldots, \psi_n) \in \mathcal{S}_k$. Therefore by Theorem 2.8 and (3.6) we get that E admits $\mathcal{S}_k$-partitions of unity.

Thus, given f and ε as in the statement, we may find an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of E and a partition of unity $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ subordinated to $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ such that each ψ_α locally is of the form (3.7) and

$$\|f(x) - f(y)\| < \varepsilon(y)/8 \quad \text{and} \quad \varepsilon(x) < \frac{5}{3}\varepsilon(y) \quad \text{for all } x, y \in U_\alpha \text{ and for each } \alpha \in \mathcal{A}. \tag{3.8}$$

Without loss of generality we may assume that for every $\alpha \in \mathcal{A}$ the diameter of the set U_α is less than $\varepsilon(x_\alpha)/8$, where x_α is a point in U_α that we fix from now on.

Our approximating function g will be of the form $g = p \circ d$, with d a diffeomorphism close to the identity that extracts the critical set of a previous approximation p defined by

$$p(x) = \sum_{\alpha\in\mathcal{A}} \psi_\alpha(x) \left(f(x_\alpha) + T(x - x_\alpha)\right), \tag{3.9}$$

where $T : E \to F$ is a continuous linear operator that we define next. Note that $\sharp(\Gamma \times \mathbb{N}) = \sharp\Gamma$, so we may write

$$\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n,$$

where $\sharp\Gamma_n = \sharp\Gamma$ and $\Gamma_n \cap \Gamma_m = \emptyset$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Hence we have

$$E = \bigoplus_{n\in\mathbb{N}} E_n,$$

where $E_n = \ell_p(\Gamma_n)$ for all $n \in \mathbb{N}$, and in this case the direct sum is understood in the ℓ_p sense. For every $n \in \mathbb{N}$, we let $P_n : E \to E_n$ denote the canonical projection. Since $\ell_p(\Gamma_n)$ is isometric to $\ell_p(\Gamma)$ and F is a linear quotient of the latter, for every $n \in \mathbb{N}$

[2] By replacing $\|\cdot\|^2$ with $\|\cdot\|^p$, this result can be proved exactly as in Lemma 2.2 above.

there exists a continuous linear surjection $T_n : E_n \to F$; by dividing by the norm of T_n, if necessary, we may assume that $\|T_n\| = 1$. We define $T : E \to F$ by

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} T_{2n-1} \circ P_{2n-1}.$$

It is routine to check that

$$\|f(x) - p(x)\| < \varepsilon(x)/2 \quad \text{for all } x \in E. \tag{3.10}$$

Claim 3.1. *The critical set $\mathcal{C}_p := \{x \in E : Dp(x)$ is not surjective$\}$ is locally contained in a closed subspace of infinite codimension in E.*

Proof. Since the sum defining p is locally finite and the functions $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ locally are of the form (3.7), given $z \in E$ there exists an open neighbourhood V_z of z, there exist $\alpha_1, ..., \alpha_m \in \mathcal{A}$, $\gamma_{1,j}, ..., \gamma_{n_j,j} \in \Gamma$ for $j = 1, ..., m$, and C^k smooth functions $\varphi_{\alpha_j} : \mathbb{R}^{n_j+1} \mapsto \mathbb{R}$ for $j = 1, \ldots, m$, such that

$$p(x) = \sum_{j=1}^{m} \psi_{\alpha_j}(x) \left(f(x_{\alpha_j}) + T(x - x_{\alpha_j}) \right) \quad \text{for all } x \in V_z,$$

where

$$\psi_{\alpha_j}(x) = \varphi_{\alpha_j}\left(\|x\|^p, e^*_{\gamma_{1,j}}(x), ..., e^*_{\gamma_{n_j,j}}(x) \right) \quad \text{for all } x \in V_z.$$

A straightforward calculation shows that

$$Dp(x) = T + \sum_{j=1}^{m} \left[r_j(x) J(x) + \sum_{i=1}^{n_j} \mu_{i,j}(x) e^*_{\gamma_{i,j}} \right], \tag{3.11}$$

where

$$J(x) := D\| \cdot \|(x),$$

$$r_j(x) := p\|x\|^{p-1} \frac{\partial \varphi_{\alpha_j}}{\partial y_1} (\|x\|^p, e^*_{\gamma_{1,j}}(x), ..., e^*_{\gamma_{n_j,j}}(x)) \left(f(x_{\alpha_j}) + T(x - x_{\alpha_j}) \right),$$

and

$$\mu_{i,j}(x) := \frac{\partial \varphi_{\alpha_j}}{\partial y_{i+1}} (\|x\|^p, e^*_{\gamma_{1,j}}(x), ..., e^*_{\gamma_{n_j,j}}(x)) \left(f(x_{\alpha_j}) + T(x - x_{\alpha_j}) \right),$$

for all $x \in V_z$ (where $\frac{\partial \varphi_{\alpha_j}}{\partial y_i}$ is the partial derivative of φ_{α_j} with respect to the i-th variable y_i). Let us define

$$E_{(1,z)} := \overline{\text{span}}\left(\{e_{\gamma_{i,j}} : 1 \leq i \leq n_j,\, 1 \leq j \leq m\} \cup \left\{e_i \,:\, i \in \bigcup_{k\in\mathbb{N}} \Gamma_{2k-1}\right\}\right). \tag{3.12}$$

Suppose that $x \in V_z$ and $x \notin E_{(1,z)}$. Then there exists

$$\gamma_x \in \Gamma \setminus \left(\{\gamma_{i,j} : 1 \leq i \leq n_j, 1 \leq j \leq m\} \cup \bigcup_{k\in\mathbb{N}} \Gamma_{2k-1}\right) \tag{3.13}$$

such that $x_{\gamma_x} \neq 0$, implying that

$$J(x)(e_{\gamma_x}) \neq 0 \tag{3.14}$$

(checking this fact just involves a straightforward calculation of the differential of the $\ell_p(\Gamma)$ norm), and also

$$T(e_{\gamma_x}) = 0. \tag{3.15}$$

We may now choose and odd number $n \in \mathbb{N}$ large enough so that

$$\Gamma_n \cap \{\gamma_{i,j} : 1 \leq i \leq n_j, 1 \leq j \leq m\} = \emptyset,$$

which implies that

$$E_n \subset \bigcap_{j=1}^{m} \bigcap_{i=1}^{n_j} \mathrm{Ker}(e^*_{\gamma_{i,j}}). \tag{3.16}$$

Then, given any $w \in F$, since $T_n : E_n \to F$ is surjective, we may find $v \in E_n$ with $T_n(v) = 2^s w$, where $n = 2s - 1$, hence $T(v) = w$, and for

$$t_v := -\frac{J(x)(v)}{J(x)(e_{\gamma_x})}$$

we have that

$$J(x)(v + t_v e_{\gamma_x}) = 0,$$

and also, bearing in mind (3.13) and (3.16),

$$e^*_{\gamma_{i,j}}(v + t_v e_{\gamma_x}) = 0 \quad \text{for all} \quad 1 \leq i \leq n_j,\, 1 \leq j \leq m.$$

By inserting the last two equations in (3.11), and then using (3.15), we thus obtain

$$Dp(x)(v + t_v e_{\gamma_x}) = T(v + t_v e_{\gamma_x}) = T(v) = w.$$

This argument shows that

$$\mathcal{C}_p \cap V_z \subset E_{(1,z)}$$

completing the proof of our claim. □

In conclusion, following the notation of Corollary 2.10, the closed set $\mathcal{C}_p \subset E$ satisfies that for every $z \in \mathcal{C}_p$ there is an open neighbourhood V_z of z in E and there are Banach spaces $E_{(1,z)}$ (defined in (3.12)) and $E_{(2,z)}$ defined by

$$E_{(2,z)} := \overline{\operatorname{span}}\left(\left\{e_i : i \in \left(\bigcup_{k\in\mathbb{N}} \Gamma_{2k}\right)\right\} \setminus \{e_{\gamma_{i,j}} :\ 1 \le i \le n_j,\, 1 \le j \le m\}\right)$$

satisfying the conditions given in Corollary 2.10, that is, (1) $E = E_{(1,z)} \oplus E_{(2,z)}$, (2) $E_{(1,z)}$ has C^k smooth partitions of unity, (3) $E_{(2,z)}$ is infinite-dimensional and has an equivalent C^k smooth norm and (4) $\mathcal{C}_p \cap V_z \subset E_{(1,z)} \oplus \{0\}$.

Then, we may apply Corollary 2.10 for the open cover $\mathcal{G} = \{\operatorname{supp}_0 \psi_\alpha : \alpha \in \mathcal{A}\}$ of E, to find a C^k diffeomorphism $d : E \to E \setminus \mathcal{C}_p$ which is limited by $\mathcal{G}$, that is, for every $x \in E$ there is $\alpha \in \mathcal{A}$ such that $\{x, d(x)\} \subset \operatorname{supp}_0 \psi_\alpha$.

Let us define

$$g = p \circ d.$$

Recall that according to (3.10), $\|p(x) - f(x)\| < \varepsilon(x)/2$ for all $x \in E$. Since d is limited by $\mathcal{G}$ we have that, for any given $x \in E$, there exists $\alpha \in \mathcal{A}$ such that $x, d(x) \in \operatorname{supp}_0 \psi_\alpha \subset U_\alpha$, and by (3.8) we have $\|f(x) - f(d(x))\| < \varepsilon(x)/8$. By combining this inequality with (3.8) and (3.10), we obtain that

$$\begin{aligned}\|g(x) - f(x)\| &\le \|p(d(x)) - f(d(x))\| + \|f(d(x)) - f(x)\| \le \frac{\varepsilon(d(x))}{2} + \frac{\varepsilon(x)}{8} \\ &< \frac{5}{6}\varepsilon(x) + \frac{\varepsilon(x)}{8} < \varepsilon(x)\end{aligned}$$

for all $x \in E$. Besides, it is clear that g does not have any critical point: since $d(x) \notin \mathcal{C}_p$, we have that $Dp(d(x)) : E \to F$ is surjective, and $Dd(x) : E \to E$ is a linear isomorphism, so $Dg(x) = Dp(d(x)) \circ Dd(x) : E \to F$ is surjective. □

4. Results for a general target space

Here we prove the theorems for a general target space F (finite or infinite dimensional). The proofs of Theorem 1.5, Theorem 1.6 and Proposition 1.7 have a very similar structure, so we present a unified proof until the moment that we must split the proofs. The main difference between Theorems 1.5 and 1.6 is the reflexivity assumption on the space Y in Theorem 1.5, which does not appear in Theorem 1.6, where in contrast we have to make stronger decomposability assumptions on the space Y, namely assumption (5). Also, Proposition 1.7 will follow closely the proof of Theorem 1.6.

Proofs of Theorem 1.5, Theorem 1.6 and Proposition 1.7. We begin by noting that item (3) of Theorem 1.5 implies that each subspace Y_n of Y is reflexive and therefore each Y_n has a shrinking M-basis (see [27, Theorem 6.1]). It can be checked that by joining all these M-bases, and because of the decomposition of Y assumed in condition (3), we get an M-basis of Y, that we call $\{u_i, u_i^*\}_{i\in I}$ and that is contained in $\bigcup_{n\in\mathbb{N}}(Y_n \times Y_n^*)$. Since every M-basis in a reflexive Banach space is shrinking (see [27, Theorem 6.1]), we have that $\{u_i, u_i^*\}_{i\in I}$ is also a shrinking M-basis of Y. For a subset $Z \subset E$, the term $Z^\perp$ denotes the annihilator of Z in Y^*, that is $Z^\perp = \{y^* \in Y^* : y^*(z) = 0 \text{ for all } z \in Z\}$. Here and throughout the paper, we are identifying Y_n^* with $\big(\bigoplus_{j\in\mathbb{N}\setminus\{n\}} Y_j\big)^\perp$ for all $n \in \mathbb{N}$. Also, we may assume without loss of generality that the functionals $\{u_i^*\}_{i\in I}$ are normalized.

Let us first establish the existence of adequate smooth partitions of unity on the space $X \oplus Y$. This is the content of the next lemmas, following the ideas of Toruńczyk in [40].

Lemma 4.1. *Let X be a Banach space with C^k smooth partitions of unity and let Y be a Banach space with a shrinking M-basis $\{u_i, u_i^*\}_{i\in I} \subset Y \times Y^*$ and a C^k smooth LUR norm $\|\cdot\|$. Then there exists a set A and a homeomorphic embedding $u : X \oplus Y \to c_0(A \cup \mathbb{N} \cup I)$ defined by*

$$u(x, y) = (\widehat{u}(x), \varphi(y), L(y)), \qquad \text{for all } x \in X \text{ and } y \in Y, \tag{4.1}$$

where

(1) *$\widehat{u} : X \to c_0(A)$ is a homeomorphic embedding with $e_a^* \circ \widehat{u} \in C^k(X)$ for every $a \in A$. The functionals $e_\gamma^* : c_0(A \cup \mathbb{N} \cup I) \to \mathbb{R}$ are defined by $e_\gamma^*(x) = x_\gamma$ (the γ-th coordinate of x).*
(2) *$\varphi : Y \to c_0(\mathbb{N})$ is defined by $e_n^* \circ \varphi = \frac{1}{n}(\varphi_n \circ \|\cdot\|) : Y \to \mathbb{R}$, where, for every $n \in \mathbb{N}$, $\varphi_n \in C^\infty(\mathbb{R})$ is a nondecreasing function with $\varphi_n(t) = t$ for all $t \geq 1/n$, $\varphi_n(t) = 0$ for all $t \leq 1/2n$ and $|\varphi_n'(t)| \leq 3$ for all t.*
(3) *$L : Y \to c_0(I)$ is a continuous linear injective operator defined by $e_i^* \circ L = u_i^*$ for every $i \in I$.*

In particular, $e_\gamma^* \circ u \in C^k(X \oplus Y)$ for every $\gamma \in \Gamma := A \cup \mathbb{N} \cup I$. (We are assuming the sets $A, \mathbb{N}$ and I are pairwise disjoint).

Proof. We follow the ideas of Toruńczyk from [40]. We include here the details for the sake of completeness. Let us begin by checking the validity of items (1)–(3).

Firstly, the existence of $\widehat{u}$ follows from Theorem 2.8, which proves (1).

Secondly, the existence of the C^∞ smooth functions $\varphi_n : \mathbb{R} \to \mathbb{R}$ is clear. Thus we have (2). Observe in addition that $\varphi : Y \to c_0(\mathbb{N})$ is a continuous function. This follows from the equicontinuity of the family of real-valued functions $\{\varphi_n\}_{n\in\mathbb{N}}$ since they are 3-Lipschitz. Namely if for a given $y_0 \in Y$ we have a sequence $(y_m)_m \subset Y$ converging to y_0, then $\lim_{m\to\infty} \|y_m\| = \|y_0\|$ and hence

$$\|\varphi(y_m) - \varphi(y_0)\| = \sup_{n\in\mathbb{N}} \left|\frac{1}{n}\varphi_n(\|y_m\|) - \frac{1}{n}\varphi_n(\|y_0\|)\right| \leq 3\big|\,\|y_m\| - \|y_0\|\,\big| \overset{m\to\infty}{\longrightarrow} 0.$$

And lastly, to prove (3), note that the linear operator $L : Y \to c_0(I)$ is injective and continuous because of the following facts: The linear operator $L : Y \to \ell_\infty(I)$ defined by $L(y)_i = u_i^*(y)$ for all $i \in I$ ($L(y)_i$ being the i-coordinate of $L(y)$) is continuous (recall that the functionals $\{u_i^*\}_{i\in I}$ are normalized) and $\operatorname{Ker}(L) = \{0\}$ (because $\{u_i, u_i^*\}$ is M-basis in Y, so $\{u_i^*\}$ is a total set over Y). Moreover, since $\overline{\operatorname{span}}\{u_i : i \in I\} = Y$, $L(\operatorname{span}\{u_i : i \in I\}) \subset c_0(I)$ and $c_0(I)$ is a closed subspace of $\ell_\infty(I)$, we have that $L(Y) \subset c_0(I)$.

