ON DIFFEOMORPHISMS DELETING WEAK COMPACTA IN BANACH SPACES

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Abstract. We prove that if X is an infinite-dimensional Banach space with C^p smooth partitions of the unity then X and $X \setminus K$ are C^p diffeomorphic, for every weakly compact set $K \subset X$.

1. Introduction, main results and preliminaries

In 1953 Victor L. Klee [20] proved that, if X is a non-reflexive Banach space or an infinite-dimensional L^p space and K is a compact subset of X, there exists a homeomorphism between X and $X \setminus K$ which is the identity outside a given neighborhood of K. Klee also proved that for those infinite-dimensional Banach spaces X the unit sphere and the unit ball are homeomorphic to any of the closed hyperplanes in X, and gave a topological classification of convex bodies in Hilbert spaces. In subsequent papers, Bessaga and Klee generalized those results to every infinite-dimensional normed space [8, 9, 12]. Klee's original proofs were of a strong geometrical flavor: very beautiful, but rather difficult to handle in an analytical way. Nevertheless, C. Bessaga found elegant explicit formulas for deleting homeomorphisms, based on the existence of continuous noncomplete (nonequivalent) norms in every infinite-dimensional Banach space. This discovery allowed him in 1966 to construct diffeomorphisms which delete points in the Hilbert space, and to prove that the Hilbert space is diffeomorphic to its unit sphere [10]. These striking results have been highly celebrated and they remain a key ingredient in the proofs of the already classic fundamental theorems on Hilbert manifolds (e.g., that every two homotopic Hilbert manifolds are diffeomorphic, see [13, 17, 22]). These kinds of results about *topological negligibility* have also found many interesting applications in several branches of mathematics, which include fixed point theory, smooth topological classification of convex bodies, strange phenomena concerning ordinary differential equations and dynamical systems in infinite dimensions, the failure of Rolle's theorem in infinite dimensions and many more things, see [4, 5, 11, 3, 7] and the references therein. Very recently, Manuel Cepedello and the first-named author have used smooth topological negligibility to prove the following approximate strong

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version of the Morse-Sard theorem: the smooth functions *with no critical points* are dense in the space of continuous functions on every Hilbert manifold [2].

In view of the interest of such applications, it is natural to try to extend these results to Banach spaces other than the Hilbert space.

The real-analytic and smooth negligibility of compact sets in Banach spaces was first studied by Tadeusz Dobrowolski [15], who showed that for every infinite-dimensional Banach space X having a C^p non-complete norm, and for every compact set K in X, the space X is C^p diffeomorphic to $X \setminus K$. Unfortunately, it is still unknown whether every Banach space with a C^p smooth equivalent norm possesses a noncomplete C^p smooth norm as well. Nevertheless, without showing the existence of smooth non-complete norms, it was recently proved in [4] that every Banach space $(X, \|\cdot\|)$ with a C^p smooth norm ϱ is C^p diffeomorphic to $X \setminus K$.

Despite all these efforts, the natural question as to the characterization of those Banach spaces in which compact sets are topologically negligible remains open. This is due to a surprising (and rather uncomfortable) theorem proved by R. Haydon [18, 19]: there are Banach spaces which have C^∞ smooth bump functions, and even C^∞ smooth partitions of unity, but do not possess any equivalent C^1 smooth norm.

In this paper we deal with the following natural question: what can be said about smooth negligibility of compacta in those Banach spaces with smooth partitions of unity? As we have just pointed out, there are Banach spaces with smooth partitions of unity which have no equivalent smooth norms, and therefore the known results on diffeomorphisms deleting compacta are useless in this setting. Nevertheless, we will prove the following.

Theorem 1.1. *Let X be an infinite-dimensional Banach space which has C^p smooth partitions of the unity, and $p \in \mathbb{N} \cup \{\infty\}$. Then, for every weakly compact set $K \subset X$ and every starlike body A such that $dist(K, X \setminus A) > 0$, there exists a C^p diffeomorphism $h : X \longrightarrow X \setminus K$ such that h is the identity outside A.*

In particular, when K is compact and $K \subset int(A)$, there always exists such a deleting diffeomorphism h.

The class of Banach spaces which admit smooth partitions of unity is quite large, see [14]. On the other hand, it is an open problem to know whether every Banach space with a C^p smooth equivalent norm has C^p smooth partitions of unity. If a positive answer to this question is ever reached, then Theorem 1.1 will be an extension of the main theorem in [4]. Otherwise and for the time being, by combining Theorem 1.1 with the main result of [4], one can easily show the following.

