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Approximation by smooth functions with no critical points on separable Banach spaces

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Abstract

We characterize the class of separable Banach spaces X such that for every continuous function $f : X \to \mathbb{R}$ and for every continuous function $\varepsilon : X \to (0, +\infty)$ there exists a C^1 smooth function $g : X \to \mathbb{R}$ for which $|f(x) - g(x)| \leqslant \varepsilon(x)$ and $g'(x) \neq 0$ for all $x \in X$ (that is, g has no critical points), as those infinite-dimensional Banach spaces X with separable dual X^*. We also state sufficient conditions on a separable Banach space so that the function g can be taken to be of class C^p, for $p = 1, 2, \dots, +\infty$. In particular, we obtain the optimal order of smoothness of the approximating functions with no critical points on the classical spaces $\ell_p(\mathbb{N})$ and $L_p(\mathbb{R}^n)$. Some important consequences of the above results are (1) the existence of *a non-linear Hahn–Banach theorem* and the *smooth approximation of closed sets*, on the classes of spaces considered above; and (2) versions of all these results for a wide class of *infinite-dimensional Banach manifolds*.

Keywords: Morse–Sard theorem; Smooth bump functions; Critical points; Approximation by smooth functions; Sard functions

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1. Introduction and main results

The Morse–Sard theorem [25,26] states that if $f:\mathbb{R}^n \to \mathbb{R}^m$ is a C^r smooth function, with $r > \max\{n-m, 0\}$, and C_f is the set of critical points of f, then the set of critical values $f(C_f)$ is of Lebesgue measure zero in $\mathbb{R}^m$. This result has proven to be very valuable in a large number of areas, especially in differential topology and analysis (see for instance [19,30] and the references therein). Additional geometric and analytical properties of the set of critical values in different versions of the Morse–Sard theorem, together with a study on the sharpness of the hypothesis of the Morse–Sard theorem, have been obtained in [5–9,22].

For many important applications of the Morse–Sard theorem, it is enough to know that any given continuous function can be uniformly approximated by a smooth map whose set of critical values has empty interior [19,30]. We refer to this as an *approximate Morse–Sard theorem*. The same type of approximation could prove key to the study of related problems in the infinite-dimensional domain.

In this paper, we will prove the strongest version of an approximate Morse–Sard theorem that one can expect to be true for a general infinite-dimensional separable Banach space, namely that every continuous function $f:X \to \mathbb{R}$, where X is an infinite-dimensional Banach space X with separable dual X^*, can be uniformly approximated by a C^1 smooth function $g:X \to \mathbb{R}$ which does not have any critical point. In some cases where more information about the structure of the Banach space X is known, we will extend our result to higher order of differentiability, C^p ($p > 1$).

Our result will also allow us to demonstrate one important corollary: the existence of *a non-linear Hahn–Banach theorem* which shows that two disjoint closed subsets in X can be separated by a 1-codimensional C^p smooth manifold of X (which is the set of zeros of a C^p smooth function with no critical points on X). This implies that every closed subset of X can be approximated by C^p smooth open subsets of X.

To put our work in context, let us briefly review some of the work established for the infinite-dimensional version of the Morse–Sard theorem. Smale [29] proved that if X and Y are separable connected smooth manifolds modeled on Banach spaces and $f:X \to Y$ is a C^r Fredholm map then $f(C_f)$ is of first Baire category and, in particular, $f(C_f)$ has no interior points provided that $r > \max\{\text{index}(df(x)), 0\}$ for all $x \in X$. Here, $\text{index}(df(x))$ stands for the index of the Fredholm operator $df(x)$, that is, the difference between the dimension of the kernel of $df(x)$ and the codimension of the image of $df(x)$, which are both finite. These assumptions are very strong as they impose that when X is infinite-dimensional then Y is necessarily infinite-dimensional too (in other words, there is no Fredholm map $f:X \to \mathbb{R}$). In fact, as Kupka proved in [20], there are C^∞ smooth functions $f:\ell_2 \to \mathbb{R}$ (where ℓ_2 is the separable Hilbert space) such that their sets of critical values $f(C_f)$ contain intervals and hence have non-empty interiors and positive Lebesgue measure. Bates and Moreira [9,22] showed that this function f can even be taken to be a polynomial of degree three. Azagra and Cepedello Boiso [2] have shown that every continuous mapping from the separable Hilbert space into $\mathbb{R}^m$ can be uniformly approximated by C^∞ smooth mappings with no critical points. Unfortunately, since part of their proof requires the use of the properties of the Hilbertian norm, this cannot be extended to non-Hilbertian Banach spaces. P. Hájek and M. Johanis [18] established the same kind of result in the case when X is a separable Banach space which contains c_0 and admits a C^p-smooth bump function. In this case, the approximating functions are of class C^p, $p = 1, 2, \dots, \infty$. This method is based on the result that the set of real-valued, C^∞ smooth functions defined on c_0 that locally depend on finitely many coordinates, is dense in the space of real-valued, continuous functions defined on c_0 [11].

However, as the authors noted, their method is not applicable when the space X has the Radon–Nikodým property (e.g., when X is reflexive), which leaves out all the classical Banach spaces $\ell_p(\mathbb{N})$ and $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

As stated above, we prove that for any infinite-dimensional Banach space X with a separable dual X^*, the set of C^1 smooth, real-valued functions with no critical points is uniformly dense in the space of all continuous, real-valued functions on X. This solves completely the problem of the approximation on separable Banach spaces by smooth, real-valued functions with no critical points when the order of smoothness of the approximating functions is one. Hence, we obtain the following characterization. For a separable infinite-dimensional Banach space X, the following are equivalent: (i) X^* is separable, and (ii) the set of C^1 smooth, real-valued functions on X with no critical points is uniformly dense in the space of all continuous, real-valued functions on X.

This result can be included in our main theorem which also applies to higher order of differentiability. Before stating our main theorem, recall that a norm $\|\cdot\|$ in a Banach space X is LUR (locally uniformly rotund [11]) if $\lim_n \|x_n - x\| = 0$ whenever the sequence $\{x_n\}_n$ and the point x are included in the unit sphere of the norm $\|\cdot\|$ and $\lim_n \|x_n + x\| = 2$. A norm $\|\cdot\|$ in X is C^p smooth (Gâteaux smooth) if it is C^p smooth (Gâteaux smooth, respectively) in $X \setminus \{0\}$.

Theorem 1.1. *Let X be an infinite-dimensional separable Banach space X with a LUR and C^p smooth norm $\|\cdot\|$, where $p \in \mathbb{N} \cup \{\infty\}$. Then, for every continuous mapping $f : X \to \mathbb{R}$ and for every continuous function $\varepsilon : X \to (0, \infty)$, there exists a C^p smooth mapping $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leqslant \varepsilon(x)$ for all $x \in X$ and g has no critical points.*

Our proof involves: (i) the use of renormings in $Y = X \oplus \mathbb{R}$ with good properties of smoothness and convexity; (ii) a special construction of carefully perturbed partitions of unity in an open subset denoted by S^+ of the unit sphere of the Banach space Y by means of a sequence of linear functionals in Y^*; (iii) the study and use of the properties of the range of the derivative of the norm in Y, Y^* and their finite-dimensional subspaces (Lemmas 2.1 and 2.2 below); and (iv) the use of the stereographic projection from X to S^+ and C^p deleting diffeomorphisms from X onto $X \setminus O$, where O is a bounded, closed, convex and C^p smooth subset of X.

Recall that the stereographic projections on LUR spheres were used in approximation results in [21] and later in [17,31].

The following example gives the optimal order of smoothness of the approximation functions with no critical points for $\ell_p(\mathbb{N})$ and $L_p(\mathbb{R}^n)$.

Example 1.2. It follows immediately from Theorem 1.1 that one can approximate every continuous, real-valued function on $\ell_p(\mathbb{N})$ and $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$ with $C^{\overline{p}}$ smooth, real-valued functions with no critical points, where $\overline{p} = [p]$ if p is not an integer, $\overline{p} = p - 1$ if p is an odd integer, and $\overline{p} = \infty$ if p is an even integer. Indeed, the standard norms of the classical separable Banach spaces $\ell_p(\mathbb{N})$ and $L_p(\mathbb{R}^n)$ are LUR and $C^{\overline{p}}$ smooth [11].

Since every Banach space with separable dual admits an equivalent LUR and C^1 smooth norm [11], we immediately deduce from Theorem 1.1 the announced characterization of the property of approximation by C^1 smooth functions with no critical points.

Corollary 1.3. *Let X be an infinite-dimensional separable Banach space. The following are equivalent*:

(1) *the dual space* X^* *is separable*;
(2) *for every continuous mapping* $f : X \to \mathbb{R}$ *and for every continuous function* $\varepsilon : X \to (0, \infty)$, *there exists a* C^1 *smooth mapping* $g : X \to \mathbb{R}$ *such that* $|f(x) - g(x)| \leqslant \varepsilon(x)$ *and* g *has no critical points.*

Next, we establish a similar statement for higher order smoothness on separable Banach spaces with a C^p smooth bump function ($p \geqslant 2$) and unconditional basis. We combine Theorem 1.1 and the results on C^1-fine approximation given in [4], to obtain the optimal order of smoothness of the approximating functions with no critical points on a large class within the Banach spaces with separable dual. In particular the following corollary applies even when the space X lacks a norm which is simultaneously LUR and C^2 smooth.

Corollary 1.4. *Let* X *be an infinite-dimensional separable Banach space with unconditional basis. Assume that* X *has a* C^p *smooth Lipschitz bump function, where* $p \in \mathbb{N} \cup \{\infty\}$. *Then, for every continuous mapping* $f : X \to \mathbb{R}$ *and for every continuous function* $\varepsilon : X \to (0, \infty)$, *there exists a* C^p *smooth mapping* $g : X \to \mathbb{R}$ *such that* $|f(x) - g(x)| \leqslant \varepsilon(x)$ *and* g *has no critical points.*

Proof. Since X is separable and admits a C^p smooth bump function, the dual space X^* is separable. Thus we obtain, from Corollary 1.3, a C^1 smooth function $h : X \to \mathbb{R}$ such that $h'(x) \neq 0$ and $|f(x) - h(x)| < \frac{\varepsilon(x)}{2}$ for every $x \in X$. Let us denote by $\|\cdot\|$ the dual norm on X^*. Define the continuous function $\bar{\varepsilon} : X \to (0, \infty)$, $\bar{\varepsilon}(x) = \frac{1}{2}\min\{\varepsilon(x), \|h'(x)\|\}$, for $x \in X$. Now, by the main result of [4], there is a C^p smooth function $g : X \to \mathbb{R}$ such that $|h(x) - g(x)| < \bar{\varepsilon}(x)$ and $\|h'(x) - g'(x)\| < \bar{\varepsilon}(x)$, for every $x \in X$. The latter implies that $\|h'(x)\| - \|g'(x)\| < \frac{1}{2}\|h'(x)\|$, and therefore $0 < \frac{1}{2}\|h'(x)\| < \|g'(x)\|$ for every $x \in X$. Hence, g is a C^p smooth function with no critical points and $|f(x) - g(x)| < |f(x) - h(x)| + |h(x) - g(x)| < \varepsilon(x)$ for every $x \in X$. □

Let us mention that the first one to study the problem of C^p-fine approximations ($p \in \mathbb{N}$) in infinite-dimensional Banach spaces was N. Moulis in her seminal paper [24]. In particular, she establishes results on C^1-fine approximations by C^∞ smooth functions of Sard type on $\ell_2(\mathbb{N})$ (that is, the set of the critical values has empty interior).

The proof of the above corollary yields to the following remark.

Remark 1.5. Assume that an infinite-dimensional separable Banach space X satisfies the C^1-fine approximation property by C^p smooth, real-valued functions, i.e., for every C^1 smooth function $f : X \to \mathbb{R}$ and every continuous function $\varepsilon : X \to (0, \infty)$ there is a C^p smooth function $h : X \to \mathbb{R}$ such that $|f(x) - h(x)| \leqslant \varepsilon(x)$ and $|f'(x) - h'(x)| \leqslant \varepsilon(x)$, for every $x \in X$. Then, the conclusion of Corollary 1.4 holds.

Furthermore, our results allow us to make the following conclusions.

Remark 1.6.

(1) All of the results presented above hold in the case when one replaces X with an open subset U of X. Actually, the same proof given in the section to follow (with obvious modifications) can be used.

(2) Whenever X has the property that every continuous, real-valued function on X can be approximated by C^p smooth, real-valued functions with no critical points, one can deduce the following corollaries.

Recall that an open subset U of X is said to be C^p smooth provided its boundary ∂U is a C^p smooth one-codimensional submanifold of X.

Corollary 1.7.

(i) (A nonlinear Hahn–Banach theorem.) *Let X be any of the Banach spaces considered in the above results. Then, for every two disjoint closed subsets C_1, C_2 of X, there exists a C^p smooth function $\varphi : X \to \mathbb{R}$ with no critical points, such that the level set $M = \varphi^{-1}(0)$ is a 1-codimensional C^p smooth submanifold of X that separates C_1 and C_2, in the following sense*: *define $U_1 = \{x \in X\colon \varphi(x) < 0\}$ and $U_2 = \{x \in X\colon \varphi(x) > 0\}$, then U_1 and U_2 are disjoint C^p smooth open subsets of X with common boundary $\partial U_1 = \partial U_2 = M$, and such that $C_i \subset U_i$ for $i = 1, 2$.*

(ii) (Smooth approximation of closed sets.) *Every closed subset of any of the Banach spaces X considered above can be approximated by C^p smooth open subsets of X in the following sense*: *for every closed set $C \subset X$ and every open set $W \subset X$ such that $C \subset W$ there is a C^p smooth open set $U \subset X$ so that $C \subset U \subseteq W$.*

The following corollary should be compared with the main result of [3].

Corollary 1.8 *(Failure of Rolle's theorem). For every open subset U of any of the above Banach spaces X there is a real-valued, Fréchet differentiable function f defined on X whose support is the closure of U, and such that f is C^p smooth on U and yet f has no critical points in U.*

We refer to [3,10,16,23,27,28] for previous results on the study of the set of critical points of a function and related topics.

Corollary 1.9. *The conclusion of Theorem* 1.1 *and Corollaries* 1.7 *and* 1.8 *remain true, in the case $p = \infty$, if we replace X with a parallelizable connected and metrizable Banach manifold M modeled on a separable infinite-dimensional Banach space E satisfying one of the following properties*:

(1) *E has a Schauder basis and a C^∞ smooth and LUR norm*; *or*
(2) *E has an unconditional basis and a C^∞ smooth Lipschitz bump function.*

Proof. If E satisfies (1) or (2) then all the above results are applicable to any open subset U of E. On the other hand, according to [14, Theorem 1A], any parallelizable connected and metrizable manifold M modelled on E is C^∞-diffeomorphic to an open subset U of E. Since the property of approximation by smooth functions with no critical points is preserved by diffeomorphisms, the result follows. □

2. Proof of Theorem 1.1

Recall that a norm $N(\cdot)$, in a Banach space E, is (1) *strictly convex* if the unit sphere of the norm $N(\cdot)$ does not include any line segment. Equivalently, $N(\frac{x+y}{2}) < 1$ for every x, y in the unit sphere with $x \neq y$; (2) *WUR* (*weakly uniformly rotund*) if $\lim_n (x_n - y_n) = 0$ in the weak topology whenever the sequences $\{x_n\}_n$ and $\{y_n\}_n$ are included in the unit sphere and $\lim_n N(x_n + y_n) = 2$.

We let $[u_1, \dots, u_n]$ stand for the linear span of the vectors $u_1, \dots, u_n$. Let us denote by $S_{\|\cdot\|}$ and $S_{\|\cdot\|^*}$ the unit sphere of a Banach space $(Z, \|\cdot\|)$ and its dual $(Z^*, \|\cdot\|^*)$, respectively.

The following two geometrical lemmas will be essential to the proof of Theorem 1.1.

Lemma 2.1. *Let $Z = [u_1, \dots, u_n]$ be a n-dimensional space $(n > 1)$ with a differentiable norm $\|\cdot\|$ (Gâteaux or Fréchet differentiable, since both notions coincide for convex functions defined on finite-dimensional spaces). Let us consider real numbers α_i, for $i = 1, \dots, n-1$. Then, the cardinal of the set*

$$\big\{T \in S_{\|\cdot\|^*}\colon\ T(u_i) = \alpha_i,\ i = 1, \dots, n-1\big\} \tag{2.1}$$

is at most two.

Before we give the proof, let us observe that the condition of differentiability of the norm in Lemma 2.1 as well as in Lemma 2.2 below is only required in the case that there is $\alpha_i \neq 0$ such that

$$\left\| \frac{u_i}{\alpha_i} \right\| = 1. \tag{$*$}$$

Moreover, in this case we only need the differentiability of $\|\cdot\|$ at the points u_i satisfying $(*)$. The case $(*)$ does not appear in the proof of Theorem 1.1. Nevertheless, to simplify the lemmas we use the condition of differentiability of the norm.

Proof. Consider the $n-1$ affine hyperplanes of Z^*, $H_i = \{T \in Z^*\colon\ T(u_i) = \alpha_i\}$, $i = 1, \dots, n-1$. Since $\{u_1, \dots, u_{n-1}\}$ are linearly independent, the intersection $H = H_1 \cap \dots \cap H_{n-1}$ is an affine 1-dimensional subspace of Z^*. The dual norm $\|\cdot\|^*$ is strictly convex (because the norm $\|\cdot\|$ is differentiable and Z is finite-dimensional, see [11]), and therefore there are at most two points in the intersection $H \cap S_{\|\cdot\|^*}$ and the proof concludes. □

From the proof of Lemma 2.1 we obtain the following.

Lemma 2.2. *Let $Z = [u_1, \dots, u_n]$ be a n-dimensional space $(n > 1)$ with a differentiable norm $\|\cdot\|$. Consider real numbers α_i, for $i = 1, \dots, n-1$ and define the set $\mathcal{S}$ of real numbers α such that*

$$\big\{T \in S_{\|\cdot\|^*}\colon\ T(u_i) = \alpha_i,\ i = 1, \dots, n-1,\ T(u_n) = \alpha\big\} = \emptyset. \tag{2.2}$$

Then, the cardinal of $\mathbb{R} \setminus \mathcal{S}$ is at most two.

Proof. From Lemma 2.1, we know that the cardinal of the set $\{T \in S_{\|\cdot\|^*}\colon\ T(u_i) = \alpha_i,\ i = 1, \dots, n-1\}$ is at most two. Assume that there are two elements T_i, $i = 1, 2$ in this set (the other cases are similar). Then, we have that $\mathbb{R} \setminus \mathcal{S} = \{T_1(u_n), T_2(u_n)\}$. □

The general strategy of the proof of Theorem 1.1 is as follows. We consider the space $Y = X \oplus \mathbb{R}$ and define the following norm on Y:

(a) If $p > 1$, for every $y = (x, r) \in Y$, put $N(y) = N(x, r) = (\|x\|^2 + r^2)^{1/2}$, where $\|\cdot\|$ is a LUR and C^p smooth norm on X. Then, clearly the norm N is LUR and C^1 smooth on Y. Moreover, N is C^p smooth on the open set $Y \setminus \{(0, \lambda)\colon\ \lambda \in \mathbb{R}\}$. Define $v = (0, 1) \in Y$ and take $\beta \in Y^* \setminus \{0\}$ such that $X = \ker\beta$. Select $\beta_1 \in Y^* \setminus [\beta]$ such that $\beta_1(\nu) \neq 0$ and $\omega \in \ker\beta \setminus \ker\beta_1$. Consider the closed hyperplane of Y, $X_1 = \ker\beta_1$. Then, the restriction of the norm N to X_1 is a C^p smooth and LUR norm on X_1. Now, the (equivalent) norm considered in $Y = X_1 \oplus [\omega]$, defined as $|z + \lambda\omega| = (N(z)^2 + \lambda^2)^{1/2}$, where $z \in X_1$ and $\lambda \in \mathbb{R}$, is LUR and C^1 smooth on Y and C^p smooth on $Y \setminus [\omega]$. In particular, the norm $|\cdot|$ is C^p smooth on the open set $\mathcal{U} = Y \setminus \ker\beta = \{(x, r)\colon\ x \in X,\ r \neq 0\}$.

It could also be proved that the Banach space Y admits an equivalent LUR and C^p smooth norm on Y with bounded derivatives up to the order p. Nevertheless, a LUR and C^1 smooth norm on Y and C^p smooth on $\mathcal{U}$, is sufficient to prove our result. Recall that if X has a LUR and C^p smooth norm and $p > 1$, then X is superreflexive [11].

(b) If $p = 1$, since the dual space Y^* is separable, there is a norm $|\cdot|$ on Y which is LUR, C^1 smooth and WUR whose dual is strictly convex [11]. Recall that if the norm $|\cdot|$ is WUR, then the dual norm $|\cdot|^*$ is uniformly Gâteaux smooth, and thus, Gâteaux smooth.

