

# The Failure of Rolle's Theorem in Infinite-Dimensional Banach Spaces

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*Communicated by G. Pisier*

Received July 18, 2000; revised October 10, 2000; accepted October 25, 2000

We prove the following new characterization of  $C^p$  (Lipschitz) smoothness in Banach spaces. An infinite-dimensional Banach space  $X$  has a  $C^p$  smooth (Lipschitz) bump function if and only if it has another  $C^p$  smooth (Lipschitz) bump function  $f$  such that its derivative does not vanish at any point in the interior of the support of  $f$  (that is,  $f$  does not satisfy Rolle's theorem). Moreover, the support of this bump can be assumed to be a smooth starlike body. The "twisted tube" method we use in the proof is interesting in itself, as it provides other useful characterizations of  $C^p$  smoothness related to the existence of a certain kind of deleting diffeomorphisms, as well as to the failure of Brouwer's fixed point theorem even for smooth self-mappings of starlike bodies in all infinite-dimensional spaces. © 2001 Academic Press

## 1. INTRODUCTION AND MAIN RESULTS

Rolle's theorem in finite-dimensional spaces states that, for every bounded open subset  $U$  of  $\mathbb{R}^n$  and for every continuous function  $f: \bar{U} \rightarrow \mathbb{R}$  such that  $f$  is differentiable in  $U$  and constant on the boundary  $\partial U$ , there exists a point  $x \in U$  such that  $f'(x) = 0$ . Unfortunately, Rolle's theorem does not remain valid in infinite dimensions. It was S. A. Shkarin [33] that first showed the failure of Rolle's theorem in superreflexive infinite-dimensional spaces and in non-reflexive spaces which have smooth norms. The class of spaces for which Rolle's theorem fails was substantially enlarged in [6], where it was also shown that an approximate version of Rolle's theorem remains nevertheless true in all Banach spaces. In fact, as a consequence of the existence of diffeomorphisms deleting points in infinite-dimensional spaces (see [1, 5]), it is easy to see that Rolle's theorem fails in all infinite-dimensional Banach spaces which have smooth norms [7].

Of course, Rolle's theorem is trivially true in the Banach spaces which do not have any smooth bumps (if  $X$  is such a space then every function on  $X$  satisfying the hypothesis of Rolle's theorem must be a constant). Thus, in many infinite-dimensional Banach spaces, Rolle's theorem is either false or trivial, depending on the smoothness properties of the spaces considered. In this setting, it does not seem too risky to conjecture, as it was done in [6], that Rolle's theorem should fail in an infinite-dimensional Banach space if and only if our space has a  $C^1$  smooth bump function.

However, none of the results quoted above allows to characterize the spaces for which Rolle's theorem fails. Indeed, what makes the problem difficult is that the spaces are not assumed to be separable, nor even to have smooth norms. As shown by R. Haydon [27], there are Banach spaces with smooth bump functions which possess no equivalent smooth norms. Besides, it is natural to demand that the smooth bumps which do not satisfy Rolle's theorem be Lipschitz whenever smooth Lipschitz bumps are available in the space considered, and this requirement makes the problem even more delicate.

In this paper we will prove the above conjecture to be right, thus providing an interesting new characterization of smoothness in Banach spaces. Our main result is the following

**THEOREM 1.1.** *Let  $X$  be an infinite-dimensional Banach space which has a  $C^p$  smooth (Lipschitz) bump function. Then there exists another  $C^p$  smooth (Lipschitz) bump function  $f: X \rightarrow [0, 1]$  with the property that  $f'(x) \neq 0$  for every  $x \in \text{int}(\text{supp } f)$ .*

Here, as in the whole paper,  $1 \leq p \leq \infty$ , and  $\text{supp } f$  denotes the support of  $f$ , that is,  $\text{supp } f = \overline{\{x \in X : f(x) \neq 0\}}$ . Let us recall that  $b: X \rightarrow \mathbb{R}$  is said to be a bump function on  $X$  provided  $b$  is not constantly zero and  $b$  has a bounded support.

From this result it is easily deduced the following

**COROLLARY 1.2.** *Let  $X$  be an infinite-dimensional Banach space. The following statements are equivalent.*

- (1)  *$X$  has a  $C^p$  smooth (and Lipschitz) bump function.*
- (2) *There exist a bounded contractible open subset  $U$  of  $X$  and a continuous function  $f: \bar{U} \rightarrow \mathbb{R}$  such that  $f$  is  $C^p$  smooth (and Lipschitz) in  $U$ ,  $f = 0$  on  $\partial U$ , and yet  $f'(x) \neq 0$  for all  $x \in U$ , that is, Rolle's theorem fails in  $X$ .*
- (3) *There exist a  $C^p$  smooth (and Lipschitz) function  $f: X \rightarrow [0, 1]$  and a bounded contractible open subset  $U$  of  $X$  such that  $f = 0$  precisely on  $X \setminus U$  and yet  $f'(x) \neq 0$  for all  $x \in U$ .*

Just in order to complete the picture of Rolle's theorem in infinite-dimensional Banach spaces let us quote the two positive results from [3, 6] on approximate and subdifferential substitutes of Rolle's theorem, which guarantee the existence of arbitrarily small derivatives (instead of vanishing ones) for every function satisfying (in an approximate manner) the conditions of the classical Rolle's theorem. Here, Baire category arguments make up for the lack of compactness, but one has to pay an  $\varepsilon$ , as is usual in such cases.

**THEOREM 1.3** (Azagra–Gómez–Jaramillo). *Let  $U$  be a bounded connected open subset of a Banach space  $X$ . Let  $f: \bar{U} \rightarrow \mathbb{R}$  be a bounded continuous function which is (Gâteaux) differentiable in  $U$ . Let  $R > 0$  and  $x_0 \in U$  be such that  $\text{dist}(x_0, \partial U) = R$ . Suppose that  $f(\partial U) \subseteq [-\varepsilon, \varepsilon]$  for some  $\varepsilon > 0$ . Then there exists some  $x_\varepsilon \in U$  such that  $\|f'(x_\varepsilon)\| \leq \frac{\varepsilon}{R}$ .*

**THEOREM 1.4** (Azagra–Deville). *Let  $U$  be a bounded connected open subset of a Banach space  $X$  which has a  $C^1$  smooth Lipschitz bump function. Let  $f: \bar{U} \rightarrow \mathbb{R}$  be a bounded continuous function, and let  $R > 0$  and  $x_0 \in U$  be such that  $\text{dist}(x_0, \partial U) = R$ . Suppose that  $f(\partial U) \subseteq [-\varepsilon, \varepsilon]$  for some  $\varepsilon > 0$ . Then*

$$\inf \{ \|p\| : p \in D^-f(x) \cup D^+f(x), x \in U \} \leq \frac{2\varepsilon}{R}.$$

(Here  $D^-f(x)$  and  $D^+f(x)$  denote the subdifferential and superdifferential sets of  $f$  at  $x$ , respectively; see [17, p. 339] for the definitions.)

In [16, 22], some results are shown which are related to Theorem 1.4; in these papers R. Deville and G. Godefroy provide mean value inequalities for non-differentiable functions.