Corollary 1.2. *Let X be an infinite-dimensional Banach space. Assume that either X possesses a C^p smooth norm or else X has C^p smooth partitions of unity. Then, for every compact set $K \subset X$ and every C^p smooth starlike body A such that $K \subset int(A)$, there exists a C^p diffeomorphism $h : X \longrightarrow X \setminus K$ such that h is the identity outside A.*

It should be noted that, for the time being, no one knows of an infinite-dimensional Banach space with a C^1 bump function which does not have either a C^1 smooth norm or C^1 smooth partitions of unity (hence which does not fall into the category to which the above Corollary applies). On the other hand, it is easy to see that the existence of a C^1 smooth bump is a necessary condition for a Banach space X to have a diffeomorphism from X onto $X \setminus \{0\}$ which restricts to the identity outside some ball.

At this point we need to introduce some terminology and notation concerning starlike bodies, which, apart from the statements of the preceding results, will play a key role in our proofs.

A closed subset A of a Banach space X is said to be a starlike body if there exists a point a_0 in the interior of A such that every ray emanating from a_0 meets ∂A, the boundary of A, at most once. We will say that a_0 is a center of A. There can obviously exist many centers for a given starlike body. Up to a suitable translation, we can always assume that $a_0 = 0$ is the origin of X, and we will often do so, unless otherwise stated. For a starlike body A with center a_0, we define the characteristic cone of A as

$$ccA = \{x \in X | a_0 + r(x - a_0) \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A with respect to the center a_0 as

$$\mu_{A,a_0}(x) = \mu_A(x) = \inf\{t > 0 \mid x - a_0 \in t(-a_0 + A)\} \text{ for all } x \in X.$$

Note that $\mu_A(x) = \mu_{-a_0+A}(x - a_0)$ for all $x \in X$. It is easily seen that μ_A is a continuous function which satisfies $\mu_A(a_0 + rx) = r\mu_A(a_0 + x)$ for every $r \geq 0$ and $x \in X$, and $\mu_A^{-1}(0) = ccA$. Moreover, $A = \{x \in X | \mu_A(x) \leq 1\}$, and $\partial A = \{x \in X \mid \mu_A(x) = 1\}$. Conversely, if $\psi : X \to [0, \infty)$ is continuous and satisfies $\psi(a_0 + \lambda x) = \lambda\psi(a_0 + x)$ for all $\lambda \geq 0$, then $A_\psi = \{x \in X \mid \psi(x) \leq 1\}$ is a starlike body. More generally, for a continuous function $\psi : X \to [0, \infty)$ such that $\psi_x(\lambda) = \psi(a_0 + \lambda x)$, $\lambda > 0$, is increasing and $\sup\{\psi_x(\lambda) : \lambda > 0\} > \varepsilon$ for every $x \in X \setminus \psi^{-1}(0)$, the set $\psi^{-1}([0, \varepsilon])$ is a starlike body whose characteristic cone is $\psi^{-1}(0) \ni a_0$.

A familiar important class of starlike bodies are *convex bodies*, that is, starlike bodies that are convex. For a convex body U, ccU is always a convex set, but in general the characteristic cone of a starlike body is not convex.

We will say that A is a C^p smooth starlike body provided its Minkowski functional μ_A is C^p smooth on the set $X \setminus ccA = X \setminus \mu_A^{-1}(0)$. This is equivalent to saying that ∂A is a C^p smooth one-codimensional submanifold of X such that no affine hyperplane tangent to ∂A contains a ray emanating from the center a_0. Throughout this paper, $p = 0, 1, 2, ..., \infty$, and C^0 smooth means just continuous.

We will also say that A is Lipschitz if μ_A is a Lipschitz function on X. It is easy to see that every *convex* body is Lipschitz with respect to any point in its interior (but this is no longer true if we drop convexity: even in the plane $\mathbb{R}^2$ there are starlike bodies which are not Lipschitz).

All the starlike bodies that we will deal with in this paper are *radially bounded.* A starlike body A is said to be radially bounded provided that, for every ray emanating from the center a_0 of A, the intersection of this ray with A is a bounded set. This amounts to saying that $ccA = \{a_0\}$. In finite dimensions every radially bounded starlike body is in fact bounded (because the Minkowski functional of the body attains an absolute minimum on the unit sphere, which is compact), but this is no longer true in infinite-dimensional Banach spaces. For instance, $A = \{x \in \ell_2 : \sum_{n=1}^{\infty} x_n^2/2^n \leq 1\}$ is a radially bounded convex body which is not bounded in the Hilbert space ℓ_2; the body A is the unit ball of the nonequivalent C^∞ smooth norm $\omega(x) = \left(\sum_{n=1}^{\infty} x_n^2/2^n\right)^{1/2}$ in ℓ_2. For every *bounded* starlike body A in a Banach space $(X, \|\cdot\|)$ there are constants $M, m > 0$ such that $m\|x\| \leq \mu_A(x) \leq M\|x\|$ for all $x \in X$. If A is just *radially bounded* then we can only ensure that $\mu_A(x) \leq M\|x\|$ for all $x \in X$, for some $M > 0$. As is shown implicitly in [14, Proposition II.5.1], a Banach space X has a C^p smooth bump function if and only if there is a bounded C^p smooth starlike body in X.

