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Rolle's Theorem and Negligibility of Points in Infinite Dimensional Banach Spaces*

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In this note we prove that if a differentiable function oscillates between $-\varepsilon$ and ε on the boundary of the unit ball then there exists a point in the interior of the ball in which the differential of the function has norm equal or less than ε. This kind of approximate Rolle's theorem is interesting because an exact Rolle's theorem does not hold in many infinite dimensional Banach spaces. A characterization of those spaces in which Rolle's theorem does not hold is given within a large class of Banach spaces. This question is closely related to the existence of C^1 diffeomorphisms between a Banach space X and $X \setminus \{0\}$ which are the identity out of a ball, and we prove that such diffeomorphisms exist for every C^1 smooth Banach space which can be linearly injected into a Banach space whose dual norm is locally uniformly rotund (LUR).

1. INTRODUCTION

Rolle's theorem in finite dimensional spaces states that for every open connected and bounded subset U in $\mathbb{R}^n$ and every continuous function $f: \overline{U} \to \mathbb{R}$ such that f is differentiable in U and constant on ∂U, there exists an x in U such that $df(x) = 0$. In a paper published in 1992, S. A. Shkarin [10] proved that Rolle's theorem fails in a large class of infinite dimensional Banach spaces, including all super-reflexive and all non-reflexive Banach spaces having a Fréchet differentiable norm—although he did not study the reflexive but non-super-reflexive case. Other explicit examples

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were found in c_0 and l_2 by J. Ferrera and J. Bés [7] and independently by J. Ferrer [8]. On the other hand it is clear that Rolle's theorem trivially holds in all non-Asplund Banach spaces because of the harmonic behaviour of differentiable maps in such spaces. It is natural to conjecture that a reasonable version of Rolle's theorem in infinite dimensional Banach spaces holds if and only if our space does not have a C^1 bump function and we prove this conjecture to be true within the class of those Banach spaces X which can be linearly injected into a Banach space Y with an equivalent norm whose dual norm is locally uniformly rotund (LUR) in Y^*. This geometrical condition, which we shall call $(*)$ for short, is satisfied by every (WCG) Banach space, every space which can be injected into some $c_0(\Gamma)$, and even by every space injectable into some $C(K)$, where K is a scattered compact with $K^{(\omega_1)} = \emptyset$. This conjecture is closely related to the question posed in [4] whether for every Banach space X having a C^1 bump function there exists a C^1 diffeomorphism $\varphi: X \to X \setminus \{0\}$ such that φ is the identity out of a ball. We give an affirmative answer to this question within the class of all Banach spaces X verifying $(*)$.

An interesting approximate version of Rolle's theorem remains nevertheless true in all Banach spaces, as we prove in this note. By an approximate Rolle's theorem we mean that if a differentiable function oscillates between $-\varepsilon$ and ε on the boundary of the unit ball then there exists a point in the interior of the ball in which the differential of the function has norm less than or equal to ε.

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2. THE APPROXIMATE ROLLE'S THEOREM

In order to prove the approximate Rolle's theorem we need the following lemmas, which are themselves interesting.

LEMMA 2.1. *Let X be a Banach space and U be an open bounded connected subset of X. Let $f: \overline{U} \to \mathbb{R}$ be a continuous bounded function such that*:

(1) *f is Gâteaux differentiable in U*

(2) $\inf f(\overline{U}) < \inf f(\partial U)$ *or* $\sup f(\overline{U}) > \sup f(\partial U)$.

Then, for every $\alpha > 0$ there exists $x \in U$ such that $\|df(x)\| \leq \alpha$.

Proof. We may suppose $\inf f(\overline{U}) < \inf f(\partial U)$. Let us choose $x_0 \in U$ such that $f(x_0) < \inf f(\partial U)$, and let α, λ be such that $0 < \alpha < \inf f(\partial U) - f(x_0)$ and $0 < \lambda < \alpha/R$, where $R = \sup\{\|x_0 - x\| : x \in \overline{U}\} + 1$. From Ekeland's Variational Principle (see Lemma 3.13 in [9], or [5]) it follows that there exists $x_1 \in \overline{U}$ such that

$$f(x_1) < f(x) + \lambda\|x - x_1\| \tag{1}$$

for all $x \neq x_1$. In particular

$$f(x_1) \leq f(x_0) + \lambda\|x_0 - x_1\| \leq f(x_0) + \lambda R < \inf f(\partial U)$$

and therefore $x_1 \in U$. On the other hand, inequality (1) implies that for every h such that $\|h\| = 1$,

$$df(x_1)(h) = \lim_{t \to 0^+} \frac{f(x_1 + th) - f(x_1)}{t} \geq -\lambda,$$

which proves $\|df(x_1)\| \leq \lambda < \alpha$. ∎

LEMMA 2.2. *Let X be a Banach space and U be an open bounded connected subset of X. Let $f : \overline{U} \to \mathbb{R}$ be a continuous bounded function such that*:

(1) *f is Gâteaux differentiable in U*

(2) *$f(\overline{U}) \subseteq [a, b]$, where $a < b$.*

Then, for every $x_0 \in U$ and $R > 0$ such that $B(x_0, R) \subseteq U$, there exists $x_1 \in B(x_0, R)$ such that $\|df(x_1)\| \leq (b - a)/2R$.

Proof. We may suppose that $[a, b] = [-\varepsilon, \varepsilon]$. Two cases will be considered.

Case I. $f(x_0) \neq 0$. We may suppose $f(x_0) < 0$ (the case $f(x_0) > 0$ is analogous). From Ekeland's Variational Principle (see Lemma 3.13 in [9], or [5]) it follows that there exists $x_1 \in \overline{U}$ such that

(1) $\|x_0 - x_1\| \leq (f(x_0) + \varepsilon)/(\varepsilon/R) < R$, and

(2) $f(x_1) < f(x) + (\varepsilon/R)\|x - x_1\|$ for all $x \neq x_1$.

From (1) we get $x_1 \in U$ and (2) implies that for every h with $\|h\| = 1$

$$df(x_1)(h) = \lim_{t \to 0^+} \frac{f(x_1 + th) - f(x_1)}{t} \geq -\varepsilon/R,$$

which proves $\|df(x_1)\| \leq \varepsilon/R$.

Case II. $f(x_0) = 0$. We may suppose $\|df(x_0)\| > \varepsilon/R$, since we would have finished otherwise. If $\|df(x_0)\| > \varepsilon/R$ there exists h with $\|h\| = 1$ such that $df(x_0)(h) < -\varepsilon/R$ and therefore there exists $\delta > 0$ such that $f(x_0 + \delta h)/\delta < -\varepsilon/R$. Applying Ekeland's Variational Principle again we obtain $x_1 \in \overline{U}$ such that:

(1) $\|x_1 - (x_0 + \delta h)\| \leq (f(x_0 + \delta h) + \varepsilon)/(\varepsilon/R) < (-\varepsilon\delta/R + \varepsilon)/(\varepsilon/R) = R - \delta$ and

(2) $f(x_1) < f(x) + (\varepsilon/R)\|x - x_1\|$ for all $x \neq x_1$.

From (1) it follows that $\|x_1 - x_0\| \leq \|x_1 - (x_0 + \delta h)\| + \delta < R$, so that $x_1 \in B(x_0, R) \subseteq U$, and (2) implies $\|df(x_1)\| \leq \varepsilon/R$. ∎

The following result is immediately deduced as a consequence of Lemmas 2.1 and 2.2.

THEOREM 2.3 (Approximate Rolle's Theorem). *Let X be a Banach space and U be an open bounded connected subset of X. Let $f : \overline{U} \to \mathbb{R}$ be a continuous bounded function. Suppose that f is Gâteaux differentiable in U and $\overline{f(\partial U)} \subseteq [a, b]$, with $a < b$. Then, for every $R > 0$ and $x_0 \in U$ such that $B(x_0, R) \subseteq U$, there exists $x_1 \in U$ such that*

$$\|df(x_1)\| \leq \frac{b - a}{2R}.$$

From this we can immediately deduce the following

COROLLARY 2.4. *Let U be an open connected bounded subset of a Banach space X. Let $f : \overline{U} :\to \mathbb{R}$ be continuous, bounded, and Gâteaux differentiable in U. Suppose that f is constant on ∂U. Then,*

$$\inf_{x \in U} \|f'(x)\| = 0.$$

It is easy to see, using Ekeland's Variational Principle, that if X is a Banach space and $f : X \to \mathbb{R}$ is continuous, Gâteaux differentiable, and bounded below (or bounded above), then $\inf_{x \in X} \|f'(x)\| = 0$. Alternatively, if we assume that f is bounded, this is an immediate consequence of Theorem 2.3.

3. DIFFEOMORPHISMS BETWEEN X AND $X \setminus \{0\}$

We use in this section Bessaga's non-complete norm technique to prove that every Banach space X verifying the condition

$(*)$ There exists a Banach space Y with an equivalent norm $\|.\|$ whose dual norm $\|.\|^*$ is locally uniformly rotund (LUR) in Y^* and a continuous linear injection $T : X \to Y$

is C^1 diffeomorphic to $X \setminus \{0\}$. If moreover X has a differentiable bump function then there exits a C^1 diffeomorphism $\varphi: X \to X \setminus \{0\}$ such that φ is the identity out of a ball centered at 0.