Therefore, in any of the cases (a) or (b) the norm $|\cdot|$ is LUR and C^1 smooth. If in addition, X is reflexive, then it can be proved [15, p. 272] that $|\cdot|^*$ is LUR and C^1 smooth as well. If X is not reflexive, we know from the conditions given in (b) that the dual norm $|\cdot|^*$ is strictly convex and Gâteaux smooth.

Let us denote $S := S_{|\cdot|}$, the unit sphere of $(Y, |\cdot|)$ and $S^* := S_{|\cdot|^*}$, the unit sphere of $(Y^*, |\cdot|^*)$. Let us consider, the duality mapping of the norm $|\cdot|$ defined as

$$D : S \longrightarrow S^*$$
$$D(x) = |\cdot|'(x),$$

which is $|\cdot| - |\cdot|^*$ continuous because the norm $|\cdot|$ is of class C^1.

We establish a C^p diffeomorphism Φ between X and half unit sphere in Y, $S^+ := \{y = (x, r) \in Y\colon\ r > 0\}$, as follows: $\Phi : X \to S^+$ is the composition $\Phi = \Pi \circ i$, where i is the inclusion $i : X \to Y$, $i(x) = (x, 1)$ and Π is defined by $\Pi : Y \setminus \{0\} \to S$, $\Pi(y) = \frac{y}{|y|}$.

In order to simplify the notation, we will make the proof for the case of a constant $\varepsilon > 0$. By taking some standard technical precautions the same proof will work in the case of a positive continuous function $\varepsilon : X \to (0, +\infty)$ (at the end of the proof we will explain what small changes should be made).

Now, given a continuous function $f : X \to \mathbb{R}$, we consider the composition $F := f \circ \Phi^{-1} : S^+ \to \mathbb{R}$, which is continuous as well. For any given $\varepsilon > 0$ we will 3ε-approximate F by a C^p smooth function $H : S^+ \to \mathbb{R}$ with the properties that:

- the set of critical points of H is the countable union of a family of disjoint sets $\{K_n\}_n$;
- there are countable families of open slices $\{O_n\}_n$ and $\{B_n\}_n$ in S^+, such that $\bigcup_n \overline{B}_n$ is relatively closed in S^+, $\operatorname{dist}(B_n, X \times \{0\}) > 0$, $K_n \subset O_n \subset B_n$, $\operatorname{dist}(O_n, S^+ \setminus B_n) > 0$ and $\operatorname{dist}(B_n, \bigcup_{m \neq n} B_m) > 0$, for every $n \in \mathbb{N}$;
- the oscillation of F in every B_n is less than ε.

(We will consider slices of S^+ of the form $\{x \in S\colon\ f(x) > r\}$, where f is a continuous linear functional of norm one and $0 < r < 1$. Recall also that the distance between two sets A and A' in a Banach space $(M, |\cdot|_M)$ is defined as the real number $\operatorname{dist}(A, A') := \inf\{|a - a'|_M\colon\ a \in A,\ a' \in A'\}$.)

Then we will prove that the function $h := H \circ \Phi$ is a C^p smooth function on X, which 3ε-approximates f, and the set of critical points of h, $\mathcal{C} = \{x \in X\colon h'(x) = 0\}$, can be written as $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$, where, for every $n \in \mathbb{N}$, the set $\mathcal{K}_n := \Phi^{-1}(K_n)$ is contained in the *open, convex, bounded and* C^p *smooth body* $\mathcal{O}_n := \Phi^{-1}(O_n)$, which in turn is contained in the open, convex, bounded and C^p smooth body $\mathcal{B}_n := \Phi^{-1}(B_n)$, in such a way that $\operatorname{dist}(\mathcal{O}_n, X \setminus \mathcal{B}_n) > 0$, the oscillation of f in $\mathcal{B}_n$ is less than ε, $\bigcup_n \overline{\mathcal{B}}_n$ is closed and $\operatorname{dist}(\mathcal{B}_n, \bigcup_{m \neq n} \mathcal{B}_m) > 0$. Once we have done this, we will compose the function h with a sequence of deleting diffeomorphisms which will eliminate the critical points of h. More precisely, for each set $\mathcal{O}_n$ we will find a C^p diffeomorphism Ψ_n from X onto $X \setminus \overline{\mathcal{O}}_n$ so that Ψ_n is the identity off $\mathcal{B}_n$. Then, by defining $g := h \circ \bigcirc_{n=1}^{\infty} \Psi_n$, we will get a C^p smooth function which 4ε-approximates f and which has no critical points.

The most difficult part in this scheme is the construction of the function H. We will inductively define linearly independent functionals $g_k \in Y^*$, open subsets U_k of S^+, points $x_k \in U_k$, real numbers $a_k \neq 0$ satisfying $|a_k - F(x_k)| < \varepsilon$, real numbers γ_k and $\gamma_{i,j}$ in the interval $(0,1)$ (with $i + j = k$), functions h_k of the form $h_k = \varphi_k(g_k)\phi_{k-1,1}(g_{k-1}) \cdots \phi_{1,k-1}(g_1)$, where the $\varphi_k, \phi_{k-1,1}, \ldots, \phi_{1,k-1}$ are suitably chosen C^∞ functions on the real line, and functions r_k of the form $r_k = s_k g_k + (1 - s_k g_k(x_k))$ (with very small $s_k \neq 0$), and put

$$\mathbf{H_k} = \frac{\sum_{i=1}^{k} a_i r_i h_i}{\sum_{i=1}^{k} h_i},$$

where $\mathbf{H_k} : U_1 \cup \cdots \cup U_k \to \mathbb{R}$. The interior of the support of h_k will be the set

$$U_k = \bigl\{x \in S^+\colon\ g_1(x) < \gamma_{1,k-1}, \ldots, g_{k-1}(x) < \gamma_{k-1,1} \text{ and } g_k(x) > \gamma_k\bigr\},$$

where the oscillation of the function F will be less than ε. Denote by T_x the (vectorial) tangent hyperplane to S^+ at the point x, that is $T_x := \ker D(x)$. The derivative of $\mathbf{H_k}$ at every point $x \in U_1 \cup \cdots \cup U_k$ will be shown to be the restriction to T_x of a *nontrivial linear combination of the linear functionals* $g_1, \ldots, g_k$. Then, by making use of Lemmas 2.1 and 2.2 and choosing the $\gamma_{i,j}$ close enough to γ_i, we will prove that the set of critical points of $\mathbf{H_k}$ is a finite union of pairwise disjoint sets which are contained in a finite union of pairwise disjoint slices, with positive distance between any two slices (see Fig. 1). These slices will be determined by functionals in finite sets $N_k \subset Y^*$ defined by a repeated application of Lemmas 2.1 and 2.2. The function H will be then defined as

$$H = \frac{\sum_{k=1}^{\infty} a_k r_k h_k}{\sum_{k=1}^{\infty} h_k}.$$

Let us begin with the formal construction of the functions $\mathbf{H_k}$. We will use the notation H_k and H_k' when the function $\frac{\sum_{i=1}^{k} a_i r_i h_i}{\sum_{i=1}^{k} h_i}$ and its derivative are thought to be defined on the open subset of Y where $\sum_{i=1}^{k} h_i \neq 0$ (we consider the functions r_i and h_i defined on Y) and reserve the symbols $\mathbf{H_k}$ and $\mathbf{H_k'}(x)$ for the restriction of H_k and $H_k'(x)$ to a subset of S and to the tangent space T_x of S at x, respectively.

Since the norm $|\cdot|$ is LUR we can find, for every $x \in S^+$, open slices $R_x = \{y \in S\colon f_x(y) > \delta_x\} \subset S^+$ and $P_x = \{y \in S\colon f_x(y) > \delta_x^4\} \subset S^+$, where $0 < \delta_x < 1$ and $|f_x| = 1 = f_x(x)$, so that the oscillation of F in every P_x is less than ε. We also assume, for technical reasons, and with no loss of generality, that $\operatorname{dist}(P_x, X \times \{0\}) > 0$.

Since Y is separable we can select a countable subfamily of $\{R_x\}_{x \in S^+}$, which covers S^+. Let us denote this countable subfamily by $\{S_n\}_n$, where $S_n := \{y \in S\colon\ f_n(y) > \delta_n\}$. Recall that the oscillation of F in every $P_n := \{y \in S\colon\ f_n(y) > \delta_n^4\}$ is less than ε and $\operatorname{dist}(P_n, X \times \{0\}) > 0$.

• *For* $k = 1$, define

$$h_1 : S^+ \longrightarrow \mathbb{R}$$
$$h_1 = \varphi_1(f_1),$$

where φ_1 is a C^∞ function on $\mathbb{R}$ satisfying

$$\varphi_1(t) = 0 \quad \text{if } t \leqslant \delta_1,$$
$$\varphi_1(1) = 1,$$
$$\varphi_1'(t) > 0 \quad \text{if } t > \delta_1.$$

Notice that the interior of the support of h_1 is the open set S_1. Denote by x_1 the point of S^+ satisfying $f_1(x_1) = 1$. Now select $a_1 \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ with $|a_1 - F(x_1)| < \varepsilon$ and define the auxiliary function

$$r_1 : S^+ \longrightarrow \mathbb{R}$$
$$r_1 = s_1 f_1 + \big(1 - s_1 f_1(x_1)\big),$$

where we have selected s_1 so that $a_1 s_1 > 0$ and $|s_1|$ small enough so that the oscillation of r_1 on S_1 is less than $\frac{\varepsilon}{|a_1|}$. Notice that $r_1(x_1) = 1$. Define

$$\mathbf{H_1} : S_1 \longrightarrow \mathbb{R}$$
$$\mathbf{H_1} = \frac{a_1 r_1 h_1}{h_1} = a_1 r_1.$$

The function $\mathbf{H_1}$ is C^p smooth in S_1 and the set of critical points of $\mathbf{H_1}$,

$$Z_1 = \big\{x \in S_1\colon\ H_1'(x) = 0 \text{ on } T_x\big\}$$

consists of the unique point x_1. Indeed, $\mathbf{H_1'}(x) = H_1'(x)|_{T_x} = a_1 s_1 f_1|_{T_x} \equiv 0$ iff $D(x) = f_1$. This implies that $Z_1 = \{x_1\}$. Now select real numbers $\gamma_{1,1}'$, t_1 and l_1 such that $\delta_1 < \gamma_{1,1}' < t_1 < l_1 < 1$ and define the open slices

$$O_{f_1} = \big\{x \in S\colon\ f_1(x) > l_1\big\} \quad \text{and} \quad B_{f_1} = \big\{x \in S\colon\ f_1(x) > t_1\big\}.$$

Clearly the above sets satisfy that $Z_1 \subset O_{f_1} \subset B_{f_1} \subset S_1$, $\operatorname{dist}(O_{f_1}, S \setminus B_{f_1}) > 0$ and $\operatorname{dist}(B_{f_1}, \{x \in S\colon\ f_1(x) \leqslant \gamma_{1,1}'\}) > 0$.

In order to simplify the notation in the rest of the proof, let us denote by $\gamma_1 = \delta_1$, $U_1 = R_1 = S_1$, $g_1 = f_1$, $z_1 = x_1$ and $\Gamma_1 = N_1 = \{g_1\}$. Let us define $\sigma_{1,1} = a_1 s_1$ and write $\mathbf{H_1'} = \sigma_{1,1}\mathbf{g_1}$

on U_1 where $\mathbf{g_1}$ is the restriction of g_1 to T_x whenever we evaluate $\mathbf{H}_1'(x)$. In addition, if $A \subset S$, we denote by $A^c = S \setminus A$.

• *For* $k = 2$. Let us denote by $y_2 \in S^+$ the point satisfying $f_2(y_2) = 1$. If either $\{g_1, D(y_2) = f_2\}$ are linearly dependent (this only occurs when $g_1 = f_2$) or $g_1(y_2) = \gamma_1$, we use the density of the norm attaining functionals (Bishop–Phelps theorem) and the continuity of D to modify y_2 and find $z_2 \in S^+$ so that $\{g_1, D(z_2) := g_2\}$ are linearly independent, $g_1(z_2) \neq \gamma_1$ and

$$\{x \in S\colon\ f_2(x) > \delta_2^2\} \subset \{x \in S\colon\ g_2(x) > \nu_2\} \subset \{x \in S\colon\ f_2(x) > \delta_2^3\},$$

for some $\nu_2 \in (0,1)$. If $g_1(y_2) \neq \gamma_1$ and $\{g_1, f_2\}$ are linearly independent, define $g_2 = f_2$ and $z_2 = y_2$. Then, apply Lemma 2.2 to the 2-dimensional space $[g_1, g_2]$ with the norm $|\cdot|^*$ (the restriction to $[g_1, g_2]$ of the dual norm $|\cdot|^*$ considered in Y^*) and the real number $\gamma_1 \in (0,1)$ to obtain $\gamma_2 \in (0,1)$ close enough to ν_2 so that

$$S_2 = \{x \in S\colon\ f_2(x) > \delta_2\} \subset \{x \in S\colon\ g_2(x) > \gamma_2\} \subset \{x \in S\colon\ f_2(x) > \delta_2^4\} = P_2$$

and

$$\{T \in [g_1, g_2]^*\colon\ |T| = 1,\ T(g_1) = \gamma_1 \text{ and } T(g_2) = \gamma_2\} = \emptyset. \tag{2.3}$$

Recall that the norm $|\cdot|^*$ is Gâteaux differentiable on Y^* and therefore the restriction of this norm to $[g_1, g_2]$, which we shall denote by $|\cdot|^*$ as well, is a differentiable norm on the space $[g_1, g_2]$ (Gâteaux and Fréchet notions of differentiability are equivalent in the case of *convex* functions defined on *finite-dimensional* spaces). Therefore, we can apply Lemma 2.2 to the norm $|\cdot|^*$ in the space $[g_1, g_2]$. In fact, the same argument works for any finite-dimensional subspace of Y^* and we will apply Lemma 2.2 in the next steps to larger finite-dimensional subspaces of Y^*. Define the sets

$$R_2 = \{x \in S\colon\ g_2(x) > \gamma_2\}, \quad \text{and}$$
$$U_2' = \{x \in S\colon\ g_1(x) < \gamma_{1,1}',\ g_2(x) > \gamma_2\}.$$

Assume that $U_1 \cap R_2 \neq \emptyset$ and consider the set

$$M_2 = D^{-1}([g_1, g_2]) \cap U_2' \cap U_1.$$

In the case that $M_2 = \emptyset$, we select as $\gamma_{1,1}$ any point in $(\gamma_1, \gamma_{1,1}')$. In the case that $M_2 \neq \emptyset$ and $\gamma_1 < \inf\{g_1(x)\colon x \in M_2\}$, we select $\gamma_{1,1}$ so that

$$\gamma_1 < \gamma_{1,1} < \inf\{g_1(x)\colon\ x \in M_2\}.$$

In the case that $\gamma_1 = \inf\{g_1(x)\colon x \in M_2\}$ and in order to obtain an appropriate $\gamma_{1,1}$ we need to study the limits of the sequences $\{x_n\} \subset M_2$ such that $\lim_n g_1(x_n) = \gamma_1$. Define

$$F_2' = \{T \in [g_1, g_2]^*\colon\ |T| = 1 \text{ and } T(g_1) = \gamma_1\}.$$

From Lemma 2.1, we deduce that the cardinal of the set F_2' is at most two. Furthermore, since $|\cdot|^*$ is strictly convex, the cardinal of the set

$$N_2' = \{g \in S^* \cap [g_1, g_2]\colon\ T(g) = 1 \text{ for some } T \in F_2'\}$$

is at most two.

Let us take any sequence $\{x_n\} \subset M_2$ with $\lim_n g_1(x_n) = \gamma_1$. Consider every x_n as an element of X^{**} and denote by $\mathbf{x_n}$ its restriction to $[g_1, g_2]$. Recall that if $x_n \in M_2$, then $D(x_n) \in S^* \cap [g_1, g_2]$, for every $n \in \mathbb{N}$. Moreover, the sequence of restrictions $\{\mathbf{x_n}\} \subset [g_1, g_2]^*$ satisfies that

$$1 = |x_n| \geqslant |\mathbf{x_n}| = \max\{\mathbf{x_n}(h)\colon\ h \in S^* \cap [g_1, g_2]\} \geqslant \mathbf{x_n}(D(x_n)) = D(x_n)(x_n) = 1,$$

for every $n \in \mathbb{N}$. Thus, there is a subsequence $\{\mathbf{x_{n_j}}\}$ converging to an element $T \in [g_1, g_2]^*$ with $|T| = 1$. Since $\lim_j g_1(x_{n_j}) = \lim_j \mathbf{x_{n_j}}(g_1) = \gamma_1$, then $T(g_1) = \gamma_1$ and this implies that $T \in F_2'$. Furthermore, if $g \in N_2'$ and $T(g) = 1$, then $\lim_j \mathbf{x_{n_j}}(g) = 1$. In addition, $T(g_2) = \lim_j \mathbf{x_{n_j}}(g_2) = \lim_j g_2(x_{n_j}) \geqslant \gamma_2$. Then, from condition (2.3), we deduce that $T(g_2) > \gamma_2$. Let us define

$$F_2 = \Big\{T \in F_2'\colon \text{ there is a sequence } \{x_n\} \subset M_2 \text{ with } \lim_n \mathbf{x_n} = T \text{ and } \lim_n \mathbf{x_n}(g_1) = \gamma_1\Big\},$$
$$N_2 = \{g \in N_2'\colon\ T(g) = 1 \text{ for some } T \in F_2\}.$$

Select a real number γ_2' satisfying $\gamma_2 < \gamma_2' < \min\{T(g_2)\colon\ T \in F_2\}$ (recall that F_2 is finite). Let us prove the following fact.

Fact 2.3.

(1) *There are numbers* $0 < t_2 < l_2 < 1$ *such that for every* $g \in N_2$*, the slices*

$$O_g := \{x \in S\colon\ g(x) > l_2\} \quad \textit{and} \quad B_g := \{x \in S\colon\ g(x) > t_2\}$$

satisfy that

$$O_g \subset B_g \subset \{x \in S\colon\ g_1(x) < \gamma_{1,1}',\ g_2(x) > \gamma_2'\} \quad \textit{and} \tag{2.4}$$

$$\operatorname{dist}(B_g, B_{g'}) > 0, \quad \textit{whenever } g, g' \in N_2,\ g \neq g'. \tag{2.5}$$

(2) *There is* $\gamma_{1,1} \in (\gamma_1, \gamma_{1,1}')$*, such that if* $x \in M_2$ *and* $g_1(x) < \gamma_{1,1}$*, then* $x \in O_g$ *for some* $g \in N_2$*.*

Proof. (1) First, if X is reflexive, we know that for every $g \in N_2$ there is $x_g \in S$ such that $D(x_g) = g$. Since $\mathbf{x_g}(g) = 1$ and $|\cdot|^*$ is Gâteaux smooth, then $\mathbf{x_g} \in F_2$. This implies that $\mathbf{x_g}(g_1) = \gamma_1 < \gamma_{1,1}'$ and $\mathbf{x_g}(g_2) > \gamma_2'$. Hence, $x_g \in \{x \in S\colon\ g_1(x) < \gamma_{1,1}',\ g_2(x) > \gamma_2'\}$. Now, since the norm $|\cdot|$ is LUR and $D(x_g) = g$, the functional g strongly exposes S at the point x_g. Taking into account that N_2 is finite we can hence obtain real numbers $0 < t_2 < l_2 < 1$ and slices O_g and B_g satisfying conditions (2.4) and (2.5) for every $g \in N_2$.