We will finish these preliminaries with some nonstandard notation concerning strict inclusions between starlike bodies. In our proofs we will often require that, for a couple of starlike bodies $A \subset B$, the boundaries of A and B are well separated. There are at least two nonequivalent natural notions of separation between boundaries of starlike bodies, and we will need to use both of them, as each one has its own advantages. The strongest and most natural notion corresponds to the fact that the distance between A and $X \setminus B$ is positive. We will use the notation $A \subset_d B$ to mean that $\text{dist}(A, X \setminus B) > 0$, and we will say that B strictly contains A in the distance sense. Notice that this notion makes sense even though A and B do not have the same center, or even if A and B are mere sets, not necessarily starlike.

The other useful notion is that the Minkowski functionals of A and B are well separated, in the following sense. First, note that if $A \subseteq B$ are starlike with respect to the same center a_0 then we always have that $\mu_B(x) \leq \mu_A(x)$ for all $x \in X$. If we also know that $\sup_{x \in A} \mu_B(x) < 1$ then we will denote $A \subset_\mu B$, saying that B strictly contains A in the gauge sense. This is equivalent to saying that there exists some $\delta > 0$ such that $a_0 + (1+\delta)(-a_0 + A) \subseteq B$. Of course, this notion only makes sense when A and B have at least one center a_0 in common. It is immediate to see that $A \subset_d B$ implies that $A \subset_\mu B$. The converse is false in general, unless A is Lipschitz. When $A \subset B$ have the same center and A is Lipschitz we have that $A \subset_d B$ if and only if $A \subset_\mu B$ (see Lemma 2.6 below).

2. Proof of the main result

In contrast with Bessaga-type constructions [10, 15, 1, **?**, 4, 5], our proof does not provide an explicit elegant formula for the deleting diffeomorphism. We rather turn to the origins and find inspiration in the geometrical ideas of the pioneering work of Klee's [20] (see also [23]). We will need to consider an infinite composition of carefully constructed self-diffeomorphisms of X.

The main ingredient of our proof is the following Proposition, which implies that if our infinite-dimensional space X has enough smooth starlike bodies then every weakly compact set K can be removed by means of a diffeomorphism $h : X \to X \setminus K$ which is the identity outside some starlike body.

Proposition 2.1. *Let X be a Banach space, and K a subset of X. Assume that there are sequences (P_n), (C_n), (A_n), (B_n), (Q_n), (D_n), (E_n) of subsets of X and a sequence (c_n) of points of X satisfying the following conditions for each $n \in \mathbb{N}$:*

(1) *A_n, B_n, Q_n, D_n, E_n are radially bounded C^p smooth starlike bodies with respect to c_{n+2};*
(2) $C_{n+2} \subset D_n \subset_\mu E_n \subset_\mu A_n \subset C_{n+1} \subset P_{n+1} \subset B_n \subset_\mu Q_n \subset P_n$
(3) $\bigcap_{n=1}^\infty C_n = \emptyset$
(4) $\bigcap_{n=1}^\infty P_n = K$.

Then there exists a C^p diffeomorphism $\Psi : X \longrightarrow X \setminus K$ such that Ψ is the identity on $X \setminus P_1$.

In order to prove this Proposition we will only require a simple geometrical Lemma. The purely topological version of this result is very easy (see [12, 23], where the authors do not even bother to write the formulas), but the smooth case is a little more difficult and requires a proof.

Lemma 2.2 (The four bodies lemma)**.** *Let X be a Banach space, and let A, B, C, D be four radially bounded C^p smooth starlike bodies with respect to the same point $a_0 \in int(A)$. Assume that*

$$A \subset_\mu B \subset C \subset_\mu D.$$

Then there exists a C^p diffeomorphism $h : X \to X$ such that

(1) $h(B) = C$
(2) *h is the identity on $A \cup (X \setminus D)$.*

Proof. We may assume $a_0 = 0$. Since $A \subset_\mu B$ and $C \subset_\mu D$, there exists some $\delta \in (0, 1)$ such that $A \subset (1 - \delta)B$ and $(1 + \delta)C \subset D$. Take a C^∞ smooth function $\lambda : \mathbb{R} \to \mathbb{R}$ such that λ is non-decreasing, $\lambda(t) = 0$ if $t \leq 1 - \delta$, and $\lambda(t) = 1$ for $t \geq 1$. Define then $f : X \to X$ by

$$f(x) = \big[\lambda(\mu_B(x))\frac{\mu_B(x)}{\mu_C(x)} + 1 - \lambda(\mu_B(x))\big]x, \text{ if } x \neq 0,$$

and $f(0) = 0$. It is easy to check that f is a C^p diffeomorphism of X such that $f(B) = C$ and f is the identity on A.