Now we will conclude the proof verifying that (1)–(3) together with the definition of u given in (4.1) yields a homeomorphic embedding with $e_\gamma^* \circ u \in C^k(X \oplus Y)$ for every $\gamma \in \Gamma$. We mainly follow [40, Page 48] but we provide details for the sake of completeness.

The continuity of u follows by the continuity of $\widehat{u}$, φ and L given by assumptions (1)–(3).

Moreover u is injective: take two points (x_1, y_1), $(x_2, y_2) \in X \oplus Y$ so that $u(x_1, y_1) = u(x_2, y_2)$; then we have $\widehat{u}(x_1) = \widehat{u}(x_2)$ and $L(y_1) = L(y_2)$. Since $\widehat{u}$ and L are injective, by (1) and (3) we get that $x_1 = x_2$ and $y_1 = y_2$.

We now prove that u has a continuous inverse $u^{-1} : u(X \oplus Y) \to X \oplus Y$. To do so, let us take a sequence $(x_n, y_n) \in X \oplus Y$ and $(x_0, y_0) \in X \oplus Y$ so that $\lim_{n\to\infty} \|u(x_n, y_n) - u(x_0, y_0)\| = 0$ and check that $\lim_{n\to\infty} \|(x_n, y_n) - (x_0, y_0)\| = 0$. First we have that

$$\begin{cases} \lim_{n\to\infty} \|\widehat{u}(x_n) - \widehat{u}(x_0)\| = 0 \\ \lim_{n\to\infty} \|\varphi(y_n) - \varphi(y_0)\| = 0 \\ \lim_{n\to\infty} \|L(y_n) - L(y_0)\| = 0 \end{cases}.$$

In particular, using that $\widehat{u}$ has a continuous inverse we get that $\lim_{n\to\infty} \|x_n - x_0\| = 0$. Moreover $\lim_{n\to\infty} \|\varphi(y_n) - \varphi(y_0)\| = 0$ implies that

$$\lim_{n\to\infty} \|y_n\| = \|y_0\| \tag{4.2}$$

and in particular $(\|y_n\|)_n$ is bounded. On the other hand for every $i \in I$ we have

$$\lim_{n\to\infty} |u_i^*(y_n) - u_i^*(y_0)| = 0, \tag{4.3}$$

so by using that $Y^* = \overline{\operatorname{span}}\{u_i^* : i \in I\}$ we can join the previous two facts (4.2) and (4.3) to conclude that $y_n \xrightarrow{n\to\infty} y_0$ weakly. Weak convergence together with convergence of norms (4.2) imply strong convergence and $\lim_{n\to\infty} y_n = y_0$ because our norm is LUR (and thus has the Kadec-Klee property).

Finally, the C^k smoothness of each $e^*_\gamma \circ u : X \oplus Y \to \mathbb{R}$ is clear too. □

Definition 4.2. Let X and Y be Banach spaces, let $\{u_i, u_i^*\}_{i\in I} \subset Y \times Y^*$ be an M-basis in Y, let $\|\cdot\|$ be a C^k smooth norm on Y and let $\mathcal{S}_k$ be the family of all functions $\psi : X \oplus Y \to \mathbb{R}$ satisfying:

(i) $\psi \in C^k(X \oplus Y)$.

(ii) ψ is locally of the form $\psi(x,y) = \varphi(x, \varphi_{k_1}(\|y\|), \dots, \varphi_{k_m}(\|y\|), u^*_{i_1}(y), \dots, u^*_{i_n}(y))$ for certain indexes $k_1, \dots, k_m \in \mathbb{N}$, $i_1, \dots, i_n \in I$ and a certain function $\varphi \in C^k(X \oplus \mathbb{R}^{m+n})$. Here $\{\varphi_j\}_j$ denotes the family of C^∞ smooth functions defined in Lemma 4.1.

Notice that conditions (i) and (ii) yield

(iii) If $\Psi \in C^\infty(\mathbb{R}^j)$ and $\psi_1, \dots, \psi_j \in \mathcal{S}_k$, then $\Psi(\psi_1, \dots, \psi_j) \in \mathcal{S}_k$.

Lemma 4.3. *Let X be a Banach space with C^k smooth partitions of unity. Let Y be a Banach space with a shrinking M-basis $\{u_i, u_i^*\}_{i\in I} \subset Y \times Y^*$ and a C^k smooth LUR norm. Then $X \oplus Y$ admits $\mathcal{S}_k$-partitions of unity.*

Proof. By Lemma 4.1 there is a homeomorphic embedding $u : X \oplus Y \to c_0(A \cup \mathbb{N} \cup I)$ for some set of indexes A. Moreover, by the definition of u given in Lemma 4.1 one can easily check that for every $\gamma \in A \cup \mathbb{N} \cup I$ we have $e^*_\gamma \circ u \in \mathcal{S}_k$. Now we simply apply Theorem 2.8 and get that $X \oplus Y$ admits $\mathcal{S}_k$-partitions of unity. □

Let us now begin the proof of Theorems 1.6 and 1.5. Let $f : X \oplus Y \to F$ and $\varepsilon : X \oplus Y \to (0,\infty)$ be continuous functions. By continuity we may find an open covering $\{U_i\}_{i\in\Delta}$ of $X \oplus Y$ of the form $U_i := \mathring{B}_X(x_i, r_i) \times \mathring{B}_Y(y_i, r_i)$ (where $\mathring{B}_X(x_i, r_i) \subset X$ is the open ball in X centered at x_i with radius $r_i > 0$ and $\mathring{B}_Y(y_i, r_i) \subset Y$ is the open ball in Y centered at y_i with radius $r_i > 0$) with $r_i < \frac{\varepsilon(x_i,y_i)}{8}$ and such that

$$\begin{cases} \|f(x,y) - f(x',y')\| < \varepsilon(x,y)/4 \\ |\varepsilon(x,y) - \varepsilon(x',y')| < \varepsilon(x,y)/8 \end{cases} \tag{4.4}$$

for all $(x,y), (x',y') \in U_i$.

By applying Lemma 4.3 there exists a partition of unity $\{\psi_i\}_{i\in\Delta} \subset \mathcal{S}_k$ subordinated to $\{U_i\}_{i\in\Delta}$. Recall also that $Y = \bigoplus_n Y_n$ and that F is a quotient of Y_n for every $n \in \mathbb{N}$. Then we choose a surjective mapping $T : Y \to F$ of norm one satisfying $T(Y_n) = F$ for n odd and $T(Y_n) = \{0\}$ for n even. For instance, if $T_n : Y_n \to F$ are surjective continuous linear operators, one may take

$$T(y) = \sum_{n=1}^{\infty} \frac{T_{2n-1}(P_{2n-1}(y))}{2^n \|P_{2n-1}\| \, \|T_{2n-1}\|}, \tag{4.5}$$

where $P_n : Y \to Y_n$ is the canonical projection onto Y_n. Now define the function

$$p(x,y) = \sum_{i \in \Delta} \psi_i(x,y)(f(x_i,y_i) + T(y-y_i)), \qquad (x,y) \in X \oplus Y. \tag{4.6}$$

Observe that by the C^k smoothness of the bump functions $\{\psi_i\}_{i\in\Delta}$ we have that $p \in C^k(X \oplus Y, F)$.

It is easy to check that $\|p(x,y) - f(x,y)\| \le \frac{\varepsilon(x,y)}{2}$ for all $(x,y) \in X \oplus Y$. That is, for every $(x,y) \in X \oplus Y$, using (4.4)

$$\begin{aligned} \|p(x,y) - f(x,y)\| &\le \sum_{i\in\Delta} \psi_i(x,y)(\|f(x_i,y_i) - f(x,y)\| + \|T(y-y_i)\|) \qquad (4.7) \\ &\le \sum_{i\in\Delta} \psi_i(x,y)(\varepsilon(x,y)/4 + \|y - y_i\|) \\ &\le \sum_{i\in\Delta} \psi_i(x,y)(\varepsilon(x,y)/4 + \varepsilon(x_i,y_i)/8) \\ &\le \sum_{i\in\Delta} \psi_i(x,y)(\varepsilon(x,y)/4 + \varepsilon(x,y)/4) \le \varepsilon(x,y)/2. \end{aligned}$$

Next, to inspect the critical set of points of p let us differentiate p with respect to y. If we prove that $(\partial p/\partial y)(x,y)$ is surjective in particular $Dp(x,y)$ will be surjective.

Observe that since $\{\psi_i\}_{i\in\Delta} \subset \mathcal{S}_k$, for each $(a_0,b_0) \in X \oplus Y$ and for each ψ_i there is a neighbourhood of (a_0,b_0) where ψ_i locally depends on $x \in X$, on $\varphi_k(\|y\|)$ for finitely many indexes k and on the 'coordinates' $u_j^*(y)$ of $y \in Y$ for finitely many indexes j. Recall that $\{\varphi_k\}_k$ is the family of C^∞ smooth functions defined in Lemma 4.1. Together with the local finiteness of $\{\text{supp}_0(\psi_i)\}_{i\in\Delta}$ we get that for every $(a_0,b_0) \in X \oplus Y$ there is an open and bounded neighbourhood $U_{(a_0,b_0)}$ and two finite set of indexes $\Delta(a_0,b_0) \subset \Delta$, $I(a_0,b_0) = \{n_1, \cdots, n_k\} \subset I$, depending on (a_0,b_0), such that for all $(x,y) \in U_{(a_0,b_0)}$

$$p(x,y) = \sum_{i \in \Delta(a_0,b_0)} \psi_i(x,y)(f(x_i,y_i) + T(y-y_i))$$

where

$$\begin{aligned} \frac{\partial p}{\partial y}(x,y) &= \sum_{i\in\Delta(a_0,b_0)} \psi_i(x,y)T + \sum_{i\in\Delta(a_0,b_0)} (f(x_i,y_i) + T(y-y_i))\frac{\partial \psi_i}{\partial y}(x,y) \\ &= T + \sum_{i\in\Delta(a_0,b_0)} (f(x_i,y_i) - T(y_i))\frac{\partial \psi_i}{\partial y}(x,y) \\ &= T + z(x,y)\cdot \|\cdot\|'(y) + \sum_{j=1}^{k} z_j(x,y)u_{n_j}^*. \qquad (4.8) \end{aligned}$$

Here $\{z(x,y), z_1(x,y), \dots, z_k(x,y)\} \subset F$ are just some vectors where $z(x,0) = 0$ whenever $(x,0) \in U_{(a_0,b_0)}$ (this comes from the fact that the functions $\varphi_n(t)$ are zero locally around $t = 0$ and therefore for every $\psi \in \mathcal{S}_k$ we have $\frac{\partial \psi}{\partial y}(x,0) \in \operatorname{span}\{u_i^* : i \in I\}$). Also note that in the second equality we are using that $\sum_{i\in\Delta} \psi_i(x,y) = 1$ and $\sum_{i\in\Delta} \frac{\partial \psi_i}{\partial y}(x,y) = 0$.

⋆ Assume first that $(x,y) \in U_{(a_0,b_0)}$ with $z(x,y) = 0$. In such case (4.8) becomes

$$\frac{\partial p}{\partial y}(x,y) = T + \sum_{j=1}^{k} z_j(x,y) u_{n_j}^*.$$

Since the shrinking M-basis $\{u_i, u_i^*\}_{i\in I}$ of Y is contained in $\cup_{n\in\mathbb{N}}(Y_n \times Y_n^*)$, by defining for every n $I_n = \{i \in I : u_i \in Y_n\}$ and by letting $m \in \mathbb{N}$ be a natural number so that $\{n_1, \dots, n_k\} \subset I_1 \cup \dots \cup I_m$, we have

$$\frac{\partial p}{\partial y}(x,y)|_{Y_{2m+1}} = T|_{Y_{2m+1}},$$

which implies that the operator $(\partial p/\partial y)(x,y)$ is surjective. We fix such $m \in \mathbb{N}$ from now on, that only depends on (a_0, b_0).

⋆ Assume now that $(x,y) \in U_{(a_0,b_0)}$ with $z(x,y) \neq 0$, and let us study the surjectivity of $(\partial p/\partial y)(x,y) : Y \to F$.

Now, Definition 1.4, together with assumption (5) in Theorem 1.5 and Theorem 1.6, allows us to define the closed and complemented subspaces of Y

$$\begin{cases} Z_0 = (\oplus_{j=1}^m Y_j) \oplus (\bigoplus\limits_{\substack{j \text{ odd} \\ j>m}} Y_j) \\ Z_1 = \bigoplus\limits_{\substack{j \text{ even} \\ j>m}} Y_j \end{cases},$$

and we have $Y = Z_0 \oplus Z_1$. This is where the proof forks. As we already observed, in Theorem 1.5 we have Y reflexive, while in Theorem 1.6 we cannot rely on reflexivity of Y but we can make use of assumption (5). We first present the proof of Theorem 1.6 because assumption (5) makes the argument easier. Thereafter we prove Theorem 1.5 using the reflexivity of Y.

Since $Y = Z_0 \oplus Z_1$ we can identify $Y^* = Z_0^* \oplus Z_1^* = Z_1^\perp \oplus Z_0^\perp$ (i.e. $Z_0^* = Z_1^\perp$ and $Z_1^* = Z_0^\perp$, where $Z_i^\perp$ denotes the annihilator of Z_i in Y^*, that is $Z_i^\perp = \{y^* \in Y^* : y^*(z) = 0 \text{ for all } z \in Z_i\}$).

Proof of Theorem 1.6 (Y is not necessarily reflexive). By assumption (5) of Theorem 1.6 we know that the projection $Q_m : Y \to Z_0$ has norm one.

Fact 4.4. *Let us consider the C^1 smooth LUR norm $\|\cdot\|$ on Y. We have that $y \in Z_0$ if and only if $\|\cdot\|'(y) \in Z_0^*$.*

Proof. Note that it is enough to prove the fact for $y \in S_{\|\cdot\|}$ (the unit sphere of Y). Let us then take $y \in S_{\|\cdot\|}$ satisfying $\|\cdot\|'(y) \in Z_0^*$. Since $y = z_0 + z_1$ with $z_0 = Q_m(y) \in Z_0$ and $z_1 \in Z_1$, $\|z_0\| = \|Q_m(y)\| \leq \|y\| = 1$ and $1 = \|\cdot\|'(y)(y) = \|\cdot\|'(y)(z_0)$ we get that $\|z_0\| = 1$. We have $\|y\| = \|z_0\|$. If $y \neq z_0$ then by the strict convexity of the norm and because Q_m has norm one,

$$\|z_0\| = \left\|Q_m\left(z_0 + \frac{z_1}{2}\right)\right\| \leq \left\|z_0 + \frac{z_1}{2}\right\| = \frac{1}{2}\|2z_0 + z_1\| = \frac{1}{2}\|y + z_0\| < 1 = \|z_0\|,$$

which is a contraction. Therefore $y = z_0$ and $y \in Z_0$.