Now consider a non-reflexive Banach space X. Let us first prove (2.4). Assume, on the contrary, that there is a point $g \in N_2$ and there is a sequence $\{y_n\} \subset S$ satisfying $g(y_n) > 1 - \frac{1}{n}$ with either $g_1(y_n) \geqslant \gamma'_{1,1}$ or $g_2(y_n) \leqslant \gamma'_2$, for every $n \in \mathbb{N}$. Since $g \in N_2$ there is a sequence $\{x_n\} \subset M_2$ with $\lim_n g_1(x_n) = \gamma_1$, $\lim_n g_2(x_n) > \gamma'_2$ and $\lim_n g(x_n) = 1$. In particular,

$$\frac{g(x_n) + 1 - \frac{1}{n}}{2} < g\left(\frac{x_n + y_n}{2}\right) \leqslant \left|\frac{x_n + y_n}{2}\right| \leqslant 1,$$

and thus $\lim_n |\frac{x_n+y_n}{2}| = 1$. Recall that, in this case the norm $|\cdot|$ is WUR, and hence $x_n - y_n \xrightarrow{\omega} 0$ (weakly converges to zero). This last assertion gives a contradiction since either $\limsup_n g_1(x_n - y_n) \leqslant \gamma_1 - \gamma'_{1,1} < 0$ or $\liminf_n g_2(x_n - y_n) \geqslant \lim_n g_2(x_n) - \gamma'_2 > 0$. Therefore we can find real numbers $0 < t_2 < l_2 < 1$ and slices O_g and B_g for every $g \in N_2$, satisfying condition (2.4). In order to obtain (2.5) we just need to modify t_2 and l_2 and select them close enough to 1. Indeed, assume on the contrary, that there are sequences $\{y_n\} \subset S$ and $\{z_n\} \subset S$ and $g, g' \in N_2$, $g \neq g'$, such that $\lim_n g(y_n) = 1$, $\lim_n g'(z_n) = 1$ and $\lim_n |y_n - z_n| = 0$. Then,

$$\begin{aligned}\frac{g(y_n) + g'(z_n)}{2} &= \frac{(g + g')(y_n) + g'(z_n - y_n)}{2} \leqslant \frac{(g + g')(y_n) + |z_n - y_n|}{2} \\ &\leqslant \frac{|g + g'|^*}{2} + \frac{|z_n - y_n|}{2} \leqslant 1 + \frac{|z_n - y_n|}{2}.\end{aligned}$$

Since the limit of the first and last terms in the above chain of inequalities is 1, we deduce that $|g + g'|^* = 2$. Since the norm $|\cdot|^*$ is strictly convex, we deduce that $g = g'$, a contradiction.

(2) Assume, on the contrary, that for every $n \in \mathbb{N}$, there is $x_n \in M_2$ with $g_1(x_n) \leqslant \gamma_1 + \frac{1}{n}$ and $\{x_n\colon n \in \mathbb{N}\} \cap (\bigcup_{g \in N_2} O_g) = \emptyset$. Since $\lim_n g_1(x_n) = \gamma_1$ and $\{x_n\} \subset M_2$, from the comments preceding Fact 2.3, we know that there is a subsequence $\{x_{n_j}\}$ and $g \in N_2$ satisfying that $\lim_j g(x_{n_j}) = 1$, which is a contradiction. This finishes the proof of Fact 2.3. □

If $U_1 \cap R_2 = \emptyset$, we can select as $\gamma_{1,1}$ any number in $(\gamma_1, \gamma'_{1,1})$. Now, we define,

$$\begin{aligned}&h_2 : S^+ \longrightarrow \mathbb{R} \\ &h_2 = \varphi_2(g_2)\phi_{1,1}(g_1),\end{aligned}$$

where φ_2 and $\phi_{1,1}$ are C^∞ functions on $\mathbb{R}$ satisfying:

$$\begin{aligned}&\varphi_2(t) = 0 \quad \text{if } t \leqslant \gamma_2, \\ &\varphi_2(1) = 1, \\ &\varphi'_2(t) > 0 \quad \text{if } t > \gamma_2,\end{aligned}$$

and

$$\begin{aligned}&\phi_{1,1}(t) = 1 \quad \text{if } t \leqslant \frac{\gamma_1 + \gamma_{1,1}}{2}, \\ &\phi_{1,1}(t) = 0 \quad \text{if } t \geqslant \gamma_{1,1}, \\ &\phi'_{1,1}(t) < 0 \quad \text{if } t \in \left(\frac{\gamma_1 + \gamma_{1,1}}{2}, \gamma_{1,1}\right).\end{aligned}$$

Notice that the interior of the support of h_2 is the open set

$$U_2 = \{x \in S\colon\ g_1(x) < \gamma_{1,1}, g_2(x) > \gamma_2\}.$$

Select one point $x_2 \in U_2$, a real number $a_2 \in \mathbb{R}^*$ with $|a_2 - F(x_2)| < \varepsilon$ and define the auxiliary function

$$r_2 : S^+ \longrightarrow \mathbb{R}$$
$$r_2 = s_2 g_2 + \big(1 - s_2 g_2(x_2)\big),$$

where we have selected s_2 so that $s_2 a_2 > 0$ and $|s_2|$ is small enough so that the oscillation of r_2 on U_2 is less than $\frac{\varepsilon}{|a_2|}$. Notice that $r_2(x_2) = 1$.

Let us study the critical points Z_2 of the function

$$\mathbf{H_2} : U_1 \cup U_2 \longrightarrow \mathbb{R}$$
$$\mathbf{H_2} = \frac{a_1 r_1 h_1 + a_2 r_2 h_2}{h_1 + h_2}. \tag{2.6}$$

Let us prove that $Z_2 = \{x \in U_1 \cup U_2\colon\ H_2'(x) = 0 \text{ on } T_x\}$ can be included in a finite number of pairwise disjoint slices within $U_1 \cup U_2$ by splitting it conveniently into up to four sets.

First, if $x \in U_1 \setminus U_2$, from (2.6), we obtain that $\mathbf{H_2}(x) = a_1 r_1(x)$ and the derivative $\mathbf{H_2'}(x) = H_2'(x)|_{T_x} = a_1 s_1 g_1|_{T_x} \equiv 0$ iff $D(x) = g_1$. Thus, $Z_2 \cap (U_1 \setminus U_2) \subseteq \{z_1\}$. Similarly, if $x \in U_2 \setminus U_1$, from (2.6), we obtain that $\mathbf{H_2}(x) = a_2 r_2(x)$ and the derivative $\mathbf{H_2'}(x) = H_2'(x)|_{T_x} = a_2 s_2 g_2|_{T_x} \equiv 0$ iff $D(x) = g_2$. Then, if $z_2 \in U_2 \setminus U_1$, $\mathbf{H_2}$ has one critical point in $U_2 \setminus U_1$, namely z_2; in this case, since $g_1(z_2) \neq \gamma_1$, the point z_2 actually belongs to $U_2 \setminus \overline{U}_1$.

Now, let us study the critical points of $\mathbf{H_2}$ in $U_1 \cap U_2$. In order to simplify the notation, let us put $\Lambda_1 = \frac{h_1}{h_1 + h_2}$, and denote by $\mathbf{g_1}$ and $\mathbf{g_2}$ the restrictions $g_1|_{T_x}$ and $g_2|_{T_x}$, respectively, whenever we consider $\mathbf{H_2'}(x)$ and $\Lambda_1'(x)$. Then, $\mathbf{H_2} = a_1 r_1 \Lambda_1 + a_2 r_2 (1 - \Lambda_1)$ and

$$\begin{aligned}\mathbf{H_2'} &= a_1 s_1 \Lambda_1 \mathbf{g_1} + a_2 s_2 (1 - \Lambda_1)\mathbf{g_2} + (a_1 r_1 - a_2 r_2)\Lambda_1' \\ &= \sigma_{1,1} \Lambda_1 \mathbf{g_1} + a_2 s_2 (1 - \Lambda_1)\mathbf{g_2} + (H_1 - a_2 r_2)\Lambda_1'.\end{aligned}$$

By computing Λ_1', we obtain $\Lambda_1' = \xi_{1,1}\mathbf{g_1} + \xi_{1,2}\mathbf{g_2}$, where the coefficients $\xi_{1,1}$ $\xi_{1,2}$ are continuous functions on $U_1 \cup U_2$ and have the following form,

$$\begin{aligned}\xi_{1,1} &= \frac{\varphi_1'(g_1) h_2 - h_1 \varphi_2(g_2) \phi_{1,1}'(g_1)}{(h_1 + h_2)^2}, \\ \xi_{1,2} &= \frac{-h_1 \varphi_2'(g_2) \phi_{1,1}(g_1)}{(h_1 + h_2)^2}.\end{aligned}$$

Thus $\mathbf{H_2'} = \sigma_{2,1}\mathbf{g_1} + \sigma_{2,2}\mathbf{g_2}$, where $\sigma_{2,1}$ and $\sigma_{2,2}$ are continuous functions on $U_1 \cup U_2$ and have the following form:

$$\begin{aligned}\sigma_{2,1} &= \sigma_{1,1}\Lambda_1 + (H_1 - a_2 r_2)\xi_{1,1}, \qquad (2.7) \\ \sigma_{2,2} &= a_2 s_2 (1 - \Lambda_1) + (H_1 - a_2 r_2)\xi_{1,2}.\end{aligned}$$

Notice that if $x \in U_1 \cap U_2$, then $\sigma_{1,1} > 0$, $a_2 s_2 > 0$, $\Lambda_1 > 0$, $1 - \Lambda_1 > 0$, $\xi_{1,1} > 0$ and $\xi_{1,2} < 0$. Therefore, on $U_1 \cap U_2$, the coefficient $\sigma_{2,1}$ is strictly positive whenever $H_1 - a_2 r_2 \geqslant 0$, and the coefficient $\sigma_{2,2}$ is strictly positive whenever $H_1 - a_2 r_2 \leqslant 0$. Since the vectors g_1 and g_2 are linearly independent, if $x \in U_1 \cap U_2$ and $\mathbf{H}_\mathbf{2}'(x) : T_x \to \mathbb{R}$ is identically zero, there is necessarily $\varrho \neq 0$ with $D(x) = \varrho(\sigma_{2,1}(x) g_1 + \sigma_{2,2}(x) g_2)$. Thus, $D(x) \in [g_1, g_2]$.

The set Z_2 can be split into the disjoint sets $Z_2 = Z_1 \cup Z_{2,1} \cup Z_{2,2}$, where

$$Z_{2,1} = \begin{cases} \{z_2\} & \text{if } z_2 \in U_2 \setminus \overline{U}_1, \\ \emptyset & \text{otherwise} \end{cases}$$

and $Z_{2,2}$ is a subset (possibly empty) within $U_1 \cap U_2 \cap D^{-1}([g_1, g_2])$. Now, let us check that $Z_{2,2} \subset \bigcup_{g \in N_2} O_g$. Indeed, if $x \in Z_{2,2}$, then $x \in U_1 \cap U_2 \subset U_1 \cap U_2'$, $D(x) \in [g_1, g_2]$ and $g_1(x) < \gamma_{1,1}$. This implies, according to Fact 2.3, that $x \in \bigcup_{g \in N_2} O_g$.

In the case that $Z_{2,1} = \{z_2\}$ and $z_2 \notin \bigcup_{g \in N_2} \overline{O}_g$, we select, if necessary, a larger t_2 with $t_2 < l_2$, so that $z_2 \notin \bigcup_{g \in N_2} \overline{B}_g$. Since the norm $|\cdot|$ is LUR and $D(z_2) = g_2$, the functional g_2 strongly exposes S at the point z_2 and we may select numbers $0 < t_2' < l_2' < 1$ and open slices, which are neighborhoods of z_2, defined by

$$O_{g_2} := \{x \in S\colon\ g_2(x) > l_2'\} \quad \text{and} \quad B_{g_2} := \{x \in S\colon\ g_2(x) > t_2'\},$$

satisfying $O_{g_2} \subset B_{g_2} \subset \{x \in S\colon\ g_1(x) < \gamma_{1,1}',\ g_2(x) > \gamma_2'\}$ and $\operatorname{dist}(B_{g_2}, B_g) > 0$ for every $g \in N_2$. In this case, we define $\Gamma_2 = N_2 \cup \{g_2\}$.

Now, if $Z_{2,1} = \{z_2\} \in \bigcup_{g \in N_2} \overline{O}_g$, we select, if necessary, a smaller constant l_2, with $0 < t_2 < l_2 < 1$, so that $Z_{2,1} = \{z_2\} \in \bigcup_{g \in N_2} O_g$. In this case, and also when $Z_{2,1} = \emptyset$, we define $\Gamma_2 = N_2$.

Notice that, in any of the cases mentioned above, Fact 2.3 clearly holds for the (possibly) newly selected real numbers t_2 and l_2.

Notice that the distance between any two sets B_g, $B_{g'}$, $g, g' \in \Gamma_1 \cup \Gamma_2$, $g \neq g'$, is positive. Moreover, $Z_1 \subset O_{g_1} \subset B_{g_1} \subset U_1 = R_1$, and $Z_{2,1} \cup Z_{2,2} \subset \bigcup_{g \in \Gamma_2} O_g \subset \bigcup_{g \in \Gamma_2} B_g \subset U_2' \subset R_2$. Therefore, $Z_2 = Z_1 \cup Z_{2,1} \cup Z_{2,2} \subset \bigcup_{g \in \Gamma_1 \cup \Gamma_2} O_g \subset \bigcup_{g \in \Gamma_1 \cup \Gamma_2} B_g \subset U_1 \cup U_2 = R_1 \cup R_2$. In addition, we have $\operatorname{dist}(\bigcup_{g \in \Gamma_1 \cup \Gamma_2} B_g, (U_1 \cup U_2)^c) > 0$.

It is worth remarking that $\mathbf{H}_\mathbf{2}' = \sigma_{2,1} \mathbf{g_1} + \sigma_{2,2} \mathbf{g_2}$ in $U_1 \cup U_2$, where $\sigma_{2,1}$ and $\sigma_{2,2}$ are continuous functions and at least one of the coefficients $\sigma_{2,1}(x), \sigma_{2,2}(x)$ is strictly positive, for every $x \in U_1 \cup U_2$. Moreover, since $H_2' = \sigma_{2,1} g_1 + \sigma_{2,2} g_2$, $H_2'(x) = a_2 s_2 g_2$ whenever $x \in U_2 \setminus U_1$ and $H_2'(x) = a_1 s_1 g_1$ whenever $x \in U_1 \setminus U_2$, then $\sigma_{2,1}(x) = 0$ whenever $x \in U_2 \setminus U_1$, and $\sigma_{2,2}(x) = 0$ whenever $x \in U_1 \setminus U_2$.

In order to clarify the construction in the general case, let us also explain in detail the construction of the function h_3 and locate the critical points of the function $\mathbf{H}_\mathbf{3}'$.

• *For $k = 3$,* let us denote by $y_3 \in S$ the point satisfying $f_3(y_3) = 1$. If either $\{g_1, g_2, f_3\}$ are linearly dependent or $g_1(y_3) = \gamma_1$ or $g_2(y_3) = \gamma_2$, we can use the density of the norm attaining functionals (Bishop–Phelps theorem) and the continuity of D to modify y_3 and find $z_3 \in S$ so that: $g_1(z_3) \neq \gamma_1$, $g_2(z_3) \neq \gamma_2$, $\{g_1, g_2, g_3 := D(z_3)\}$ are linearly independent, and

$$\{x \in S\colon\ f_3(x) > \delta_3^2\} \subset \{x \in S\colon\ g_3(x) > \nu_3\} \subset \{x \in S\colon\ f_3(x) > \delta_3^3\}$$

for some $\nu_3 \in (0, 1)$. If $\{g_1, g_2, f_3\}$ are linearly independent, $g_1(y_3) \neq \gamma_1$, and $g_2(y_3) \neq \gamma_2$, we define $g_3 = f_3$ and $z_3 = y_3$. Then, we apply Lemma 2.2 to the linearly independent vectors,

$\{g_1, g_2, g_3\}$ and the real numbers $\gamma_1 \in (0,1)$, $\gamma_2 \in (0,1)$ and obtain $\gamma_3 \in (0,1)$ close enough to ν_3 so that

$$S_3 = \{x \in S\colon\ f_3(x) > \delta_3\} \subset \{x \in S\colon\ g_3(x) > \gamma_3\} \subset \{x \in S\colon\ f_3(x) > \delta_3^4\} = P_3,$$

$$\{T \in [g_1, g_2, g_3]^*\colon\ T(g_1) = \gamma_1,\ T(g_2) = \gamma_2,\ T(g_3) = \gamma_3 \text{ and } |T| = 1\} = \emptyset, \tag{2.8}$$

$$\{T \in [g_1, g_3]^*\colon\ T(g_1) = \gamma_1,\ T(g_3) = \gamma_3 \text{ and } |T| = 1\} = \emptyset, \tag{2.9}$$

$$\{T \in [g_2, g_3]^*\colon\ T(g_2) = \gamma_2,\ T(g_3) = \gamma_3 \text{ and } |T| = 1\} = \emptyset. \tag{2.10}$$

Select $\gamma'_{2,1} \in (\gamma_2, \gamma'_2)$ and define

$$R_3 = \{x \in S\colon\ g_3(x) > \gamma_3\} \quad \text{and}$$
$$U'_3 = \{s \in S\colon\ g_1(x) < \gamma'_{1,2},\ g_2(x) < \gamma'_{2,1} \text{ and } g_3(x) > \gamma_3\},$$

where $\gamma'_{1,2}$ is a number in $(\gamma_1, \frac{\gamma_1+\gamma_{1,1}}{2})$.

Notice that $\operatorname{dist}(B_g, U'_3) > 0$ for every $g \in \Gamma_1 \cup \Gamma_2$. Assume that $R_3 \cap (U_1 \cup U_2) \neq \emptyset$, and consider the sets

$$M_{3,1} = \{x \in (U_1 \cap U'_3) \setminus U_2\colon\ D(x) \in [g_1, g_3]\},$$
$$M_{3,2} = \{x \in (U_2 \cap U'_3) \setminus U_1\colon\ D(x) \in [g_2, g_3]\},$$
$$M_{3,1,2} = \{x \in U_1 \cap U_2 \cap U'_3\colon\ D(x) \in [g_1, g_2, g_3]\},$$

and $M_3 = M_{3,1} \cup M_{3,2} \cup M_{3,1,2}$.

In the case that $M_3 = \emptyset$, we select as $\gamma_{2,1}$ any point in $(\gamma_2, \gamma'_{2,1})$ and $\gamma_{1,2}$ any point in $(\gamma_1, \gamma'_{1,2})$.

In the case that $M_3 \neq \emptyset$ and $\operatorname{dist}(M_3, (U_1 \cup U_2)^c) > 0$ we can easily find $\gamma_{2,1} \in (\gamma_2, \gamma'_{2,1})$ and $\gamma_{1,2} \in (\gamma_1, \gamma'_{1,2})$ with $M_3 \subset \{x \in S\colon\ g_1(x) > \gamma_{1,2}\} \cup \{x \in S\colon\ g_2(x) > \gamma_{2,1}\}$.

In the case that $\operatorname{dist}(M_3, (U_1 \cup U_2)^c) = 0$ and in order to obtain suitable constants $\gamma_{2,1}$ and $\gamma_{1,2}$, we need to study the limits of the sequences $\{x_n\} \subset M_3$ such that

$$\lim_n \operatorname{dist}(x_n, (U_1 \cup U_2)^c) = 0.$$

Define the sets

$$F'_{3,i} = \{T \in [g_i, g_3]^*\colon\ T(g_i) = \gamma_i \text{ and } |T| = 1\} \quad \text{for } i = 1, 2,$$
$$F'_{3,1,2} = \{T \in [g_1, g_2, g_3]^*\colon\ T(g_1) = \gamma_1,\ T(g_2) = \gamma_2 \text{ and } |T| = 1\},$$

and

$$N'_{3,i} = \{g \in S^* \cap [g_i, g_3]\colon\ T(g) = 1 \text{ for some } T \in F'_{3,i}\} \quad \text{for } i = 1, 2,$$
$$N'_{3,1,2} = \{g \in S^* \cap [g_1, g_2, g_3]\colon\ T(g) = 1 \text{ for some } T \in F'_{3,1,2}\}.$$

Since the norm $|\cdot|^*$ is Gâteaux smooth, we apply Lemma 2.1 to the finite-dimensional space $[g_1, g_2, g_3]$ and the restriction of the norm $|\cdot|^*$ to $[g_1, g_2, g_3]$ (which is a differentiable norm on the space $[g_1, g_2, g_3]$), and deduce that the cardinal of any of the sets $F'_{3,i}$, $F'_{3,1,2}$ is at most two. Furthermore, from the strict convexity of the norm $|\cdot|^*$ we obtain that the cardinal of any of the sets $N'_{3,i}$ and $N'_{3,1,2}$, is at most two. Let us consider, for $i = 1, 2$, the norm-one extensions to $[g_1, g_2, g_3]$ of the functionals of $F'_{3,i}$, that is,

$$F''_{3,i} = \bigl\{T \in [g_1, g_2, g_3]^*\colon\ T|_{[g_i, g_3]} \in F'_{3,i} \text{ and } |T| = 1\bigr\}.$$

Since the norm $|\cdot|^*$ is Gâteaux smooth, for every $G \in F'_{3,i}$ there is exactly *one* norm-one extension T to $[g_1, g_2, g_3]$. Therefore the cardinal of the set $F''_{3,i}$ is at most two. Hence the sets $F'_3 := F''_{3,1} \cup F''_{3,2} \cup F'_{3,1,2}$ and $N'_3 := N'_{3,1} \cup N'_{3,2} \cup N'_{3,1,2}$ are finite. In addition, as a consequence of the equalities (2.8), (2.9) and (2.10), we deduce that $T(g_3) \neq \gamma_3$ for every $T \in F'_3$. Indeed, if $T \in F'_{3,1,2}$ the assertion follows immediately from (2.8). If $T \in F''_{3,i}$ for some $i \in \{1, 2\}$, then $T|_{[g_i, g_3]} \in F'_{3,i}$, that is, $|T|_{[g_i, g_3]}| = 1$ and $T(g_i) = \gamma_i$. From (2.9) for $i = 1$, and (2.10) for $i = 2$, we obtain that $T(g_3) \neq \gamma_3$. We can restrict our study to one of the following kind of sequences:

(1) Fix $i \in \{1, 2\}$. Consider any sequence $\{x_n\} \subset M_{3,i}$ such that $\lim_n \operatorname{dist}(x_n, (U_1 \cup U_2)^c) = 0$. Then, it easily follows that $\lim_n g_i(x_n) = \gamma_i$. Indeed,
   - if $\{x_n\} \subset M_{3,1}$, then in particular $\{x_n\} \subset U_1 = R_1$. Therefore, $\operatorname{dist}(x_n, (U_1 \cup U_2)^c) \geqslant \operatorname{dist}(x_n, R_1^c)$. Thus, $\lim_n \operatorname{dist}(x_n, R_1^c) = 0$ and this implies that $\lim_n g_1(x_n) = \gamma_1$;
   - if $\{x_n\} \subset M_{3,2}$, then in particular $\{x_n\} \subset U_2 \subset R_2$. Recall that $U_1 \cup U_2 = R_1 \cup R_2$. Therefore, $\operatorname{dist}(x_n, (U_1 \cup U_2)^c) \geqslant \operatorname{dist}(x_n, R_2^c)$. Thus, $\lim_n \operatorname{dist}(x_n, R_2^c) = 0$ and this implies that $\lim_n g_2(x_n) = \gamma_2$.