On the other hand, pick $\theta : \mathbb{R} \to \mathbb{R}$ a C^∞ smooth function such that θ is non-increasing, $\theta(t) = 1$ if $t \leq 1 + \delta/4$, and $\theta(t) = 0$ if $t \geq 1 + \delta/2$. Consider the mapping $g : X \setminus \{0\} \to X \setminus \{0\}$ defined by

$$g(x) = \big[\theta(\mu_C(x))\frac{\mu_C(x)}{\mu_B(x)} + 1 - \theta(\mu_C(x))\big]x,$$

which is a C^p diffeomorphism as well. Now define $h : X \to X$ by

$$h(x) = \begin{cases} f(x) & \text{if } \mu_B(x) < 1 + \frac{\delta}{4}; \\ g^{-1}(x) & \text{if } 1 < \mu_B(x) \end{cases}$$

Observe that if $1 \leq \mu_B(x) \leq 1 + \delta/4$ then $f(x) = \big[\mu_B(x)/\mu_C(x)\big]x = g^{-1}(x)$; hence h is well-defined and locally a C^p diffeomorphism. Moreover, it is easy to see that

$h(X \setminus (1+\delta/4)B) = X \setminus (1+\delta/4)C$, which (bearing in mind the definition of h) implies that h is one-to-one. On the other hand, since $h((1+\delta/4)B) = (1+\delta/4)C$ and $h(X \setminus B) = g^{-1}(X \setminus B) = X \setminus C$, it follows that h is a surjection. Therefore $h : X \to X$ is a C^p diffeomorphism. Finally, it is clear that $h(B) = C$, and h is the identity on $A \cup \big(X \setminus (1+\delta/2)B\big) \supset A \cup \big(X \setminus D\big)$. □

Proof of Proposition 2.1

The proof of this Proposition, as well as some parts of that of Proposition 2.3 below, resembles the arguments included in [23] (which in turn are inspired, like the rest of the present paper, by Klee's seminal work [20]).

Fix any $n \in \mathbb{N}$. Consider the inclusions of bodies

$$D_n \subset_\mu E_n \subset B_n \subset_\mu Q_n$$
$$D_n \subset_\mu E_n \subset A_n \subset_\mu Q_n.$$

According to the Four Bodies Lemma there exist C^p diffeomorphisms $f_n, g_n : X \to X$ such that

$$f_n(E_n) = B_n, \text{ and } f_n \text{ is the identity on } D_n \cup (X \setminus Q_n),$$
$$g_n(E_n) = A_n, \text{and } g_n \text{ is the identity on } D_n \cup (X \setminus Q_n).$$

Define then $h_n = g_n \circ f_n^{-1} : X \to X$, which is a C^p diffeomorphism of X satisfying that

$$h_n(B_n) = A_n, \text{ and } h_n \text{ is the identity on } D_n \cup (X \setminus Q_n).$$

Now consider the family of C^p diffeomorphisms (h_n). For each $n \in \mathbb{N}$ define the mapping $\psi_n : X \to X$ by the composition

$$\psi_n(x) = (h_1 \circ h_2 \circ ... \circ h_{n-1} \circ h_n)(x),$$

which is obviously a C^p diffeomorphism of X. Since h_n is the identity on $X \setminus Q_n$ and $Q_n \subset P_n$, we have that h_n is the identity on $X \setminus P_n$. It follows that

$$\psi_{n|X\setminus P_n} = \psi_{n-1|X\setminus P_n} \text{ for all } n \geq 2. \tag{1}$$

Note that, from the conditions in the statement of Proposition 2.1, we know that

$$X \setminus P_n \subset X \setminus P_{n+1} \subset X \setminus K, \text{ for all } n, \text{ and } X \setminus K = \bigcup_{n=1}^{\infty} X \setminus P_n. \tag{2}$$

Then we can define $\psi : X \setminus K \to X$ by letting

$$\psi_{|X\setminus P_{n+1}} = \psi_{n|X\setminus P_{n+1}} \text{ for each } n \in \mathbb{N}. \tag{3}$$

Taking equations (1) and (2) above into account, it is clear that the mapping ψ is well defined, one-to-one, and is locally a C^p diffeomorphism. Let us see that ψ is surjective and therefore a C^p diffeomorphism from $X \setminus K$ onto X.