For the other implication let us take $y \in Z_0 \cap S_{\|\cdot\|}$ and consider $\|\cdot\|'(y) = z^* = z_0^* + z_1^*$, where $z_i^* \in Z_i^*$, $i = 0, 1$. In particular, $1 = \|\cdot\|'(y)(y) = z_0^*(y) \leq \|z_0^*\|^*\|y\| = \|z_0^*\|^*$. Here $\|\cdot\|^*$ denotes the dual norm on Y^* and $\|z_0^*\|^*$ is the dual norm of z_0^* as an element of Y^*. On the other hand, for every $y' = z_0' + z_1' \in S_{\|\cdot\|}$ with $z_i' \in Z_i$, $i = 1, 2$, it holds $z_0^*(y') = z_0^*(z_0') = z_0^*(Q_m(y')) = (z_0^* + z_1^*)(Q_m(y')) = \|\cdot\|'(y)(Q_m(y')) \leq \|Q_m\|\|y'\| = 1$ so $\|z_0^*\|^* = 1$. Since $1 = z_0^*(y) = z^*(y) = \|z_0^*\|^* = \|z^*\|^* = \|y\|$ and the norm $\|\cdot\|$ is Gâteaux smooth, we get that $z^* = z_0^* \in Z_0^*$. □

Let us show that

$$\{(x,y) \in U_{(a_0,b_0)} : \frac{\partial p}{\partial y}(x,y) \text{ is not surjective } \} \subset U_{(a_0,b_0)} \cap \big(X \oplus Z_0\big). \tag{4.9}$$

Indeed, if $y \notin Z_0$, by Fact 4.4 $\|\cdot\|'(y) \notin Z_0^* = Z_1^\perp$, so there is an even integer $t > m$ and an index $i_0 \in I_t$ such that $\|\cdot\|'(y)(u_{i_0}) \neq 0$. We fix from now on this $i_0 \in I_t$ that depends only on y. In particular, for every $w \in Y_{2m+1}$ there is $\lambda_w \in \mathbb{R}$ such that $\|\cdot\|'(y)(w - \lambda_w u_{i_0}) = 0$. This way, from (4.8) we get

$$\frac{\partial p}{\partial y}(x,y)(w - \lambda_w u_{i_0}) = T(w - \lambda_w u_{i_0}) = T(w),$$

because $T(u_{i_0}) = 0$. It is then clear that $(\partial p/\partial y)(x,y)$ is surjective. The inclusion (4.9) is now proved.

As a conclusion of (4.9), the closed sets of critical points $\mathcal{C}_p$ and $\mathcal{CP}_p$,

$$\begin{aligned}\mathcal{C}_p := \{(x,y) \in X \oplus Y : p'(x,y) \text{ is not surjective}\}\\ \subset \mathcal{CP}_p := \{(x,y) \in X \oplus Y : \frac{\partial p}{\partial y}(x,y) \text{ is not surjective}\}\end{aligned}$$

satisfy that

$$U_{(a_0,b_0)} \cap \mathcal{C}_p \subset U_{(a_0,b_0)} \cap \mathcal{CP}_p \subset X \oplus \Big((\oplus_{j=1}^m Y_j) \oplus (\bigoplus_{\substack{j \text{ odd} \\ j>m}} Y_j)\Big) = X \oplus Z_0 \oplus \{0\}.$$

Observe that $X \oplus Z_0$ and Z_1 are closed subspaces of $X \oplus Y$, $X \oplus Z_0$ has C^1 smooth partitions of unity and Z_1 is infinite-dimensional and has a C^1 smooth norm. Notice that, by the construction, the spaces $X \oplus Z_0$ and Z_1 depend on the point (a_0, b_0). Then by applying Corollary 2.10, there exists a C^1 diffeomorphism

$$d : X \oplus Y \to \big(X \oplus Y\big) \setminus \mathcal{CP}_p$$

limited by the open cover $\{\operatorname{supp}_0 \psi_i\}_{i\in\Delta}$ where $\operatorname{supp}_0 \psi_i = \{(x, y) \in X \oplus Y : \psi_i(x, y) > 0\}$. Define

$$g = p \circ d.$$

We now verify that g has no critical points. Observe that for all $(x, y) \in X \oplus Y$ we have that $d'(x, y)$ is an isomorphism and since $d(x, y) \notin \mathcal{CP}_p$ it follows that $(\partial p/\partial y)(d(x, y))$ is a surjective operator from Y onto F, which makes $p'(d(x, y))$ a surjective operator from $X \oplus Y$ onto F as well. Therefore for every $(x, y) \in X \oplus Y$

$$(p \circ d)'(x, y) = p'(d(x, y)) \circ d'(x, y)$$

is a surjective operator.

Finally let us verify that $\|f(x, y) - g(x, y)\| \le \varepsilon(x, y)$ for all $(x, y) \in X \oplus Y$. Indeed, for every $(x, y) \in X \oplus Y$, using (4.4), (4.7) and that d is limited by the open cover $\{\operatorname{supp}_0 \psi_i\}_{i\in\Delta}$,

$$\begin{aligned} \|f(x, y) - g(x, y)\| &\le \|f(x, y) - f(d(x, y))\| + \|f(d(x, y)) - p(d(x, y))\| \\ &\le \varepsilon(x, y)/4 + \varepsilon(d(x, y))/2 \\ &\le \varepsilon(x, y). \end{aligned}$$

The proof of Theorem 1.6 is now complete.

Now let us briefly mention how to modify the proof of Theorem 1.6 to obtain the proof of Proposition 1.7.

Proof of Proposition 1.7. Note that taking $c_0(\Gamma) = Y$ in the above proof and using a norm $\|\cdot\|$ in $c_0(\Gamma)$ that is C^∞ smooth and locally depends on finitely many coordinates, we define the mapping $u : X \oplus c_0(\Gamma) \to c_0(A \cup \Gamma)$ by $u(x, y) = (\widehat{u}(x), y)$ for every $(x, y) \in X \oplus c_0(\Gamma)$, being $\widehat{u} : X \to c_0(A)$ a homeomorphic embedding of X into $c_0(A)$ with C^k smooth coordinate functions. Then u is a homeomorphic embedding with C^k smooth coordinate functions satisfying that each coordinate function locally depends on $x \in X$ and a finite number of 'coordinates' of y, so by Lemma 2.7 the space $X \oplus c_0(\Gamma)$ admits C^k smooth partitions of unity satisfying the same local property. Thus following the above proof, we define p in such a way that for any $(a_0, b_0) \in X \oplus c_0(\Gamma)$ there is a bounded neighbourhood $U_{(a_0,b_0)}$ of (a_0,b_0) and there is a set of indexes $\{n_1,\cdots,n_k\}\subset I$, depending on (a_0,b_0), such that for all $(x,y)\in U_{(a_0,b_0)}$ the expression (4.8) of the partial derivative of p is

$$\frac{\partial p}{\partial y}(x,y)=T+\sum_{j=1}^{k} z_j(x,y)e^*_{n_j}, \tag{4.10}$$

for some vectors $z_j(x,y)\in F$. In this case, the use of deleting diffeomorphisms in the last step of the above proof is not required. Indeed, by the definition of T from (4.5), which required a decomposition of $c_0(\Gamma)=\bigoplus_{n\in\mathbb{N}} Y_n$ with $\{e_\gamma,e^*_\gamma\}_{\gamma\in\Gamma}\subset\bigcup_{n\in\mathbb{N}}(Y_n\times Y_n^*)$, and calling $I_n=\{\gamma\in\Gamma:\ e_\gamma\in Y_n\}$ we have that taking $m\in\mathbb{N}$ so that $\{n_1,\dots,n_k\}\subset I_1\cup\cdots\cup I_m$ then

$$\frac{\partial p}{\partial y}(x,y)|_{Y_{2m+1}}=T|_{Y_{2m+1}}.$$

This yields that $(\partial p/\partial y)(x,y)$ is surjective. Notice that, as in the proof of Theorem 1.1, the closed subspaces Y_n can be considered to be $Y_n=c_0(\Gamma_n)$, where $\Gamma=\cup_n\Gamma_n$ with $|\Gamma_n|=|\Gamma|$ and $\Gamma_n\cap\Gamma_m=\emptyset$ for $m\neq n$. Therefore, using (4.7) and letting $g=p$ the proof of Proposition 1.7 is finished with no use of deleting diffeomorphisms.

Proof of Theorem 1.5 (Y is reflexive). As in the non reflexive case, let us consider the closed sets $\mathcal{C}_p$ and $\mathcal{CP}_p$,

$$\begin{aligned}\mathcal{C}_p:=\{(x,y)\in X\oplus Y:\ p'(x,y)\text{ is not surjective }\}\\ \subset\mathcal{CP}_p:=\{(x,y)\in X\oplus Y:\ \frac{\partial p}{\partial y}(x,y)\text{ is not surjective}\}.\end{aligned}$$

Let us prove first that

$$\begin{aligned}U_{(a_0,b_0)}\cap\mathcal{CP}_p&\subset U_{(a_0,b_0)}\cap\left\{(x,y)\in X\oplus Y:\ \|\cdot\|'(y)\in Z_0^*\right\}\\&=U_{(a_0,b_0)}\cap\left\{(x,y)\in X\oplus Y:\ \|\cdot\|'(y)\in Z_0^*\cap S_{\|\cdot\|^*}\right\}\\&=U_{(a_0,b_0)}\cap\left\{(x,y)\in X\oplus Y:\ Z_1\subset\operatorname{Ker}(\|\cdot\|'(y))\right\},\end{aligned}\tag{4.11}$$

where here $\|\cdot\|$ denotes the C^k smooth LUR norm on Y. The last two equalities are clear. For the first inclusion we have that whenever $\|\cdot\|'(y)\notin Z_0^*=Z_1^{\perp}$, there is an even integer $t>m$ and an index $i_0\in I_t$ with $\|\cdot\|'(y)(u_{i_0})\neq 0$. In particular for every $w\in Y_{2m+1}$ there is $\lambda_w\in\mathbb{R}$ such that $\|\cdot\|'(y)(w-\lambda_w u_{i_0})=0$, so from (4.8) we get

$$\frac{\partial p}{\partial y}(x,y)(w-\lambda_w u_{i_0})=T(w-\lambda_w u_{i_0})=T(w),$$

because $T(u_{i_0}) = 0$. Thus $(\partial p/\partial y)(x, y)$ is a surjective operator from Y onto F.

For the following lemma it is a key point the reflexivity of Y.

Lemma 4.5. *Let us consider the C^k smooth LUR norm $\|\cdot\|$ on the reflexive Banach space Y. The set $\{y \in Y : \|\cdot\|'(y) \in Z_0^*\}$ is contained in the graph of a continuous function $\eta : Z_0 \to Z_1$.*

Proof. The following argument is almost identical to that of [5, Claim 3.7]; we reproduce it for completeness and for the readers' convenience.

Definition of η: Pick a point $w \in Z_0$. Note that the function $Z_1 \ni v \mapsto \xi_w(v) := \|w + v\|^2$ is convex and continuous, and satisfies $\lim_{\|v\| \to \infty} \xi_w(v) = \infty$, hence, since Z_1 is reflexive, ξ_w attains a minimum at some point $v_w \in Z_1$; in fact this minimum point v_w is unique because the norm $\|\cdot\|$ is strictly convex. Let us denote

$$\eta(w) := v_w$$

and let us prove that

$$\{y \in Y : Z_1 \subset \mathrm{Ker}(\|\cdot\|'(y))\} \subset G(\eta) = \{(w, \eta(w)) : w \in Z_1\}.$$

Take a point $w+v \in \{y \in Y : Z_1 \subset \mathrm{Ker}(\|\cdot\|'(y))\}$, with $w \in Z_1$ and $v \in Z_0$. In particular $w + v \neq 0$ and

$$\|\cdot\|'(w + v)(e) = 0 \text{ for every } e \in Z_1.$$

But this means that v is a critical point for the function $\xi_w : Z_1 \to \mathbb{R}$, therefore $v = \eta(w)$ by definition of η.

Continuity of η: Now let us see that the function $\eta : Z_0 \to Z_1$ is continuous. Suppose η is discontinuous at w_0 and let $v_0 := \eta(w_0)$. Then there exist a sequence $\{w_i\}_i$ with $\lim_i w_i = w_0$ in Z_0, a sequence $v_i := \eta(w_i)$ in Z_1 and a number $\varepsilon_0 > 0$ so that

$$\|v_i - v_0\| \geq \varepsilon_0 \text{ for all } i \in \mathbb{N}. \tag{4.12}$$

From the previous argument we know that the points $v_i, v_0 \in Z_1$ are characterized as being the unique points in Z_1 for which we have

$$\|w_i + v_i\| \leq \|w_i + v_i + e\| \text{ for all } e \in Z_1, \tag{4.13}$$

$$\|w_0 + v_0\| \leq \|w_0 + v_0 + e\| \text{ for all } e \in Z_1. \tag{4.14}$$

By taking $e = -v_i$ in (4.13) we learn that $\|v_i\| - \|w_i\| \leq \|w_i + v_i\| \leq \|w_i\|$, hence $\|v_i\| \leq 2\|w_i\|$, and because $\{\|w_i\|\}_i$ converges to $\|w_0\|$ we deduce that $\{v_i\}_i$ is bounded. Since Z_1 is reflexive, this implies that $\{v_i\}_i$ has a subsequence that weakly converges to a point $v_0' \in Z_1$. We keep denoting this subsequence by $\{v_i\}_i$.

Now, if we take $e = -v_i + e'$ in (4.13), with $e' \in Z_1$, we obtain

$$\|w_i + v_i\| \leq \|w_i + e'\| \text{ for all } e' \in Z_1.$$

This implies (using the facts that $\{v_i\}_i$ weakly converges to v_0', $\{w_i\}_i$ converges in norm to w_0, and the weak lower semicontinuity of the norm) that

$$\begin{aligned} \|w_0 + v_0'\| &\leq \liminf_{i\to\infty} \|w_i + v_i\| \leq \limsup_{i\to\infty} \|w_i + v_i\| \\ &\leq \limsup_{i\to\infty} \|w_i + e'\| = \|w_0 + e'\| \text{ for all } e' \in Z_1. \end{aligned} \tag{4.15}$$

That is, we have shown that

$$\|w_0 + v_0'\| \leq \|w_0 + e'\| \text{ for all } e' \in Z_1. \tag{4.16}$$

By taking $e' = v_0' + \xi$ with $\xi \in Z_1$ we conclude that

$$\|w_0 + v_0'\| \leq \|w_0 + v_0' + \xi\| \text{ for all } \xi \in Z_1.$$

According to (4.14), v_0 is the only point which can satisfy this inequality. Hence $v_0' = v_0$.

But (4.15) tells us even more: by taking $e' = v_0'$ we also learn that $\lim_i \|w_i + v_i\| = \|w_0 + v_0'\|$. Since we also know that $\{w_i + v_i\}_i$ weakly converges to $w_0 + v_0'$ and the norm $\|\cdot\|$ is LUR (hence $\|\cdot\|$ has the Kadec-Klee property), this implies that $\{w_i + v_i\}_i$ converges to $w_0 + v_0' = w_0 + v_0$ in the norm topology as well. As we also have $\lim_{i\to\infty} w_i = w_0$ in the norm topology, we deduce that $\lim_{i\to\infty} \|v_i - v_0\| = 0$, which contradicts (4.12) and complete the proof of Lemma 4.5. □

As a consequence of (4.11) and Lemma 4.5 we have that

$$U_{(a_0,b_0)} \cap \mathcal{CP}_p \subset X \times \{y = (z_0, z_1) \in Z_0 \oplus Z_1 : \, z_1 = \eta(z_0)\}$$

for some continuous function η (depending on (a_0, b_0)). In particular, the function $\widetilde{\eta} : X \oplus Z_0 \to Z_1$ defined by $\widetilde{\eta}(x, z_0) = \eta(z_0)$ is continuous and we have

$$U_{(a_0,b_0)} \cap \mathcal{C}_p \subset U_{(a_0,b_0)} \cap \mathcal{CP}_p \subset \{(x, z_0, z_1) \in X \oplus Z_0 \oplus Z_1 : \, z_1 = \widetilde{\eta}(x, z_0)\}.$$

Observe that $X \oplus Z_0$ and Z_1 are closed subspaces of $X \oplus Y$, $X \oplus Z_0$ has C^1 smooth partitions of unity and Z_1 is infinite-dimensional and has a C^1 smooth norm. Notice that, by the construction, the spaces $X \oplus Z_0$ and Z_1 depend on the point (a_0, b_0).