It is not difficult to see that condition $(*)$ is equivalent to saying that X admits a continuous (not necessarily equivalent) norm whose dual norm is (LUR). Recall that a norm ρ in a Banach space $(X, \|.\|)$ is said to be non-complete provided the normed space (X, ρ) is not complete.

THEOREM 3.1. *Let X be an infinite dimensional Banach space that verifies condition $(*)$. Then*

(1) *X admits a $C^1(X \setminus \{0\})$ non-complete norm ω;*

(2) *there exists a C^1 diffeomorphism $\varphi: X \to X \setminus \{0\}$ such that $\varphi(x) = x$ if $\omega(x) \geq 1$.*

Proof. First of all let us see that every Banach space Y with an equivalent norm $\|.\|$ whose dual norm $\|.\|^*$ is (LUR) admits a $C^1(Y \setminus \{0\})$ non-complete norm ω. It is known that every infinite dimensional Banach space admits a continuous non-complete norm (see [2, Chap. III, Lemma 5.1]). Let $g: Y \to \mathbb{R}$ be such a norm in Y. Define

$$\omega(y) = \left[\inf\{g^2(u) + \|y - u\|^2 : u \in Y\}\right]^{1/2}, \qquad y \in Y.$$

It is easy to check that ω is a continuous norm in Y. As $\omega(y) \leq g(y)$ for all $y \in Y$ and g is non-complete, it is obvious that ω is also non-complete. On the other hand, it is known (see [6, Proposition 2.3]) that if $(Y, \|.\|)$ is a Banach space such that the dual norm $\|.\|^*$ is LUR then for every proper convex lsc function $f: Y \to (-\infty, +\infty]$ the infimal convolution with the squared norm

$$f_n(y) = \inf\{f(u) + n\|y - u\|^2 : u \in Y\}, \qquad y \in Y,$$

is C^1 smooth and convex (and if moreover f is bounded on bounded sets, then $f_n \to f$ uniformly on bounded sets as $n \to \infty$). Taking $f = g^2$ and $n = 1$, from this result we obtain that ω^2 is $C^1(Y)$, so that ω is $C^1(Y \setminus \{0\})$.

Now we should note that every subspace Z in Y has an equivalent norm whose dual norm is LUR. Indeed, considering the projection $\pi: Y^* \to Z^*$, $\pi(y^*) = y^*_{|Z}$, and using Theorem 2.1(ii) of [3, Chap. II], we get that Z^* has an equivalent LUR dual norm. Therefore, if Y^* has an equivalent LUR dual norm, then every closed subspace of Y admits a C^1 non-complete norm.

So let X, Y, and $T: X \to Y$ be as in condition $(*)$ and consider $Z = \overline{T(X)}$. If $T(X) = Z$, since Z has a C^1 non-complete norm ω and $T: X \to Z$ is a linear isomorphism, $\omega_0(x) = \omega(T(x))$ defines a C^1 non-complete norm on X. If, on the contrary, $T(X)$ is a dense but not closed subspace of Z, it is clear that $\omega_0(x) = \|T(x)\|$ defines a C^1 non-complete norm on X. In any case we get a C^1 non-complete norm on X. This proves (1).

Now one can prove (2) using Bessaga's non-complete norm technique as T. Dobrowolski does in [4]. In fact (2) is immediately deduced from Theorem 3.3 in [4]. Nevertheless we will say a few words about the way one can construct the diffeomorphism φ. There exists a linearly independent sequence $\{y_k\}_{k=2}^{\infty}$ in X such that $\sum_{k=1}^{\infty} 2^{k+6}\omega(y_{k+1} - y_k) < 1/2$, where $y_1 = 0$, and a point $\bar{y}$ in the completion of (X, ω) such that $\bar{y} \notin X$ and $\lim_k \omega(y_k - \bar{y}) = 0$. Let $\gamma: \mathbb{R} \to [0,1]$ be a C^∞ function with $\gamma = 1$ in $(-\infty, 1/2]$, $\gamma^{-1}(0) = [1, \infty)$, and $\|\gamma'\|_\infty \le 4$. Define $p: (0, \infty) \to X$ by