   Now, let us take any sequence $\{x_n\} \subset M_{3,i}$ such that $\lim_n g_i(x_n) = \gamma_i$. Consider every x_n as an element of X^{**} and denote by $\mathbf{x_n}$ its restriction to $[g_1, g_2, g_3]$. Recall that $D(x_n) \in S^* \cap [g_i, g_3]$ for every $n \in \mathbb{N}$. Then, the sequence of restrictions $\{\mathbf{x_n}\} \subset [g_1, g_2, g_3]^*$ satisfies that

$$\begin{aligned} 1 = |x_n| \geqslant |\mathbf{x_n}| \geqslant |\mathbf{x_n}|_{[g_i, g_3]}| = \max\bigl\{\mathbf{x_n}(h)\colon\ h \in S^* \cap [g_i, g_3]\bigr\} \\ \geqslant \mathbf{x_n}\bigl(D(x_n)\bigr) = D(x_n)(x_n) = 1 \end{aligned}$$

   for every $n \in \mathbb{N}$. Thus, there is a subsequence $\{\mathbf{x_{n_j}}\}$ converging to an element $T \in [g_1, g_2, g_3]^*$ with $|T| = |T|_{[g_i, g_3]}| = 1$. Since $\lim_j g_i(x_{n_j}) = \lim_j \mathbf{x_{n_j}}(g_i) = \gamma_i$, we have that $T(g_i) = \gamma_i$ and this implies that $T|_{[g_i, g_3]} \in F'_{3,i}$ and $T \in F''_{3,i}$. Furthermore, if $g \in N'_{3,i}$ and $T(g) = 1$, then $\lim_j \mathbf{x_{n_j}}(g) = 1$. In addition, $T(g_3) = \lim_j \mathbf{x_{n_j}}(g_3) = \lim_j g_3(x_{n_j}) \geqslant \gamma_3$ because $\{x_{n_j}\} \subset U'_3$. Then, from condition (2.10) if $i = 1$ and condition (2.9) if $i = 2$, we deduce that $T(g_3) > \gamma_3$. Finally, let us check that $T(g_s) = \lim_j \mathbf{x_{n_j}}(g_s) \leqslant \gamma_s$, where $s \in \{1, 2\}$ and $s \neq i$:
   - if $i = 1$, the sequence $\{x_{n_j}\} \subset M_{3,1}$ and thus $\{x_{n_j}\} \subset (U_1 \cap U'_3) \setminus U_2$. In particular $\{x_{n_j}\} \subset U'_3$ and $g_1(x_{n_j}) < \gamma'_{1,2} < \gamma_{1,1}$ for every $j \in \mathbb{N}$. Therefore, if $x_{n_j} \notin U_2$ for all j, we must have $\mathbf{x_{n_j}}(g_2) = g_2(x_{n_j}) \leqslant \gamma_2$ for every $j \in \mathbb{N}$;
   - if $i = 2$, the sequence $\{x_{n_j}\} \subset M_{3,2}$ and thus $\{x_{n_j}\} \subset (U_2 \cap U'_3) \setminus U_1$. In particular $x_{n_j} \notin U_1 = R_1$, for every $j \in \mathbb{N}$ and this implies $\mathbf{x_{n_j}}(g_1) = g_1(x_{n_j}) \leqslant \gamma_1$ for every $j \in \mathbb{N}$.

(2) Consider a sequence $\{x_n\} \subset M_{3,1,2}$, such that $\lim_n \operatorname{dist}(x_n, (U_1 \cup U_2)^c) = 0$. Then, it easily follows that $\lim_n g_i(x_n) = \gamma_i$ for $i = 1, 2$. Indeed, $U_1 \cup U_2 = R_1 \cup R_2$ and then $\operatorname{dist}(x_n, (R_1 \cup R_2)^c) \geqslant \operatorname{dist}(x_n, R_i^c)$ for every $n \in \mathbb{N}$ and $i = 1, 2$. Hence $\lim_n \operatorname{dist}(x_n, R_i^c) = 0$. Since $\{x_n\} \subset R_i$ for $i = 1, 2$, we obtain that $\lim_n g_i(x_n) = \gamma_i$ for $i = 1, 2$.

Now, let us take any sequence $\{x_n\} \subset M_{3,1,2}$ such that $\lim_n g_i(x_n) = \gamma_i$, for every $i = 1, 2$. Consider every x_n as an element of X^{**} and denote by $\mathbf{x_n}$ its restriction to $[g_1, g_2, g_3]$. Then, the sequence of restrictions $\{\mathbf{x_n}\} \subset [g_1, g_2, g_3]^*$ satisfies that

$$1 = |x_n| \geqslant |\mathbf{x_n}| = \max\bigl\{\mathbf{x_n}(h)\colon\ h \in S^* \cap [g_1, g_2, g_3]\bigr\} \geqslant \mathbf{x_n}\bigl(D(x_n)\bigr) = D(x_n)(x_n) = 1$$

for every $n \in \mathbb{N}$. Thus, there is a subsequence $\{\mathbf{x_{n_j}}\}$ converging to an element $T \in [g_1, g_2, g_3]^*$ with $|T| = 1$. Since $\lim_j g_i(x_{n_j}) = \lim_j \mathbf{x_{n_j}}(g_i) = \gamma_i$ for $i = 1, 2$, then $T(g_i) = \gamma_i$ for $i = 1, 2$, and this implies that $T \in F'_{3,1,2}$. Furthermore, if $g \in N'_{3,1,2}$ and $T(g) = 1$, then $\lim_j \mathbf{x_{n_j}}(g) = 1$. In addition, $T(g_3) = \lim_j \mathbf{x_{n_j}}(g_3) = \lim_j g_3(x_{n_j}) \geqslant \gamma_3$ because $\{x_{n_j}\} \subset U'_3$. Then, from condition (2.8), we deduce that $T(g_3) > \gamma_3$.

Let us define, for $i = 1, 2$,

$$F_{3,i} = \Bigl\{T \in F''_{3,i}\colon \text{ there is } \{x_n\} \subset M_{3,i} \text{ with } \lim_n \mathbf{x_n}(g_i) = \gamma_i, \text{ and } \lim_n \mathbf{x_n} = T\Bigr\},$$

$$F_{3,1,2} = \Bigl\{T \in F'_{3,1,2}\colon \text{ there is } \{x_n\} \subset M_{3,1,2} \text{ with } \lim_n \mathbf{x_n}(g_1) = \gamma_1,\ \lim_n \mathbf{x_n}(g_2) = \gamma_2 \text{ and } \lim_n \mathbf{x_n} = T\Bigr\},$$

and

$$F_3 = F_{3,1} \cup F_{3,2} \cup F_{3,1,2}. \tag{2.11}$$

Select a real number γ'_3 satisfying $\gamma_3 < \gamma'_3 < \min\{T(g_3)\colon\ T \in F_3\}$ (recall that F_3 is finite), and define,

$$N_{3,i} = \bigl\{g \in N'_{3,i}\colon \text{ there is } T \in F_{3,i} \text{ with } T(g) = 1\bigr\} \quad \text{for } i = 1, 2,$$
$$N_{3,1,2} = \bigl\{g \in N'_{3,1,2}\colon \text{ there is } T \in F_{3,1,2} \text{ with } T(g) = 1\bigr\},$$

and $N_3 = N_{3,1} \cup N_{3,2} \cup N_{3,1,2}$. Let us prove the following fact.

Fact 2.4.

(1) *There are numbers* $0 < t_3 < l_3 < 1$ *such that for every* $g \in N_3$*, the slices*

$$O_g := \bigl\{x \in S\colon\ g(x) > l_3\bigr\} \quad \textit{and} \quad B_g := \bigl\{x \in S\colon\ g(x) > t_3\bigr\}$$

satisfy that

$$O_g \subset B_g \subset \bigl\{x \in S\colon\ g_1(x) < \gamma'_{1,2},\ g_2(x) < \gamma'_{2,1},\ g_3(x) > \gamma'_3\bigr\} \quad \textit{and} \tag{2.12}$$
$$\operatorname{dist}(B_g, B_{g'}) > 0, \quad \textit{whenever } g, g' \in N_3,\ g \neq g'. \tag{2.13}$$

(2) *There are numbers* $\gamma_{1,2} \in (\gamma_1, \gamma'_{1,2})$ *and* $\gamma_{2,1} \in (\gamma_2, \gamma'_{2,1})$ *such that if* $x \in M_3$, $g_1(x) < \gamma_{1,2}$ *and* $g_2(x) < \gamma_{2,1}$, *then* $x \in O_g$ *for some* $g \in N_3$.

Proof. (1) First, if X is reflexive, we know that for every $g \in N_3$ there is $x_g \in S$ such that $D(x_g) = g$. Let us study the three possible cases:

- If $g \in F_{3,1}$, denote by $\mathbf{x_g}$ the restriction of x_g to $[g_1, g_2, g_3]$. Since $\mathbf{x_g}(g) = 1$ and $|\cdot|^*$ is Gâteaux smooth, then $\mathbf{x_g} = T$ for some $T \in F_{3,1}$. This implies that $\mathbf{x_g}(g_1) = \gamma_1 < \gamma'_{1,2}$, $\mathbf{x_g}(g_3) > \gamma'_3$ and $\mathbf{x_g}(g_2) \leqslant \gamma_2 < \gamma'_{2,1}$. Hence, $x_g \in \{x \in S\colon\ g_1(x) < \gamma'_{1,2}$, $g_2(x) > \gamma'_{2,1}$ and $g_3(x) > \gamma'_3\}$.
- If $g \in F_{3,2}$, denote by $\mathbf{x_g}$ the restriction of x_g to $[g_1, g_2, g_3]$. Since $\mathbf{x_g}(g) = 1$ and $|\cdot|^*$ is Gâteaux smooth, then $\mathbf{x_g} = T$ for some $T \in F_{3,2}$. This implies that $\mathbf{x_g}(g_2) = \gamma_2 < \gamma'_{2,1}$, $\mathbf{x_g}(g_3) > \gamma'_3$ and $\mathbf{x_g}(g_1) \leqslant \gamma_1 < \gamma'_{1,2}$. Hence, $x_g \in \{x \in S\colon\ g_1(x) < \gamma'_{1,2}$, $g_2(x) > \gamma'_{2,1}$ and $g_3(x) > \gamma'_3\}$.
- If $g \in F_{3,1,2}$, denote by $\mathbf{x_g}$ the restriction of x_g to $[g_1, g_2, g_3]$. Since $\mathbf{x_g}(g) = 1$ and $|\cdot|^*$ is Gâteaux smooth, then $\mathbf{x_g} = T$ for some $T \in F_{3,1,2}$. This implies that $\mathbf{x_g}(g_1) = \gamma_1 < \gamma'_{1,2}$, $\mathbf{x_g}(g_2) = \gamma_2 < \gamma'_{2,1}$ and $\mathbf{x_g}(g_3) > \gamma'_3$. Hence, $x_g \in \{x \in S\colon\ g_1(x) < \gamma'_{1,2}$, $g_2(x) > \gamma'_{2,1}$ and $g_3(x) > \gamma'_3\}$.

Now, since the norm $|\cdot|$ is LUR and $D(x_g) = g$, the functional g strongly exposes S at the point x_g for every $g \in N_3$. Since N_3 is finite, we can hence obtain real numbers $0 < t_3 < l_3 < 1$ and slices O_g and B_g, for every $g \in N_3$, satisfying conditions (2.12) and (2.13).

Now consider a non-reflexive Banach space X. Let us first prove (2.12). Assume, on the contrary, that there is a point $g \in N_3$ and there is a sequence $\{y_n\} \subset S$ satisfying $g(y_n) > 1 - \frac{1}{n}$ and such that for every $n \in \mathbb{N}$ either $g_1(y_n) \geqslant \gamma'_{1,2}$ or $g_2(y_n) \geqslant \gamma'_{2,1}$ or $g_3(y_n) \leqslant \gamma'_3$. If $g \in N_3$ there is a sequence $\{x_n\} \subset M_3$ with $\lim_n g_i(x_n) \leqslant \gamma_i$, for $i = 1, 2$, $\lim_n g_3(x_n) > \gamma'_3$ and $\lim_n g(x_n) = 1$. In particular,

$$\frac{g(x_n) + 1 - \frac{1}{n}}{2} \leqslant g\left(\frac{x_n + y_n}{2}\right) \leqslant \left|\frac{x_n + y_n}{2}\right| \leqslant 1,$$

and thus $\lim_n |\frac{x_n + y_n}{2}| = 1$. Recall that in the non-reflexive case, the norm $|\cdot|$ is WUR, and then $x_n - y_n \xrightarrow{\omega} 0$ (weakly converges to zero). This last assertion gives a contradiction since we have either $\limsup_n g_1(x_n - y_n) \leqslant \gamma_1 - \gamma'_{1,2} < 0$ or $\limsup_n g_2(x_n - y_n) \leqslant \gamma_2 - \gamma'_{2,1} < 0$ or $\liminf_n g_3(x_n - y_n) \geqslant \lim_n g_3(x_n) - \gamma'_3 > 0$. Therefore we can find real numbers $0 < t_2 < l_2 < 1$ and slices O_g and B_g for every $g \in N_3$, satisfying condition (2.12). The proof of (2.13) is the same as the one given in Fact 2.3, where the only property we need is the strict convexity of $|\cdot|^*$.

(2) Assume, on the contrary, that for every $n \in \mathbb{N}$, there is $x_n \in M_3$ with $g_i(x_n) \leqslant \gamma_i + \frac{1}{n}$, for $i = 1, 2$ and $\{x_n\colon\ n \in \mathbb{N}\} \cap (\bigcup_{g \in N_3} O_g) = \emptyset$. Then, there is a subsequence of $\{x_n\}$, which we keep denoting by $\{x_n\}$, such that either $\{x_n\} \subset M_{3,1}$ or $\{x_n\} \subset M_{3,2}$ or $\{x_n\} \subset M_{3,1,2}$. In the first case, $\lim_n g_1(x_n) = \gamma_1$. In the second case, $\lim_n g_2(x_n) = \gamma_2$. In the third case, $\lim_n g_i(x_n) = \gamma_i$, for every $i = 1, 2$. From the definition of F_3 and N_3 and the comments preceding Fact 2.4, we know that there is a subsequence $\{x_{n_j}\}$ and $g \in N_3$ satisfying that $\lim_j g(x_{n_j}) = 1$, which is a contradiction. This finishes the proof of Fact 2.4. □

If $R_3 \cap (U_1 \cup U_2) = \emptyset$ we may select as $\gamma_{1,2}$ any number in $(\gamma_1, \gamma'_{1,2})$ and $\gamma_{2,1}$ any number in $(\gamma_2, \gamma'_{2,1})$.

Now we define h_3 as follows:

$$h_3 : S^+ \longrightarrow \mathbb{R}$$
$$h_3 = \varphi_3(g_3)\phi_{2,1}(g_2)\phi_{1,2}(g_1),$$

where φ_3, $\phi_{2,1}$ and $\phi_{1,2}$ are C^∞ functions on $\mathbb{R}$ satisfying that

$$\varphi_3(t) = 0 \quad \text{if } t \leqslant \gamma_3,$$
$$\varphi_3(1) = 1,$$
$$\varphi_3'(t) > 0 \quad \text{if } t > \gamma_3,$$

and

$$\phi_{1,2}(t) = 1 \quad \text{if } t \leqslant \frac{\gamma_1 + \gamma_{1,2}}{2}, \qquad \phi_{2,1}(t) = 1 \quad \text{if } t \leqslant \frac{\gamma_2 + \gamma_{2,1}}{2},$$
$$\phi_{1,2}(t) = 0 \quad \text{if } t \geqslant \gamma_{1,2}, \qquad \phi_{2,1}(t) = 0 \quad \text{if } t \geqslant \gamma_{2,1},$$
$$\phi_{1,2}'(t) < 0 \quad \text{if } t \in \left(\frac{\gamma_1 + \gamma_{1,2}}{2}, \gamma_{1,2}\right), \qquad \phi_{2,1}'(t) < 0 \quad \text{if } t \in \left(\frac{\gamma_2 + \gamma_{2,1}}{2}, \gamma_{2,1}\right).$$

Clearly the interior of the support of h_3 is the set

$$U_3 = \{x \in S^+ \colon\ g_1(x) < \gamma_{1,2},\ g_2(x) < \gamma_{2,1} \text{ and } g_3(x) > \gamma_3\}.$$

Select one point $x_3 \in U_3$, a real number $a_3 \in \mathbb{R}^*$ with $|a_3 - F(x_3)| < \varepsilon$ and define the auxiliary function

$$r_3 : S^+ \longrightarrow \mathbb{R}$$
$$r_3 = s_3 g_3 + (1 - s_3 g_3(x_3)),$$

where we have selected s_3 so that $s_3 a_3 > 0$ and $|s_3|$ is small enough so that the oscillation of r_3 on U_3 is less than $\frac{\varepsilon}{|a_3|}$. Notice that $r_3(x_3) = 1$.

Let us study the critical points Z_3 of the C^p smooth function

$$\mathbf{H_3} : U_1 \cup U_2 \cup U_3 \longrightarrow \mathbb{R}$$
$$\mathbf{H_3} = \frac{a_1 r_1 h_1 + a_2 r_2 h_2 + a_3 r_3 h_3}{h_1 + h_2 + h_3}. \tag{2.14}$$

Let us prove that $Z_3 := \{x \in U_1 \cup U_2 \cup U_3\colon\ H_3'(x) = 0 \text{ on } T_x\}$ can be included in a finite number of disjoint slices within $U_1 \cup U_2 \cup U_3$ by splitting it conveniently into the (already defined) Z_1, Z_2 and up to four more disjoint sets within U_3, as Fig. 1 suggests.

The function $\mathbf{H_3'}$ can be written as $\mathbf{H_3'} = \sigma_{3,1}\mathbf{g_1} + \sigma_{3,2}\mathbf{g_2} + \sigma_{3,3}\mathbf{g_3}$, where $\sigma_{3,i}$ are continuous and real functions on $U_1 \cup U_2 \cup U_3$ and $\mathbf{g_i}$ denotes the restriction $g_i|_{T_x}$, $i = 1, 2, 3$, whenever we evaluate $\mathbf{H_3'}(x)$ on T_x.