Bearing in mind that h_j is the identity on $D_j \supset C_{j+2}$ and $A_j \subset C_{j+1}$, we have that $h_j(A_n) = A_n$ if $j \leq n-1$, and since $h_n(B_n) = A_n$ we may deduce that

$$\psi_n(B_n) = h_1 \circ ... \circ h_n(B_n) = h_1 \circ ... \circ h_{n-1}(A_n) = h_1 \circ ... \circ h_{n-2}(A_n) = ... = A_n;$$

and in particular $\psi_n(X \setminus B_n) = X \setminus A_n$. But, by the hypothesis on the bodies, $P_{n+1} \subset B_n \subset P_n$, that is $X \setminus P_n \subset X \setminus B_n \subset X \setminus P_{n+1}$, and hence

$$\psi(X \setminus B_n) = \psi_n(X \setminus B_n) = X \setminus A_n. \tag{4}$$

Now, note that the hypothesis of Proposition 2.1 imply that $C_{n+2} \subset A_n \subset C_{n+1}$, $\bigcap_{n=1}^{\infty} C_n = \emptyset$, which yield

$$X = \bigcup_{n=1}^{\infty} (X \setminus A_n). \tag{5}$$

On the other hand, since $K = \bigcap_{n=1}^{\infty} P_{n+1} \subset \bigcap_{n=1}^{\infty} B_n \subset \bigcap_{n=1}^{\infty} P_n = K$, we have that

$$X \setminus K = \bigcup_{n=1}^{\infty} (X \setminus B_n). \tag{6}$$

Now, by combining equations (4), (5) and (6), we get that

$$\psi(X \setminus K) = \psi\big(\bigcup_{n=1}^{\infty} (X \setminus B_n)\big) = \bigcup_{n=1}^{\infty} (X \setminus A_n) = X,$$

hence ψ is a C^p diffeomorphism from $X \setminus K$ onto X. Moreover, if $x \in X \setminus P_1 \subset X \setminus P_2$, from the definition of ψ, and bearing in mind that h_1 is the identity on $X \setminus P_1$, we conclude that $\psi(x) = \psi_1(x) = h_1(x) = x$. Finally, if we define $\Psi = \psi^{-1}$, it is clear that Ψ is a C^p diffeomorphism from X onto $X \setminus K$ which is the identity off P_1. □

The next step in the proof of our main theorem is of course to ensure that if an infinite-dimensional Banach space X has C^p smooth partitions of unity then, for every weakly compact set $K \subset X$, there are families of C^p smooth starlike bodies satisfying the conditions of Proposition 2.1.

Proposition 2.3. *Let X be an infinite-dimensional Banach space which admits C^p smooth partitions of unity. There exists B, a radially bounded C^p smooth starlike body with respect to the origin, such that, for every weakly compact set $K \subset X$ and every $r > 0$ such that $K \subset rB$, there are sequences (P_n), (C_n), (A_n), (B_n), (Q_n), (D_n), (E_n) of subsets of X and a sequence (c_n) of points of X satisfying the following conditions for each $n \in \mathbb{N}$:*

1. *A_n, B_n, Q_n, D_n, E_n are radially bounded C^p smooth starlike bodies with respect to c_{n+2};*
2. *$C_{n+2} \subset D_n \subset_\mu E_n \subset_\mu A_n \subset C_{n+1} \subset P_{n+1} \subset B_n \subset_\mu Q_n \subset P_n$;*
3. *$\bigcap_{n=1}^{\infty} C_n = \emptyset$;*
4. *$\bigcap_{n=1}^{\infty} P_n = K$;*
5. *$P_1 \subset 4rB$.*

The proof of Proposition 2.3 is quite long and will be split into several lemmas.

Notation 2.4. If X is a Banach space and $B_X = \{x \in X : \|x\| \le 1\}$ is its unit ball, for all subsets A, B of X and for every $\varepsilon > 0$, we will denote

$$[A, B] = \{tx + (1-t)y : x \in A, y \in B, t \in [0,1]\},$$

and $N(A,\varepsilon)=\{x\in X:\text{dist}(x,A)\le\varepsilon\}=\overline{A+\varepsilon B_X}$. When $A=\{a\}$ is a singleton we will simply write $[A,B]=[a,B]$.

Lemma 2.5. *Let X be a Banach space, C a bounded convex body in X, and K a weakly compact subset of X. Then $V:=[K,C]$ is a starlike body with respect to every interior point of C. Moreover, V is bounded and $\mu_V:X\to[0,\infty)$ is Lipschitz.*

Proof. Since $C\subseteq V$, it is obvious that V has nonempty interior. By using the (weak) compactness of K and $[0,1]$, it is easy to see that V is closed.