The “twisted tube” method that we develop in Section 2 in order to prove Theorem 1.1 is interesting in itself and, with little more work, provides a useful characterization of  $C^p$  smoothness in infinite-dimensional Banach spaces related to the existence of a certain kind of *deleting diffeomorphisms*. Namely, we have the following

**THEOREM 1.5.** *Let  $X$  be an infinite-dimensional Banach space. The following assertions are equivalent.*

- (1)  *$X$  has a  $C^p$  smooth bump function.*
- (2) *There exists a nonempty contractible closed subset  $D$  of the unit ball  $B_X$  and a  $C^p$  diffeomorphism  $f: X \rightarrow X \setminus D$  so that  $f$  restricts to the identity outside  $B_X$ .*

When  $X$  has a (not necessarily equivalent)  $C^p$  smooth norm this result was already known [1, 5, 7] and, moreover, one can take for  $D$  a single point, or a small ball. Theorem 1.5 provides a new result in the case when  $X$  possesses a  $C^p$  smooth bump but has no equivalent  $C^p$  smooth norm. Unfortunately, it is still unknown whether Theorem 1.5 is true in full generality when  $D$  is a single point. The proof we give here does not clarify this question (in our proof  $D$  is nothing but a small “twisted tube” inside  $B_X$ ). Nevertheless, some important applications of smooth negligibility do not require such accurate instruments as a diffeomorphism deleting just a single point, and it is often enough to use diffeomorphisms which remove a small bounded set, as in the statement of Theorem 1.5. Indeed, this theorem will allow us to deduce two interesting corollaries.

First, the celebrated Brouwer’s fixed point theorem fails even for smooth self-mappings of balls or starlike bodies in all infinite-dimensional Banach spaces. Let us recall that Brouwer’s theorem states that every continuous self-map of the unit ball of a finite-dimensional normed space admits a fixed point. This is the same as saying that there is no continuous retraction from the unit ball onto the unit sphere, or that the unit sphere is not contractible (the identity map on the sphere is not homotopic to a constant map). In infinite dimensions the situation is completely different and Brouwer’s theorem is no longer true (see [2, 8, 9, 13, 24, 31, 32]). Theorem 1.5 yields a trivial proof that Brouwer’s theorem is false in infinite dimensions even for smooth self-mappings of balls or starlike bodies; this is a particular case (the non-Lipschitz one) of the main result in [2].

Second, we deduce from the above characterization that the support of the bump functions which violate Rolle’s theorem can always be assumed to be a smooth starlike body. This is all shown in Section 3.

In Section 2 we give the proofs of Theorems 1.1 and 1.5. A much simpler proof of Theorem 1.1 for the non-Lipschitz case is included in this section too.

## 2. THE PROOFS

The idea behind the proof of Theorem 1.1 is as simple as this. First we build a twisted tube  $T$  of infinite length in the interior of the unit ball  $B_X$ , with a beginning but with no end. This twisted tube can be thought of as directed by an ever-winding infinite path  $p$  that gets lost in the infinitely many dimensions of our space  $X$ . In technical words, one can construct a diffeomorphism  $\pi$  between a straight (unbounded) half-cylinder  $C$  and a twisted (bounded) tube  $T$  contained in  $B_X$ . The tube  $T$  is going to be the support of a smooth bump function  $f$  that does not satisfy Rolle’s theorem. In order to define such a function  $f$  we only have to make it strictly

increase in the direction which is tangent to the leading path  $p$  at each point of the tube  $T$ . The graph of  $f$  would thus represent an ever-ascending stairway built upon our twisted tube, with a beginning but no end.

The spirit of the proof that (1) implies (2) in Theorem 1.5 is not very different. We will make use of the diffeomorphism  $\pi$  between a straight (unbounded) half-cylinder  $C$  and a bounded twisted tube  $T$  contained in  $B_X$ . If we consider a straight closed half-cylinder  $C'$  contained in the interior of  $C$  and directed by the same line as  $C$ , it is elementary that there is a diffeomorphism  $g: X \rightarrow X \setminus C'$  so that  $g$  restricts to the identity outside  $C$ . In fact this is true even in the plane. Now, by composing this diffeomorphism  $g$  with the diffeomorphisms  $\pi$  and  $\pi^{-1}$  that give us an appropriate coordinate system in the twisted tube  $T = \pi(C)$ , we get a diffeomorphism  $f: X \rightarrow X \setminus T'$ , where  $T' = \pi(C')$  is a smaller closed twisted tube inside  $T$ , and  $f$  restricts to the identity outside the unit ball. The precise definition of  $f$  would be  $f(x) = \pi(g(\pi^{-1}(x)))$  if  $x \in T$ , and  $f(x) = x$  if  $x \in X \setminus T$ . If we take  $D = T'$  we are done.

In the rest of this section we will be involved in the task of formalizing these ideas.

The following lemma guarantees the existence of bounded infinite twisted tubes in all infinite-dimensional Banach spaces.

**LEMMA 2.1.** *There are universal constants  $M > 0$  (large) and  $\varepsilon > 0$  (small) such that, for every infinite-dimensional Banach space  $X$ , if we consider the decomposition  $X = H \oplus [z]$  (where  $H = \text{Ker } z^*$  for some  $z^* \in X^*$  with  $z^*(z) = \|z^*\| = \|z\| = 1$ ) and the open half-cylinder  $C$  of diameter  $2\varepsilon$ , directed by  $z$ , and with base on  $H$ ,  $C = \{x + tz \in X : \|x\| < \varepsilon, t > 0\}$ , then there exists an injection  $\pi: C \rightarrow B_X$  which is a  $C^\infty$  diffeomorphism onto its image. The image  $T = \pi(C)$  is thus a bounded open set which we will call a bounded open infinitely twisted tube in  $X$ . Moreover, the first derivatives of the mappings  $\pi: C \rightarrow T$  and  $\pi^{-1}: T \rightarrow C$  are both uniformly bounded by  $M$ .*

Assume for a while that Lemma 2.1 is already established and let us explain how Theorems 1.1 and 1.5 can be deduced.

*Proof of Theorem 1.1.* Consider the diffeomorphism  $\pi: C \rightarrow T \subset B_X$  from Lemma 2.1. Take a  $C^p$  smooth (Lipschitz) non-negative bump function  $\varphi$  on  $H$  so that the support of  $\varphi$  is contained in the base of  $C$ , that is,  $\varphi(x) = 0$  whenever  $\|x\| \geq \frac{\varepsilon}{2}$ , for instance. Pick a  $C^\infty$  smooth real function  $\mu: \mathbb{R} \rightarrow [0, 1]$  such that  $\mu(t) = 0$  for  $t \leq 1$ ,  $0 < \mu(t) < 1$  for  $t > 1$  and  $0 < \mu'(t) < 1$  for all  $t > 1$ . Then define  $g: X = H \oplus [z] \rightarrow \mathbb{R}$  by

$$g(x, t) = \varphi(x) \mu(t).$$

It is plain that  $g$  is a  $C^p$  smooth (Lipschitz) function such that  $g'(x, t) \neq 0$  for every  $(x, t) \in \text{int}(\text{supp } g)$ , that is, for every  $(x, t)$  such that  $g(x, t) \neq 0$  (take into account that the interior of the support of  $g$  coincides in this case with the open support of  $g$ , that is the set of points at which  $g$  does not vanish). Indeed,

$$g'(x, t)(0, 1) = \frac{\partial g}{\partial t}(x, t) = \varphi(x) \mu'(t)$$

and therefore  $g'(x, t)(0, 1) = 0$  if and only if  $\varphi(x) = 0$  or  $\mu'(t) = 0$ , which happens if and only if  $\varphi(x) = 0$  or  $\mu(t) = 0$ , that is to say,  $g(x, t) = 0$ . Now let us define  $f: X \rightarrow \mathbb{R}$  by

$$f(y) = \begin{cases} g(\pi^{-1}(y)) & \text{if } y \in T; \\ 0 & \text{if } y \notin T. \end{cases}$$