The last step is to compose p with a deleting diffeomorphism d defined as in the non reflexive case (Theorem 1.6) and check that we still keep the uniform control on the approximation. By using Theorem 2.9 we define a C^k diffeomorphism $d : X \oplus Y \to$

$(X \oplus Y) \setminus \mathcal{CP}_p$ limited by the open cover $\{\mathrm{supp}_0\, \psi_i\}_{i\in\Delta}$, where $\mathrm{supp}_0\, \psi_i = \{(x,y) \in X \oplus Y : \psi_i(x,y) > 0\}$. Define

$$g = p \circ d.$$

We now verify that g has no critical points. Observe that for all $(x,y) \in X \oplus Y$ we have that $d'(x,y)$ is an isomorphism and since $d(x,y) \notin \mathcal{CP}_p$ it follows that $(\partial p/\partial y)(d(x,y))$ is a surjective operator from Y onto F, which makes $p'(d(x,y))$ a surjective operator from $X \oplus Y$ onto F as well. Therefore for every $(x,y) \in X \oplus Y$

$$(p \circ d)'(x,y) = p'(d(x,y)) \circ d'(x,y)$$

is a surjective operator from $X \oplus Y$ onto F.

Finally let us check that $\|f(x,y) - g(x,y)\| \le \varepsilon(x,y)$ for all $(x,y) \in X \oplus Y$. Indeed, for every $(x,y) \in X \oplus Y$, using (4.4), (4.7) and the fact that d is limited by the open cover $\{\mathrm{supp}_0\, \psi_i\}_{i\in\Delta}$,

$$\begin{aligned}\|f(x,y) - g(x,y)\| &\le \|f(x,y) - f(d(x,y))\| + \|f(d(x,y)) - p(d(x,y))\| \\ &\le \varepsilon(x,y)/4 + \varepsilon(d(x,y))/2 \\ &\le \varepsilon(x,y).\end{aligned}$$

The proof of Theorem 1.5 is now complete. □

5. Results for a finite dimensional target space

As mentioned in the Introduction, before proving the results on approximations by C^k smooth functions with no critical points in the case of a finite dimensional target space F in Section 5.2, we will show the renorming results stated in Proposition 1.12 and Corollary 1.14 in Section 5.1.

5.1. Residuality of certain norms in Banach spaces

Let us denote by $(\mathcal{N}_Y, \rho)$ the metric space of all norms on the Banach space Y which are equivalent to the given norm $\|\cdot\|$ on Y, endowed with the metric defined for $p, q \in (\mathcal{N}_Y, \rho)$ by

$$\rho(p,q) = \sup\{|p(x) - q(x)| : x \in B_{\|\cdot\|}\}. \tag{5.1}$$

The set $\mathcal{N}_Y$ is an open subset in the space $(\mathcal{Q}_Y, \rho)$ of all continuous seminorms on Y with the metric defined for $p, q \in \mathcal{Q}_Y$ by the same expression (5.1). It is well known that $(\mathcal{Q}_Y, \rho)$ is a complete metric space and thus $(\mathcal{N}_Y, \rho)$ is a Baire space (i.e. the intersection of countably many open dense subsets of $\mathcal{N}_Y$ is dense in $\mathcal{N}_Y$). Analogously, $(\mathcal{N}^*_{Y^*}, \rho^*)$

denotes the metric space of all dual norms on Y^* which are equivalent to $\|\cdot\|^*$ (the dual norm of $\|\cdot\|$) with the metric ρ^* for $p^*, q^* \in \mathcal{N}^*_{Y^*}$ defined by

$$\rho^*(p^*, q^*) = \sup\{|p^*(x^*) - q^*(x^*)| : x^* \in B_{\|\cdot\|^*}\}. \tag{5.2}$$

Also, it is well known that the mapping

$$\Phi : (\mathcal{N}_Y, \rho) \to (\mathcal{N}^*_{Y^*}, \rho^*), \quad \Phi(p) = p^*, \tag{5.3}$$

where p^* is the dual norm from p is an homeomorphism of $(\mathcal{N}_Y, \rho)$ onto $(\mathcal{N}^*_{Y^*}, \rho^*)$. So $(\mathcal{N}^*_{Y^*}, \rho^*)$ is a Baire space as well. See [20] and [17] for more details about the above facts.

Recall that a subset A of a metric space M is called G_δ if it is a countable intersection of open sets, and it is called K_σ if it is a countable union of compact sets. Additionally, A is called residual if A contains a G_δ dense subset of M. A subset C of a Banach space Y is called a cone if $\lambda x \in C$ for all $\lambda \geq 0$ whenever $x \in C$. For a set V in a Banach space Y, cone(V) denotes the cone generated by V, that is cone$(V) = \{\lambda x : x \in V, \lambda \geq 0\}$. For a norm $p \in \mathcal{N}_Y$, we denote by NA_p the set of elements $x^* \in Y^*$ such that x^* attains its p^*-norm, that is, there is $x \in S_p = \{x \in Y : p(x) = 1\}$ satisfying $p^*(x^*) = x^*(x)$.

Proof of Proposition 1.12. Step 1. Firstly, let us prove that for any compact set $W \subset S_{\|\cdot\|^*}$, the set of norms $p \in \mathcal{N}_Y$ such that its dual norm p^* is Fréchet differentiable at the points of W is residual in $(\mathcal{N}_Y, \rho)$. Let us consider, for every $n \in \mathbb{N}$, the sets $\mathcal{F}_n \subset \mathcal{N}^*_{Y^*}$ of dual norms $p^* \in \mathcal{N}^*_{Y^*}$ such that there is $\delta > 0$ satisfying

$$\sup\left\{\frac{p^*(x^* + h^*) + p^*(x^* - h^*) - 2p^*(x^*)}{\|h^*\|^*} : h^* \in Y^*,\ \|h^*\|^* \leq \delta;\ x^* \in W\right\} < \frac{1}{n}.$$

Because of the convexity of p^*, if $0 < |t| \leq 1$

$$\frac{p^*(x^* + th^*) + p^*(x^* - th^*) - 2p^*(x^*)}{\|th^*\|^*} \leq \frac{p^*(x^* + h^*) + p^*(x^* - h^*) - 2p^*(x^*)}{\|h^*\|^*},$$

so $\mathcal{F}_n$ is the set of dual norms $p^* \in \mathcal{N}^*_{Y^*}$ such that there is $\delta \in (0, 1)$ satisfying

$$\sup\left\{\frac{p^*(x^* + h^*) + p^*(x^* - h^*) - 2p^*(x^*)}{\|h^*\|^*} : h^* \in Y^*,\ \|h^*\|^* = \delta;\ x^* \in W\right\} < \frac{1}{n}.$$

Clearly if $p^* \in \cap_n \mathcal{F}_n$, then p^* is Fréchet differentiable at each point $x^* \in W$ and by homogeneity of the norm at each point of cone$(W) \setminus \{0\}$. It remains to check that $\mathcal{F}_n$ is open and dense in (P^*, ρ^*).

(i) $\mathcal{F}_n$ is open. Let us consider $p^* \in \mathcal{F}_n$, $\delta \in (0, 1)$ and $m \in \mathbb{N}$ satisfying that

$$\sup\left\{\frac{p^*(x^* + h^*) + p^*(x^* - h^*) - 2p^*(x^*)}{\|h^*\|^*} : h^* \in Y^*,\ \|h^*\|^* = \delta;\ x^* \in W\right\}$$

$$< \frac{1}{n} - \frac{1}{m}.$$

Now, if $q^* \in \mathcal{N}^*_{Y^*}$ satisfies $\rho^*(q^*, p^*) \leq \frac{\delta}{12m}$ then for every $x^* \in W$ and $\|h^*\|^* = \delta$ we have that $\|x^* \pm h^*\|^* \leq 2$ and, by (5.2), $|(q^* - p^*)(x^* \pm h^*)| \leq 2\rho^*(q^*, p^*)$ and $|(q^* - p^*)(x^*)| \leq \rho^*(q^*, p^*)$ so

$$\begin{aligned}
&\frac{q^*(x^* + h^*) + q^*(x^* - h^*) - 2q^*(x^*)}{\|h^*\|^*} \leq \\
&\leq \frac{p^*(x^* + h^*) + p^*(x^* - h^*) - 2p^*(x^*)}{\|h^*\|^*} \\
&+ \frac{(q^* - p^*)(x^* + h^*) + (q^* - p^*)(x^* - h^*) - 2(q^* - p^*)(x^*)}{\|h^*\|^*} \leq \\
&\leq \frac{1}{n} - \frac{1}{m} + \frac{2\rho^*(q^*, p^*) + 2\rho^*(q^*, p^*) + 2\rho^*(q^*, p^*)}{\delta} \\
&< \frac{1}{n} - \frac{1}{m} + \frac{6\delta}{12m\delta} = \frac{1}{n} - \frac{1}{2m} < \frac{1}{n}.
\end{aligned}$$

We have then proved that $B(p^*, \frac{\delta}{12m})$, the closed ball of center p^* and radius $\frac{\delta}{12m}$ in $(\mathcal{N}^*_{Y^*}, \rho^*)$, is contained in $\mathcal{F}_n$.

(ii) $\mathcal{F}_n$ is dense in $(\mathcal{N}^*_{Y^*}, \rho^*)$. Indeed, we will directly prove that $\cap_n \mathcal{F}_n$ is dense. The proof follows some ideas of [22] and [14]. Let us consider $p^* \in \mathcal{N}^*_{Y^*}$ and $\varepsilon \in (0, 1)$.

Denote by B_{p^*} and S_{p^*} the closed unit ball and unit sphere respectively of Y^* for the norm p^*. Let us keep using standard notation: denote by p the predual norm of p^* on Y, and let B_p and S_p stand for the closed unit ball and unit sphere respectively of Y for the norm p. Let us denote $V := \operatorname{cone}(W)$. For every $x^* \in S_{p^*} \cap V$, we select $x \in S_p$, which we shall denote by $\psi(x^*)$, such that $x^*(\psi(x^*)) > 1 - \frac{\varepsilon}{2}$. Let us consider the slices of B_{p^*},

$$S\big(B_{p^*}, \psi(x^*), \frac{\varepsilon}{2}\big) := \big\{y^* \in B_{p^*} : y^*(\psi(x^*)) > 1 - \frac{\varepsilon}{2}\big\}, \quad \text{for all } x^* \in S_{p^*} \cap V.$$

Now, the union of the slices

$$\bigcup_{x^* \in S_{p^*} \cap V} S\big(B_{p^*}, \psi(x^*), \frac{\varepsilon}{2}\big)$$

is a relatively open set in B_{p^*} and a covering of the compact set $S_{p^*} \cap V$ (notice that $S_{p^*} \cap V$ is homeomorphic to W because p^* and $\|\cdot\|^*$ are equivalent), so there is a finite family $\{S(B_{p^*}, \psi(x^*_k), \frac{\varepsilon}{2})\}_{k=1}^m$ such that

$$S_{p^*} \cap V \subset \bigcup_{k=1}^{m} S\big(B_{p^*}, \psi(x^*_k), \frac{\varepsilon}{2}\big).$$

We may assume without loss of generality that the set $\{\psi(x_k^*)\}_{k=1}^m$ is symmetric (otherwise we consider the union of slices $(\cup_{k=1}^m S(B_{p^*}, \psi(x_k^*), \frac{\varepsilon}{2})) \cup (\cup_{k=1}^m S(B_{p^*}, -\psi(x_k^*), \frac{\varepsilon}{2})))$. Thus, the set

$$C = B_{p^*} \setminus \Big(\bigcup_{k=1}^m S\big(B_{p^*}, \psi(x_k^*), \frac{\varepsilon}{2}\big)\Big) = B_{p^*} \bigcap \Big(\bigcap_{k=1}^m \{x^* \in Y^* : x^*(\psi(x_k^*)) \leq 1 - \frac{\varepsilon}{2}\}\Big)$$

is bounded, weak* closed, symmetric and

$$(1 - \frac{\varepsilon}{2}) B_{p^*} \subset C \subset B_{p^*}.$$

So C is the unit ball of a dual equivalent norm on Y^* (see [17, Fact 5.4]).

Now, let us consider

$$\begin{aligned} \alpha(x^*) &= \max\{x^*(\psi(x_k^*)) : k = 1, \dots, m\} = \max\{|x^*(\psi(x_k^*))| : k = 1, \dots, m\} = \\ &= \|(x^*(\psi(x_1^*)), \dots, x^*(\psi(x_m^*)))\|_\infty, \qquad \text{for all } x^* \in Y^*, \end{aligned} \tag{5.4}$$

where $\|\cdot\|_\infty$ is the infinity norm in $\mathbb{R}^m$ (the second identity in (5.4) is due to the fact that the set $\{\psi(x_k^*)\}_{k=1}^m$ is symmetric). Since $\|\cdot\|_\infty$ may be approximated uniformly on bounded sets of $\mathbb{R}^m$ by the C^∞ norms $\|\cdot\|_s$ with s even integer for $s \to \infty$, being $\|w\|_s = (\sum_{i=1}^m |w_i|^s)^{1/s}$ for $w = (w_1, \cdots, w_m) \in \mathbb{R}^m$, we may select an even integer s large enough so that

$$0 \leq \|w\|_s - \|w\|_\infty \leq \frac{\varepsilon}{2}, \qquad \text{whenever } w \in \mathbb{R}^m \text{ and } \|w\|_\infty \leq 2. \tag{5.5}$$

Let us define

$$\beta(x^*) := \|(x^*(\psi(x_1^*)), \dots, x^*(\psi(x_m^*)))\|_s, \qquad \text{for } x^* \in Y^*$$

and

$$B = B_{p^*} \cap \big\{x^* \in Y^* : \beta(x^*) \leq 1 - \frac{\varepsilon}{2}\big\}. \tag{5.6}$$

The set B is bounded, weak* closed and symmetric. Moreover,

$$(1 - \varepsilon) B_{p^*} \subset B \subset B_{p^*}. \tag{5.7}$$

Indeed, if $x^* \in (1-\varepsilon) B_{p^*}$ then $|x^*(\psi(x_k^*))| \leq (1-\varepsilon)$ for all $k = 1, \dots, m$, so by (5.5), we have $\beta(x^*) = \alpha(x^*) + (\beta(x^*) - \alpha(x^*)) \leq (1-\varepsilon) + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2}$. Therefore, B is the unit ball of a dual equivalent norm, which we shall denote by q^*, defined on Y^* (see [17, Fact 5.4]). In fact, from (5.6) we have

$$q^*(x^*) = \max\big\{p^*(x^*), (1 - \frac{\varepsilon}{2})^{-1} \beta(x^*)\big\}, \qquad \text{for } x^* \in Y^*.$$

Our candidate for approximating p^* in $(\mathcal{N}^*_{Y^*}, \rho^*)$ is the norm q^*. We now prove that $q^* \in \cap_n \mathcal{F}_n$. The norm q^* is Fréchet differentiable at every point $x^* \in S_{p^*} \cap V$ (and thus, by homogeneity of the norm, at every point of $V \setminus \{0\}$). This is a consequence of the fact that for every $x^* \in S_{p^*} \cap V$ there is an open neighbourhood U_{x^*} of x^* where the norm q^* satisfies $q^*(y^*) = (1 - \frac{\varepsilon}{2})^{-1}\beta(y^*) > p^*(y^*)$ for all $y^* \in U_{x^*}$. Let us check this assertion: if $x^* \in S_{p^*} \cap V$ there is $k \in \{1, \dots, m\}$ such that $x^* \in S(B_{p^*}, \psi(x^*_k), \frac{\varepsilon}{2})$ and thus $x^*(\psi(x^*_k)) > 1 - \frac{\varepsilon}{2}$. Therefore, $\beta(x^*) \geq \alpha(x^*) > 1 - \frac{\varepsilon}{2}$ and $(1 - \frac{\varepsilon}{2})^{-1}\beta(x^*) > 1 = p^*(x^*)$. By continuity there is an open neighbourhood U_{x^*} of x^* in Y^* satisfying $(1 - \frac{\varepsilon}{2})^{-1}\beta(y^*) > p^*(y^*) > 0$ for all $y^* \in U_{x^*}$. So $q^*(y^*) = (1 - \frac{\varepsilon}{2})^{-1}\beta(y^*) > 0$ for all $y^* \in U_{x^*}$. Since β is Fréchet differentiable at the points $y^* \in Y^*$ such that $\beta(y^*) > 0$, we get that q^* is Fréchet differentiable at every point of U_{x^*}.