Clearly, from (2.14), $\mathbf{H_3}$ and $\mathbf{H_3'}$ restricted to $(U_1 \cup U_2) \setminus U_3$ coincide with $\mathbf{H_2}$ and $\mathbf{H_2'}$, respectively. Then, $Z_3 \setminus U_3 = Z_3 \setminus \overline{U}_3 = Z_2$. Let us study the set $Z_3 \cap U_3$. First, if $x \in U_3 \setminus (U_1 \cup U_2)$, from (2.14), we obtain that $\mathbf{H_3}(x) = a_3 r_3(x)$ and $H_3'(x) = a_3 r_3'(x) = a_3 s_3 g_3$. Therefore $\mathbf{H_3'}(x) =$

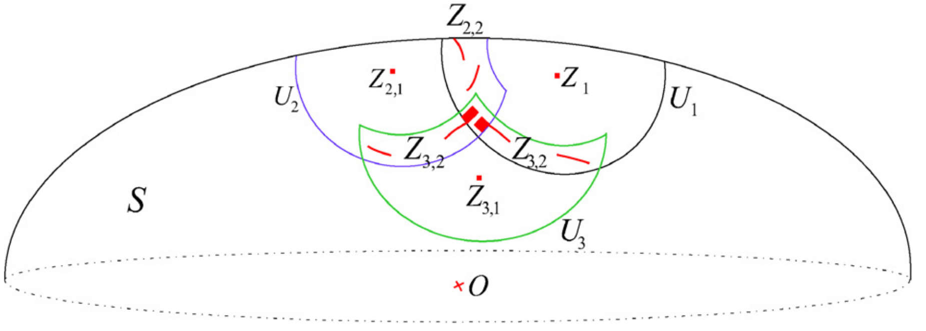

Fig. 1. Case $n = 3$: the decomposition of Z_3.

$a_3 s_3 g_3|_{T_x} \equiv 0$ iff $D(x) = g_3$. If the point $z_3 \in U_3 \setminus (U_1 \cup U_2)$ then, $\mathbf{H_3}$ has exactly one critical point in $U_3 \setminus (U_1 \cup U_2)$; in this case, since $g_1(z_3) \neq \gamma_1$ and $g_2(z_3) \neq \gamma_2$, the point z_3 actually belongs to $U_3 \setminus (\overline{U}_1 \cup \overline{U}_2)$.

Secondly, let us study the critical points of $\mathbf{H_3}$ in $U_3 \cap (U_1 \cup U_2)$. If we define $\Lambda_2 = \frac{h_1+h_2}{h_1+h_2+h_3}$, then we can rewrite $\mathbf{H_3}$ in $U_3 \cap (U_1 \cup U_2)$ as

$$\mathbf{H_3} = \frac{a_1 r_1 h_1 + a_2 r_2 h_2}{h_1 + h_2} \cdot \frac{h_1 + h_2}{h_1 + h_2 + h_3} + \frac{a_3 r_3 h_3}{h_1 + h_2 + h_3} = \mathbf{H_2}\Lambda_2 + a_3 r_3 (1 - \Lambda_2),$$

and

$$\mathbf{H_3}' = \mathbf{H_2}'\Lambda_2 + a_3 s_3 (1 - \Lambda_2)\mathbf{g_3} + (\mathbf{H_2} - a_3 r_3)\Lambda_2'.$$

By computing Λ_2', we obtain $\Lambda_2' = \xi_{2,1}\mathbf{g_1} + \xi_{2,2}\mathbf{g_2} + \xi_{2,3}\mathbf{g_3}$, where the coefficients $\xi_{2,1}, \xi_{2,2}$ and $\xi_{2,3}$ are continuous functions of the following form:

$$\begin{aligned}
\xi_{2,1} &= \frac{-\varphi_3(g_3)\phi_{2,1}(g_2)\phi_{1,2}'(g_1)(h_1 + h_2) + h_3\varphi_1'(g_1) + h_3\varphi_2(g_2)\phi_{1,1}'(g_1)}{(h_1 + h_2 + h_3)^2}, \\
\xi_{2,2} &= \frac{-\varphi_3(g_3)\phi_{2,1}'(g_2)\phi_{1,2}(g_1)(h_1 + h_2) + h_3\varphi_2'(g_2)\phi_{1,1}(g_1)}{(h_1 + h_2 + h_3)^2}, \\
\xi_{2,3} &= \frac{-\varphi_3'(g_3)\phi_{2,1}(g_2)\phi_{1,2}(g_1)(h_1 + h_2)}{(h_1 + h_2 + h_3)^2}.
\end{aligned} \tag{2.15}$$

Since $g_1(x) < \gamma_{1,2} < \frac{\gamma_1 + \gamma_{1,1}}{2}$ for every $x \in U_3$, we have that $\phi_{1,1}'(g_1(x)) = 0$ for every $x \in U_3$, and we can drop the term $h_3\varphi_2(g_2)\phi_{1,1}'(g_1)$ in the above expression of $\xi_{2,1}$. Thus, if $x \in U_3 \cap (U_1 \cup U_2)$, the coefficients $\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}$ for $\mathbf{H_3'}$ have the following form,

$$\begin{aligned}
\sigma_{3,1} &= \sigma_{2,1}\Lambda_2 + (\mathbf{H_2} - a_3 r_3)\xi_{2,1}, \\
\sigma_{3,2} &= \sigma_{2,2}\Lambda_2 + (\mathbf{H_2} - a_3 r_3)\xi_{2,2}, \\
\sigma_{3,3} &= a_3 s_3 (1 - \Lambda_2) + (\mathbf{H_2} - a_3 r_3)\xi_{2,3},
\end{aligned}$$

where $a_3 s_3 > 0$, $\Lambda_2 > 0$, $1 - \Lambda_2 > 0$, $\xi_{2,1} \geqslant 0$, $\xi_{2,2} \geqslant 0$, $\xi_{2,1} + \xi_{2,2} > 0$ and $\xi_{2,3} < 0$ on $U_3 \cap (U_1 \cup U_2)$. Therefore, if $H_2 - a_3 r_3 \leqslant 0$, the coefficient $\sigma_{3,3} > 0$. When $H_2 - a_3 r_3 \geqslant 0$ and

$\sigma_{2,2} > 0$, we have that $\sigma_{3,2} > 0$. Finally, when $H_2 - a_3 r_3 \geqslant 0$ and $\sigma_{2,1} > 0$, we have $\sigma_{3,1} > 0$ (recall that for every $x \in U_1 \cup U_2$, there is $j \in \{1,2\}$ such that $\sigma_{2,j}(x) > 0$). Since the vectors $\{g_1, g_2, g_3\}$ are linearly independent we get that, if $\mathbf{H}_3'(x) = 0$ for some $x \in U_3 \cap (U_1 \cup U_2)$, then there necessarily exists $\varrho \neq 0$ such that $D(x) = \varrho(\sigma_{3,1}(x)g_1 + \sigma_{3,2}(x)g_2 + \sigma_{3,3}(x)g_3)$, that is $D(x) \in [g_1, g_2, g_3]$.

In fact we can be more accurate and obtain that if $x \in (U_3 \cap U_2) \setminus U_1$ and $\mathbf{H}_3'(x) = 0$ then $D(x) \in [g_2, g_3]$. Indeed, in step 2 we proved that $\sigma_{2,1} = 0$ in $U_2 \setminus U_1$. Moreover, the functions $\varphi_1(g_1)$, $\phi_{1,1}(g_1)$ and $\phi_{1,2}(g_1)$ are constant outside U_1, thus their derivatives vanish outside U_1. This implies $\xi_{2,1} = 0$ and consequently $\sigma_{3,1} = 0$ in $(U_3 \cap U_2) \setminus U_1$. Similarly, if $x \in (U_3 \cap U_1) \setminus U_2$ and $\mathbf{H}_3'(x) = 0$, then $D(x) \in [g_1, g_3]$. Indeed, from step 2 we know that $\sigma_{2,2} = 0$ on $U_1 \setminus U_2$. Moreover, the function $\varphi_2'(g_2)\phi_{1,1}(g_1)$ vanishes outside U_2. In addition, if $x \in (U_3 \cap U_1) \setminus U_2$ then $g_1(x) < \gamma_{1,2} < \gamma_{1,1}$ and hence $g_2(x) \leqslant \gamma_2$. Thus $\phi_{2,1}'(g_2(x)) = 0$, which implies $\xi_{2,2}(x) = 0$. Consequently $\sigma_{3,2}(x) = 0$ if $x \in (U_3 \cap U_1) \setminus U_2$.

Define the sets

$$Z_{3,1} = \begin{cases} \{z_3\} & \text{if } z_3 \in U_3 \setminus (\overline{U}_1 \cup \overline{U}_2), \\ \emptyset & \text{otherwise,} \end{cases}$$
$$Z_{3,2} = Z_3 \cap U_3 \cap (U_1 \cup U_2).$$

Now, let us check that $Z_{3,2} \subset \bigcup_{g \in N_3} O_g$. Indeed, if $x \in Z_{3,2}$, then $x \in (U_1 \cup U_2) \cap U_3$. Now,

- if $x \in (U_1 \cap U_3) \setminus U_2$, then $D(x) \in [g_1, g_3]$. Since $(U_1 \cap U_3) \setminus U_2 \subset (U_1 \cap U_3') \setminus U_2$ we can deduce that $x \in M_{3,1} \subset M_3$;
- if $x \in (U_2 \cap U_3) \setminus U_1$, then $D(x) \in [g_2, g_3]$. Since $(U_2 \cap U_3) \setminus U_1 \subset (U_2 \cap U_3') \setminus U_1$ we can deduce that $x \in M_{3,2} \subset M_3$;
- if $x \in U_1 \cap U_2 \cap U_3$, then $D(x) \in [g_1, g_2, g_3]$. Since $U_1 \cap U_2 \cap U_3 \subset U_1 \cap U_2 \cap U_3'$ we can deduce that $x \in M_{3,1,2} \subset M_3$.

Finally, since $x \in U_3$, we have that $g_1(x) < \gamma_{1,2}$ and $g_2(x) < \gamma_{2,1}$. We apply Fact 2.4(2) to conclude that there is $g \in N_3$ such that $x \in O_g$.

In the case that $Z_{3,1} = \{z_3\} \notin \bigcup_{g \in N_3} \overline{O}_g$, we select if necessary, a larger t_3, with $t_3 < l_3$, so that $z_3 \notin \bigcup_{g \in N_3} \overline{B}_g$. Since the norm is LUR and $D(z_3) = g_3$ we may select numbers $0 < t_3' < l_3' < 1$ and open slices, which are neighborhoods of z_3 defined by

$$O_{g_3} := \{x \in S\colon\ g_3(x) > l_3'\} \quad \text{and} \quad B_{g_3} := \{x \in S\colon\ g_3(x) > t_3'\},$$

satisfying $O_{g_3} \subset B_{g_3} \subset \{x \in S\colon\ g_1(x) < \gamma_{1,2}',\ g_2(x) < \gamma_{2,1}',\ g_3(x) > \gamma_3'\}$ and $\operatorname{dist}(B_{g_3}, B_g) > 0$ for every $g \in N_3$. In this case, we define $\Gamma_3 = N_3 \cup \{g_3\}$.

Now, if $Z_{3,1} = \{z_3\} \in \bigcup_{g \in N_3} \overline{O}_g$, we select, if necessary, a smaller constant l_3, with $0 < t_3 < l_3 < 1$, so that $Z_{3,1} = \{z_3\} \in \bigcup_{g \in N_3} O_g$. In this case, and also when $Z_{3,1} = \emptyset$, we define $\Gamma_3 = N_3$.

Notice that, in any of the cases mentioned above, Fact 2.4 clearly holds for the (possibly) newly selected real numbers t_3 and l_3.

Then, the distance between any two sets B_g, $B_{g'}$, $g, g' \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $g \neq g'$, is strictly positive. Moreover $Z_{3,1} \cup Z_{3,2} \subset \bigcup_{g \in \Gamma_3} O_g \subset \bigcup_{g \in \Gamma_3} B_g \subset U_3' \subset R_3$. Therefore, $Z_3 = Z_1 \cup Z_2 \cup Z_{3,1} \cup Z_{3,2} \subset \bigcup_{g \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} O_g \subset \bigcup_{g \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} B_g \subset U_1 \cup U_2 \cup U_3 = R_1 \cup R_2 \cup R_3$. Finally, recall that $\operatorname{dist}(B_g, R_3^c) > 0$, for every $g \in \Gamma_3$ and $\operatorname{dist}(B_g, (U_1 \cup U_2 \cup U_3)^c) > 0$ for every $g \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

It is worth mentioning that, by combining all the results obtained in the step $k=3$, we can deduce that $\mathbf{H_3'} = \sigma_{3,1}\mathbf{g_1} + \sigma_{3,2}\mathbf{g_2} + \sigma_{3,3}\mathbf{g_3}$, where $\sigma_{3,i}$ are continuous functions on $U_1 \cup U_2 \cup U_3$, $H_3' = \sigma_{3,1}g_1 + \sigma_{3,2}g_2 + \sigma_{3,3}g_3$, where $\sigma_{3,i}$ are continuous functions in the corresponding open subset of Y, $\sigma_{3,i}(x) = 0$ whenever $x \in (U_1 \cup U_2 \cup U_3) \setminus U_i$, and for every $x \in U_1 \cup U_2 \cup U_3$ there is at least one coefficient $\sigma_{3,i}(x) > 0$.

• Assume that, in the steps $j = 2, \dots, k$, with $k \geqslant 2$, we have selected points $z_j \in S^+$ and constants $\gamma_j \in (0,1)$, with $g_1(z_j) \neq \gamma_1, \dots, g_{j-1}(z_j) \neq \gamma_{j-1}$, $\{g_1, \dots, g_k := D(z_k)\}$ linearly independent functionals such that

$$S_j = \{x \in S\colon f_j(x) > \delta_j\} \subset \{x \in S\colon g_j(x) > \gamma_j\} \subset \{x \in S\colon f_j(x) > \delta_j^4\} = P_j \tag{2.16}$$

for all $j = 2, \dots, k$, and

$$\{T \in [g_{i_1}, \dots, g_{i_s}, g_j]^*\colon g_{i_1}(x) = \gamma_{i_1}, \dots, g_{i_s}(x) = \gamma_{i_s}, g_j(x) = \gamma_j \text{ and } |T| = 1\} = \emptyset$$

for every $1 \leqslant i_1 < \cdots < i_s \leqslant j-1$, and $1 \leqslant s \leqslant j-1$, $2 \leqslant j \leqslant k$. Assume we have defined the functions $h_j = \varphi_j(g_j)\phi_{j-1,1}(g_{j-1}) \cdots \phi_{1,j-1}(g_1)$, where $\varphi_j, \phi_{j-1,1}, \dots, \phi_{1,j-1}$ are C^∞ functions on $\mathbb{R}$ satisfying

$$\begin{aligned} &\varphi_j(t) = 0 \quad \text{if } t \leqslant \gamma_j, \\ &\varphi_j(1) = 1, \\ &\varphi_j'(t) > 0 \quad \text{if } t > \gamma_j, \end{aligned}$$

and

$$\begin{aligned} &\phi_{1,j-1}(t) = 1 \quad \text{if } t \leqslant \frac{\gamma_1 + \gamma_{1,j-1}}{2}, \qquad \dots, \qquad \phi_{j-1,1}(t) = 1 \quad \text{if } t \leqslant \frac{\gamma_{j-1} + \gamma_{j-1,1}}{2}, \\ &\phi_{1,j-1}(t) = 0 \quad \text{if } t \geqslant \gamma_{1,j-1}, \qquad \dots, \qquad \phi_{j-1,1}(t) = 0 \quad \text{if } t \geqslant \gamma_{j-1,1}, \\ &\phi_{1,j-1}'(t) < 0 \quad \text{if } t \in \left(\frac{\gamma_1 + \gamma_{1,j-1}}{2}, \gamma_{1,j-1}\right), \quad \dots, \\ &\phi_{j-1,1}'(t) < 0 \quad \text{if } t \in \left(\frac{\gamma_{j-1} + \gamma_{j-1,1}}{2}, \gamma_{j-1,1}\right), \end{aligned}$$

where $\gamma_1 < \gamma_{1,j-1}, \dots, \gamma_{j-1} < \gamma_{j-1,1}$, and $2 \leqslant j \leqslant k$.

The interior of the support of h_j is the set

$$U_j = \{x \in S\colon g_1(x) < \gamma_{1,j-1}, \dots, g_{j-1}(x) < \gamma_{j-1,1} \text{ and } g_j(x) > \gamma_j\}.$$

Assume we have also defined the C^p smooth functions r_j and $\mathbf{H_j}$:

$$\begin{aligned} &r_j : S^+ \longrightarrow \mathbb{R} \qquad && \mathbf{H_j} : U_1 \cup U_2 \cup \cdots \cup U_j \longrightarrow \mathbb{R} \\ &r_j = s_j g_j + (1 - s_j g_j(x_j)), \qquad && \mathbf{H_j} = \frac{\sum_{i=1}^j a_i r_i h_i}{\sum_{i=1}^j h_i}, \end{aligned}$$

for $2 \leqslant j \leqslant k$, where $x_j \in U_j$ the numbers $a_j, s_j \in \mathbb{R}^*$ satisfy that $|a_j - F(x_j)| < \varepsilon$, $s_j a_j > 0$, and the oscillation of r_j on U_j is less than $\frac{\varepsilon}{|a_j|}$.

Assume that for $2 \leqslant j \leqslant k$ the set of critical points Z_j of $\mathbf{H_j}$ is a union of the form $Z_j = Z_{j-1} \cup Z_{j,1} \cup Z_{j,2}$, where Z_{j-1} is the set of critical points of $\mathbf{H_{j-1}}$, the sets $Z_{j-1}, Z_{j,1}, Z_{j,2}$ are pairwise disjoint, $Z_j \subset D^{-1}([g_1, \dots, g_j])$, $Z_{j-1} \subset (U_1 \cup \dots \cup U_{j-1}) \setminus \overline{U}_j$ and $Z_{j,1} \cup Z_{j,2} \subset U_j$. Furthermore, assume that (i) there is an open subset U_j' such that

$$U_j \subset U_j' \subset R_j := \{x \in S\colon\ g_j(x) > \gamma_j\}$$

and $\operatorname{dist}(B_g, U_j') > 0$, for every $g \in \Gamma_1 \cup \dots \cup \Gamma_{j-1}$, (ii) there is a finite subset $\Gamma_j \subset S^*$ and open slices of S,

$$B_g := \{x \in S\colon\ g(x) > t_j\} \quad \text{and} \quad O_g := \{x \in S\colon\ g(x) > l_j\}, \quad 0 < t_j < l_j < 1,$$

satisfying $B_g \subset U_j'$ for every $g \in \Gamma_j$, $\operatorname{dist}(B_g, B_{g'}) > 0$ whenever $g, g' \in \Gamma_j$, $g \neq g'$ and there is $\gamma_j' \in (\gamma_j, 1)$ such that

$$Z_{j,1} \cup Z_{j,2} \subset \bigcup_{g \in \Gamma_j} O_g \subset \bigcup_{g \in \Gamma_j} B_g \subset U_j' \cap \{x \in S\colon\ g_j(x) > \gamma_j'\} \subset U_j.$$

Assume also that for $2 \leqslant j \leqslant k$, $\mathbf{H_j'} = \sigma_{j,1}\mathbf{g_1} + \dots + \sigma_{j,j}\mathbf{g_j}$ on $U_1 \cup \dots \cup U_j$, where $\sigma_{j,i}$ are continuous functions on $U_1 \cup \dots \cup U_j$, and $H_j' = \sigma_{j,1}g_1 + \dots + \sigma_{j,j}g_j$, where $\sigma_{j,i}$ are continuous functions on the corresponding open subset of Y. Finally, assume that for $2 \leqslant j \leqslant k$, and for every $x \in U_1 \cup \dots \cup U_j$ there is at least one index $m \in \{1, \dots, j\}$ such that $\sigma_{j,m}(x) > 0$, and that if $x \in (U_1 \cup \dots \cup U_j) \setminus U_m$, with $m \in \{1, \dots, j\}$, then $\sigma_{j,m}(x) = 0$.

• Now, let us denote by $y_{k+1} \in S$ the point satisfying $f_{k+1}(y_{k+1}) = 1$. If either $\{g_1, \dots, g_k, f_{k+1}\}$ are linearly dependent or $g_i(y_{k+1}) = \gamma_i$ for some $i \in \{1, \dots, k\}$, we can use the density of the norm attaining functionals (Bishop–Phelps theorem) and the continuity of D to modify y_{k+1} and find $z_{k+1} \in S$ so that $g_i(z_{k+1}) \neq \gamma_i$, for every $i = 1, \dots, k$, $\{g_1, \dots, g_k, g_{k+1} := D(z_{k+1})\}$ are linearly independent and

$$\{x \in S\colon\ f_{k+1}(x) > \delta_{k+1}^2\} \subset \{x \in S\colon\ g_{k+1}(x) > \nu_{k+1}\} \subset \{x \in S\colon\ f_{k+1}(x) > \delta_{k+1}^3\}$$

for some $\nu_{k+1} \in (0,1)$. If $g_i(y_{k+1}) \neq \gamma_i$ for every $i \in \{1, \dots, k\}$ and $\{g_1, \dots, g_k, f_{k+1}\}$ are linearly independent, we define $z_{k+1} = y_{k+1}$ and $g_{k+1} = f_{k+1}$. Then we apply Lemma 2.2 to the linearly independent vectors, $\{g_1, \dots, g_{k+1}\}$ and the real numbers $\gamma_1, \dots, \gamma_k$ and obtain $\gamma_{k+1} \in (0,1)$ close enough to ν_{k+1} so that

$$\begin{aligned} S_{k+1} = \{x \in S\colon\ f_{k+1}(x) > \delta_{k+1}\} &\subset \{x \in S\colon\ g_{k+1}(x) > \gamma_{k+1}\} \\ &\subset \{x \in S\colon\ f_{k+1}(x) > \delta_{k+1}^4\} = P_{k+1} \end{aligned}$$

and

$$\begin{aligned} \{T \in [g_{i_1}, \dots, g_{i_s}, g_{k+1}]^*\colon\ &T(g_{i_1}) = \gamma_{i_1}, \dots, T(g_{i_s}) = \gamma_{i_s}, \\ &T(g_{k+1}) = \gamma_{k+1} \text{ and } |T| = 1\} = \emptyset \end{aligned} \tag{2.17}$$

for every $1 \leqslant i_1 < \dots < i_s \leqslant k$ and $1 \leqslant s \leqslant k$.