Now let us see that V is starlike with respect to every point $x_0\in\text{int}(C)$. Take two points $x_1,x_2\in\partial V\subset V$ with $x_1\in[x_0,x_2]$. Assuming that $x_1\ne x_2$ we will get a contradiction. Indeed, since $V=\bigcup_{y\in K}[y,C]$ and $x_1\in\partial V$, we have that $x_1\in X\setminus\text{int}([y,C])$ for every $y\in K$. Hence, for every $y\in K$, either $x_1\in\partial[y,C]$ or $x_1\notin[y,C]$; in either case, since $[y,C]$ is a starlike body with respect to $x_0\in\text{int}(C)$, and $x_2\ne x_1\in[x_0,x_2]$, we get that $x_2\notin[y,C]$. But then we have that $x_2\notin\bigcup_{y\in K}[y,C]=V$, a contradiction.

It is obvious that V is bounded. It only remains to show that μ_V (with respect to any point $x_0\in\text{int}(C)$) is Lipschitz. Without loss of generality we may assume that the given center is $x_0=0$. Let $M>0$ be such that $\mu_C(x)\le M\|x\|$ for all $x\in X$. Since $C\subseteq[y,C]$ we have that $\mu_{[y,C]}(x)\le\mu_C(x)\le M\|x\|$ for all $x\in X$ and, bearing in mind that $[y,C]$ is a convex body, this means that $\mu_{[y,C]}$ is M-Lipschitz for all $y\in K$. On the other hand, it is easily seen that $\mu_V(x)=\inf_{y\in K}\mu_{[y,C]}(x)$. Since the infimum of M-Lipschitz functions is always an M-Lipschitz function, we have that μ_V is M-Lipschitz. □

Lemma 2.6. *Let X be a Banach space, A a Lipschitz starlike body with respect to the origin. Then, for every $\varepsilon>0$ there exists $\delta>0$ so that $A+\delta B_X\subset(1+\varepsilon)A$.*

Proof. Let M be a Lipschitz constant for μ_A. For a given $\varepsilon>0$ choose $\delta>0$ with $\delta M<\varepsilon$. Take $x=y+z$, with $y\in A$, $z\in\delta B_X$. Then we have

$$\mu_A(x)=\mu_A(y+z)-\mu_A(y)+\mu_A(y)\le M\|z\|+\mu_A(y)\le M\delta+1<1+\varepsilon.$$

□

Lemma 2.7. *Let C a bounded convex body in a Banach space X, with $0\in int(C)$. Then, for every $\delta\in(0,1)$, $dist\big((1-\delta)C,X\setminus C\big)>0$, that is, $(1-\delta)C\subset_d C$.*

Proof. This is an easy consequence of the preceding lemma and the fact that μ_C is Lipschitz because C is a convex body. □

Lemma 2.8. *Let $T:X\longrightarrow Y$ be a continuous linear injection between two Banach spaces. Then, for every radially bounded C^p smooth body B' in Y which is starlike with respect to a point $b'\in T(X)$, we have that $B=T^{-1}(B')$ is a radially bounded C^p smooth starlike body in X with respect to $b=T^{-1}(b')$.*

Proof. Let $b'=T(b)$ be the center of B'. Then $A':=-b'+B'$ is starlike with respect to the origin, radially bounded and C^p smooth. Consider the function $\psi:X\longrightarrow[0,\infty)$ defined by $\psi(x)=\mu_{A'}(T(x))$. Then $A:=\{x\in X:\psi(x)\le1\}$ is a C^p smooth starlike body in X (with respect to the origin); besides, since $\psi(x) > 0$ whenever $x \neq 0$, we have that $ccA = \{0\}$, that is, A is radially bounded. It is obvious that $A = T^{-1}(A')$. Then we see that $B = T^{-1}(B') = T^{-1}(b' + A') = b + A$ is a radially bounded C^p smooth starlike body with respect to $b \in X$. □

Lemma 2.9. *Let $T : X \longrightarrow Y$ be a continuous linear injection between two Banach spaces. Assume that A' and B' are starlike bodies with respect to $y_0 = T(x_0) \in T(X)$, and $A' \subset_\mu B'$. Then $A := T^{-1}(A') \subset_\mu T^{-1}(B') := B$.*

Proof. Left to the reader. □

The following lemmas show how one can approximate and interpolate starlike bodies with smooth starlike bodies, provided the space has smooth partitions of unity.