Since norms are positively homogeneous, the fact that q^* is Fréchet differentiable on $S_{p^*} \cap V$ implies that it is so in $V \setminus \{0\}$ and therefore on the compact set W. Now, by standard arguments it can be checked that the Fréchet differentiability of q^* on the compact set W yields

$$\lim_{\delta \to 0} \sup \left\{ \frac{q^*(x^* + h^*) + q^*(x^* - h^*) - 2q^*(x^*)}{\|h^*\|^*} : h^* \in Y^*, \|h^*\|^* = \delta;\ x^* \in W \right\} = 0, \tag{5.8}$$

which is equivalent to the uniform Fréchet differentiability of q^* on W. Now, (5.8) implies that $q^* \in \cap_n \mathcal{F}_n$. Moreover, because of (5.7) we have that

$$p^*(x^*) \leq q^*(x^*) \leq \frac{1}{1 - \varepsilon} p^*(x^*), \qquad \text{for all } x^* \in Y^*.$$

Thus

$$0 \leq q^*(x^*) - p^*(x^*) \leq \frac{\varepsilon}{1 - \varepsilon} p^*(x^*) \leq \frac{\varepsilon}{1 - \varepsilon} K_p, \qquad \text{for all } x^* \in B_{\|\cdot\|^*},$$

being $K_p = \sup\{p^*(x^*) : x^* \in B_{\|\cdot\|^*}\}$. Since ε varies over $(0, 1)$ we get that p^* can be approximated in $(\mathcal{N}^*_{Y^*}, \rho^*)$ by norms in $\cap_n \mathcal{F}_n$. Since $p^* \in \mathcal{N}^*_{Y^*}$ is arbitrary we get the denseness of $\cap_n \mathcal{F}_n$ in $(\mathcal{N}^*_{Y^*}, \rho^*)$. If we combine (i) and (ii) we obtain the residuality in $(\mathcal{N}^*_{Y^*}, \rho^*)$ of the set of dual norms which are Fréchet differentiable at the points of W (and, by the homogeneity of the norm, at the points of $V \setminus \{0\}$). Now, because of the homeomorphism (5.3) we get that the set of norms in $\mathcal{N}_Y$ whose dual norms are Fréchet differentiable at the points of $V \setminus \{0\}$ is residual in $(\mathcal{N}_Y, \rho)$.

Step 2. Now, if W is a K_σ subset of $S_{\|\cdot\|^*}$, then $W = \cup_n W_n$ where each W_n is a compact set of $S_{\|\cdot\|^*}$. We apply Step 1 to every compact subset W_n to get the residuality in $(\mathcal{N}_Y, \rho)$ of the set of norms whose dual norms are Fréchet differentiable at the points of $\text{cone}(W_n) \setminus \{0\}$. Hence, because $(\mathcal{N}_Y, \rho)$ is a Baire space, we get the residuality in $(\mathcal{N}_Y, \rho)$ of the set of norms whose dual norms are Fréchet differentiable at the points of $\text{cone}(W) \setminus \{0\}$.

Finally, let us check the assertion related to the norm attaining set NA_p. It is well known that if a dual norm p^* is Fréchet differentiable at a point $x^* \in Y^* \setminus \{0\}$, then in particular, x^* attains its p^*-norm. Let us prove it here for the sake of completeness. We may assume that $p^*(x^*) = 1$ and let us take $x^{**} \in Y^{**}$ such that $(p^*)'(x^*) = x^{**}$ and in particular $x^{**}(x^*) = 1 = p^{**}(x^{**})$ (by a standard notation, p^{**} is the dual norm of p^*). Also, let us take a sequence $\{x_n\}_n \subset Y$ with $p(x_n) = 1$ for all n, being p the predual norm of p^*, and $\lim_n x^*(x_n) = 1$. Since p^* is Fréchet differentiable at x^*, by the Smulyan Lemma, $\lim_n p^{**}(x_n - x^{**}) = 0$. Since p^{**} and $\|\cdot\|^{**}$ are equivalent, $\lim_n \|x_n - x^{**}\|^{**} = 0$ and $x^{**} \in Y$, so x^* attains its p^*-norm at $x^{**} \in Y$. $\square$

The next corollary is a consequence of Proposition 1.12 and the well-known theorem of Fabian, Zajicek and Zizler [20] (see also [17, Page 53]) stating that the set of LUR norms (LUR dual norms) in a Banach space Y (Y^*) is empty or residual in $(\mathcal{N}_Y, \rho)$ $((\mathcal{N}^*_{Y^*}, \rho^*)$, respectively).

Corollary 1.14. *Let Y be a Banach space with a LUR norm $\|\cdot\|$ whose dual norm $\|\cdot\|^*$ is LUR. Let $W \subset S_{\|\cdot\|^*}$ be a K_σ subset. Then the set of norms $p \in \mathcal{N}_Y$ such that both p and its dual norm p^* are LUR and p^* is Fréchet differentiable at the points of* $\mathrm{cone}(W) \setminus \{0\}$ *is residual in $(\mathcal{N}_Y, \rho)$. In particular, the set of norms $p \in \mathcal{N}_Y$ such that both p and its dual norm p^* are LUR and* $\mathrm{cone}(W) \subset \mathrm{NA}_p$ *is residual in $(\mathcal{N}_Y, \rho)$.*

In Section 5.2, we will need the following property of the duality mapping $\|\cdot\|' : S_{\|\cdot\|} \to S_{\|\cdot\|^*}$, whenever we work with a Banach space Y with C^1 norm $\|\cdot\|$. Note that $\|\cdot\|^*$ being LUR implies the C^1 smoothness of $\|\cdot\|$ ([18, Page 344]).

Claim 5.1. *Let Y be a Banach space with a LUR norm $\|\cdot\|$ whose dual norm $\|\cdot\|^*$ is LUR. Then the duality mapping $\|\cdot\|' : S_{\|\cdot\|} \to S_{\|\cdot\|^*} \cap \mathrm{NA}_{\|\cdot\|}$ is a homeomorphism, where* $\mathrm{NA}_{\|\cdot\|}$ *is the set of norm attaining functionals of Y^*.*

Proof. Firstly, the function $\|\cdot\|' : S_{\|\cdot\|} \to S_{\|\cdot\|^*}$ is one-to-one because of the rotundity of the norm $\|\cdot\|$ and $\|\cdot\|'$ is continuous because of the C^1 smoothness of the norm $\|\cdot\|$.

If Y is reflexive then $\mathrm{NA}_{\|\cdot\|} = Y^*$ and $\|\cdot\|'$ is surjective. The continuity of the inverse function is a consequence of the C^1 smoothness property of the dual norm $\|\cdot\|^*$.

If Y is not reflexive, then $\mathrm{NA}_{\|\cdot\|} \neq Y^*$. So $\|\cdot\|' : S_{\|\cdot\|} \to S_{\|\cdot\|^*}$ is not surjective. Thus, we should consider the bijection $\|\cdot\|' : S_{\|\cdot\|} \to S_{\|\cdot\|^*} \cap \mathrm{NA}_{\|\cdot\|}$. In this case the dual norm $\|\cdot\|^*$ is Fréchet differentiable at every point of the set $\mathrm{NA}_{\|\cdot\|} \setminus \{0\}$. Indeed, recall that a dual norm $\|\cdot\|^*$ is Fréchet differentiable at a point $x^* \in S_{\|\cdot\|^*}$ if and only if x^* strongly exposes $B_{\|\cdot\|}$ (at some point $x \in S_{\|\cdot\|}$) [18, Corollary 7.21]. Now, if x^* attains its norm (at some point $x \in S_{\|\cdot\|}$), it is straightforward to verify that the LUR condition of the norm $\|\cdot\|$ (at the point x) yields $\lim_{\varepsilon \to 0} \mathrm{diam}(S(B_{\|\cdot\|}, x^*, \varepsilon)) = 0$, that is x^* strongly exposes $B_{\|\cdot\|}$ (at x). So $\|\cdot\|^*$ is Fréchet differentiable at x^* (with $(\|\cdot\|^*)'(x^*) = x$) and thus the subdifferential $\partial \|\cdot\|^* : S_{\|\cdot\|^*} \to 2^{S_{\|\cdot\|^{**}}}$ is $\|\cdot\|^* - \|\cdot\|^{**}$ upper semicontinuous at

x^* and, in particular, the restriction $\partial\|\cdot\|^*|_{S_{\|\cdot\|^*}\cap \mathrm{NA}_{\|\cdot\|^*}} : S_{\|\cdot\|^*}\cap \mathrm{NA}_{\|\cdot\|^*} \to S_{\|\cdot\|}$, which is univaluated, is $\|\cdot\|^* - \|\cdot\|$ continuous at x^* (see [35, Corollary 2.6 and Proposition 2.8]). Notice that this restriction is the inverse of $\|\cdot\|' : S_{\|\cdot\|} \to S_{\|\cdot\|^*}\cap \mathrm{NA}_{\|\cdot\|^*}$, so the claim is proved. □

Remark 5.2. Notice that the conclusion of Claim 5.1 also holds under the weaker conditions that the norm $\|\cdot\|$ on Y is LUR and C^1 smooth since the proof given above is also valid for this case.

5.2. Approximation for the case of a finite dimensional target space

Firstly, we note that our proof below will show that, for the case $k = 1$, we can take the space Y to just be Asplund and WCG because then Y has automatically a C^1 smooth and LUR norm [17].

First of all, let us see how Theorem 1.8 yields Corollary 1.9 and Corollary 1.10.

Proof of Corollary 1.9. Since $E = X \oplus Y$ for some closed subspace X and the property of having C^1-partitions of unity is hereditary, X and Y have C^1-partitions of unity as well. In particular Y is a separable infinite dimensional Asplund space (see [17, Page 58]). Thus Y has a C^1 smooth and LUR norm and Theorem 1.8 applies to $X \oplus Y$. □

Proof of Corollary 1.10. The assumptions given in the statement of Corollary 1.10 for $k = 1$ state that Y is a WCG Banach space. This yields that Y has the 1-Separable Complementation Property (1-SCP, for short), that is every closed separable subspace of Y is contained in a closed separable and 1-complemented subspace of Y (see [27, Theorem 3.42 and Proposition 3.43, pages. 105-106]; here we are using that a WCG Banach space is WLD, that is, Weakly Lindelöf Determined). In particular, there is a separable infinite dimensional closed and complemented subspace Z_1 of Y. Thus Y can be decomposed into a direct sum of the form $Y = Z_0 \oplus Z_1$ for a suitable closed subspace Z_0 of Y. Now, since Y has a C^1 smooth norm, it is Asplund. Recall that being Asplund WCG is an hereditary property (see [17, Corollary VI.4.4], where we are using that a WCG Banach space is WCD, that is, Weakly Countably Determined, which is an hereditary property too). In particular, Z_0 is Asplund and WCG as well. Since Z_0 is WCG and has a C^1 smooth norm, by [17, Theorem VIII.3.2] Z_0 has C^1 smooth partitions of unity. Therefore, the Banach space $\widetilde{X} = X \oplus Z_0$ has C^1 smooth partitions of unity. So we can apply Theorem 1.8 to the direct sum $\widetilde{X} \oplus \widetilde{Y}$, where $\widetilde{Y} = Z_1$ is a separable space with a C^1 smooth and LUR norm.

Now, the assumptions given in the statement of Corollary 1.10 for $k > 1$ imply that Y is a suppereflexive space as it was mentioned in the introduction (see [19] or [17, Chapter V, Proposition 1.3]). So in particular Y is an Asplund WCG Banach space. Following the reasoning of case $k = 1$ we can decompose Y into $Y = Z_0 \oplus Z_1$, where Z_0 is a superreflexive Banach space and has a C^k smooth norm (thus Z_0 has C^k smooth partitions of unity) and Z_1 is a separable and superreflexive Banach space with a C^k smooth and LUR norm. Again, we can apply Theorem 1.8 to the direct sum $\widetilde{X} \oplus \widetilde{Y}$, where $\widetilde{X} = X \oplus Z_0$ has C^k smooth partition of unity and $\widetilde{Y} = Z_1$ is a separable Banach space with a C^k smooth and LUR norm. $\square$

Proof of Theorem 1.8. Since we are assuming that Y is a separable Banach space with a C^k smooth norm, in particular Y is a separable Asplund space so Y admits a shrinking M-basis $\{u_i, u_i^*\}_{i\in\mathbb{N}} \subset Y \times Y^*$ (see, for example, [27, Theorem 6.3]). We may assume without loss of generality that $\|u_i^*\| = 1$ for all $i \in \mathbb{N}$. In addition, let us choose any sequence of normalized vectors $\{v_i^* : i \in \mathbb{N}\} \subset Y^*$ such that $\{v_i^*,\, u_i^* : i \in \mathbb{N}\}$ are linearly independent.

Next, we will renorm Y for the case $k = 1$ and Y non reflexive: since Y is a separable Asplund space, Y has an equivalent LUR norm whose dual norm is LUR (see, for example, [17]). Thus, we may assume, by applying Corollary 1.14 and renorming Y, that the initial norm $\|\cdot\|$ on Y is LUR, its dual norm $\|\cdot\|^*$ is LUR (thus $\|\cdot\|$ is C^1 smooth) and the norm $\|\cdot\|^*$ is Fréchet differentiable at the non zero points of the space

$$V = \operatorname{span}(\{v_i^* : i \in \mathbb{N}\} \cup \{u_i^* : i \in \mathbb{N}\}),$$

and thus $V \subset NA_{\|\cdot\|}$ (where $NA_{\|\cdot\|}$ is the set of norm attaining functionals of Y^* for the norm $\|\cdot\|$). Notice that we apply Corollary 1.14 to $W = V \cap S_{\|\cdot\|^*}$ because W can be written as the union $W = \cup_{n\in\mathbb{N}} W_n$, where $W_n = \operatorname{span}(\{v_i^*,\, u_i^* : i = 1,\dots,n\}) \cap S_{\|\cdot\|^*}$ is a compact set for every n, thus $W \subset S_{\|\cdot\|^*}$ is a K_σ subset and $\operatorname{cone}(W) = V$. For the remaining cases it is not necessary to renorm Y.