Define

$$R_{k+1} = \{x \in S\colon\ g_{k+1}(x) > \gamma_{k+1}\}.$$

Recall that $\bigcup_{g\in\Gamma_k} B_g \subset U'_k \cap \{x \in S\colon\ g_k(x) > \gamma'_k\}$ and select $\gamma'_{k,1} \in (\gamma_k, \gamma'_k)$. In addition, we select numbers

$$\gamma'_{k-1,2} \in \left(\gamma_{k-1}, \frac{\gamma_{k-1} + \gamma_{k-1,1}}{2}\right), \quad \ldots, \quad \gamma'_{1,k} \in \left(\gamma_1, \frac{\gamma_1 + \gamma_{1,k-1}}{2}\right), \tag{2.18}$$

and define the open set

$$U'_{k+1} = \{x \in S\colon\ g_1(x) < \gamma'_{1,k}, \ldots, g_k(x) < \gamma'_{k,1} \text{ and } g_{k+1}(x) > \gamma_{k+1}\}. \tag{2.19}$$

Notice that $\operatorname{dist}(B_g, U'_{k+1}) > 0$ for every $g \in \Gamma_1 \cup \cdots \cup \Gamma_k$.

Assume that $R_{k+1} \cap (U_1 \cup \cdots \cup U_k) \neq \emptyset$ and define, for every $1 \leqslant i_1 < \cdots < i_s \leqslant k$ and $1 \leqslant s \leqslant k$, the set

$$M_{k+1,i_1,\ldots,i_s} = \{x \in U_{i_1} \cap \cdots \cap U_{i_s} \cap U'_{k+1}\colon\ x \notin U_j \text{ for every } j \in \{1,\ldots,k\} \setminus \{i_1,\ldots,i_s\}, \\ \text{and } D(x) \in [g_{i_1},\ldots,g_{i_s},g_{k+1}]\}$$

and

$$M_{k+1} = \bigcup \{M_{k+1,i_1,\ldots,i_s}\colon\ 1 \leqslant i_1 < \cdots < i_s \leqslant k \text{ and } 1 \leqslant s \leqslant k\}.$$

In the case when $M_{k+1} = \emptyset$ we select as $\gamma_{1,k}$ any point in $(\gamma_1, \gamma'_{1,k}), \ldots,$ and $\gamma_{k,1}$ any point in $(\gamma_k, \gamma'_{k,1})$.

Notice that $U_1 \cup \cdots \cup U_k = R_1 \cup \cdots \cup R_k$. In the case when $M_{k+1} \neq \emptyset$ and $\operatorname{dist}(M_{k+1}, (U_1 \cup \cdots \cup U_k)^c) = \operatorname{dist}(M_{k+1}, (R_1 \cup \cdots \cup R_k)^c) > 0$, we can immediately find $\gamma_{1,k} \in (\gamma_1, \gamma'_{1,k}), \ldots,$ $\gamma_{k,1} \in (\gamma_k, \gamma'_{k,1})$ with $M_{k+1} \subset \{x \in S\colon\ g_1(x) > \gamma_{1,k}\} \cup \cdots \cup \{x \in S\colon\ g_k(x) > \gamma_{k,1}\}$.

In the case when $\operatorname{dist}(M_{k+1}, (U_1 \cup \cdots \cup U_k)^c) = 0$ and in order to find suitable positive numbers $\gamma_{1,k}, \ldots, \gamma_{k,1}$, we need to study the limits of the sequences $\{x_n\} \subset M_{k+1}$ such that $\lim_n \operatorname{dist}(x_n, (U_1 \cup \cdots \cup U_k)^c) = 0$. Define, for every $1 \leqslant i_1 < \cdots < i_s \leqslant k$ and $1 \leqslant s \leqslant k$, the sets

$$F'_{k+1,i_1,\ldots,i_s} = \{T \in [g_{i_1},\ldots,g_{i_s},g_{k+1}]^*\colon\ T(g_i) = \gamma_i \text{ for every } i \in \{i_1,\ldots,i_s\} \text{ and } |T| = 1\},$$
$$N'_{k+1,i_1,\ldots,i_s} = \{g \in S^* \cap [g_{i_1},\ldots,g_{i_s},g_{k+1}]\colon\ T(g) = 1 \text{ for some } T \in F'_{k+1,i_1,\ldots,i_s}\}.$$

Since the norm $|\cdot|^*$ is Gâteaux smooth, we can apply Lemma 2.1 to the finite-dimensional space $[g_{i_1},\ldots,g_{i_s},g_{k+1}]$ with the norm $|\cdot|^*$ restricted to this finite-dimensional space, and deduce that the cardinal of any of the sets $F'_{k+1,i_1,\ldots,i_s}$ is at most two. Moreover, since the norm is strictly convex, the cardinal of each set $N'_{k+1,i_1,\ldots,i_s}$ is at most two. Let us consider the *norm-one* extensions to $[g_1,\ldots,g_k,g_{k+1}]$ of the elements of $F'_{k+1,i_1,\ldots,i_s}$, that is,

$$F''_{k+1,i_1,\ldots,i_s} = \{T \in [g_1,\ldots,g_{k+1}]^*\colon\ T|_{[g_{i_1},\ldots,g_{i_s},g_{k+1}]} \in F'_{k+1,i_1,\ldots,i_s} \text{ and } |T| = 1\}.$$

Since the norm $|\cdot|^*$ is Gâteaux smooth, for every $G \in F'_{k+1,i_1,\dots,i_s}$ there is a *unique norm-one* extension T defined on $[g_1,\dots,g_{k+1}]$. Thus, the cardinal of the every set $F''_{k+1,i_1,\dots,i_s}$ is at most two. Therefore the sets

$$F'_{k+1} = \bigcup \bigl\{ F''_{k+1,i_1,\dots,i_s}\colon\ 1 \leqslant i_1 < \cdots < i_s \leqslant k \text{ and } 1 \leqslant s \leqslant k \bigr\},$$
$$N'_{k+1} = \bigcup \bigl\{ N'_{k+1,i_1,\dots,i_s}\colon\ 1 \leqslant i_1 < \cdots < i_s \leqslant k \text{ and } 1 \leqslant s \leqslant k \bigr\}$$

are finite. As a consequence of equality (2.17), we deduce that $T(g_{k+1}) \neq \gamma_{k+1}$ for every $T \in F'_{k+1}$. We can restrict our study to the following kind of sequences. Fix $1 \leqslant s \leqslant k$ and $1 \leqslant i_1 < \cdots < i_s \leqslant k$ and consider a sequence $\{x_n\} \subset M_{k+1,i_1,\dots,i_s}$ such that $\lim_n \operatorname{dist}(x_n,(U_1 \cup \cdots \cup U_k)^c) = 0$. Let us prove that for every $i \in \{i_1,\dots,i_s\}$, $\lim_n g_i(x_n) = \gamma_i$. Indeed, if $\{x_n\} \subset M_{k+1,i_1,\dots,i_s}$, then in particular $\{x_n\} \subset U_i \subset R_i$ for every $i \in \{i_1,\dots,i_s\}$. Recall that $U_1 \cup \cdots \cup U_k = R_1 \cup \cdots \cup R_k$. Therefore, $\operatorname{dist}(x_n,(U_1 \cup \cdots \cup U_k)^c) \geqslant \operatorname{dist}(x_n, R_i^c)$ for every $i \in \{i_1,\dots,i_s\}$. Thus, $\lim_n \operatorname{dist}(x_n, R_i^c) = 0$ for every $i \in \{i_1,\dots,i_s\}$. Since $\{x_n\} \subset R_i$, this implies that $\lim_n g_i(x_n) = \gamma_i$, for every $i \in \{i_1,\dots,i_s\}$.

Now, let us take any sequence $\{x_n\} \subset M_{k+1,i_1,\dots,i_s}$ such that $\lim_n g_i(x_n) = \gamma_i$, for $i \in \{i_1,\dots,i_s\}$. Consider every x_n as an element of X^{**} and denote by $\mathbf{x_n}$ its restriction to $[g_1,\dots,g_{k+1}]$. Recall that $D(x_n) \in S^* \cap [g_{i_1},\dots,g_{i_s},g_{k+1}]$ for every $n \in \mathbb{N}$. Then, the sequence of restrictions $\{\mathbf{x_n}\} \subset [g_1,\dots,g_{k+1}]^*$ satisfies that

$$\begin{aligned} 1 = |x_n| \geqslant |\mathbf{x_n}| \geqslant \bigl|\mathbf{x_n}|_{[g_{i_1},\dots,g_{i_s},g_{k+1}]}\bigr| = \max\bigl\{\mathbf{x_n}(h)\colon\ h \in S^* \cap [g_{i_1},\dots,g_{i_s},g_{k+1}]\bigr\} \\ \geqslant \mathbf{x_n}\bigl(D(x_n)\bigr) = D(x_n)(x_n) = 1 \end{aligned}$$

for every $n \in \mathbb{N}$. Thus, there is a subsequence $\{\mathbf{x_{n_j}}\}$ converging to an element $T \in [g_1,\dots,g_{k+1}]^*$ with $|T| = 1$ and $|T|_{[g_{i_1},\dots,g_{i_s},g_{k+1}]}| = 1$. Since $\lim_j g_i(x_{n_j}) = \gamma_i$ for every $i \in \{i_1,\dots,i_s\}$, we have that $T(g_i) = \gamma_i$ for every $i \in \{i_1,\dots,i_s\}$. This implies that $T|_{[g_{i_1},\dots,g_{i_s},g_{k+1}]} \in F'_{k+1,i_1,\dots,i_s}$ and $T \in F''_{k+1,i_1,\dots,i_s}$. Furthermore, if $g \in N'_{k+1,i_1,\dots,i_s}$ and $T(g) = 1$, then $\lim_j \mathbf{x_{n_j}}(g) = 1$. In addition, $T(g_{k+1}) = \lim_j \mathbf{x_{n_j}}(g_{k+1}) = \lim_j g_{k+1}(x_{n_j}) \geqslant \gamma_{k+1}$ because $\{x_{n_j}\} \subset U'_{k+1}$. Then, from condition (2.17), we deduce that $T(g_{k+1}) > \gamma_{k+1}$. Finally, let us check that $g_i(x_n) \leqslant \gamma_i$ for every $i \in \{1,\dots,k\} \setminus \{i_1,\dots,i_s\}$ and $n \in \mathbb{N}$. Indeed, since $\{x_n\} \subset U'_{k+1}$, we have $g_1(x) < \gamma'_{1,k} < \gamma_{1,i-1},\dots,g_{i-1}(x) < \gamma'_{i-1,k+2-i} < \gamma_{i-1,1}$. Now, from the definition of U_i and the fact that $\{x_n\colon n \in \mathbb{N}\} \cap U_i = \emptyset$, we deduce that $g_i(x_n) \leqslant \gamma_i$, for every $n \in \mathbb{N}$. Finally, if $T = \lim_j \mathbf{x_{n_j}}$ in $[g_1,\dots,g_{k+1}]$, then $T(g_i) = \lim_j \mathbf{x_{n_j}}(g_i) \leqslant \gamma_i$, for every $i \in \{1,\dots,k\} \setminus \{i_1,\dots,i_s\}$.

Let us define, for every $1 \leqslant s \leqslant k$ and $1 \leqslant i_1 < \cdots < i_s \leqslant k$, the sets

$$\begin{aligned} F_{k+1,i_1,\dots,i_s} = \Bigl\{ T \in F''_{k+1,i_1,\dots,i_s}\colon \text{ there is } \{x_n\} \subset M_{k+1,i_1,\dots,i_s}, \text{ with } \lim_n \mathbf{x_n}(g_i) = \gamma_i, \\ \text{for } i \in \{i_1,\dots,i_s\} \text{ and } \lim_n \mathbf{x_n} = T \Bigr\}, \end{aligned}$$
$$N_{k+1,i_1,\dots,i_s} = \bigl\{ g \in N'_{k+1,i_1,\dots,i_s}\colon \text{ there is } T \in F_{k+1,i_1,\dots,i_s} \text{ with } T(g) = 1 \bigr\},$$

and

$$F_{k+1} = \bigcup \{F_{k+1,i_1,\ldots,i_s}\colon\ 1 \leqslant s \leqslant k \text{ and } 1 \leqslant i_1 < \cdots < i_s \leqslant k\},$$
$$N_{k+1} = \bigcup \{N_{k+1,i_1,\ldots,i_s}\colon\ 1 \leqslant s \leqslant k \text{ and } 1 \leqslant i_1 < \cdots < i_s \leqslant k\},$$

which are all finite. Select a real number γ'_{k+1} satisfying $\gamma_{k+1} < \gamma'_{k+1} < \min\{T(g_{k+1})\colon\ T \in F_{k+1}\}$.

Fact 2.5.

(1) *There are numbers* $0 < t_{k+1} < l_{k+1} < 1$ *such that for every* $g \in N_{k+1}$, *the slices*

$$O_g := \{x \in S\colon\ g(x) > l_{k+1}\} \quad \textit{and} \quad B_g := \{x \in S\colon\ g(x) > t_{k+1}\}$$

satisfy that

$$O_g \subset B_g \subset \{x \in S\colon\ g_1(x) < \gamma'_{1,k}, \ldots, g_k(x) < \gamma'_{k,1},\ g_{k+1}(x) > \gamma'_{k+1}\} \quad \textit{and} \tag{2.20}$$
$$\operatorname{dist}(B_g, B_{g'}) > 0, \quad \textit{whenever } g, g' \in N_{k+1},\ g \neq g'. \tag{2.21}$$

(2) *There are numbers* $\gamma_{1,k} \in (\gamma_1, \gamma'_{1,k}), \ldots, \gamma_{k,1} \in (\gamma_k, \gamma'_{k,1})$ *such that if* $x \in M_{k+1}$, $g_1(x) < \gamma_{1,k}1, \ldots, g_k(x) < \gamma_{k,1}$, *then* $x \in O_g$, *for some* $g \in N_{k+1}$.

Proof. (1) First, if X is reflexive, we know that for every $g \in N_{k+1}$ there is $x_g \in S$ such that $D(x_g) = g$. There is $1 \leqslant s \leqslant k$ and $1 \leqslant i_1 < \cdots < i_s \leqslant k$ such that $g \in F_{k+1,i_1,\ldots,i_s}$. Denote by $\mathbf{x_g}$ the restriction of x_g to $[g_1, \ldots, g_{k+1}]$. Since $\mathbf{x_g}(g) = 1$ and $|\cdot|^*$ is Gâteaux smooth, we have that $\mathbf{x_g} = T$ for some $T \in F_{k+1,i_1,\ldots,i_s}$. This implies that $\mathbf{x_g}(g_i) = \gamma_i < \gamma'_{i,k+1-i}$ whenever $i \in \{i_1, \ldots, i_s\}$, $\mathbf{x_g}(g_{k+1}) > \gamma'_{k+1}$ and $\mathbf{x_g}(g_i) \leqslant \gamma_i < \gamma'_{i,k+1-i}$, whenever $i \in \{1, \ldots, k\} \setminus \{i_1, \ldots, i_s\}$. Hence, $x_g \in \{x \in S\colon\ g_1(x) < \gamma'_{1,k}, \ldots, g_k(x) < \gamma'_{k,1} \text{ and } g_{k+1}(x) > \gamma'_{k+1}\}$.

Now, since the norm $|\cdot|$ is LUR and $D(x_g) = g$, the functional g strongly exposes S at the point x_g for every $g \in N_{k+1}$. Since N_{k+1} is finite, we can obtain real numbers $0 < t_{k+1} < l_{k+1} < 1$ and slices O_g and B_g, for every $g \in N_{k+1}$, satisfying conditions (2.20) and (2.21).

Now consider a non-reflexive Banach space X. Let us first prove (2.20). Assume, on the contrary, that there is a point $g \in N_{k+1}$ and there is a sequence $\{y_n\} \subset S$ satisfying $g(y_n) > 1 - \frac{1}{n}$ with either $g_1(y_n) \geqslant \gamma'_{1,k}, \ldots,$ or $g_k(y_n) \geqslant \gamma'_{k,1}$, or $g_{k+1}(y_n) \leqslant \gamma'_{k+1}$, for every $n \in \mathbb{N}$. If $g \in N_{k+1}$ there is a sequence $\{x_n\} \subset M_{k+1}$ with $\lim_n g_i(x_n) \leqslant \gamma_i$, for every $i \in \{1, \ldots, k\}$, $\lim_n g_{k+1}(x_n) > \gamma'_{k+1}$ and $\lim_n g(x_n) = 1$. In particular,

$$\frac{g(x_n) + 1 - \frac{1}{n}}{2} \leqslant g\left(\frac{x_n + y_n}{2}\right) \leqslant \left|\frac{x_n + y_n}{2}\right| \leqslant 1,$$

and thus $\lim_n |\frac{x_n+y_n}{2}| = 1$. Since in this case the norm $|\cdot|$ is WUR, we have that $x_n - y_n \xrightarrow{\omega} 0$ (weakly converges to zero). This last assertion gives a contradiction since either $\limsup_n g_i(x_n - y_n) \leqslant \gamma_i - \gamma'_{i,k+1-i} < 0$ for some $i \in \{1, \ldots, k\}$ or $\liminf_n g_{k+1}(x_n - y_n) \geqslant \lim_n g_{k+1}(x_n) - \gamma'_{k+1} > 0$. Therefore, we can find real numbers $0 < t_{k+1} < l_{k+1} < 1$ and slices O_g and B_g for every $g \in N_{k+1}$, satisfying condition (2.20). The proof of (2.21) is the same as the one given in Fact 2.3, where the only property we need is the strict convexity of $|\cdot|^*$.

(2) Assume, on the contrary, that for every $n \in \mathbb{N}$, there is $x_n \in M_{k+1}$ with $g_i(x_n) \leqslant \gamma_i + \frac{1}{n}$ for every $i \in \{1, \dots, k\}$ and $\{x_n\colon n \in \mathbb{N}\} \cap (\bigcup_{g \in N_{k+1}} O_g) = \emptyset$. Then there is a subsequence of $\{x_n\}$, which we denote by $\{x_n\}$ as well, and there are numbers $1 \leqslant s \leqslant k$ and $1 \leqslant i_1 < \cdots < i_s < k$ such that $\{x_n\} \subset M_{k+1,i_1,\dots,i_s}$. In particular, $\{x_n\} \subset U_i \subset R_i$ and then $g_i(x_n) > \gamma_i$ for every $i \in \{i_1, \dots, i_s\}$ and $n \in \mathbb{N}$. Hence, $\lim_n g_i(x_n) = \gamma_i$ for every $i \in \{i_1, \dots, i_s\}$. Since $\{x_n\} \subset M_{k+1,i_1,\dots,i_s}$, from the comments preceding Fact 2.5, we know that there is a subsequence $\{x_{n_j}\}$ and $g \in N_{k+1,i_1,\dots,i_s}$ satisfying that $\lim_j g(x_{n_j}) = 1$, which is a contradiction. This finishes the proof of Fact 2.5. □

If $R_{k+1} \cap (U_1 \cup \dots \cup U_k) = \emptyset$ we may select as $\gamma_{1,k}$ any number in $(\gamma_1, \gamma'_{1,k}), \dots,$ and $\gamma_{k,1}$ any number in $(\gamma_k, \gamma'_{k,1})$.