It is clear that  $f$  is a well defined  $C^p$  smooth (Lipschitz) function, and  $\text{supp}(f) = \pi(\text{supp}(g)) \subset T$ , from which it follows that  $f$  has a bounded support. We claim that  $f'(y) \neq 0$  whenever  $y \in \text{int}(\text{supp } f)$ , that is,  $f$  does not satisfy Rolle's theorem. Indeed, if  $y \in \text{int}(\text{supp } f)$  then  $\pi^{-1}(y) = (x, t) \in \text{int}(\text{supp } g)$  and therefore  $g'(x, t)(0, 1) \neq 0$ . But then

$$f'(y) = g'(x, t) \circ D\pi^{-1}(y) \neq 0,$$

because  $D\pi^{-1}(y)$  is a linear isomorphism. ■

Now we will turn our attention to the proof of Theorem 1.5. Before proceeding with the proof, let us fix some standard terminology and notation used throughout this section and the following one. A closed subset  $A$  of a Banach space  $X$  is said to be a starlike body provided  $A$  has a non-empty interior and there exists a point  $x_0 \in \text{int } A$  such that each ray emanating from  $x_0$  meets the boundary of  $A$  at most once. In this case we will say that  $A$  is *starlike with respect to*  $x_0$ . When dealing with starlike bodies, we can always assume that they are starlike with respect to the origin (up to a suitable translation), and we will do so unless otherwise stated.

For a starlike body  $A$ , the characteristic cone of  $A$  is defined as

$$ccA = \{x \in X \mid rx \in A \text{ for all } r > 0\},$$

and the Minkowski functional of  $A$  as

$$q_A(x) = \inf \left\{ \lambda > 0 \mid \frac{1}{\lambda} x \in A \right\}$$

for all  $x \in X$ . It is easily seen that for every starlike body  $A$  its Minkowski functional  $q_A$  is a continuous function which satisfies  $q_A(rx) = rq_A(x)$  for every  $r \geq 0$  and  $q_A^{-1}(0) = ccA$ . Moreover,  $A = \{x \in X \mid q_A(x) \leq 1\}$ , and  $\partial A = \{x \in X \mid q_A(x) = 1\}$ , where  $\partial A$  stands for the boundary of  $A$ . Conversely, if  $\psi: X \rightarrow [0, \infty)$  is continuous and satisfies  $\psi(\lambda x) = \lambda\psi(x)$  for all  $\lambda \geq 0$ , then  $A_\psi = \{x \in X \mid \psi(x) \leq 1\}$  is a starlike body. Convex bodies (that is, closed convex sets with nonempty interior) are an important kind of starlike bodies. We will say that  $A$  is a  $C^p$  smooth (Lipschitz) starlike body provided its Minkowski functional  $q_A$  is  $C^p$  smooth (and Lipschitz) on the set  $X \setminus q_A^{-1}(0)$ .

It is worth noting that for every Banach space  $(X, \|\cdot\|)$  with a  $C^p$  smooth (Lipschitz) bump function there exist a functional  $\psi$  and constants  $a, b > 0$  such that  $\psi$  is  $C^p$  smooth (Lipschitz) away from the origin,  $\psi(\lambda x) = |\lambda| \psi(x)$  for every  $x \in X$  and  $\lambda \in \mathbb{R}$ , and  $a \|x\| \leq \psi(x) \leq b \|x\|$  for every  $x \in X$  (see [17, Proposition II.5.1]). The level sets of this function are precisely the boundaries of the smooth bounded starlike bodies  $A_c = \{x \in X \mid \psi(x) \leq c\}$ ,  $c \in \mathbb{R}$ . This shows in particular that every Banach space having a  $C^p$  smooth (Lipschitz) bump function has a  $C^p$  smooth (Lipschitz) bounded starlike body as well. The converse is clearly true.

*Proof of Theorem 1.5.* First of all let us choose a number  $\varepsilon > 0$ , a cylinder  $C$ , a bounded twisted tube  $T$ , and a diffeomorphism  $\pi: C \rightarrow T$  from Lemma 2.1.

Let  $B$  be a  $C^\infty$  smooth convex body in the plane  $\mathbb{R}^2$  whose boundary contains the set

$$\left\{ (s, t): t = -1, |s| \leq \frac{\varepsilon}{4} \right\} \cup \left\{ (s, t): |s| = \frac{\varepsilon}{2}, t \geq -1 + \frac{\varepsilon}{4} \right\},$$

and let  $q_B$  be the Minkowski functional of  $B$ . Define  $B' = \frac{1}{2} B = \{(s, t): q_B(s, t) \leq \frac{1}{2}\}$ . Let  $\theta: (\frac{1}{2}, \infty) \rightarrow [0, \infty)$  be a  $C^\infty$  smooth real function so that  $\theta'(t) < 0$  for  $\frac{1}{2} < t < 1$ ,  $\theta(t) = 0$  for  $t \geq 1$ , and  $\lim_{t \rightarrow 1/2^+} \theta(t) = +\infty$ . Now define  $\varphi: \mathbb{R}^2 \setminus B' \rightarrow \mathbb{R}^2$  by

$$\varphi(s, t) = (\varphi_1(s, t), \varphi_2(s, t)) = (s, t + \theta(q_B(s, t))).$$

It is elementary to check that  $\varphi$  is a  $C^\infty$  diffeomorphism from  $\mathbb{R}^2 \setminus B'$  onto  $\mathbb{R}^2$  so that  $\varphi$  restricts to the identity outside the band  $B$ .