Now, we can apply Lemma 4.1 and Lemma 4.3 to obtain $\mathcal{S}_k$-partitions of unity on $X \oplus Y$, where the family $\mathcal{S}_k$ is given in Definition 4.2, that is, $\mathcal{S}_k$ is the family of functions $\psi : X \oplus Y \to \mathbb{R}$ satisfying:

(i) $\psi \in C^k(X \oplus Y, \mathbb{R})$.
(ii) ψ is locally of the form $\psi(x, y) = \varphi(x, \varphi_{k_1}(\|y\|), \dots, \varphi_{k_m}(\|y\|), u_{i_1}^*(y), \dots, u_{i_n}^*(y))$ for certain indexes $i_1, \dots, i_n \in I$, $k_1, \dots, k_m \in \mathbb{N}$ and certain C^k smooth function $\varphi \in C^k(X \oplus \mathbb{R}^{m+n})$. Here $\{\varphi_n\}_n$ denotes the family of C^∞ functions defined in Lemma 4.1.
(iii) If $\Psi \in C^\infty(\mathbb{R}^n)$ and $\psi_1, \dots, \psi_n \in \mathcal{S}_k$, then $\Psi(\psi_1, \dots, \psi_n) \in \mathcal{S}_k$.

In order to clarify the ideas, let us split the proof into two steps: the real-valued case and the general finite dimensional case.

Step 1. The real-valued case. Let us consider a continuous function $f : X \oplus Y \to \mathbb{R}$ and a continuous function $\varepsilon : X \oplus Y \to (0, \infty)$. Let us consider in $X \oplus Y$ an open covering of the form $\{U_i := \mathring{B}_X(x_i, r_i) \times \mathring{B}_Y(y_i, r_i)\}_{i\in\Delta}$ (with $\mathring{B}_X(x_i, r_i) \subset X$ the open ball in X centered at x_i and radius $r_i > 0$ and $\mathring{B}_Y(y_i, r_i) \subset Y$ the open ball in Y centered at y_i and radius $r_i > 0$ for all $i \in \Delta$) such that

$$|f(x,y)-f(x',y')| < \frac{\varepsilon(x,y)}{4} \quad \text{for all } (x,y),(x'y') \in U_i \text{ and } i \in \Delta, \tag{5.9}$$

$$|\varepsilon(x,y)-\varepsilon(x',y')| < \frac{\varepsilon(x,y)}{8} \quad \text{for all } (x,y),(x'y') \in U_i \text{ and } i \in \Delta,$$

$$\text{and} \quad r_i < \frac{\varepsilon(x_i,y_i)}{8} \quad \text{for every } i \in \Delta.$$

Then, there exists a partition of unity $\{\psi_i\}_{i\in\Delta} \subset \mathcal{S}_k$ satisfying $\operatorname{supp}_0 \psi_i \subset U_i$ for all $i \in \Delta$. Let us define the function

$$p(x,y) = \sum_{i\in\Delta} (f(x_i,y_i) + v_1^*(y-y_i))\,\psi_i(x,y), \quad \text{for all } (x,y) \in X \oplus Y. \tag{5.10}$$

A simple calculation shows that

$$\begin{aligned} |p(x,y)-f(x,y)| &\le \sum_{i\in\Delta} (|f(x_i,y_i)-f(x,y)| + |v_1^*(y-y_i)|)\,\psi_i(x,y) \qquad (5.11)\\ &< \sum_{i\in\Delta} (\varepsilon(x,y)/4 + \|y-y_i\|)\,\psi_i(x,y) \\ &< \sum_{i\in\Delta} (\varepsilon(x,y)/4 + \varepsilon(x_i,y_i)/8)\,\psi_i(x,y) \\ &< \sum_{i\in\Delta} (\varepsilon(x,y)/4 + \varepsilon(x,y)/4)\,\psi_i(x,y) \le \varepsilon(x,y)/2. \end{aligned}$$

Since the sum over all $i \in \Delta$ in (5.10) is locally finite and each $\psi_i(x,y)$ locally depends on x, on $\varphi_k(\|y\|)$ for finitely many indexes k and on the 'coordinates' $u_j^*(y)$ of $y \in Y$ for finitely many indexes j, for every $(a_0,b_0) \in X \oplus Y$ there is an open and bounded neighbourhood $U_{(a_0,b_0)}$ and two finite number of indexes $\Delta(a_0,b_0) \subset \Delta$, $I(a_0,b_0) \subset \mathbb{N}$, depending on (a_0,b_0), such that

$$p(x,y) = \sum_{i\in\Delta(a_0,b_0)} (f(x_i,y_i) + v_1^*(y-y_i))\,\psi_i(x,y), \quad \text{for } (x,y) \in U_{(a_0,b_0)}$$

and

$$\begin{aligned} \frac{\partial p}{\partial y}(x,y) &= \sum_{i\in\Delta(a_0,b_0)} \psi_i(x,y)\,v_1^* + \sum_{i\in\Delta(a_0,b_0)} (f(x_i,y_i)+v_1^*(y-y_i))\,\frac{\partial\psi_i}{\partial y}(x,y) \\ &= v_1^* + \sum_{i\in\Delta(a_0,b_0)} (f(x_i,y_i)+v_1^*(y-y_i))\,\frac{\partial\psi_i}{\partial y}(x,y), \quad \text{for } (x,y) \in U_{(a_0,b_0)}. \end{aligned}$$

Let us call for every $(x,y) \in U_{(a_0,b_0)}$,

$$k(x,y) = \sum_{i\in\Delta(a_0,b_0)} (f(x_i,y_i)+v_1^*(y-y_i))\,\frac{\partial\psi_i}{\partial y}(x,y) =$$

$$= z(x,y)\,\|\cdot\|'(y) + \sum_{j\in I(a_0,b_0)} w_j(x,y)\,u_j^* \;\in\; \mathrm{span}(\{\|\cdot\|'(y),\, u_j^* : j\in I(a_0,b_0)\}),$$

where $z(x,y)$, $w_j(x,y) \in \mathbb{R}$ for all $(x,y) \in U_{(a_0,b_0)}$ and all $j \in I(a_0,b_0)$ and we may assume that $z(x,0) = 0$ for $(x,0) \in U_{(a_0,b_0)}$. The latter fact is explained because the functions $\varphi_n(t)$ are zero locally around $t = 0$. In particular, for every $\psi \in \mathcal{S}_k$ we have $\frac{\partial\psi}{\partial y}(x,0) \in \mathrm{span}\{u_i^* : i \in \mathbb{N}\}$. Now we consider the following cases:

⋆ If $(x,y) \in U_{(a_0,b_0)}$ and $z(x,y) = 0$, then

$$\frac{\partial p}{\partial y}(x,y) = v_1^* + k(x,y) = v_1^* + \sum_{j\in I(a_0,b_0)} w_j(x,y)\,u_j^* \neq 0$$

because $\{v_1^*,\, u_j^* : j \in \mathbb{N}\}$ are linearly independent.

⋆ If $(x,y) \in U_{(a_0,b_0)}$, $z(x,y) \neq 0$ and $\dfrac{\partial p}{\partial y}(x,y) = 0$, then

$$\|\cdot\|'(y) = -\frac{1}{z(x,y)} v_1^* - \sum_{j\in I(a_0,b_0)} \frac{w_j(x,y)}{z(x,y)}\, u_j^*$$

so

$$\|\cdot\|'(y) \in \mathrm{span}(\{v_1^*,\, u_j^* : j \in I(a_0,b_0)\}) \cap S_{\|\cdot\|^*} := K.$$

Notice that K is compact and $K \subset S_{\|\cdot\|^*} \cap NA_{\|\cdot\|}$ because of the properties of the norm considered in Y. Recall that, by Claim 5.1 and Remark 5.2, $\|\cdot\|' : S_{\|\cdot\|} \to S_{\|\cdot\|^*} \cap NA_{\|\cdot\|}$ is a homeomorphism. So the set $(\|\cdot\|')^{-1}(K)$ is compact, where we denote by $(\|\cdot\|')^{-1}$ the inverse function of $\|\cdot\|' : S_{\|\cdot\|} \to S_{\|\cdot\|^*} \cap NA_{\|\cdot\|}$. Also,

$$y = \|y\| \left((\|\cdot\|')^{-1}\left(-\frac{1}{z(x,y)} v_1^* - \sum_{j\in I(a_0,b_0)} \frac{w_j(x,y)}{z(x,y)} u_j^* \right) \right) \in \|y\| \cdot (\|\cdot\|')^{-1}(K). \tag{5.12}$$

Call $A := (\|\cdot\|')^{-1}(K)$ and

$$A' := \bigcup_{\lambda \geq 0} \lambda A.$$

Next we will show that $A' \subset Y$ is the graph of a continuous function $\eta : Y_0 \to Y_1$ where Y_0, Y_1 are closed subspaces of Y, Y_0 has finite dimension, Y_1 has infinite dimension and $Y_0 \oplus Y_1 = Y$. Therefore by (5.12) we would get that

$$\{y \in Y : (x,y) \in U_{(a_0,b_0)} \text{ and } \frac{\partial p}{\partial y}(x,y) = 0\}$$

$$\subset \left\{ y \in Y : (x,y) \in U_{(a_0,b_0)} \text{ and } \frac{y}{\|y\|} \in A \right\}$$
$$\subset \{ y \in Y : (x,y) \in U_{(a_0,b_0)} \text{ and } y \in A' \}$$

is contained in the graph of η.

Define $Y_1 = \text{span}(\{v_1^*, u_j^* : j \in I(a_0,b_0)\})_\perp$, that is the annihilator of $\text{span}(\{v_1^*, u_j^* : j \in I(a_0,b_0)\})$ in Y. Since the dimension of $\text{span}(\{v_1^*, u_j^* : j \in I(a_0,b_0)\})$ is finite, we have that the codimension of Y_1 is finite so there is a subspace $Y_0 \subset Y$ of finite dimension such that $Y = Y_0 \oplus Y_1$. Then, we can identify $Y^* = Y_0^* \oplus Y_1^* = Y_1^\perp \oplus Y_0^\perp$ with $Y_1^* = Y_0^\perp$ and

$$Y_0^* = Y_1^\perp = ((\text{span}(\{v_1^*, u_j^* : j \in I(a_0,b_0)\}))_\perp)^\perp = \text{span}(\{v_1^*, u_j^* : j \in I(a_0,b_0)\}).$$

Define the canonical projections

$$\pi_i : Y = Y_0 \oplus Y_1 \to Y_i \text{ as } \pi(y = y_0 + y_1) = y_i \text{ where } y \in Y,\ y_i \in Y_i \text{ for } i = 0,1.$$

Let us consider the restriction

$$(\|\cdot\|')^{-1}|_K : K = S_{\|\cdot\|^*} \cap Y_1^\perp \to A \subset S_{\|\cdot\|}$$

and the composition

$$\pi_0 \circ (\|\cdot\|')^{-1}|_K : K \to Y_0.$$

Lemma 5.3. *Under the previous assumptions:*

- **(i)** $\pi_0 \circ (\|\cdot\|')^{-1}|_K$ *is one-to-one;*
- **(ii)** *thus* $\pi_0|_A : A \to Y_0$ *is one-to-one;*
- **(iii)** $\pi_0(A) \subset Y_0$ *is closed;*
- **(iv)** $(\pi_0|_A)^{-1} : \pi_0(A) \to A$ *is continuous;*
- **(v)** *thus* $\pi_1 \circ (\pi_0|_A)^{-1} : \pi_0(A) \subset Y_0 \to Y_1$ *is continuous;*
- **(vi)** A *is the graph of a continuous function, that is*

$$A = \{(y_0, \pi_1 \circ (\pi_0|_A)^{-1}(y_0)) : y_0 \in \pi_0(A)\}.$$

Proof. **(i)** Let us take $y^*, z^* \in K$ and $y = (\|\cdot\|')^{-1}(y^*) \in A$, $z = (\|\cdot\|')^{-1}(z^*) \in A$, such that $\pi_0 \circ (\|\cdot\|')^{-1}(y^*) = \pi_0 \circ (\|\cdot\|')^{-1}(z^*)$ and thus $\pi_0(y) = \pi_0(z)$. Now, if $y = y_0 + y_1$ and $z = z_0 + z_1$ with $y_0, z_0 \in Y_0$ and $y_1, z_1 \in Y_1$, we have that $\pi_0(y) = y_0 = z_0 = \pi_0(z)$. Since $y^* = \|\cdot\|'(y)$, $z^* = \|\cdot\|'(z)$ and $z^* \in Y_1^\perp$, we have

$$1 = y^*(y) \text{ and } 1 = z^*(z) = z^*(z_0) = z^*(y_0) = z^*(y_0 + y_1) = z^*(y).$$

Since the norm $\|\cdot\|$ is Gâteaux differentiable at y, we get that $y^* = z^*$ and thus $\pi_0 \circ (\|\cdot\|')^{-1}|_K$ is one-to-one.

(ii) Assertion (i) yields $\pi_0|_A : A \to Y_0$ is one-to-one.

(iii) Since $\pi_0 : Y \to Y_0$ is continuous and A is compact we have that $\pi_0(A)$ is compact (thus closed) in Y_0.

(iv) Since $\pi_0|_A : A \to \pi_0(A)$ is a continuous bijection between the compact (Hausdorff) sets A and $\pi_0(A)$, we have that $\pi_0|_A$ is a homeomorphism between A and $\pi_0(A)$, so $(\pi_0|_A)^{-1} : \pi_0(A) \to A$ is continuous.

(v) Now, it is enough to compose $(\pi_0|_A)^{-1}$ with the (continuous) canonical projection π_1 from Y onto Y_1 to get the continuity of $\pi_1 \circ (\pi_0|_A)^{-1} : \pi_0(A) \subset Y_0 \to Y_1$.