Lemma 2.10. *Let X be a Banach space with C^p smooth partitions of unity, and C a starlike body with $ccC = \{0\}$. Then, for every $\delta > 0$, there exists $A \subset X$, a C^p smooth starlike body, with $ccA = \{0\}$ and such that $(1-\delta)C \subset A \subset (1+\delta)C$.*

Proof. Since X has C^p smooth partitions of unity, it has a C^p smooth bump as well, and in particular there exists B, a bounded C^p smooth starlike body with respect to the origin [14, Proposition II.5.1]. Choose $\varepsilon_0 \in (0,1)$ such that $\frac{1}{1-\varepsilon_0} < 1+\delta$, and $1+\varepsilon_0 < \frac{1}{1-\delta}$. Define $\varepsilon : X \setminus \{0\} \to (0,\infty)$ by $\varepsilon(x) = \varepsilon_0 \mu_C(x)$ for all $x \neq 0$, which is a continuous strictly positive function. Since X has C^p smooth partitions of unity, so does its open subset $X \setminus \{0\}$, and therefore every continuous function on $X \setminus \{0\}$ can be ε-approximated by a C^p smooth function on $X \setminus \{0\}$. Hence, given the continuous function $\mu_C : X \setminus \{0\} \to (0,\infty)$, there exists a C^p smooth function $g : X \setminus \{0\} \to \mathbb{R}$ such that $|\mu_C(x) - g(x)| \leq \varepsilon(x)$ for all $x \neq 0$. Now define $\psi : X \longrightarrow \mathbb{R}$ by

$$\psi(x) = \mu_B(x) g\big(\frac{x}{\mu_B(x)}\big) \text{ if } x \neq 0,$$

and $\psi(0) = 0$. The function ψ is clearly continuous on X, ψ is of class C^p on $X \setminus \{0\}$, and ψ is positively homogeneous. Moreover,

$$|\psi(x) - \mu_C(x)| = \big|\mu_B(x) g\big(\frac{x}{\mu_B(x)}\big) - \mu_C(x)\big| =$$
$$\big|\mu_B(x) g\big(\frac{x}{\mu_B(x)}\big) - \mu_B(x)\mu_C\big(\frac{x}{\mu_B(x)}\big)\big| \leq \mu_B(x)\varepsilon\big(\frac{x}{\mu_B(x)}\big) = \varepsilon_0 \mu_C(x)$$

for all $x \neq 0$. In particular, $\psi(x) \geq (1-\varepsilon_0)\mu_C(x) > 0$ if $x \neq 0$. Therefore, $A := \{x \in X : \psi(x) \leq 1\}$ is a C^p smooth starlike body with respect to 0. Let us check that A approximates C as required. We have

$$x \in A \iff \psi(x) \leq 1 \implies \mu_C(x) \leq 1 + \varepsilon_0 \mu_C(x) \implies (1-\varepsilon_0)\mu_C(x) \leq 1 \implies$$
$$x \in \frac{1}{1-\varepsilon_0} C \subset (1+\delta)C,$$

so $A \subset (1+\delta)C$. On the other hand, if $x \in (1-\delta)C$, that is, $\mu_C(x) \leq 1-\delta$, then we have $\psi(x) \leq (1+\varepsilon_0)\mu_C(x) \leq (1+\varepsilon_0)(1-\delta) < 1$, hence $x \in A$. □

Lemma 2.11. *Let X be a Banach space with C^p smooth partitions of unity, K a closed subset of X, and D a bounded starlike body with respect to 0, such that $K \subset_d D$. Then there exist D_1 and D_2, C^p smooth starlike bodies with respect to 0, such that*

$$K \subset D_1 \subset_\mu D_2 \subset D.$$

Moreover, if K is a bounded starlike body with respect to 0, the above is true for any set D, and the starlike body D_2 satisfies $D_2 \subset_d D$.

Proof. Since $K \subset_d D$ we can take $0 < \theta < 1/2$ so that $K \subset (1-2\theta)D$. Choose $\delta \in (0,1)$ with $(1-2\theta)/(1-\theta) < 1-\delta$ and $(1+\delta)(1-\theta) < 1$. Applying the preceding lemma to $C := (1-\theta)D$, we get a C^p smooth starlike body with respect to 0, D_1, such that $(1-\delta)C \subset D_1 \subset (1+\delta)C$. In particular, taking into account that $1-2\theta < (1-\theta)(1-\delta)$, we deduce $K \subset (1-2\theta)D \subset (1-\theta)(1-\delta)D = (1-\delta)C \subset D_1$. Now pick $\varepsilon > 0$ such that $(1+\varepsilon)(1+\delta)(1-\theta) < 1$, and set $D_2 := (1+\varepsilon)D_1$. The body D_2 is C^p smooth and starlike with respect to 0, and $D_1 \subset_\mu D_2$. Finally, we also have $D_2 = (1+\varepsilon)D_1 \subset (1+\varepsilon)(1+\delta)C \subset (1+\varepsilon)(1+\delta)(1-\theta)D \subset D$.