Now we define h_{k+1},

$$h_{k+1} : S^+ \longrightarrow \mathbb{R}$$
$$h_{k+1} = \varphi_{k+1}(g_{k+1})\phi_{k,1}(g_k) \cdots \phi_{1,k}(g_1),$$

with $\varphi_{k+1}, \phi_{k,1}, \dots, \phi_{1,k}$ C^∞ functions on $\mathbb{R}$ satisfying

$$\begin{aligned} &\varphi_{k+1}(t) = 0 \quad \text{if } t \leqslant \gamma_{k+1}, \\ &\varphi_{k+1}(1) = 1, \\ &\varphi'_{k+1}(t) > 0 \quad \text{if } t > \gamma_{k+1}, \end{aligned}$$

and

$$\begin{aligned} &\phi_{1,k}(t) = 1 \quad \text{if } t \leqslant \frac{\gamma_1 + \gamma_{1,k}}{2}, \qquad \dots, \qquad \phi_{k,1}(t) = 1 \quad \text{if } t \leqslant \frac{\gamma_k + \gamma_{k,1}}{2}, \\ &\phi_{1,k}(t) = 0 \quad \text{if } t \geqslant \gamma_{1,k}, \qquad \dots, \qquad \phi_{k,1}(t) = 0 \quad \text{if } t \geqslant \gamma_{k,1}, \\ &\phi'_{1,k}(t) < 0 \quad \text{if } t \in \left(\frac{\gamma_1 + \gamma_{1,k}}{2}, \gamma_{1,k}\right), \qquad \dots, \qquad \phi'_{k,1}(t) < 0 \quad \text{if } t \in \left(\frac{\gamma_k + \gamma_{k,1}}{2}, \gamma_{k,1}\right). \end{aligned}$$

Clearly the interior of the support of h_{k+1} is the set

$$U_{k+1} = \{x \in S\colon\ g_1(x) < \gamma_{1,k}, \dots, g_k(x) < \gamma_{k,1} \text{ and } g_{k+1}(x) > \gamma_{k+1}\}.$$

Select one point $x_{k+1} \in U_{k+1}$, a real number $a_{k+1} \in \mathbb{R}^*$ with $|a_{k+1} - F(x_{k+1})| < \varepsilon$ and define the auxiliary function

$$r_{k+1} : S^+ \longrightarrow \mathbb{R}$$
$$r_{k+1} = s_{k+1} g_{k+1} + \big(1 - s_{k+1} g_{k+1}(x_{k+1})\big),$$

where we have selected s_{k+1} so that $s_{k+1} a_{k+1} > 0$ and $|s_{k+1}|$ is small enough so that the oscillation of r_{k+1} on U_{k+1} is less than $\frac{\varepsilon}{|a_{k+1}|}$. Notice that $r_{k+1}(x_{k+1}) = 1$.

Let us study the set of critical points Z_{k+1} of the C^p smooth function

$$\mathbf{H_{k+1}} : U_1 \cup \cdots \cup U_{k+1} \longrightarrow \mathbb{R}$$
$$\mathbf{H_{k+1}} = \frac{\sum_{i=1}^{k+1} a_i r_i h_i}{\sum_{i=1}^{k+1} h_i}. \tag{2.22}$$

Let us prove that $Z_{k+1} := \{x \in U_1 \cup \cdots \cup U_{k+1}\colon\ H'_{k+1}(x) = 0 \text{ on } T_x\}$ can be included in a finite union of disjoint slices within $U_1 \cup \cdots \cup U_{k+1}$ by splitting Z_{k+1} conveniently into the (already defined) set Z_k and a finite number of disjoint sets within U_{k+1}.

It is straightforward to verify that $\mathbf{H'_{k+1}} = \sigma_{k+1,1}\mathbf{g_1} + \cdots + \sigma_{k+1,k+1}\mathbf{g_{k+1}}$, where $\sigma_{k+1,i}$ are continuous functions on $U_1 \cup \cdots \cup U_{k+1}$ and $\mathbf{g_i}$ denotes the restriction $g_i|_{T_x}$, $i = 1, \ldots, k+1$, whenever we evaluate $\mathbf{H'_{k+1}}(x)$.

From (2.22), the restriction of $\mathbf{H_{k+1}}$ and $\mathbf{H'_{k+1}}$ to $(U_1 \cup \cdots \cup U_k) \setminus U_{k+1}$ coincide with $\mathbf{H_k}$ and $\mathbf{H'_k}$, respectively. Then, $Z_{k+1} \setminus U_{k+1} = Z_k = Z_{k+1} \setminus \overline{U}_{k+1}$. Let us study the set $Z_{k+1} \cap U_{k+1}$. First, if $x \in U_{k+1} \setminus (U_1 \cup \cdots \cup U_k)$, from (2.22), we obtain that $\mathbf{H_{k+1}}(x) = a_{k+1} r_{k+1}(x)$ and the derivative $H'_{k+1}(x) = a_{k+1} r'_{k+1}(x)$. Therefore $\mathbf{H'_{k+1}}(x) = a_{k+1} s_{k+1} g_{k+1}|_{T_x} = 0$ iff $D(x) = g_{k+1}$. If the point $z_{k+1} \in U_{k+1} \setminus (U_1 \cup \cdots \cup U_k)$, then $\mathbf{H_{k+1}}$ has exactly one critical point in $U_{k+1} \setminus (U_1 \cup \cdots \cup U_k)$; in this case, since $g_i(z_{k+1}) \neq \gamma_i$ for every $i = 1, \ldots, k$, the point z_{k+1} actually belongs to $U_{k+1} \setminus (\overline{U}_1 \cup \cdots \cup \overline{U}_k)$.

Secondly, let us study the critical points of $\mathbf{H_{k+1}}$ in $U_{k+1} \cap (U_1 \cup \cdots \cup U_k)$. If we define $\Lambda_k = \frac{\sum_{i=1}^{k} h_i}{\sum_{i=1}^{k+1} h_i}$, then we can rewrite $\mathbf{H_{k+1}}$ on $U_{k+1} \cap (U_1 \cup \cdots \cup U_k)$ as

$$\mathbf{H_{k+1}} = \frac{\sum_{i=1}^{k} a_i r_i h_i}{\sum_{i=1}^{k} h_i} \cdot \frac{\sum_{i=1}^{k} h_i}{\sum_{i=1}^{k+1} h_i} + \frac{a_{k+1} r_{k+1} h_{k+1}}{\sum_{i=1}^{k+1} h_i} = \mathbf{H_k}\Lambda_k + a_{k+1} r_{k+1}(1 - \Lambda_k),$$

and

$$\mathbf{H'_{k+1}} = \mathbf{H'_k}\Lambda_k + a_{k+1} s_{k+1}(1 - \Lambda_k)\mathbf{g_{k+1}} + (\mathbf{H_k} - a_{k+1} r_{k+1})\Lambda'_k.$$

Notice that, on the open set U_{k+1}, we have that $\phi_{i,j}(g_i) \equiv 1$, whenever $i + j \leqslant k$. Indeed, on the one hand, if $x \in U_{k+1}$, and $i \in \{1, \ldots, k\}$, then $g_i(x) < \gamma_{i,k+1-i} \leqslant \frac{\gamma_i + \gamma_{i,j}}{2}$, whenever $i + j \leqslant k$. On the other hand, $\phi_{i,j}(t) \equiv 1$ if $t \leqslant \frac{\gamma_i + \gamma_{i,j}}{2}$. Therefore $h_i|_{U_{k+1}} = \varphi_i(g_i)$, for every $i = 1, \ldots, k$, and

$$\Lambda_k = \frac{\sum_{i=1}^{k} \varphi_i(g_i)}{\sum_{i=1}^{k} \varphi_i(g_i) + h_{k+1}}.$$

By computing Λ'_k in U_{k+1}, we obtain $\Lambda'_k = \xi_{k,1}\mathbf{g_1} + \cdots + \xi_{k,k+1}\mathbf{g_{k+1}}$, where the coefficients $\xi_{k,1}, \ldots, \xi_{k,k+1}$ are continuous functions of the following form:

$$\xi_{k,j} = \frac{-\varphi_{k+1}(g_{k+1})\phi'_{j,k+1-j}(g_j)(\prod_{i=1;i\neq j}^{k} \phi_{i,k+1-i}(g_i))(\sum_{i=1}^{k} h_i) + h_{k+1}\varphi'_j(g_j)}{(\sum_{i=1}^{k+1} h_i)^2},$$

$j = 1, \ldots, k,$

$$\xi_{k,k+1} = \frac{-\varphi'_{k+1}(g_{k+1})(\prod_{i=1}^{k} \phi_{i,k+1-i}(g_i))(\sum_{i=1}^{k} h_i)}{(\sum_{i=1}^{k+1} h_i)^2}.$$

Thus, if $x \in U_{k+1} \cap (U_1 \cup \cdots \cup U_k)$, the coefficients $\sigma_{k+1,1}, \ldots, \sigma_{k+1,k+1}$ for $\mathbf{H}'_{\mathbf{k+1}}$ have the following form,

$$\sigma_{k+1,j} = \sigma_{k,j} \Lambda_k + (\mathbf{H_k} - a_{k+1} r_{k+1}) \xi_{k,j} \quad \text{for } j = 1, \ldots, k,$$
$$\sigma_{k+1,k+1} = a_{k+1} s_{k+1} (1 - \Lambda_k) + (\mathbf{H_k} - a_{k+1} r_{k+1}) \xi_{k,k+1}.$$

Notice that in $U_{k+1} \cap (U_1 \cup \cdots \cup U_k)$, $a_{k+1} s_{k+1} > 0$, $\Lambda_k > 0$, $1 - \Lambda_k > 0$, $\xi_{k,j} \geqslant 0$, for every $j = 1, \ldots, k$, $\sum_{j=1}^{k} \xi_{k,j} > 0$ and $\xi_{k,k+1} < 0$. Therefore, if $H_k - a_{k+1} r_{k+1} \leqslant 0$, the coefficient $\sigma_{k+1,k+1} > 0$. When $H_k - a_{k+1} r_{k+1} \geqslant 0$ and $\sigma_{k,j} > 0$, the coefficient $\sigma_{k+1,j} > 0$ (recall that, from the step k we know that, for every $x \in U_1 \cup \cdots \cup U_k$ there exists at least one $j \in \{1, \ldots, k\}$ with $\sigma_{k,j} > 0$). Hence, if $\mathbf{H}'_{\mathbf{k+1}}(x) = 0$ for some $x \in U_{k+1} \cap (U_1 \cup \cdots \cup U_k)$, there necessarily exists $\varrho \neq 0$ such that $D(x) = \varrho(\sigma_{k+1,1}(x) g_1 + \cdots + \sigma_{k+1,k+1}(x) g_{k+1})$, that is $D(x) \in [g_1, \ldots, g_{k+1}]$.

In fact we can be more accurate and obtain that if $\mathbf{H}'_{\mathbf{k+1}}(x) = 0$, $x \in U_{k+1} \cap (U_1 \cup \cdots \cup U_k)$ and $x \notin \bigcup_{j \in F} U_j$ for some proper subset $F \subset \{1, \ldots, k\}$, then $D(x) \in \operatorname{span}\{g_j\colon j \in \{1, \ldots, k+1\} \setminus F\}$. Indeed, from step k we know that, if $x \in (U_1 \cup \cdots \cup U_k) \setminus U_j$, where $j \in \{1, \ldots, k\}$, then $\sigma_{k,j}(x) = 0$. Now, if $j \in F$ and $j = 1$, it is clear that the functions $\varphi'_1(g_1)$ and $\phi'_{1,k}(g_1)$ vanish outside U_1. This implies $\xi_{k,1}(x) = 0$ and consequently $\sigma_{k+1,1}(x) = 0$. If $j \in F$ and $2 \leqslant j \leqslant k$, since $x \in U_{k+1}$ we know that

$$g_1(x) < \gamma_{1,k} < \gamma_{1,j-1}, \quad \ldots, \quad g_{j-1}(x) < \gamma_{j-1,k+2-j} < \gamma_{j-1,1},$$

and then necessarily $g_j(x) \leqslant \gamma_j$. Since the functions $\varphi'_j(g_j)$ and $\phi'_{j,k+1-j}(g_j)$ vanish whenever $g_j \leqslant \gamma_j$, we deduce $\xi_{k,j}(x) = 0$ and thus $\sigma_{k+1,j}(x) = 0$.

Let us now define the sets

$$Z_{k+1,1} = \begin{cases} \{z_{k+1}\} & \text{if } z_{k+1} \in U_{k+1} \setminus (\overline{U}_1 \cup \cdots \cup \overline{U}_k), \\ \emptyset & \text{otherwise}, \end{cases}$$
$$Z_{k+1,2} = Z_{k+1} \cap U_{k+1} \cap (U_1 \cup \cdots \cup U_k).$$

Now, let us check that $Z_{k+1,2} \subset \bigcup_{g \in N_{k+1}} O_g$. Indeed, if $x \in Z_{k+1,2}$, there are constants $1 \leqslant s \leqslant k$ and $1 \leqslant i_1 < \cdots < i_k \leqslant k$, such that $x \in U_{k+1} \cap U_{i_1} \cap \cdots \cap U_{i_s}$ and $x \notin \bigcup_{j \in F} U_j$, where $F = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_s\}$. From the preceding assertion, $D(x) \in [g_{i_1}, \ldots, g_{i_s}, g_{k+1}]$. From the definition of $M_{k+1, i_1, \ldots, i_s}$ and the fact that $U_{k+1} \subset U'_{k+1}$, we obtain that $x \in M_{k+1, i_1, \ldots, i_s} \subset M_{k+1}$. Since $x \in U_{k+1}$, we have that $g_1(x) < \gamma_{1,k}, \ldots, g_k(x) < \gamma_{k,1}$. We apply Fact 2.5(2) to conclude that there is $g \in N_{k+1}$ such that $x \in O_g$.

In the case when $Z_{k+1,1} = \{z_{k+1}\} \notin \bigcup_{g \in N_{k+1}} \overline{O}_g$, we select, if necessary, a larger t_{k+1}, with $t_{k+1} < l_{k+1}$, so that $z_{k+1} \notin \bigcup_{g \in N_{k+1}} \overline{B}_g$ for every $g \in N_{k+1}$. Since the norm is LUR and $D(z_{k+1}) = g_{k+1}$ we may select numbers $0 < t'_{k+1} < l'_{k+1} < 1$ and open slices, which are neighborhoods of z_{k+1} defined by

$$O_{g_{k+1}} := \{x \in S\colon g_{k+1}(x) > l'_{k+1}\} \quad \text{and} \quad B_{g_{k+1}} := \{x \in S\colon g_{k+1}(x) > t'_{k+1}\},$$

satisfying $O_{g_{k+1}} \subset B_{g_{k+1}} \subset \{x \in S\colon g_1(x) < \gamma'_{1,k}, \ldots, g_k(x) < \gamma'_{k,1}, g_{k+1}(x) > \gamma'_{k+1}\}$ and $\operatorname{dist}(B_{g_{k+1}}, O_g) > 0$, for every $g \in N_{k+1}$. In this case, we define $\Gamma_{k+1} = N_{k+1} \cup \{g_{k+1}\}$.

Now, if $Z_{k+1,1} = \{z_{k+1}\} \in \bigcup_{g \in N_{k+1}} \overline{O}_g$, we select, if necessary a smaller constant l_{k+1}, with $0 < t_{k+1} < l_{k+1} < 1$, so that $Z_{k+1,1} = \{z_{k+1}\} \in \bigcup_{g \in N_{k+1}} O_g$. In this case, and also when $Z_{k+1,1} = \emptyset$, we define $\Gamma_{k+1} = N_{k+1}$.

Notice that, in any of the cases mentioned above, Fact 2.5 clearly holds for the (possibly) newly selected real numbers t_{k+1} and l_{k+1}.

Then, the distance between any two sets B_g, $B_{g'}$, where $g, g' \in \Gamma_1 \cup \cdots \cup \Gamma_{k+1}$, and $g \neq g'$, is strictly positive. Moreover $Z_{k+1,1} \cup Z_{k+1,2} \subset \bigcup_{g \in \Gamma_{k+1}} O_g \subset \bigcup_{g \in \Gamma_{k+1}} B_g \subset U'_{k+1} \subset R_{k+1}$. Therefore, $Z_{k+1} = Z_1 \cup \cdots \cup Z_k \cup Z_{k+1,1} \cup Z_{k+1,2} \subset \bigcup_{g \in \Gamma_1 \cup \cdots \cup \Gamma_{k+1}} O_g \subset \bigcup_{g \in \Gamma_1 \cup \cdots \cup \Gamma_{k+1}} B_g \subset U_1 \cup \cdots \cup U_{k+1} = R_1 \cup \cdots \cup R_{k+1}$. Also, recall that $\operatorname{dist}(B_g, R^c_{k+1}) > 0$, for every $g \in \Gamma_{k+1}$ and $\operatorname{dist}(B_g, (U_1 \cup \cdots \cup U_{k+1})^c) > 0$, for every $g \in \Gamma_1 \cup \cdots \cup \Gamma_{k+1}$.

Finally, let us notice that, by combining the results obtained in step $k+1$, we deduce that $\mathbf{H'_{k+1}} = a_{k+1} s_{k+1} \mathbf{g_{k+1}}$ in $U_{k+1} \setminus (U_1 \cup \cdots \cup U_k)$ and $\mathbf{H'_{k+1}} = \mathbf{H'_k}$ on $(U_1 \cup \cdots \cup U_k) \setminus U_{k+1}$, and in general $\mathbf{H'_{k+1}} = \sigma_{k+1,1} \mathbf{g_1} + \cdots + \sigma_{k+1,k+1} \mathbf{g_{k+1}}$ on $U_1 \cup \cdots \cup U_{k+1}$ where $\sigma_{k+1,i}$ are continuous functions on $U_1 \cup \cdots \cup U_{k+1}$ and $H'_{k+1} = \sigma_{k+1,1} g_1 + \cdots + \sigma_{k+1,k+1} g_{k+1}$ where $\sigma_{k+1,i}$ are continuous functions on the corresponding open subset of Y. Moreover, for every $x \in U_1 \cup \cdots \cup U_{k+1}$ there is at least one index $i \in \{1, \ldots, k+1\}$ such that $\sigma_{k+1,i}(x) > 0$. Furthermore, $\sigma_{k+1,j}(x) = 0$ whenever $x \in (U_1 \cup \cdots \cup U_{k+1}) \setminus U_j$, $j \in \{1, \ldots, k+1\}$.

Once we have defined, by induction, the functions h_k, r_k and the constants a_k, for all $k \in \mathbb{N}$, we define

$$H : S^+ \longrightarrow \mathbb{R}$$
$$H = \frac{\sum_{k=1}^{\infty} a_k r_k h_k}{\sum_{k=1}^{\infty} h_k}.$$

It is straightforward to verify that the family $\{U_k\}_{k \in \mathbb{N}}$ of open sets of S^+ is a locally finite open covering of S^+. Thus, for every $x \in S^+$ there is $k_x \in \mathbb{N}$ and a (relatively open in S^+) neighborhood $V_x \subset S^+$ of x, such that $V_x \cap (\bigcup_{k > k_x} U_k) = \emptyset$ and therefore $H|_{V_x} = \mathbf{H_{k_x}}|_{V_x}$. Thus H is C^p smooth whenever the functions $\{h_k\}_{k \in \mathbb{N}}$ are C^p smooth.