Next, recall that since  $X$  has a  $C^p$  smooth bump then it has a  $C^p$  bounded starlike body  $A$  as well. If  $X = H \oplus [z]$ , take  $W = A \cap H$ , which is a  $C^p$  bounded starlike body in  $H$ , and denote by  $q_W$  its Minkowski functional. We can assume that  $W \subseteq B(0, 1)$ , that is,  $\|x\| \leq q_W(x)$  for all  $x \in H$ . Let us define

$$\psi(x, t) = q_B(q_W(x), t)$$

for all  $(x, t) \in X = H \oplus [z]$ . It is clear that  $\psi$  is a continuous function which is positive-homogeneous and  $C^p$  smooth away from the half-line  $L = \{(x, t) \in X : x = 0, t \geq 0\}$ . Then the sets

$$U = \{(x, t) \in X : \psi(x, t) \leq 1\}, \quad U' = \{(x, t) \in X : \psi(x, t) \leq \frac{1}{2}\}$$

are cylindrical  $C^p$  starlike bodies whose characteristic cones are the half-line  $L$ . If we define

$$h(x, t) = (x, (\varphi^{-1})_2(q_W(x), t))$$

for  $(x, t) \in X = H \oplus [z]$ , it is not difficult to realize that  $h$  is a  $C^p$  diffeomorphism from  $X$  onto  $X \setminus U'$  so that  $h$  restricts to the identity outside  $U$ . The inverse of  $h$  is given by

$$h^{-1}(x, t) = (x, t + \theta(\psi(x, t))).$$

Now consider the point  $p_0 = (0, 2) \in X = H \oplus [z]$  and the cylindrical bodies  $V := p_0 + U$  and  $V' := p_0 + U'$ , and put  $g(x, t) = h(x, t - 2)$ . Then  $g: X \rightarrow X \setminus V'$  is a  $C^p$  diffeomorphism such that  $g$  is the identity outside  $V$ . Note that, since  $W \subseteq B(0, 1)$ , we have that  $V' \subset V \subset C = \{(x, t) \in X : \|x\| < \varepsilon, t > 0\}$ . Let us define

$$f(x) = \begin{cases} \pi(g(\pi^{-1}(x))) & \text{if } x \in T; \\ x & \text{otherwise.} \end{cases}$$

It is then clear that  $f$  is a  $C^p$  diffeomorphism from  $X$  onto  $X \setminus T'$ , where  $T' = \pi(V')$  is a smaller closed twisted tube inside  $\pi(V) \subseteq T$ , and  $f$  restricts to the identity outside the larger tube  $\pi(V) \subset T$ , which is contained in  $B_X$ . This completes the proof that (1) implies (2).

Conversely, if there is such an  $f$  as in (2), we can assume that  $f(0) \neq 0$  and take  $x^* \in X^*$  so that  $x^*(f(0)) \neq 0$ ; then the function  $b: X \rightarrow \mathbb{R}$  defined by  $b(x) = x^*(x - f(x))$  is a  $C^p$  smooth bump on  $X$ . ■

*Proof of Lemma 2.1.* We will make use of the following lemma, which guarantees the existence of an appropriate path of linear isomorphisms. Here  $\text{Isom}(X)$  stands for the set of linear isomorphisms of  $X$ , which is regarded as a subset of  $\mathcal{L}(X, X)$ , the linear continuous mappings of  $X$  into  $X$ .

**LEMMA 2.2.** *There is a universal constant  $K > 0$  such that for every infinite-dimensional Banach space  $X$  there are paths  $\beta: [0, \infty) \rightarrow \text{Isom}(X)$  and  $p: [0, \infty) \rightarrow X$  with the following properties:*



- (i) Both  $\beta$  and  $p$  are  $C^\infty$  smooth, as well as the path of inverse isomorphisms  $\beta^{-1}: [0, \infty) \rightarrow \text{Isom}(X)$ ,  $\beta^{-1}(t) = [\beta(t)]^{-1}$ .
- (ii)  $1 \leq \|\beta(t)\| \leq K$  and  $1 \leq \|\beta^{-1}(t)\| \leq K$  for all  $t \in [0, \infty)$ .
- (iii)  $\sup_{t \geq 0} \|\beta'(t)\| \leq K$  and  $\sup_{t \geq 0} \|(\beta^{-1})'(t)\| \leq K$ .
- (iv) There exists a certain  $v \in X$ , with  $1 \geq \|v\| \geq \frac{1}{K}$ , such that  $p'(t) = \beta(t)(v)$  for all  $t \geq 0$ .
- (v) For every  $t, s \in [0, \infty)$  we have that  $\|p(t) - p(s)\| \geq \frac{1}{K} \min\{1, |t - s|\}$ .

*Proof.* Let  $(x_n)_{n=0}^\infty$  be a normalized basic sequence in  $X$  with biorthogonal functionals  $(x_n^*)_{n=0}^\infty \subset X^*$  (that is,  $x_n^*(x_k) = \delta_{n,k} = 1$  if  $n=k$ , and 0 otherwise) satisfying  $\|x_n^*\| \leq 3$  (one can always take such sequences, see [15, p. 93; 18, p. 39]). For  $n \geq 1$  set  $v_n = x_n - x_{n-1}$ . Let  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^\infty$  function with the following properties:

- (a)  $\theta(t) = 0$  whenever  $t \leq -\frac{1}{2}$  or  $t \geq 1$ ;
- (b)  $\theta(t) = 1$  for  $t \in [0, \frac{1}{2}]$ ;
- (c)  $\theta'(t) > 0$  for  $t \in (-\frac{1}{2}, 0)$ ;
- (d)  $\theta(t) = 1 - \theta(t-1)$  for  $t \in [\frac{1}{2}, 1]$ ;
- (e)  $\sup_{t \in \mathbb{R}} |\theta'(t)| \leq 3$ .

For  $n \geq 1$  let us define  $\theta_n: \mathbb{R} \rightarrow \mathbb{R}$  by  $\theta_n(t) = \theta(t-n+1)$ . It is clear that the functions  $\theta_n$  are all  $C^\infty$  smooth and have Lipschitz constant less than or equal to 3,  $\theta_n = 0$  on  $(-\infty, n-1-\frac{1}{2}] \cup [n, \infty)$ ,  $\theta_n = 1$  on  $[n-1, n-\frac{1}{2}]$ , and  $\theta_n(t) = 1 - \theta_{n+1}(t)$  for all  $t \in [n-1, n+\frac{1}{2}]$ . Note that the  $\theta_n$  form a partition of unity.

Our path  $\beta$  of linear isomorphisms is going to be of the form

$$\beta(t) = \sum_{n=1}^{\infty} \theta_n(t) S_n,$$

where each  $S_n \in \text{Isom}(X)$  takes the vector  $v_1$  into  $v_n$  and for every  $\lambda \in [0, 1]$  the mapping  $L_{n,\lambda} = (1-\lambda)S_n + \lambda S_{n+1}$  is still a linear isomorphism and, moreover, the families of isomorphisms  $\{L_{n,\lambda}\}_{n \in \mathbb{N}, \lambda \in [0, 1]}$  and  $\{L_{n,\lambda}^{-1}\}_{n \in \mathbb{N}, \lambda \in [0, 1]}$  are uniformly bounded. Let us define the isomorphisms  $S_n$ . They are going to be of the form

$$S_n(x) = x + f_n(x)(v_n - v_1),$$

where  $f_n \in X^*$  satisfies  $f_n(v_1) = 1 = f_n(v_n)$ , and  $\|f_n\| \leq 18$  (the exact definition of  $f_n$  will be given later). Their inverses  $S_n^{-1}$  will be

$$S_n^{-1}(y) = y - f_n(y)(v_n - v_1).$$

We want the linear mappings  $L_{n,\lambda} = (1-\lambda)S_n + \lambda S_{n+1}$  to be linear isomorphisms. We have

$$y = L_{n,\lambda}(x) = x + (1-\lambda)f_n(x)(v_n - v_1) + \lambda f_{n+1}(x)(v_{n+1} - v_1), \quad (1)$$

from which

$$x = y - [(1-\lambda)f_n(x)(v_n - v_1) + \lambda f_{n+1}(x)(v_{n+1} - v_1)], \quad (2)$$