$$p(t) = y_1 + \sum_{k=1}^{\infty} \gamma(2^{k-1}t)(y_{k+1} - y_k)$$

for $t \ge 0$. p is a C^∞ path satisfying $\omega(p(t) - p(s)) \le 1/2|t - s|$, $\lim_{t \to 0} p(t) = \bar{y}$, $\omega(p'(t)) < 1/2$ for all $t > 0$ and $p(t) = 0$ if and only if $t \ge 1$. Let x be an arbitrary vector in X and let $F: [0, \infty) \to [0, \infty)$ be defined by $F(\alpha) = \omega(x - p(\alpha))$ for $\alpha > 0$ and $F(0) = \omega(x - \bar{y})$. We have $|F(\alpha) - F(\beta)| \le 1/2|\alpha - \beta|$, so from Banach's contraction principle applied to the interval $[0, \infty)$, it follows that the equation $F(\alpha) = \alpha$ has a unique solution. This means that for any $x \in X$, a number $\alpha(x)$ with the property

$$\omega(x - p(\alpha(x))) = \alpha(x)$$

is uniquely determined. Moreover, since x, being in X, cannot be equal to $\bar{y}$, we have $\alpha(x) \neq 0$. This implies that the mapping

$$\psi(z) = p(\omega(z)) + z$$

is one-to-one from $X \setminus \{0\}$ onto X, with

$$\psi^{-1}(x) = x - p(\alpha(x)).$$

As ω and p are C^1, so is ψ. Let $\Phi(x, \alpha) = \alpha - \omega(x - p(\alpha))$. Since for any $x \in X$ we have $x - p(\alpha(x)) \neq 0$, the mapping Φ is differentiable on a

neighbourhood of any point in $X \times (0, \infty)$. On the other hand,

$$\frac{\partial \Phi(x, \alpha)}{\partial \alpha} \geq 1 - 1/2 > 0$$

because $|F(\alpha) - F(\beta)| \leq 1/2|\alpha - \beta|$. So, using the implicit function theorem we obtain that $\psi : X \setminus \{0\} \to X$ is a C^1 diffeomorphism. Finally, it is clear that $\psi(z) = z$ whenever $\omega(z) \geq 1$. ∎

THEOREM 3.2. *For a Banach space X satisfying condition $(*)$, the following are equivalent.*

(1) *X has a C^1 bump function.*

(2) *There exists a C^1 diffeomorphism $\varphi : X \to X \setminus \{0\}$ such that φ is the identity out of a ball centered at 0.*

Proof. If $\varphi : X \to X \setminus \{0\}$ is a C^1 diffeomorphism such that $\varphi(x) = x$ whenever $\|x\| \geq r$ for some $r > 0$, then, taking $p \in X^*$ such that $p(\varphi(0)) \neq 0$ and defining $f(x) = p(\varphi(x) - x)$ we obtain a C^1 bump function f such that $f(0) \neq 0$ and $f(x) = 0$ if $\|x\| \geq r$, which proves that (2) implies (1).

Now suppose that X has a C^1 bump function. Proposition 5.1 in [3, Chap. II] gives us a function ψ on X such that ψ is C^1 smooth on $X \setminus \{0\}$, $\psi(tx) = |t|\psi(x)$ for $x \in X$ and $t \in \mathbb{R}$, and there are constants $a > 0$ and $b > 0$ such that $a\|x\| \leq \psi(x) \leq b\|x\|$ for $x \in X$. Let $\lambda : (0, \infty) \to (0, \infty)$ be a non-decreasing C^∞ function such that $\lambda(t) = 0$ for $t \leq 1/2$ and $\lambda(t) = 1$ for $t \geq 1$. Let

$$H(x) = \left[\lambda(\psi(x)) \frac{\psi(x)}{\omega(x)} + 1 - \lambda(\psi(x)) \right] x,$$

for $x \neq 0$, and $H(0) = 0$. H is a one-to-one mapping from X onto X transforming the set $\{x \in X : \psi(x) \leq 1\}$ onto $\{x \in X : \omega(x) \leq 1\}$, and H is C^1. Using the implicit function theorem as in the preceding theorem we obtain that H^{-1} is also C^1. By composing this diffeomorphism with that of Theorem 3.1 we get a C^1 diffeomorphism between X and $X \setminus \{0\}$ that is the identity out of a ball centered at 0. ∎

Now we can prove as in [1] the following

COROLLARY 3.3. *If a Banach space X verifies condition $(*)$ and has a Fréchet smooth equivalent norm then the sphere S_X is C^1 diffeomorphic to each hyperplane in X. If moreover X is isomorphic to one of its hyperplanes, then X is C^1 diffeomorphic to its sphere.*

Remark 3.4. All the results in this section remain true if we replace condition $(*)$ by the following one

$(**)$ There exist a Banach space Y with an equivalent differentiable norm $\|.\|$, an infinite-dimensional reflexive closed subspace $Z \subseteq Y$, and a continuous linear injection $T : X \to Y$ such that $Z \subseteq T(X) \subseteq Y$.