(vi) Finally, we can write A as the graph of the continuous function $\pi_1 \circ (\pi_0|_A)^{-1}$ defined in the closed subset $\pi_0(A)$ of Y_0 with values in Y_1:

$$A = \{(y_0, \pi_1 \circ (\pi_0|_A)^{-1}(y_0)) : y_0 \in \pi_0(A)\}. \quad \square$$

In fact, the previous function $\pi_1 \circ (\pi_0|_A)^{-1} : \pi_0(A) \to Y_1$ can be extended continuously to Y_0 by defining

$$\eta : Y_0 \to Y_1, \quad \eta(\lambda y_0) := \lambda \pi_1 \circ (\pi_0|_A)^{-1}(y_0) \quad \text{for every } \lambda \geq 0 \text{ and } y_0 \in \pi_0(A).$$

⋆ *The function η is well defined:*

- First, suppose that $\lambda y_0 = \lambda' y_0' \in Y_0$ with $\lambda, \lambda' \geq 0$ and $y_0, y_0' \in \pi_0(A)$. Notice that $0 \notin \pi_0(A)$ because for every $y_0 \in \pi_0(A)$ there is $y^* \in K$ such that $y^*(y_0) = 1$. Thus, if $\lambda = 0$ then $\lambda' = 0$ and $\eta(\lambda y_0) = \eta(\lambda' y_0') = 0$. If $\lambda \neq 0$, then $y_0 = \frac{\lambda'}{\lambda} y_0'$. Since there are $y^* \in K \subset Y_1^{\perp}$ and $y' = y_0' + y_1' \in A$ (with $y_1' \in Y_1$) satisfying

$$1 = y^*(y_0) = \frac{\lambda'}{\lambda} y^*(y_0') = \frac{\lambda'}{\lambda} y^*(y') \leq \frac{\lambda'}{\lambda}$$

  we get that $\lambda \leq \lambda'$. Replacing y_0 by y_0' in the preceding argument we get $\lambda' \leq \lambda$ and thus $y_0 = y_0'$ so $\eta(\lambda y_0) = \eta(\lambda' y_0')$.
- *The function η is defined in the entire space* Y_0. In order to prove this, let us check that for every $z_0 \in Y_0 \setminus \{0\}$ there is $\lambda_0 > 0$ such that $\frac{z_0}{\lambda_0} \in \pi_0(A)$. Indeed, since the set K is symmetric and compact, $\sup\{z^*(z_0) : z^* \in K\}$ is attained at some point $y^* \in K$ and satisfies $\sup\{z^*(z_0) : z^* \in K\} = y^*(z_0) > 0$ so $\sup\{z^*(\frac{z_0}{y^*(z_0)}) : z^* \in K\} = y^*(\frac{z_0}{y^*(z_0)}) = 1$. On the one hand, there is $y = y_0 + y_1 \in A$ (with $y_i \in Y_i$, $i = 0, 1$) such that $(\|\cdot\|')^{-1}(y^*) = y$. Thus, the points y_0 and $\frac{z_0}{y^*(z_0)}$ considered as functionals (i.e. as elements of Y^{**}) and restricted to $Y_1^{\perp}$ verify that $y_0|_{Y_1^{\perp}}$ and $\frac{z_0}{y^*(z_0)}|_{Y_1^{\perp}}$ are supporting functionals of $B_{\|\cdot\|^*} \cap Y_1^{\perp}$ at the point y^* with

$$z^*(y_0) \leq 1 = y^*(y_0) \quad \text{and } z^*(\frac{z_0}{y^*(z_0)}) \leq 1 = y^*(\frac{z_0}{y^*(z_0)}) \quad \text{for all } z^* \in K.$$

Since the norm $\|\cdot\|^*|_{Y_1^\perp}$ is Gâteaux differentiable at y^* we have necessarily $y_0|_{Y_1^\perp} = \frac{z_0}{y^*(z_0)}|_{Y_1^\perp}$. Since y_0 and $\frac{z_0}{y^*(z_0)}$ are points in Y_0 we have that $y_0|_{Y_0^\perp} = \frac{z_0}{y^*(z_0)}|_{Y_0^\perp} = 0$ and thus $y_0 = \frac{z_0}{y^*(z_0)}$ because $Y^* = Y_0^\perp \oplus Y_1^\perp$ and $z^*(y_0) = z^*(\frac{z_0}{y^*(z_0)})$ for all $z^* \in Y^*$. So for $\lambda_0 = y^*(z_0) > 0$, we have $\frac{z_0}{\lambda_0} = y_0 \in \pi_0(A)$.

⋆ *The function η is continuous:* Assume that $\lambda_n y_{0,n} \xrightarrow{n} \lambda y_0$ with λ, $\lambda_n \geq 0$ for all n and y_0, $y_{0,n} \in \pi_0(A)$. Notice that for every $z_0 \in \pi_0(A)$ there is $z^* \in K$ such that $z^*(z_0) = 1$ so $\|z_0\| \geq 1$. Also, $\pi_0(A)$ and $\pi_1(A)$ are bounded sets because A is bounded and π_0 and π_1 are linear and continuous operators. Now, if $\lambda = 0$ then $\{\lambda_n y_{0,n}\}_n \xrightarrow{n} 0$ and $|\lambda_n| \leq \|\lambda_n y_{0,n}\| \xrightarrow{n} 0$ so $\eta(\lambda_n y_{0,n}) = \lambda_n \eta(y_{0,n}) \xrightarrow{n} 0 = \lambda\eta(y_0)$ (because $\{\eta(y_{0,n})\}_n$ is bounded) and η is continuous at 0. Next, if $\lambda > 0$, we may assume without loss of generality that $\lambda_n > 0$ for all n. Since $\frac{\lambda_n}{\lambda} y_{0,n} \xrightarrow{n} y_0$, in order to prove the continuity of η at λy_0 it is enough to prove that $\frac{\lambda_n}{\lambda} \xrightarrow{n} 1$ and thus $y_{0,n} \xrightarrow{n} y_0$. This together with the continuity of $\pi_1 \circ (\pi_0|_A)^{-1}$ at $y_0 \in \pi_0(A)$ and the definition of η will provide $\eta(\lambda_n y_{0,n}) = \lambda_n \eta(y_{0,n}) \xrightarrow{n} \lambda\eta(y_0) = \eta(\lambda y_0)$ and the continuity of η at λy_0. Let us denote $\frac{\lambda_n}{\lambda} = t_n$ for all n. Let us take y, $y_n^* \in K$ such that $y^*(y_0) = 1$, $y_n^*(y_{0,n}) = 1$ for all n. Also, let us take y_1, $y_{1,n} \in Y_1$ such that $y_0 + y_1 \in A$ and $y_{0,1} + y_{1,n} \in A$. On the one hand,

$$\begin{aligned} t_n =& t_n y_n^*(y_{0,n}) - y_n^*(y_0) + y_n^*(y_0) \\ =& y_n^*(t_n y_{0,n} - y_0) + y_n^*(y_0 + y_1) \leq \|t_n y_{0,n} - y_0\| + 1 \end{aligned}$$

so $\limsup_n t_n \leq 1$. On the other hand,

$$t_n \geq t_n y^*(y_{0,n} + y_{1,n}) = t_n y^*(y_{0,n}) = y^*(y_0) + y^*(t_n y_{0,n} - y_0) \geq 1 - \|t_n y_{0,n} - y_0\|$$

and thus $\liminf_n t_n \geq 1$. So $\lim_n t_n = 1$ and the proof of the continuity of η is finished.

Now, consider the set

$$A' = \bigcup_{\lambda \geq 0} \lambda A = \{(\lambda y_0, \lambda\eta(y_0)) : \, y_0 \in \pi_0(A), \, \lambda \geq 0\} = \{(z, \eta(z)) : \, z \in Y_0\}.$$

We trivially have that A' is the graph of η. Moreover, recall that (5.12) yields

$$\begin{aligned} &\{y \in Y : (x, y) \in U_{(a_0,b_0)} \text{ and } \frac{\partial p}{\partial y}(x, y) = 0\} \\ &\subset \{y \in Y : (x, y) \in U_{(a_0,b_0)} \text{ and } \frac{y}{\|y\|} \in A\} \\ &\subset \{y \in Y : (x, y) \in U_{(a_0,b_0)} \text{ and } y \in A'\}. \end{aligned} \tag{5.13}$$

Thus $\{y \in Y : (x,y) \in U_{(a_0,b_0)}$ and $\frac{\partial p}{\partial y}(x,y) = 0\}$ is contained in the graph of η, as we wanted. Recall that there is an alternate proof for the construction of the function η by using some of the ideas of Lemma 4.5.

As a conclusion of (5.13), if we denote by $\mathcal{C}_p$ the closed set of $X \oplus Y$ of critical points of p and consider the closed set of $X \oplus Y$

$$\mathcal{CP}_p := \big\{(x,y) \in X \oplus Y : \frac{\partial p}{\partial y}(x,y) = 0\big\},$$

we have that

$$\begin{aligned} U_{(a_0,b_0)} \cap \mathcal{C}_p \subset U_{(a_0,b_0)} \cap \mathcal{CP}_p \subset X \times \{(z,\eta(z)) : z \in Y_0\} = \\ = \{(x,z,z') \in X \oplus Y_0 \oplus Y_1 : z' = \eta(z) := \widetilde{\eta}(x,z)\}, \end{aligned}$$

where $\widetilde{\eta} : X \oplus Y_0 \to Y_1$ is a continuous function with $X \oplus Y_0$ a Banach space with C^k smooth partitions of unity and Y_1 an infinite dimensional Banach space with a C^k smooth norm. Notice that, by the construction, the spaces $X \oplus Y_0$, Y_1 and the function $\widetilde{\eta}$ depend on the point (a_0,b_0).

Therefore, we may apply Theorem 2.9 and define a diffeomorphism

$$d : X \oplus Y \to \big(X \oplus Y\big) \setminus \mathcal{CP}_p$$

limited by the open cover $\{\operatorname{supp}_0 \psi_i\}_{i\in\Delta}$, where $\operatorname{supp}_0 \psi_i = \{(x,y) \in X \oplus Y : \psi_i(x,y) > 0\}$.

Let us define the composition $g = p \circ d$, which is clearly C^k smooth. Let us check that g has no critical points. On the one hand, for all $(x,y) \in X \oplus Y$ we have that $d(x,y) \notin \mathcal{CP}_p$ and thus $\frac{\partial p}{\partial y}(d(x,y)) \neq 0$ which makes $p'(d(x,y)) \neq 0$ as well. On the other hand, $d'(x,y)$ is an isomorphism on $X \oplus Y$ for all $(x,y) \in X \oplus Y$. So we have that $(p \circ d)'(x,y) = p'(d(x,y)) \circ d'(x,y) \neq 0$ for all $(x,y) \in X \oplus Y$.

In addition, it can be checked that $|f(x,y) - g(x,y)| < \varepsilon(x,y)$ for all $(x,y) \in X \oplus Y$. Indeed, since for all (x,y) there is $i_0 \in \Delta$ (depending on (x,y)) with $\{(x,y), d(x,y)\} \subset \operatorname{supp}_0 \psi_{i_0} \subset U_{i_0}$, by using the upper bounds given in (5.9) and (5.11) we have

$$\begin{aligned} |f(x,y) - g(x,y)| \le |f(x,y) - f(d(x,y))| + |f(d(x,y)) - p(d(x,y))| \le \\ \le \frac{\varepsilon(x,y)}{4} + \frac{\varepsilon(d(x,y))}{2} < \varepsilon(x,y). \end{aligned} \tag{5.14}$$

Step 2. The general case of a finite dimensional target space F. If the space F has dimension n, we consider the n linearly independent vectors $\{v_m^*\}_{m=1}^n \subset S_{\|\cdot\|}$ given at the beginning of the proof such that $\{v_m^* : m = 1, \cdots, n\} \cup \{u_i^* : i \in \mathbb{N}\}$ are linearly independent. Without loss of generality we may assume that $F = \mathbb{R}^n$ with the euclidean norm. Let us consider a continuous a function $f : X \oplus Y \to \mathbb{R}^n$ and a continuous function $\varepsilon : X \oplus Y \to (0, \infty)$. Let us consider a covering of $X \oplus Y$ given by $\{U_i := \mathring{B}_X(x_i, r_i) \times \mathring{B}_Y(y_i, r_i)\}_{i\in\Delta}$ (with $\mathring{B}_X(x_i, r_i) \subset X$ and $\mathring{B}_Y(y_i, r_i) \subset Y$) such that

$$\|f(x,y) - f(x',y')\| < \frac{\varepsilon(x,y)}{4\sqrt{n}} \quad \text{for all } (x,y),(x'y') \in U_i \text{ and } i \in \Delta, \tag{5.15}$$
$$|\varepsilon(x,y) - \varepsilon(x',y')| < \frac{\varepsilon(x,y)}{8} \quad \text{for all } (x,y),(x',y') \in U_i \text{ and } i \in \Delta,$$
$$\text{and} \quad r_i < \frac{\varepsilon(x_i,y_i)}{8\sqrt{n}} \quad \text{for every } i \in \Delta.$$

Then, there exists a partition of unity $\{\psi_i\}_{i\in\Delta} \subset \mathcal{S}_k$ satisfying $\operatorname{supp}_0 \psi_i \subset U_i$ for all $i \in \Delta$. We proceed in a similar way to Step 1 for each component f_m of f and get functions $p_m \in C^k(X \oplus Y, \mathbb{R})$ for $m = 1, \dots, n$ defined by

$$p_m(x,y) = \sum_{i\in\Delta} (f_m(x_i,y_i) + v_m^*(y - y_i))\, \psi_i(x,y),$$
$$\text{for all } (x,y) \in X \oplus Y \text{ and } m = 1, \dots, n. \tag{5.16}$$

Similarly to (5.11) we can check that

$$|f_m(x,y) - p_m(x,y)| < \frac{\varepsilon(x,y)}{2\sqrt{n}} \quad \text{for all } (x,y) \in X \oplus Y, \text{ and } m = 1, \dots, n.$$

If we define

$$p : X \oplus Y \to \mathbb{R}^n, \quad p(x,y) = (p_1(x,y), \dots, p_n(x,y)), \quad \text{for all } (x,y) \in X \oplus Y,$$

then $p \in C^k(X \oplus Y, \mathbb{R}^n)$ and

$$\|f(x,y) - p(x,y)\| < \frac{\varepsilon(x,y)}{2} \quad \text{for all } (x,y) \in X \oplus Y. \tag{5.17}$$

Also, reasoning as in the preceding step, for every $(a_0, b_0) \in X \oplus Y$ there is an open and bounded neighbourhood $U_{(a_0,b_0)}$ of (a_0, b_0), a finite number of indexes $I(a_0, b_0) \subset \mathbb{N}$ (depending on (a_0, b_0)) such that

$$\frac{\partial p_m}{\partial y}(x,y) = v_m^* + z_m(x,y)\|\cdot\|'(y) + \sum_{j\in I(a_0,b_0)} w_{m,j}(x,y)\, u_j^*, \quad \text{for all } (x,y) \in U_{(a_0,b_0)},$$

where $z_m(x,y) \in \mathbb{R}$, $w_{m,j}(x,y) \in \mathbb{R}$ for all $(x,y) \in U_{(a_0,b_0)}$, all $m = 1, \dots, n$ and all $j \in I(a_0, b_0)$, where $z_m(x,0) = 0$ for all $(x,0) \in U_{(a_0,b_0)}$ and all $m = 1, \dots, n$. Now, we consider the following cases:

$\star$ If $(x,y) \in U_{(a_0,b_0)}$ and $z_m(x,y)=0$ for all $m=1,\dots,n$, then

$$\frac{\partial p_m}{\partial y}(x,y) = v_m^* + \sum_{j\in I(a_0,b_0)} w_{m,j}(x,y)\, u_j^*, \qquad \text{for all } m=1,\dots,n,$$

so $\left\{\dfrac{\partial p_m}{\partial y}(x,y)\right\}_{m=1}^n$ are linearly independent and thus $\frac{\partial p}{\partial y}(x,y)$ is surjective.