and we need to write  $f_n(x)$  and  $f_{n+1}(x)$  as linear functions of  $y$ . If we apply the functionals  $f_n$  and  $f_{n+1}$  successively to Eq. (1), we denote  $A_n = f_n(x)$ ,  $B_n = f_{n+1}(x)$ ,  $C_n = f_n(y)$ ,  $D_n = f_{n+1}(y)$ , and we take into account that  $1 = f_n(v_1) = f_n(v_n) = f_{n+1}(v_1)$ , then we obtain the system

$$\begin{cases} A_n + \lambda[f_n(v_{n+1}) - 1] B_n = C_n \\ (1-\lambda)[f_{n+1}(v_n) - 1] A_n + B_n = D_n, \end{cases} \quad (3)$$

which we want to have a unique solution for  $A_n, B_n$ . The determinant of this system is

$$\Delta_{n,\lambda} = 1 - \lambda(1-\lambda)[f_{n+1}(v_n) - 1][f_n(v_{n+1}) - 1],$$

and we want  $\Delta_{n,\lambda}$  to be bounded below by a strictly positive number, and this bound has to be uniform in  $n, \lambda$ . For  $n \geq 3$  this can easily be done by setting

$$f_n = x_1^* - x_{n-1}^*$$

(so that  $f_n(v_n) = 1 = f_n(v_1)$ ,  $f_n(v_{n+1}) = 0$ ,  $f_{n+1}(v_n) = -1$ , and therefore  $\Delta_{n,\lambda} = (1-\lambda)^2 + \lambda^2 \geq \frac{1}{2}$  for all  $\lambda \in [0, 1]$ ). For  $n = 1, 2$ , put

$$f_2 = x_1^* + 2x_2^* + \frac{7}{3}x_3^*, \quad \text{and} \quad f_1 = x_1^*;$$

then  $f_2(v_3) = \frac{1}{3}$ ,  $f_2(v_2) = 1$ ,  $f_2(v_1) = 1$ ,  $f_3(v_2) = -2$ ,  $f_1(v_2) = -1$ ,  $f_1(v_1) = 1$ , and everything is fine (indeed,  $\Delta_{1,\lambda} = 1$  and  $\Delta_{2,\lambda} = (1-\lambda)^2 + \lambda^2 \geq \frac{1}{2}$  for all  $\lambda \in [0, 1]$ ).

Therefore, with these definitions, the linear system (3) has a unique solution for  $A_n, B_n$ , which can be easily calculated and estimated by Cramer's rule, of the form

$$A_n(y) = \frac{1}{\Delta_{n,\lambda}} (f_n(y) - \lambda[f_n(v_{n+1}) - 1] f_{n+1}(y))$$

$$B_n(y) = \frac{1}{\Delta_{n,\lambda}} (f_{n+1}(y) - (1-\lambda)[f_{n+1}(v_n) - 1] f_n(y)).$$

The linear forms  $y \mapsto A_n(y)$ ,  $y \mapsto B_n(y)$  satisfy that  $\|A_n\| \leq 144$  and  $\|B_n\| \leq 144$  for all  $n$ , as is easily checked. Now, by substituting  $f_n(x) = A_n(y)$  and  $f_{n+1}(x) = B_n(y)$  in (2) we get the expression for the inverse of  $L_{n,\lambda}$ , that is,

$$x = L_{n,\lambda}^{-1}(y) = y - [(1 - \lambda) A_n(y)(v_n - v_1) + \lambda B_n(y)(v_{n+1} - v_1)]. \quad (4)$$

By taking into account that  $\|A_n\| \leq 144$ ,  $\|B_n\| \leq 144$ ,  $\|f_n\| \leq 18$  and  $\|v_n - v_1\| \leq 4$  for all  $n$ , one can estimate that  $1 \leq \|L_{n,\lambda}\| \leq 73$  and  $1 \leq \|L_{n,\lambda}^{-1}\| \leq 577$  for all  $n \in \mathbb{N}$ ,  $\lambda \in [0, 1]$ .

So let us define  $\beta: [0, \infty) \rightarrow \text{Isom}(X)$  by

$$\beta(t) = \sum_{n=1}^{\infty} \theta_n(t) S_n. \quad (5)$$

This path is well defined because the sum is locally finite; in fact, from the definition of  $\theta_n$  it is clear that, for a given  $t_0 \in [0, \infty)$  there exist some  $\delta > 0$  and  $N = N(t_0) \in \mathbb{N}$  such that  $\beta(t) = \theta_N(t) S_N + \theta_{N+1}(t) S_{N+1}$  for all  $t \in (t_0 - \delta, t_0 + \delta)$ , that is,  $\beta$  is locally of the form  $\beta(t) = L_{n,\lambda(t)}$ , where  $\lambda(t) = \theta_n(t)$ . This implies that the  $\beta(t)$  are really linear isomorphisms and that the path is  $C^\infty$  smooth.

On the other hand, the path  $\beta^{-1}(t) = [\beta(t)]^{-1} \in \text{Isom}(X)$  is  $C^\infty$  smooth as well, because it is the composition of our path  $\beta$  with the mapping  $\varphi: \text{Isom}(X) \rightarrow \text{Isom}(X)$ ,  $\varphi(U) = U^{-1}$ , which is  $C^\infty$  smooth and whose derivative is given by  $\varphi'(U)(S) = -U^{-1} \circ S \circ U^{-1}$  for every  $S \in \mathcal{L}(X, X)$  (see [14, Theorem 5.4.3]). This proves condition (i) of the lemma.

Next, by bearing in mind the local expression of  $\beta$  and the above estimations for  $\|L_{n,\lambda}\|$  and  $\|L_{n,\lambda}^{-1}\|$ , we deduce that

$$1 \leq \|\beta(t)\| \leq R \quad \text{and} \quad 1 \leq \|\beta^{-1}(t)\| \leq R$$

for all  $t \in [0, \infty)$ , where  $R \geq 577$  will be fixed later. This shows condition (ii). Now, if  $t_0 \in [0, \infty)$  and we write  $\beta(t) = \theta_N(t) S_N + \theta_{N+1}(t) S_{N+1}$  for  $t \in (t_0 - \delta, t_0 + \delta)$  as above, then it is clear that  $\beta'$  is locally of the form

$$\beta'(t) = \theta'_N(t) S_N + \theta'_{N+1}(t) S_{N+1}$$

and therefore

$$\|\beta'(t)\| \leq |\theta'_N(t)| \|S_N\| + |\theta'_{N+1}(t)| \|S_{N+1}\| \leq 3(73 + 73) = 438,$$

from which we get  $\sup_{t \geq 0} \|\beta'(t)\| \leq 438 \leq R$ . Moreover, we have

$$(\beta^{-1})'(t) = -(\beta(t))^{-1} \circ \beta'(t) \circ (\beta(t))^{-1}$$

and therefore

$$\|(\beta^{-1})'(t)\| \leq \|\beta(t)^{-1}\|^2 \|\beta'(t)\| \leq (577)^2 438,$$

from which  $\sup_{t \geq 0} \|(\beta^{-1})'(t)\| \leq R$  and condition (iii) is satisfied as well provided we fix  $R = (577)^2 438$ .