Fact 2.6. *The function H 3ε-approximates F in S^+.*

Proof. Recall that the oscillation of F in U_k is less that ε, the oscillation of r_k in U_k is less than $\frac{\varepsilon}{|a_k|}$, $|a_k - F(x_k)| < \varepsilon$ and $r_k(x_k) = 1$, for every $k \in \mathbb{N}$. Now, if $h_k(x) \neq 0$, then $x \in U_k$ and

$$\begin{aligned} \left|a_k r_k(x) - F(x)\right| &\leqslant \left|a_k r_k(x) - a_k r_k(x_k)\right| + \left|a_k r_k(x_k) - F(x)\right| \\ &= |a_k| \left|r_k(x) - r_k(x_k)\right| + \left|a_k - F(x)\right| \\ &\leqslant |a_k| \frac{\varepsilon}{|a_k|} + \left|a_k - F(x_k)\right| + \left|F(x_k) - F(x)\right| \leqslant 3\varepsilon. \end{aligned} \tag{2.23}$$

Hence,

$$\left|H(x) - F(x)\right| = \frac{|\sum_{k=1}^{\infty} (a_k r_k(x) - F(x)) h_k(x)|}{\sum_{k=1}^{\infty} h_k(x)} \leqslant \frac{\sum_{k=1}^{\infty} |a_k r_k(x) - F(x)| h_k(x)}{\sum_{k=1}^{\infty} h_k(x)} \leqslant 3\varepsilon. \qquad \square$$

Let us denote by C the critical points of H in S^+. Since for every $x \in S^+$, there is $k_x \in \mathbb{N}$ and a (relatively open in S^+) neighborhood $V_x \subset S^+$ of x such that $V_x \cap (\bigcup_{k > k_x} U_k) = \emptyset$, we have that

$H|_{V_x} = \mathbf{H_{k_x}}|_{V_x}$ and $C \subset \bigcup_k Z_k$. Recall that $\bigcup_k Z_k \subset \bigcup\{O_g\colon g \in \bigcup_k \Gamma_k\} \subset \bigcup\{B_g\colon g \in \bigcup_k \Gamma_k\}$, the oscillation of F on B_g is less than ε and $\operatorname{dist}(B_g, B_{g'}) > 0$, for every $g, g' \in \bigcup_k \Gamma_k$ with $g \neq g'$. Furthermore, from the inductive construction of the sets $\{B_g\colon g \in \bigcup_k \Gamma_k\}$, it is straightforward to verify that (i) for every $k > 1$, if $g \in \Gamma_k$ and $g' \in \bigcup_{m>k} \Gamma_m$, then $\operatorname{dist}(B_g, B_{g'}) \geqslant \gamma_k' - \gamma_{k,1}' > 0$ and (ii) if $g \in \Gamma_1$ and $g' \in \bigcup_{m>1} \Gamma_m$, then $\operatorname{dist}(B_g, B_{g'}) \geqslant t_1 - \gamma_{1,1}' > 0$. Therefore, for every $g \in \bigcup_k \Gamma_k$,

$$\operatorname{dist}\left(B_g, \bigcup\left\{B_{g'}\colon g' \in \bigcup_k \Gamma_k, g' \neq g\right\}\right) > 0. \tag{2.24}$$

We relabel the countable families of open slices $\{O_g\}_{g\in\bigcup_k \Gamma_k}$ and $\{B_g\}_{g\in\bigcup_k \Gamma_k}$ as $\{O_n\}$, $\{B_n\}$, respectively. Notice that the set $\bigcup_n \overline{B}_n$ is a (relatively) closed set in S^+. Indeed, if $\{x_j\} \subset \bigcup_n \overline{B}_n$ and $\lim_j x_j = x \in S^+$, since $\bigcup_n U_n'$ is also a locally finite open covering of S^+, there is n_x and a (relatively open in S^+) neighborhood $W_x \subset S^+$ of x, such that $W_x \cap (\bigcup_{n>n_x} U_n') = \emptyset$. In addition, from the construction of the family $\{B_n\}$, there is $N \in \mathbb{N}$ such that $\bigcup_{n>N} \overline{B}_n \subset \bigcup_{n>n_x} U_n'$, and thus there is $j_0 \in \mathbb{N}$ with $\{x_j\}_{j>j_0} \subset \bigcup_{n=1}^N \overline{B}_n$. Hence $x \in \bigcup_{n=1}^N \overline{B}_n \subset \bigcup_n \overline{B}_n$.

Let us denote $\mathcal{B}_n = \Phi^{-1}(B_n)$ and $\mathcal{O}_n = \Phi^{-1}(O_n)$ for every $n \in \mathbb{N}$.

Fact 2.7. *$\mathcal{O}_n$ and $\mathcal{B}_n$ are open, convex and bounded subsets of X, for every $n \in \mathbb{N}$.*

Proof. Since Φ is continuous, it is clear that $\mathcal{O}_n$ and $\mathcal{B}_n$ are open sets. The sets O_n and B_n are slices of the form $R = \{x \in S\colon b(x) > \delta\}$ for some $b \in S^*$ and $\delta > 0$ such that $\operatorname{dist}(R, X \times \{0\}) > 0$. Let us prove that $\mathcal{R} := \Phi^{-1}(R)$ is convex and bounded in X. First, let us check that the cone in Y generated by R and defined by

$$\operatorname{cone}(R) = \{\lambda x\colon x \in R, \lambda > 0\} = \left\{x \in Y\colon b\left(\frac{x}{|x|}\right) > \delta\right\}$$

is a convex set: consider $0 \leqslant \alpha \leqslant 1$ and $x, x' \in \operatorname{cone}(R)$. Then,

$$\begin{aligned} b\big(\alpha x + (1-\alpha)x'\big) &= \alpha b(x) + (1-\alpha)b(x') > \alpha\delta|x| + (1-\alpha)\delta|x'| \\ &= \delta|\alpha x| + \delta\big|(1-\alpha)x'\big| \geqslant \delta\big|\alpha x + (1-\alpha)x'\big|, \end{aligned}$$

and this implies that $\alpha x + (1-\alpha)x' \in \operatorname{cone}(R)$. Therefore, the intersection of the two convex sets $\operatorname{cone}(R) \cap (X \times \{1\}) = \Pi^{-1}(R)$ is convex. Now, it is clear that $\mathcal{R} = \Phi^{-1}(R) = i^{-1}(\Pi^{-1}(R))$ is convex as well.

Let us prove that $\Pi^{-1}(R)$ is bounded in Y. Consider the linear bounded operator $\pi_2 : Y = X \oplus \mathbb{R} \to \mathbb{R}$, $\pi_2(x, r) = r$ for every $(x, r) \in X \oplus \mathbb{R}$. Then, $\Pi^{-1}(y) = \frac{y}{\pi_2(y)}$ for every $y \in S^+$. On the one hand, $d := \operatorname{dist}(R, X \times \{0\}) > 0$ and then

$$\pi_2(x, r) = r = \frac{|(x, r) - (x, 0)|}{|(0, 1)|} \geqslant \frac{d}{|(0, 1)|} := s > 0 \quad \text{for every } (x, r) \in R.$$

On the other hand,

$$\big|\Pi^{-1}(y)\big| = \frac{|y|}{\pi_2(y)} = \frac{1}{\pi_2(y)} \leqslant \frac{1}{s} \quad \text{for every } y \in R,$$

and thus $\Pi^{-1}(R)$ is bounded. Since the norm $\|\cdot\|$ considered on $X \times \{0\}$ (defined as $\|(x,0)\| = \|x\|$) and the restriction of the norm $|\cdot|$ to $X \times \{0\}$ are equivalent norms on $X \times \{0\}$, there exist constants $m, M > 0$ such that $m\|x - x'\| \leqslant |i(x) - i(x')| = |(x,1) - (x',1)| = |(x - x', 0)| \leqslant M\|x - x'\|$ for every $x, x' \in X$. Hence,

$$\begin{aligned}\|\Phi^{-1}(y)\| &= \|i^{-1}(\Pi^{-1}(y)) - i^{-1}(0,1)\| \leqslant \frac{1}{m}|\Pi^{-1}(y) - (0,1)| \\ &\leqslant \frac{1}{m}(|\Pi^{-1}(y)| + |(0,1)|) \leqslant \frac{1 + s|(0,1)|}{s \cdot m},\end{aligned}$$

for every $y \in R$, what shows that $\mathcal{R}$ is bounded in X. □

Fact 2.8. *$\overline{\mathcal{O}}_n$ and $\overline{\mathcal{B}}_n$ are (closed convex and bounded) C^p smooth bodies for every $n \in \mathbb{N}$.*

Proof. We already know that these sets are closed, convex and bounded bodies, hence it is enough to prove that their boundaries $\partial\mathcal{O}_n$ and $\partial\mathcal{B}_n$ are C^p smooth one-codimensional submanifolds of X. Since $\partial\mathcal{B}_n = \Phi^{-1}(\partial B_n)$, $\partial\mathcal{O}_n = \Phi^{-1}(\partial O_n)$, and Φ is a C^p diffeomorphism, this is the same as showing that ∂O_n and ∂B_n are C^p smooth one-codimensional submanifolds of S. But, if O_n is defined by $O_n = \{y \in S\colon\ g_n(y) > \beta_n\}$, we have that ∂O_n is the intersection of S with the hyperplane $X_n = \{y \in Y\colon\ g_n(y) = \beta_n\}$ of Y, and X_n is transversal to S at every point of ∂O_n (otherwise the hyperplane X_n would be tangent to S at some point of ∂O_n and, by strict convexity of S, this implies that $\partial O_n = X_n \cap S$ is a singleton, which contradicts the fact that O_n is a nonempty open slice of S), hence the intersection $\partial O_n = S \cap X_n$ is a one-codimensional submanifold of S. The same argument applies to ∂B_n. □

Fact 2.9. $\operatorname{dist}(\mathcal{O}_n, X \setminus \mathcal{B}_n) > 0$ *and* $\operatorname{dist}(\mathcal{B}_n, \bigcup_{m \neq n} \mathcal{B}_m) > 0$*, for every $n \in \mathbb{N}$.*

Proof. This is a consequence of the fact that $\operatorname{dist}(O_n, S^+ \setminus B_n) > 0$, $\operatorname{dist}(B_n, \bigcup_{m \neq n} B_m) > 0$, and Φ is Lipschitz. Indeed, on the one hand, recall that $|i(x) - i(x')| = |(x - x', 0)| \leqslant M\|x - x'\|$, for every $x, x' \in X$. On the other hand,

$$|\Pi(y) - \Pi(y')| = \frac{|y|y'| - y'|y||}{|y||y'|} = \frac{|y(|y'| - |y|) + (y - y')|y||}{|y||y'|} \leqslant \frac{2|y - y'|}{|y'|} \leqslant \frac{2}{\zeta}|y - y'|,$$

for every $y, y' \in X \times \{1\}$, where $\zeta = \operatorname{dist}(0, X \times \{1\}) > 0$. Therefore, $|\Phi(x) - \Phi(x')| \leqslant \frac{2M}{\zeta}\|x - x'\|$, for every $x, x' \in X$. Now, if two sets $A, A' \subset S^+$ satisfy that $\operatorname{dist}(A, A') > 0$, then $\operatorname{dist}(A, A') \leqslant |a - a'| \leqslant \frac{2M}{\zeta}\|\Phi^{-1}(a) - \Phi^{-1}(a')\|$, for every $a \in A$, $a' \in A'$. Therefore, $0 < \operatorname{dist}(A, A') \leqslant \frac{2M}{\zeta}\operatorname{dist}(\Phi^{-1}(A), \Phi^{-1}(A'))$. □

Fact 2.10. *For every $n \in \mathbb{N}$, there exists a C^p diffeomorphism Ψ_n from X onto $X \setminus \overline{\mathcal{O}}_n$ such that Ψ_n is the identity off $\mathcal{B}_n$.*

Proof. Assume that $0 \in \mathcal{O}_n$. Since $\operatorname{dist}(\mathcal{O}_n, X \setminus \mathcal{B}_n) > 0$, there is $\delta_n > 0$ such that $\operatorname{dist}((1 + \delta_n)\mathcal{O}_n, \mathcal{B}_n) > 0$. We can easily construct a C^p smooth radial diffeomorphism $\Psi_{n,2}$ from $X \setminus \{0\}$ onto $X \setminus \overline{\mathcal{O}}_n$ satisfying $\Psi_{n,2}(x) = x$ if $x \notin (1 + \delta_n)\mathcal{O}_n$. Indeed, take a C^∞ smooth function

$\lambda_n : [0, \infty) \to [1, \infty)$ satisfying that $\lambda_n(t) = t$ for $t \geqslant 1 + \delta_n$, $\lambda_n(0) = 1$ and $\lambda_n'(t) > 0$ for $t > 0$, and define

$$\Psi_{n,2}(x) = \lambda_n\big(\mu_n(x)\big)\frac{x}{\mu_n(x)}$$

for $x \in X \setminus \{0\}$, where μ_n is the Minkowski functional of $\overline{\mathcal{O}}_n$, which is C^p smooth on $X \setminus \{0\}$.

Now, since $0 \in \mathcal{O}_n$, there is $\alpha_n > 0$ such that $\alpha_n B_{\|\cdot\|} \subset \mathcal{O}_n$. According to [13, Proposition 3.1] and [12, Lemma 2] (see also [1]), there exists a C^p diffeomorphism $\Psi_{n,1}$ from X onto $X \setminus \{0\}$ such that $\Psi_{n,1}$ is the identity off $\alpha_n B_{\|\cdot\|}$ (this set may be regarded as the unit ball of a equivalent C^p smooth norm on X).

Then, the composition $\Psi_n := \Psi_{n,2} \circ \Psi_{n,1}$ is a C^p diffeomorphism from X onto $X \setminus \overline{\mathcal{O}}_n$ such that Ψ_n is the identity off $\mathcal{B}_n$. If $0 \notin \mathcal{O}_n$, select $\omega_n \in \mathcal{O}_n$ and repeat the above construction of the diffeomorphism with the sets $\mathcal{O}_n - \omega_n$ and $\mathcal{B}_n - \omega_n$. Then, $\Psi_n := \tau_{\omega_n} \circ \Psi_{n,2} \circ \Psi_{n,1} \circ \tau_{-\omega_n}$ is the required C^p diffeomorphism, where $\tau_\omega(x) = x + \omega$. □

Now, the infinite composition $\Psi = \bigcirc_{n=1}^{\infty} \Psi_n$ is a well-defined C^p diffeomorphism from X onto $X \setminus \bigcup_n \overline{\mathcal{O}}_n$, which is the identity outside $\bigcup_n \mathcal{B}_n$ and $\Psi(\mathcal{B}_n) \subset \mathcal{B}_n$. This follows from the fact that, for every $x \in X$, there is an open neighborhood V_x and $n_x \in \mathbb{N}$ such that $V_x \cap (\bigcup_{n \neq n_x} \mathcal{B}_n) = \emptyset$, and therefore $\Psi|_{V_x} = \Psi_{n_x}|_{V_x}$.

Finally, let us check that the C^p smooth function

$$g : X \longrightarrow \mathbb{R}$$
$$g := H \circ \Phi \circ \Psi$$

4ε-approximates f on X and g does not have critical points. Indeed, for every $x \in X$, if $\Psi(x) \neq x$ then there is $\mathcal{B}_{n_x}$ such that $x \in \mathcal{B}_{n_x}$. Since the oscillation of f in $\mathcal{B}_{n_x}$ is less than ε and $\Psi(x) \in \mathcal{B}_{n_x}$, we can deduce that $|f(\Psi(x)) - f(x)| < \varepsilon$, for every $x \in X$. Recall that $F \circ \Phi = f$ and $|H(x) - F(x)| < 3\varepsilon$, for every $x \in S^+$. Then,

$$\begin{aligned} \big|g(x) - f(x)\big| &= \big|H \circ \Phi\big(\Psi(x)\big) - F \circ \Phi(x)\big| \\ &\leqslant \big|H\big(\Phi\big(\Psi(x)\big)\big) - F\big(\Phi\big(\Psi(x)\big)\big)\big| + \big|F \circ \Phi\big(\Psi(x)\big) - F \circ \Phi(x)\big| \\ &\leqslant 3\varepsilon + \varepsilon = 4\varepsilon \end{aligned} \tag{2.25}$$

for every $x \in X$. Since Φ and Ψ are C^p diffeomorphisms, we have that $g'(x) = 0$ if and only if $H'(\Phi(\Psi(x))) = 0$. For every $x \in X$, $\Psi(x) \notin \bigcup_n \overline{\mathcal{O}}_n$ and thus $\Phi(\Psi(x)) \notin \bigcup_n \overline{O}_n$. It follows that $H'(\Phi(\Psi(x))) \neq 0$ and g does not have any critical point.

Before finishing the proof, let us say what additional precautions are required in the case when ε is a strictly positive continuous function:

- the slices $S_k = \{x \in S\colon\ f_k(x) > \delta_k\}$ $(k \in \mathbb{N})$ are selected with the additional property that the oscillations of the two functions F and $\bar{\varepsilon} = \varepsilon \circ \Phi^{-1}$ in S_k are less than $\frac{\bar{\varepsilon}(y_k)}{2}$, where y_k is the point of S^+ satisfying $f_k(y_k) = 1$; this implies, in particular, that $\frac{1}{2}\bar{\varepsilon}(y_k) < \bar{\varepsilon}(x) < \frac{3}{2}\bar{\varepsilon}(y_k)$ for every $x \in S_k$;
- the real numbers $a_k \in \mathbb{R}^*$ satisfy that $|a_k - F(x_k)| < \frac{\bar{\varepsilon}(y_k)}{2}$;

- the oscillation of r_k in S_k is less than $\frac{\bar{\varepsilon}(y_k)}{|a_k|}$.

From the above conditions and inequality (2.23), it can be deduced that if $x \in U_k$, then $|a_k r_k(x) - F(x)| \leqslant 2\bar{\varepsilon}(y_k) < 4\bar{\varepsilon}(x)$. From this, it can be obtained that $|H(x) - F(x)| \leqslant 4\bar{\varepsilon}(x)$, for every $x \in S^+$. Equivalently, $|H \circ \Phi(x) - F \circ \Phi(x)| = |H \circ \Phi(x) - f(x)| < 4\varepsilon(x)$, for every $x \in X$. Now, if $x \neq \Psi(x)$, then there is $\mathcal{B}_{n_x}$ such that $x, \Psi(x) \in \mathcal{B}_{n_x}$. Thus, $|f(\Psi(x)) - f(x)| < \frac{\varepsilon(\Phi^{-1}(y_{n_x}))}{2} < \varepsilon(x)$. Now, from inequality (2.25), we obtain: (a) if $x \in B_{n_x}$ for some n_x, then $|g(x) - f(x)| \leqslant 4\varepsilon(\Psi(x)) + \varepsilon(x) \leqslant 6\varepsilon(\Phi^{-1}(y_{n_x})) + \varepsilon(x) \leqslant 13\varepsilon(x)$, and (b) if $x \notin \bigcup_n B_n$, then $|g(x) - f(x)| \leqslant 4\varepsilon(\Psi(x)) = 4\varepsilon(x)$. This finishes the proof of Theorem 1.1. □

Remark 2.11. The construction of the function g with no critical points that approximates f with a constant $\varepsilon > 0$, is considerably shorter in the case that either (i) $X = \ell_2(\mathbb{N})$ (and we use West's theorem [32]) or (ii) X is non-reflexive and the norm $|\cdot|$ considered on Y can be constructed with the additional property that the set $\{f \in Y^*\colon f$ does not attain its norm$\}$ contains a dense subspace (except the zero functional) of Y^*.

Indeed, in the first case, we can define as $|\cdot|$ the standard norm on $\ell_2(\mathbb{N})$. In both cases, the use of the auxiliary functions r_n is not required, we can consider the slice $R_n := S_n$ (that is, the additional construction of the sequence of slices $\{R_n\}$ is not required) and we can select for every $n \in \mathbb{N}$, any strictly decreasing sequence $\{\gamma_{n,i}\}_{i\in\mathbb{N}}$ such that $\lim_i \gamma_{n,i} = \delta_n$. Then, let us choose a non-zero functional $w \in Y^* \setminus [f_n\colon n \in \mathbb{N}]$ (where $[f_n\colon n \in \mathbb{N}]$ denotes the space of all finite linear combinations of the set $\{f_n\colon n \in \mathbb{N}\}$) with $|w|^* < \varepsilon$, and define $H = \frac{\sum_i a_i h_i}{\sum_i h_i} + w$ and $\mathbf{H_n} = \frac{\sum_{i=1}^n a_i h_i}{\sum_{i=1}^n h_i} + w$, for every $n \in \mathbb{N}$. We obtain in the case (i), that Z_n (the critical points of $\mathbf{H_n}$), is included in the compact set $D^{-1}([f_1, \ldots, f_n, w] \cap S^*)$. Then, it can be proved that the set C of critical points of H and thus the set $\mathcal{C}$ of critical points of the composition $H \circ \Phi$, are closed and locally compact sets of S^+ and $\ell_2(\mathbb{N})$, respectively. Now, g is obtained, by applying West theorem [32], considering a C^∞ deleting diffeomorphism Ψ from $\ell_2(\mathbb{N})$ onto $\ell_2(\mathbb{N}) \setminus \mathcal{C}$, with the additional property that the family $\{(x, \Psi(x))\colon x \in \ell_2(\mathbb{N})\}$ refines the open covering $\{\Phi^{-1}(S_n);\ n \in \mathbb{N}\}$). Finally, we can define $g := H \circ \Phi \circ \Psi$.

In the case (ii), we can select the family $\mathcal{G} = \{f_n\colon n \in \mathbb{N}\} \cup \{w\}$ with the additional requirement that $[\mathcal{G}] \setminus \{0\}$ is included in the set of non-norm attaining functionals. Thus, it can be proved that the sets of critical points of both $\mathbf{H_n}$ and H are empty. Therefore, the set of critical points of $g := H \circ \Phi$ is empty and g approximates f. Notice that this case is particularly interesting because the use of a deleting diffeomorphism is not required.

Open Problems 2.12.

(1) The analytical case for the Hilbert space ℓ_2. *Can every continuous function $f : \ell_2 \to \mathbb{R}$ be uniformly approximated by real-analytic smooth functions with no critical points*?

(2) *Remark* 2.11(ii) *suggests the following problem*: *if X is a non-reflexive separable Banach space, can we construct an equivalent norm $|\cdot|$ on X with good properties of smoothness and convexity and the additional property that the set $NA^c \cup \{0\}$ contains a dense subspace of X^*, where $NA^c := \{f \in X^*\colon f$ does not attain its norm$\}$*?

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