$\star$ If $(x,y) \in U_{(a_0,b_0)}$ and $z_m(x,y) \neq 0$ for at least one $m \in \{1,\dots,n\}$ and $\{\dfrac{\partial p_m}{\partial y}(x,y)\}_{m=1}^n$ are linearly dependent, then there is a non trivial linear combination

$$\begin{aligned} 0 = \sum_{m=1}^n \lambda_m \frac{\partial p_m}{\partial y}(x,y) &= \sum_{m=1}^n \lambda_m\, v_m^* + \left(\sum_{m=1}^n \lambda_m z_m(x,y)\right)\|\cdot\|'(y) \\ &+ \sum_{m=1}^n \lambda_m \Bigg(\sum_{j\in I(a_0,b_0)} w_{m,j}(x,y)\, u_j^*\Bigg), \end{aligned}$$

with $\{\lambda_m\}_{m=1}^n \subset \mathbb{R}$ and at least one $\lambda_m \neq 0$. Denote $\lambda(x,y) := -\sum_{m=1}^n \lambda_m z_m(x,y)$. Notice that, in this case, $\lambda(x,y)\neq 0$. Otherwise,

$$\sum_{m=1}^n \lambda_m\, v_m^* = -\sum_{m=1}^n \lambda_m \Bigg(\sum_{j\in I(a_0,b_0)} w_{m,j}(x,y)\, u_j^*\Bigg),$$

which is impossible because the first sum is in $\operatorname{span}(\{v_m^* : m=1,\dots,n\})\setminus\{0\}$ and the second sum is in $\operatorname{span}(\{u_j^* : j\in I(a_0,b_0)\})$ and the intersection $\operatorname{span}(\{v_m^* : m = 1,\dots,n\}) \cap \operatorname{span}(\{u_j^* : j\in I(a_0,b_0)\}) = \{0\}$. Then,

$$\|\cdot\|'(y) = \sum_{m=1}^n \frac{\lambda_m}{\lambda(x,y)}\, v_m^* + \sum_{m=1}^n \frac{\lambda_m}{\lambda(x,y)} \Bigg(\sum_{j\in I(a_0,b_0)} w_{m,j}(x,y)\, u_j^*\Bigg),$$

where

$$b_1(x,y) := \sum_{m=1}^n \frac{\lambda_m}{\lambda(x,y)}\, v_m^* \in \operatorname{span}(\{v_m^* : m=1,\dots,n\})\setminus\{0\}$$

and

$$b_2(x,y) := \sum_{m=1}^n \frac{\lambda_m}{\lambda(x,y)} \Bigg(\sum_{j\in I(a_0,b_0)} w_{m,j}(x,y)\, u_j^*\Bigg) \in \operatorname{span}(\{u_j^* : j\in I(a_0,b_0)\}).$$

Thus we have

$$\|\cdot\|'(y) \in \operatorname{span}(\{v_m^*,\, u_j^* : j\in I(a_0,b_0) \text{ and } m\in\{1,\dots,n\}\}) \cap S_{\|\cdot\|^*} := K$$

with $K \subset NA_{\|\cdot\|}$ compact and thus

$$y \in \|y\| \, (\| \cdot \|')^{-1}(K).$$

Next, we follow Step 1 and define the infinite dimensional closed subspace of Y

$$Y_1 = \operatorname{span}(\{v_m^*, \, u_j^* : \, j \in I(a_0, b_0) \text{ and } m \in \{1, \ldots, n\}\})_\perp$$

and a finite dimensional subspace Y_0 such that $Y = Y_0 \oplus Y_1$ (both Y_0 and Y_1 depending on (a_0, b_0)) to get subsets A and A' satisfying

$$\begin{aligned} &\left\{ y \in Y : (x,y) \in U_{(a_0,b_0)} \text{ and } \left\{ \frac{\partial p_m}{\partial y}(x,y) \right\}_{m=1}^n \text{ are linearly dependent } \right\} \subset \\ &\subset \left\{ y \in Y : (x,y) \in U_{(a_0,b_0)} \text{ and } \frac{y}{\|y\|} \in A \right\} \subset \\ &\subset \{ y \in Y : (x,y) \in U_{(a_0,b_0)} \text{ and } y \in A' \}, \end{aligned}$$

being A' the graph of a continuous function $\eta : Y_0 \to Y_1$. Therefore, if we denote by $\mathcal{C}_p$ the closed set of $X \oplus Y$ of critical points of p and consider the closed set of $X \oplus Y$

$$\mathcal{CP}_p = \{(x,y) \in X \oplus Y : \, \frac{\partial p}{\partial y}(x,y) \text{ is not surjective } \},$$

then

$$U_{(a_0,b_0)} \cap \mathcal{C}_p \subset U_{(a_0,b_0)} \cap \mathcal{CP}_p \subset \{(x,z,z') \in X \oplus Y_0 \oplus Y_1 : \, z' = \eta(z) := \widetilde{\eta}(x,z)\},$$

with $\widetilde{\eta} : X \oplus Y_0 \to Y_1$ a continuous function. Notice that Y_0, Y_1, and $\widetilde{\eta}$ depend on (a_0, b_0).

As in Step 1, we apply Theorem 2.9 and compose p with a C^k diffeomorphism $d : X \oplus Y \to (X \oplus Y) \setminus \mathcal{CP}_p$ limited by the open cover $\{\operatorname{supp}_0 \psi_i\}_{i \in \Delta}$. The composition $g := p \circ d$, which is C^k smooth, does not have critical points: We have that $d(x,y) \notin \mathcal{CP}_p$ for all $(x,y) \in X \oplus Y$ and thus $\frac{\partial p}{\partial y}(d(x,y))$ is a surjective operator from Y onto $\mathbb{R}^n$, which also makes $p'(d(x,y))$ a surjective operator from $X \oplus Y$ onto $\mathbb{R}^n$ for all $(x,y) \in X \oplus Y$. Since $d'(x,y)$ is an isomorphism on $X \oplus Y$ for all $(x,y) \in X \oplus Y$, we have that $(p \circ d)'(x,y) = p'(d(x,y)) \circ d'(x,y)$ is a surjective operator from $X \oplus Y$ onto $\mathbb{R}^n$ for all $(x,y) \in X \oplus Y$.

In addition, it can be checked that $\|f(x,y) - g(x,y)\| < \varepsilon(x,y)$ for all $(x,y) \in X \oplus Y$. Indeed, since for all (x,y) there is $i_0 \in \Delta$ with $\{(x,y), d(x,y)\} \subset \operatorname{supp}_0 \psi_{i_0} \subset U_{i_0}$, by using the upper bounds in (5.15) and (5.17) we have $\varepsilon(d(x,y)) < \frac{9}{8}\varepsilon(x,y)$ and

$$\begin{aligned} \|f(x,y) - g(x,y)\| &\leq \|f(x,y) - f(d(x,y))\| + \|f(d(x,y)) - p(d(x,y))\| \\ &\leq \frac{\varepsilon(x,y)}{4\sqrt{n}} + \frac{\varepsilon(d(x,y))}{2} < \varepsilon(x,y). \quad \square \end{aligned}$$

6. Final comments and conclusions

As we mention in the introduction, when proving the existence of C^k smooth approximations without critical points of continuous functions from a Banach space X to any quotient F of X it is enough to consider $F = X$ as we stated in Fact 1.15, which we shall now prove.

Proof of Fact 1.15. It is enough to prove (2) ⇒ (1). Consider a quotient F of X, a continuous mapping $f : X \to F$ and a continuous function $\varepsilon : X \to (0, \infty)$. Let us denote by $\pi : X \to F$ the canonical quotient mapping taking every point x to its equivalence class in F. Let us consider a Bartle-Graves continuous selection mapping $S : F \to X$, i.e. a continuous mapping $S : F \to X$ such that $\pi \circ S = Id$, being $Id : F \to F$ the identity mapping (see [8] or [17, Chapter VII, Lemma 3.2]). Let us consider the continuous composition $S \circ f : X \to X$. By assumption, there exists $g \in C^k(X, X)$ such that $\|S \circ f(x) - g(x)\| < \varepsilon(x)$ for all $x \in X$ and g has no critical points. Then, $\|(\pi \circ S \circ f)(x) - \pi \circ g(x)\| < \varepsilon(x)$ and thus $\|f(x) - \pi \circ g(x)\| < \varepsilon(x)$ for all $x \in X$. Clearly $\pi \circ g \in C^k(X, F)$ and $D(\pi \circ g)(x) = \pi \circ Dg(x)$ is the composition of surjective linear mappings, thus surjective for all $x \in X$. □

Finally, let us mention, that, as in the separable case given in [7], the existence of C^k smooth approximations without critical points to real-valued continuous functions defined on a Banach space X provides the following corollaries.

Corollary 6.1. ***(A non linear C^k smooth Hahn-Banach separation result).*** *Let X and Y be any pair of Banach spaces considered in Theorem 1.8, Corollary 1.9 or Corollary 1.10. Then for every two disjoint closed subsets C_1 and C_2 of $X \oplus Y$ there is a C^k smooth function $\varphi : X \oplus Y \to \mathbb{R}$ with no critical points, such that the level set $A = \varphi^{-1}(0)$ is a 1-codimensional C^k smooth submanifold of $X \oplus Y$ that separates C_1 from C_2, that is to say, the open and disjoint sets $U_1 = \{z \in X \oplus Y : \varphi(z) > 0\}$ and $U_2 = \{z \in X \oplus Y : \varphi(z) < 0\}$ have a common boundary $A = \partial U_1 = \partial U_2$ and $C_i \subset U_i$ for $i = 1, 2$.*

Corollary 6.2. ***(C^k smooth approximations of closed sets).*** *Let X and Y be any pair of Banach spaces considered in Theorem 1.8, Corollary 1.9 or Corollary 1.10. Then, every closed subset of $X \oplus Y$ can be approximated by C^k smooth open subsets, that is to say, for every closed subset $C \subset X \oplus Y$ and every open subset $W \subset X \oplus Y$ such that $C \subset W$ there is a C^k smooth open set $U \subset X \oplus Y$ (i.e. ∂U is a 1-codimensional C^k smooth submanifold of $X \oplus Y$) such that $C \subset U \subset W$.*

Data availability

No data was used for the research described in the article.

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References

[1] D. Azagra, M. Cepedello, Uniform approximation of continuous mappings by smooth mappings with no critical points on Hilbert manifolds, Duke Math. J. 124 (1) (2004) 47–66.
[2] D. Azagra, R. Deville, James' theorem fails for starlike bodies, J. Funct. Anal. 180 (2) (2001) 328–346.
[3] D. Azagra, R. Deville, M. Jiménez-Sevilla, On the range of the derivatives of a smooth function between Banach spaces, Math. Proc. Camb. Philos. Soc. 134 (1) (2003) 163–185.
[4] D. Azagra, M. Fabian, M. Jiménez-Sevilla, Exact filling of figures with the derivatives of smooth mappings between Banach spaces, Can. Math. Bull. 48 (4) (2005) 481–499.
[5] D. Azagra, T. Dobrowolski, M. García Bravo, Smooth approximations without critical points of continuous mappings between Banach spaces, and diffeomorphic extractions of sets, Adv. Math. 354 (2019) 106756, 80 pp.
[6] D. Azagra, M. Jiménez-Sevilla, The failure of Rolle's theorem in infinite-dimensional Banach spaces, J. Funct. Anal. 182 (1) (2001) 207–226.
[7] D. Azagra, M. Jiménez-Sevilla, Approximation by smooth functions with no critical points on separable infinite-dimensional Banach spaces, J. Funct. Anal. 242 (2007) 1–36.
[8] Robert G. Bartle, Lawrence M. Graves, Mappings between function spaces, Trans. Am. Math. Soc. 72 (1952) 400–413.
[9] S.M. Bates, C.G. Moreira, De nouvelles perspectives sur le théorème de Morse-Sard, C. R. Acad. Sci., Sér. 1 Math. 332 (2001) 13–17.
[10] R. Bonic, J. Frampton, Smooth functions on Banach manifolds, J. Math. Mech. 15 (5) (1966) 877–898.
[11] J.M. Borwein, M. Fabian, P.D. Loewen, The range of the gradient of a Lipschitz C^1-smooth bump in infinite dimensions, Isr. J. Math. 132 (2002) 239–251.
[12] J.M. Borwein, M. Fabian, I. Kortezov, P.D. Loewen, The range of the gradient of a continuously differentiable bump, J. Nonlinear Convex Anal. 2 (1) (2001) 1–19.
[13] S. Dantas, P. Hájek, T. Russo, Smooth norms in dense subspaces of Banach spaces, J. Math. Anal. Appl. 487 (1) (2020) 123963, 16 pp.
[14] S. Dantas, P. Hájek, T. Russo, Smooth and polyhedral norms via fundamental biorthogonal systems, Int. Math. Res. Not. (16) (2023) 13909–13939.
[15] S. Dantas, P. Hájek, T. Russo, Smooth norms in dense subspaces of $\ell_p(\Gamma)$ and operator ranges, Rev. Mat. Complut. (2023), https://doi.org/10.1007/s13163-023-00479-w.
[16] R. Deville, On the range of the derivative of a smooth function and applications, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 100 (1–2) (2006) 63–74.
[17] R. Deville, G. Godefroy, V. Zizler, Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow, 1993.
[18] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, Banach Space Theory. The Basis for Linear and Nonlinear Analysis, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011.
[19] M. Fabian, J.H.M. Whitfield, V. Zizler, Norms with locally Lipschitzian derivatives, Isr. J. Math. 44 (3) (1983) 262–276.
[20] M. Fabian, L. Zajicek, V. Zizler, On residuality of the set of rotund norms on a Banach space, Math. Ann. 258 (1982) 349–351.
[21] M. García-Bravo, Extraction of critical points of smooth functions on Banach spaces, J. Math. Anal. Appl. 482 (1) (2020) 123535, 21 pp.
[22] P. Georgiev, A.S. Granero, M. Jiménez-Sevilla, J.P. Moreno, Mazur intersection properties and differentiability of convex functions in Banach spaces, J. Lond. Math. Soc. (2) 61 (2) (2000) 531–542.

[23] P. Hajek, Smooth norms that depend locally on finitely many coordinates, Proc. Am. Math. Soc. 123 (12) (1995) 3817–3821.
[24] P. Hájek, Smooth functions on c_0, Isr. J. Math. 104 (1998) 17–27.
[25] P. Hajek, M. Johanis, Smooth approximations without critical points, Cent. Eur. J. Math. 1 (3) (2003) 284–291.
[26] P. Hajek, M. Johanis, Smooth Analysis in Banach Spaces, De Gruyter Series in Nonlinear Analysis and Applications, Berlin/Boston, 2014.
[27] P. Hájek, V. Montesinos Santalucía, J. Vanderwerff, V. Zizler, Biorthogonal Systems in Banach Spaces, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 26, Springer, New York, 2008.
[28] R. Haydon, Trees in renorming theory, Proc. Lond. Math. Soc. 78 (3) (1999) 541–584.
[29] M.W. Hirsch, Differential Topology, Grad. Texts in Math., vol. 33, Springer-Verlag, New York, 1976.
[30] M. Jiménez-Sevilla, A note on the range of the derivatives of analytic approximations of uniformly continuous functions on c_0, J. Math. Anal. Appl. 348 (2) (2008) 573–580.
[31] I. Kupka, Counterexample to the Morse-Sard theorem in the case of infinite-dimensional manifolds, Proc. Am. Math. Soc. 16 (1965) 954–957.
[32] J. Lindenstrauss, On reflexive spaces having the metric approximation property, Isr. J. Math. 3 (1965) 199–204.
[33] J. Lindenstrauss, On non-separable reflexive Banach spaces, Bull. Am. Math. Soc. 72 (1966) 967–970.
[34] C.G. Moreira, Hausdorff measure and the Morse-Sard theorem, Publ. Mat. 45 (2001) 149–162.
[35] R.R. Phelps, Convex Functions, Monotone Operators and Differentiability, second edition, Lecture Notes in Mathematics, vol. 1364, Springer-Verlag, Berlin, 1993.
[36] A. Sard, The measure of the critical values of differentiable maps, Bull. Am. Math. Soc. 48 (1942) 833–890.
[37] A. Sard, Images of critical sets, Ann. Math. 68 (1958) 247–259.
[38] S. Smale, An infinite-dimensional version of Sard's theorem, Am. J. Math. 87 (1965) 861–866.
[39] A. Sobczyk, Projection of the space (m) on its subspace c_0, Bull. Am. Math. Soc. 47 (1941) 938–947.
[40] H. Toruńczyk, Smooth partitions of unity on some non-separable Banach spaces, Stud. Math. 46 (1973) 43–51.
[41] S.L. Troyanski, On locally convex and differentiable norms in certain non-separable Banach spaces, Stud. Math. 37 (1971) 173–180.
[42] J. Vanderwerff, Fréchet differentiable norms on spaces of countable dimension, Arch. Math. 58 (1992) 471–476.
[43] Y. Yomdin, G. Comte, Tame Geometry with Applications in Smooth Analysis, Lecture Notes in Math., vol. 1834, Springer-Verlag, Berlin, 2004.
[44] V. Zizler, Nonseparable Banach spaces, in: W.B. Johnson, J. Lindenstrauss (Eds.), Handbook of Banach Spaces, Volume II, Elsevier Science Publishers B.V., Amsterdam, 2003, pp. 1743–1816.