Now let us define the path  $p: [0, \infty) \rightarrow X$  by

$$p(t) = \int_{-\infty}^t \beta(s)(v_1) ds = \int_{-\infty}^t \left( \sum_{n=1}^{\infty} \theta_n(s) S_n(v_1) \right) ds.$$

It is clear that  $p$  is a  $C^\infty$  smooth path in  $X$ , and  $p'(t) = \beta(t)(v_1)$  for all  $t \geq 0$  (from which it follows that  $p$  is Lipschitz). Let us see that  $p$  is bounded. For a given  $t > 0$  there exists  $N = N(t) \in \mathbb{N}$  so that  $N - 1 - \frac{1}{2} \leq t \leq N - \frac{1}{2}$  and therefore, taking into account the definition of  $\theta_n$  and the fact that  $S_n(v_1) = v_n = x_n - x_{n-1}$  for all  $n$ , we have that

$$\begin{aligned} \|p(t)\| &= \left\| \int_{-\infty}^t \sum_{n=1}^{\infty} \theta_n(s) S_n(v_1) ds \right\| = \left\| \sum_{n=1}^{\infty} \left( \int_{-\infty}^t \theta_n(s) ds \right) v_n \right\| \\ &= \left\| \left( \int_{-\infty}^{\infty} \theta(s) ds \right) \sum_{n=1}^{N-1} v_n + \left( \int_{-\infty}^t \theta_N(s) ds \right) v_N \right\| \\ &\leq \left( \int_{-\infty}^{\infty} \theta(s) ds \right) \left\| \sum_{n=1}^{N-1} v_n \right\| + \left( \int_{-\infty}^{\infty} \theta(s) ds \right) \|v_N\| \\ &= \left( \int_{-\infty}^{\infty} \theta(s) ds \right) \left( \|x_{N-1} - x_0\| + \|x_N - x_{N-1}\| \right) \leq \frac{3}{2} (2 + 2) = 6. \end{aligned}$$

This shows that the image of  $p$  is contained in the ball  $B(0, 6)$  and  $p$  is bounded. Let us also remark that  $2 \geq \|v_1\| \geq x_1^*(x_1 - x_0) / \|x_1^*\| \geq \frac{1}{4}$ .

Finally, let us check that  $p$  satisfies the separation condition (v). Let  $0 \leq t < r$  and take  $N \in \mathbb{N}$  so that  $N - 1 - \frac{1}{2} < r \leq N - \frac{1}{2}$ ; then we have

$$\begin{aligned} p(r) - p(t) &= \sum_{n=1}^{\infty} \left( \int_t^r \theta_n(s) ds \right) v_n = \sum_{n=1}^{\infty} \left( \int_t^r \theta_n(s) ds \right) (x_n - x_{n-1}) \\ &= - \left( \int_t^r \theta_1(s) ds \right) x_0 + \sum_{k=1}^{N-1} \left( \int_t^r \theta_k(s) ds - \int_t^r \theta_{k+1}(s) ds \right) x_k \\ &\quad + \left( \int_t^r \theta_N(s) ds \right) x_N. \end{aligned}$$

By observing that  $\max\{1-s, 2s-1\} \geq \frac{1}{3}$  for all  $s \in \mathbb{R}$  and taking into account the definition of the  $\theta_n$ , it is not difficult to see that

$$\max \left\{ \int_t^r \theta_N(s) ds, \int_t^r \theta_{N-1}(s) ds - \int_t^r \theta_N(s) ds \right\} \geq \min \left\{ \frac{1}{3} |t-r|, a \right\}, \quad (6)$$

where  $a = \int_{-1/2}^0 \theta(s) ds > 0$ . Then, by applying either  $x_N^*$  or  $x_{N-1}^*$  to the expression for  $p(r) - p(t)$  above, depending on which the maximum in (6) is, and bearing in mind that  $x_n^*(x_k) = \delta_{n,k}$  and  $\|x_n^*\| \leq 3$  for all  $n, k$ , we get that

$$\max \{ x_N^*(p(r) - p(t)), x_{N-1}^*(p(r) - p(t)) \} \geq \min \left\{ \frac{1}{3} |t-r|, a \right\},$$

and it follows that  $\|p(r) - p(t)\| \geq \min \left\{ \frac{1}{9} |t-r|, \frac{a}{3} \right\}$ . Since  $a = \int_{-1/2}^0 \theta(s) ds \geq 1/8$  and  $R = (577)^2 438$ , this clearly implies that

$$\|p(r) - p(t)\| \geq \frac{1}{R} \min \{ 1, |t-r| \}$$

for all  $t, r \geq 0$ .

In order to get paths  $\beta$  and  $p$  and a vector  $v$  with properties (i)–(v) and such that  $p$  is contained in the unit ball, it is enough to multiply them all by  $\frac{1}{6}$ . ■

We now proceed with the proof of Lemma 2.1. Consider  $X = H \oplus [z] = H \times \mathbb{R}$  and  $C_\varepsilon = \{x + tz \in X : \|x\| < \varepsilon, t > 0\}$ , where  $H = \text{Ker } z^*$  for some  $z^* \in X^*$  with  $z^*(z) = \|z^*\| = \|z\| = 1$ , and  $\varepsilon > 0$  is to be fixed later. Let  $\beta$  and  $p$  be the paths from Lemma 2.2. There is no loss of generality if we assume that  $v \in [z]$ ,  $z^*(v) \geq \frac{1}{K}$ . Let us define  $\pi: C_\varepsilon \rightarrow X$  by

$$\pi(x, t) = \beta(t)(x) + p(t).$$

It is clear that  $\pi$  is  $C^\infty$  smooth and has a bounded derivative. We are going to show that  $\pi$  is a diffeomorphism onto its image,  $T_\varepsilon$ , and  $\pi^{-1}: T_\varepsilon \rightarrow C_\varepsilon$  has a bounded derivative as well. To this end let us define the path  $\alpha: [0, \infty) \rightarrow X^*$  by

$$\alpha(t) = f_t = z^* \circ \beta^{-1}(t).$$

This is a  $C^\infty$  smooth and Lipschitz path in  $X^*$ , and  $\alpha(t) = f_t$  satisfies that  $\text{Ker } f_t = \beta(t)(H)$ . It is clear from this definition and the properties of  $\beta$  and  $p$  that

- (i)  $\|\alpha'(t)\| \leq K$ , and
- (ii)  $\alpha(t)(p'(t)) = z^*(v) \geq \frac{1}{K}$

for all  $t \geq 0$ . Now, for a fixed (but arbitrary)  $y \in T_\varepsilon = \pi(C_\varepsilon)$ , let us introduce the auxiliary function  $F = F_y: [0, \infty) \rightarrow \mathbb{R}$  defined by

$$F(t) = \alpha(t)(y - p(t)).$$

We have that

$$\begin{aligned} F'_y(r) &= \alpha'(r)(y - p(r)) - \alpha(r)(p'(r)) \\ &\leq \|\alpha'(r)\| \|y - p(r)\| - \alpha(r)(p'(r)) \\ &\leq K \|y - p(r)\| - \frac{1}{K} \end{aligned}$$

for all  $r \geq 0$ . If we choose  $\varepsilon > 0$  smaller than  $1/6K^5$  this implies that  $\pi$  is a  $C^\infty$  diffeomorphism onto its image, as we next see.