Indeed, if X, Y, and Z are as in this condition, let us consider any continuous noncomplete norm ω_0 on Z and let us define

$$\omega(x) = \left[\inf\left\{w_0^2(z) + \|x - z\|^2 : z \in Z\right\}\right]^{1/2}, \qquad x \in X.$$

Since Z is reflexive the infimum defining $\omega(x)$ is attained and using the differentiability of $\|.\|$ it is easy to see that the norm ω is differentiable in Y. Moreover ω is non-complete because $\omega(z) \leq \omega_0(z)$ for all $z \in Z$ and ω_0 is non-complete on the closed subspace $Z \subseteq Y$. Now define $\omega_1(x) = \omega(T(x))$ for each $x \in X$. It is clear that ω_1 is a differentiable non-complete norm in X, and so we can construct the diffeomorphisms between X and $X \setminus \{0\}$ in the same way as before.

4. AN EXACT ROLLE'S THEOREM IN INFINITE DIMENSIONAL BANACH SPACES FAILS

In this section we use the preceding results to prove that an exact Rolle's theorem either fails or trivially holds in infinite dimensional Banach spaces verifying $(*)$. The following result, whose proof is clearly motivated by Shkarin's ideas in [10], provides a characterization of spaces that do not verify Rolle's theorem within the class of those spaces verifying $(*)$.

THEOREM 4.1. *If a Banach space X verifies condition $(*)$, the following are equivalent*:

(1) *X has a C^1 bump function.*

(2) *There exists an open connected bounded subset U and a continuous bounded function $f : \overline{U} \to \mathbb{R}$ such that f is $C^1(U)$, $f \equiv 0$ on ∂U, and yet $df(x) \neq 0$ for all $x \in U$; that is, Rolle's theorem fails in X.*

(3) *There exists a $C^1(X)$ bounded function $f : X \to \mathbb{R}$ and an open connected bounded subset U in X such that $f \equiv 0$ on $X \setminus U$ and yet $df(x) \neq 0$ for all $x \in U$.*

Proof. It is obvious that (3) implies (2) and one can easily check that (2) implies (1). Let us prove that (1) implies (3). By Proposition 5.1 in [3, Chap.

II], there exists a function ψ on X such that ψ is C^1 smooth on $X \setminus \{0\}$, $\psi(tx) = |t|\psi(x)$ for $x \in X$ and $t \in \mathbb{R}$, and there are constants $a > 0$ and $b > 0$ such that $a\|x\| \le \psi(x) \le b\|x\|$ for $x \in X$. From Theorem 3.2 we get a C^1 diffeomorphism $\varphi : X \to X \setminus \{0\}$ such that φ is the identity out of a ball centered at 0. Let $\theta : \mathbb{R} \to \mathbb{R}$ be an even C^∞ function such that $\theta(0) = 1$, $\theta'(t) < 0$ for all $t \in (0, 1)$ and $\theta(t) = 0$ for all $t \ge 1$. We define $f : X \to \mathbb{R}$ by $f = \theta \circ \psi \circ \varphi$. Since f is the composition of the C^1 functions $\varphi : X \to X \setminus \{0\}$, $\psi : X \setminus \{0\} \to \mathbb{R}$ and θ, f is C^1, and f is bounded because so is θ. Moreover, we have $f(x) = 0$ if $\psi(\varphi(x)) \ge 1$. However, $f'(x) \ne 0$ for all x such that $\psi(\varphi(x)) < 1$, because

$$f'(x)(y) = \theta'(\psi(\varphi(x)))d\psi(\varphi(x))(\varphi'(x)(y)) \ne 0$$

for some $y \in X$ since $\varphi'(x)$ is a linear isomorphism, $d\psi(z) \ne 0$ for all $z \in X \setminus \{0\}$ and $\theta'(\psi(\varphi(x))) < 0$ whenever $\psi(\varphi(x)) < 1$. So, taking $U = \{x \in X : \psi(\varphi(x)) < 1\}$, (1) implies (3) is proved. ∎

Remark 4.2. Rolle's theorem trivially holds in non-Asplund Banach spaces: if X is a non-Asplund Banach space, U is an open connected bounded subset in X, and we have a continuous bounded function $f : \overline{U} \to \mathbb{R}$ that is Fréchet differentiable in U and $f \equiv 0$ on ∂U, then necessarily $f \equiv 0$ on U (see [3, Chap. III, p. 97]).

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