Indeed, let us first prove that  $\pi$  is an injection. Assume that  $y = \pi(x, t) = \pi(w, s)$  for some  $(x, t), (w, s) \in C_\varepsilon$ . Then we have  $y - p(t) = \beta(t)(x)$  and  $y - p(s) = \beta(s)(w)$ , so that  $x = \beta^{-1}(t)(y - p(t))$  and  $w = \beta^{-1}(s)(y - p(s))$ , and, in order to conclude that  $(x, t) = (w, s)$ , it is enough to see that  $t = s$ . Note that  $\beta(t)(x) - \beta(s)(w) = p(s) - p(t)$  and therefore, by (v) of Lemma 2.2,

$$\begin{aligned} \frac{1}{K} \min\{1, |t - s|\} &\leq \|p(s) - p(t)\| = \|\beta(t)(x) - \beta(s)(w)\| \\ &\leq \|\beta(t)(x)\| + \|\beta(s)(w)\| \leq K(\|x\| + \|w\|) \leq 2K\varepsilon \leq \frac{1}{3K^4}, \end{aligned}$$

so that  $|t - s| \leq 2K^2\varepsilon \leq 1/3K^3$ . Now, since  $p$  and  $\beta$  are both  $K$ -Lipschitz, for every  $r \in [t, s]$  we have that

$$\begin{aligned} \|y - p(r)\| &\leq \|y - p(t)\| + \|p(t) - p(r)\| = \|\beta(t)(x)\| + \|p(t) - p(r)\| \\ &\leq K \|x\| + K |t - r| \leq K\varepsilon + 2K^3\varepsilon \leq 3K^3\varepsilon. \end{aligned}$$

By combining this with the above estimation for  $F'_y(r)$  we get

$$F'_y(r) \leq K \|y - p(r)\| - \frac{1}{K} \leq 3K^4\varepsilon - \frac{1}{K} \leq -\frac{1}{2K} \quad (7)$$

for every  $r \in [t, s]$ . Now suppose that  $t \neq s$ . Then, since  $x = \beta^{-1}(t)(y - p(t))$  and  $w = \beta^{-1}(s)(y - p(s))$  are both in  $H$  we have that  $0 = z^*(x) = z^*(w) = F_y(t) = F_y(s)$ , so that, by the classical Rolle's theorem, there should exist some  $r \in (t, s)$  with  $F'_y(r) = 0$ . But this contradicts (7). Therefore  $t = s$  and  $\pi$  is an injection.

If, for a given  $y \in \pi(C_\varepsilon)$ , we denote by  $t(y)$  the unique  $t = t(y)$  such that  $y = \pi(\beta^{-1}(t)(y - p(t)), t)$  then it is clear that the inverse  $\pi^{-1}: T_\varepsilon \rightarrow C_\varepsilon$  is defined by

$$\pi^{-1}(y) = (\beta^{-1}(t(y))(y - p(t(y))), t(y)). \quad (8)$$

For each  $y$  the number  $t(y)$  is uniquely determined by the equation

$$G(y, t) := F_y(t) = 0,$$

and the argument above shows that

$$\frac{\partial G}{\partial t}(y, t) = F'_y(t) \leq -\frac{1}{2K} \quad (9)$$

for every  $y \in T_\varepsilon$  and  $t$  in a neighbourhood of  $t(y)$ . Then, according to the implicit function theorem we get that the function  $y \mapsto t(y)$  is  $C^\infty$  smooth. Furthermore, we have that

$$t'(y) = \frac{-(\partial G / \partial y)(y, t(y))}{(\partial G / \partial t)(y, t(y))} = \frac{-z^* \circ \beta^{-1}(t(y))}{F'_y(t(y))},$$

and therefore, according to the above estimations,

$$\|t'(y)\| \leq \|z^* \circ \beta^{-1}(t(y))\| \frac{1}{|F'_y(t(y))|} \leq 2K^2,$$

which shows that  $y \mapsto t(y)$  has a bounded derivative as well. Then it is clear that  $\pi^{-1}$  is  $C^\infty$  and has a bounded derivative (all the functions involved in (8) have been proved to have bounded derivatives). This concludes the proof of Lemma 2.1.  $\blacksquare$

We will finish this section with a simple alternative proof of the failure of Rolle's theorem in the non-Lipschitz case.

*Remark 2.3.* If we drop the Lipschitz condition from the statement of Theorem 1.1, a much simpler proof based on the same idea is available. Let us make a sketch of this proof.

Consider the decomposition  $X = H \times \mathbb{R}$  and pick a non-negative  $C^p$  smooth bump function  $\varphi$  on  $H$  whose support is contained on the ball  $B_H(0, 1/16)$ . First, we construct a  $C^\infty$  smooth path  $q: [0, \infty) \rightarrow B_H$ , where  $B_H$  stands for the unit ball of the hyperplane  $H$ , with the property that  $q$  has no accumulation points at the infinity, that is,  $\lim_{n \rightarrow \infty} q(t_n)$  does not exist for any  $(t_n)$  going to  $\infty$ . This can easily be done by having  $q$  lost in the infinitely many dimensions of  $H$ . For instance, take a biorthogonal

sequence  $\{x_n, x_n^*\} \subseteq H \times H^*$  so that  $\|x_n\| = 1$  and  $\|x_n^*\| \leq 3$ , and consider a  $C^\infty$  function  $\theta: \mathbb{R} \rightarrow [0, 1]$  so that  $\text{supp } \theta \subseteq [-1, 1]$ ,  $\theta(0) = 1$ ,  $\theta'(t) < 0$  for  $t \in (0, 1)$ , and  $\theta(t-1) = 1 - \theta(t)$  for  $t \in [0, 1]$ . The path  $q$  may be defined as

$$q(t) = \sum_{n=1}^{\infty} \theta(t-n+1) x_n$$

for  $t \geq 0$ . Now we reparametrize  $q$  and define  $p: [0, 1) \rightarrow B_H$  by

$$p(t) = q\left(\frac{t}{1-t}\right).$$

Let  $\alpha: \mathbb{R} \rightarrow [0, 1]$  be a  $C^\infty$  smooth function so that  $\alpha(t) = 0$  for all  $t \leq 0$ , and  $\alpha'(t) > 0$  for all  $t > 0$ . Then the function  $g: X = H \times \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$g(x, t) = \begin{cases} \varphi(x - p(t)) \alpha(t) & \text{if } t \in [0, 1); \\ 0 & \text{otherwise} \end{cases}$$

is a  $C^p$  smooth bump function which does not satisfy Rolle's theorem. Indeed, it is easy to see that

$$g'(x, t)(p'(t), 1) = \varphi(x - p(t)) \alpha'(t) > 0,$$

and in particular  $g'(x, t) \neq 0$ , for all  $(x, t)$  in the interior of the support of  $g$ .

### 3. KILLING SINGULARITIES: THE FAILURE OF BROUWER'S FIXED POINT THEOREM IN INFINITE DIMENSIONS

Here we will present two applications of Theorem 1.5, both of which have in common the following principle: if you have a mapping with a single singular point or an isolated set of singularities that bother you, you can just kill them by composing your map with some deleting diffeomorphisms. In this way you obtain a new map which is as close as you want to the old one but does not have the adverse properties created by the singular points you eliminate.

*The Support of the Bumps That Violate Rolle's Theorem.* The bump function constructed in the proof of Theorem 1.1 has a weird support, namely a twisted tube. Some readers might judge this fact rather unpleasant and wonder whether it is possible to construct a bump function which does not satisfy Rolle's theorem and whose support is a nicer set, such as a ball or



a starlike body. To comfort those readers let us first recall that in infinite dimensions there is no topological difference between a tube (whether it is twisted or not) and a ball or a starlike body (see Theorem 3.1 in [5]). Furthermore, as we said above, Theorem 1.5 allows us to show that for a given  $C^p$  smooth bounded starlike body  $A$  in an infinite-dimensional Banach space  $X$ , it is always possible to construct a  $C^p$  smooth bump function on  $X$  which does not satisfy Rolle's theorem and whose support is precisely the body  $A$ .

**COROLLARY 3.1.** *Let  $X$  be an infinite-dimensional Banach space with a  $C^p$  smooth bounded starlike body  $A$ . Then there exists a  $C^p$  smooth bump function  $g$  on  $X$  whose support is precisely the body  $A$ , and with the property that  $g'(x) \neq 0$  for all  $x$  in the interior of  $A$  (that is,  $g$  does not satisfy Rolle's theorem).*

*Proof.* Let  $q_A$  be the Minkowski functional of  $A$ . We may assume that  $B_X \subseteq A$ . By Theorem 1.5 there is a closed subset  $D$  of  $A$  and a  $C^p$  diffeomorphism  $f: X \rightarrow X \setminus D$  which is the identity outside  $A$ . It can be assumed that the origin belongs to  $D$ . Then the function  $h: X \rightarrow \mathbb{R}$  defined by

$$h(x) = q_A(f(x))$$

is  $C^p$  smooth on  $X$ , restricts to the gauge  $q_A$  outside  $A$ , and has the remarkable property that  $h'(x) \neq 0$  for all  $x \in X$  (indeed,  $h'(x) = q'_A(f(x)) \cdot f'(x)$  is non-zero everywhere because  $q'_A(y) \neq 0$  whenever  $y \neq 0$ ,  $0 \notin f(X)$ , and  $f'(x)$  is a linear isomorphism at each point  $x$ ).

Now, take a  $C^\infty$  real function  $\theta: \mathbb{R} \rightarrow [0, 1]$  such that  $\theta(t) > 0$  for  $t \in (-1, 1)$ ,  $\theta = 0$  outside  $[-1, 1]$ ,  $\theta(t) = \theta(-t)$ ,  $\theta(0) = 1$ , and  $\theta'(t) < 0$  for all  $t \in (0, 1)$ . Then, if we define  $g: X \rightarrow \mathbb{R}$  by

$$g(x) = \theta(h(x)),$$

it is immediately checked that  $g$  is a  $C^p$  smooth bump on  $X$  which does not satisfy Rolle's theorem and whose support is precisely the body  $A$ . ■

*The Failure of Brouwer's Fixed Point Theorem in Infinite Dimensions.* The celebrated Brouwer's fixed point theorem tells us that every continuous self-map of the unit ball of a finite-dimensional normed space admits a fixed point. This is the same as saying that there is no continuous retraction from the unit ball onto the unit sphere, or that the unit sphere is not contractible (the identity map on the sphere is not homotopic to a constant map). However, none of the above forms of Brouwer's fixed point theorem remains valid in infinite dimensions. A nice counterexample was given by the pioneering results of Klee's on topological negligibility of points [29, 30]: for every infinite-dimensional Banach space  $X$  there always exists a homeomorphism  $h: X \rightarrow X \setminus \{0\}$  so that  $h$  restricts to the identity outside

the unit ball  $B_X$ . The required retraction of  $B_X$  onto the unit sphere  $S_X$  is then given by  $R(x) = h(x)/\|h(x)\|$  for  $x \in B_X$ . By taking into account the subsequent progress on topological negligibility of subsets made by C. Bessaga, T. Dobrowolski and the first-named author among others (see [1, 5, 11, 13, 19, 20]), this mapping  $h$  may even be assumed  $C^p$  smooth provided that the sphere  $S_X$  is  $C^p$  smooth. In [32] B. Nowak showed that for several infinite-dimensional Banach spaces Brouwer's theorem fails even for Lipschitz mappings (that is, under the strongest uniform-continuity condition), and in [9] Y. Benyamini and Y. Sternfeld generalized Nowak's result for all infinite-dimensional normed spaces. More recently, M. Cepedello and the first-named author showed that these results hold for the smooth Lipschitz category as well (see [2]).

The proof of these results in the general case is somewhat involved, but if we drop the Lipschitz condition then the fact that Brouwer's theorem is false in infinite dimensions even for smooth self-mappings of balls or starlike bodies is a trivial consequence of Theorem 1.5. The Lipschitz case is much harder to handle because the known diffeomorphisms which remove points or small balls from infinite-dimensional Banach spaces are not Lipschitz, so that the above "deleting diffeomorphisms approach" does not work in this case. For a better insight into these topics the reader should have a look at [8, Chaps. 3, 4, and 10].

**COROLLARY 3.2** (Azagra–Cepedello). *Let  $X$  be an infinite-dimensional Banach space and let  $A$  be a  $C^p$  smooth bounded starlike body. Then:*

- (1) *The boundary  $\partial A$  is  $C^p$  contractible.*
- (2) *There is a  $C^p$  smooth retraction from  $A$  onto  $\partial A$ .*
- (3) *There exists a  $C^p$  smooth mapping  $\varphi: A \rightarrow A$  without approximate fixed points.*

*Proof.* Let  $f: X \rightarrow X \setminus D$  be the diffeomorphism from Theorem 1.5. We may assume that the origin belongs to the deleted set  $D$  and that  $B_X \subseteq A$ , so that  $f$  restricts to the identity outside  $A$ . Then the formula

$$R(x) = \frac{f(x)}{q_A(f(x))},$$

where  $q_A$  is the Minkowski functional of  $A$ , defines a  $C^p$  smooth retraction from  $A$  onto the boundary  $\partial A$ . This proves (2).

Once we have such a retraction it is easy to prove parts (1) and (3): the formula  $\varphi(x) = -R(x)$  defines a  $C^p$  smooth self-mapping of  $A$  without approximate fixed points. On the other hand, if we pick a non-decreasing

$C^\infty$  function  $\zeta: \mathbb{R} \rightarrow \mathbb{R}$  so that  $\zeta(t) = 0$  for  $t \leq \frac{1}{4}$  and  $\zeta(t) = 1$  for  $t \geq \frac{3}{4}$ , then the formula

$$H(t, x) = R((1 - \zeta(t))x),$$

for  $t \in [0, 1]$ ,  $x \in \partial A$ , defines a  $C^p$  homotopy joining the identity to a constant on  $\partial A$ , that is,  $H$  contracts the pseudosphere  $\partial A$  to a point. ■

## ACKNOWLEDGMENTS

This research was carried out during a postdoctoral stay of the authors at the Equipe d'Analyse de l'Université Pierre et Marie Curie, Paris 6. The authors are indebted to the Equipe d'Analyse and very especially to Gilles Godefroy for their kind hospitality and generous advice.

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