Assume that K is a bounded starlike body with respect to 0, and D is a mere subset of X such that $K \subset_d D$. Choose real numbers $\varepsilon > 0$ and $\delta \in (0,1)$ satisfying $1 < (1-\delta)(1+\varepsilon)$ and $(1+\delta)(1+\varepsilon)K \subset_d D$. By imitating the previous paragraph, with $C := (1+\varepsilon)K$, we obtain D_1, a C^p smooth starlike body with respect to 0, such that $(1-\delta)C \subset D_1 \subset (1+\delta)C$. Bearing in mind the choice of δ and ε, we deduce that $K \subset D_1 \subset_d D$. Now it is clear how to define D_2, a C^p smooth starlike body with respect to 0 such that $D_1 \subset_\mu D_2 \subset_d D$. □

The following lemma is one of the keys to the proof of Proposition 2.3.

Lemma 2.12. *Let X be a nonreflexive Banach space, K a weakly compact set, and C a bounded convex body with $0 \in int(C)$ and $K \subset_d C$. Then there exist $\varepsilon > 0$ and a sequence (C_n) of convex bodies such that*

(1) $\bigcap_{n=1}^{\infty} C_n = \emptyset$,
(2) $C_{n+1} \subset_d C_n \subset C$ *for all* $n \in \mathbb{N}$, *and*
(3) $[K, C_1] + 3\varepsilon B_X \subset C$.

Proof. Since $K \subset_d C$, there exists $\delta_0 > 0$ such that $K \subset (1-2\delta_0)C$ and, by Lemma 2.7, $\operatorname{dist}\big((1-\delta_0)C, X \setminus C\big) \geq \delta_1$ for some $\delta_1 > 0$.

Since X is nonreflexive, according to James' theorem, there exists a continuous linear functional $T \in X^*$ such that T does not attain its sup on the body $(1-2\delta_0)C$, $\alpha := \sup\{T(x) : x \in (1-2\delta_0)C\}$. Define now $H_n := \{x \in (1-2\delta_0)C : T(x) \geq \alpha - 1/n\}$ for each $n \in \mathbb{N}$. We have that $\bigcap_{n=1}^{\infty} H_n = \emptyset$, $H_{n+1} \subset H_n$ for all n, and $H_1 \subset (1-2\delta_0)C \subset_d (1-\delta_0)C$. Take $\varepsilon > 0$ such that $H_1 + \varepsilon B_X \subset (1-\delta_0)C$ and $3\varepsilon < \delta_1$. Then, for each $n \in \mathbb{N}$ let us define

$$C_n = N(H_n, \frac{\varepsilon}{2^n}) = \{x \in X : \operatorname{dist}(x, H_n) \leq \frac{\varepsilon}{2^n}\}.$$

It is easy to see that (C_n) satisfies the three properties of the statement. □

Proof of Proposition 2.3.

Case I. Assume that X is nonreflexive.

Let E be a bounded convex body with $0 \in \operatorname{int}(E)$. By Lemma 2.7, we have that $(1/8)E \subset_d (1/4)E \subset_d (1/2)E$. According to Lemma 2.10, there exists a C^p smooth starlike body with respect to 0 such that $(1/8)E \subset B \subset (1/4)E$. This body B is the one we need.

Now take a weakly compact set $K \subset X$ such that $K \subset rB$. Hence $K \subset rB \subset (r/4)E \subset_d (r/2)E$. According to Lemma 2.12, there exists $\varepsilon > 0$ and a sequence (C_n) of convex bodies such that

$$\bigcap_{n=1}^{\infty} C_n = \emptyset,\ [K, C_1] + 3\varepsilon B_X \subset (r/2)E,\ \text{ and } C_{n+1} \subset_d C_n \subset rE \text{ for all } n \in \mathbb{N}.$$

Let us choose a sequence (c_n) of points of X such that $c_n \in \operatorname{int}(C_n)$ for every $n \in \mathbb{N}$. Set $\Delta = \operatorname{diam}(\frac{r}{2}E) > 0$. For each $n \in \mathbb{N}$, define

$$V_n = [C_n, K].$$

By Lemma 2.5, V_n is a Lipschitz starlike body with respect to every point in the interior of C_n. Let $\mu_n = \mu_{V_n}$ be the Minkowski functional of V_n with respect to the point $c_{n+1} \in \operatorname{int}(C_{n+1}) \subset \operatorname{int}(C_n)$. Note that μ_n is a Lipschitz function.

Next we are going to inductively construct a sequence of positive numbers (δ_n) such that, if we define

$$P_n := \{x \in X : \mu_n(x) \leq 1 + \delta_n\}$$

for each $n \in \mathbb{N}$, then (P_n) is a sequence of bounded starlike bodies such that

(i) $P_{n+1} \subset_d P_n \subset P_1 \subset (r/2)E$ for all $n \in \mathbb{N}$,
(ii) $\bigcap_{n=1}^{\infty} P_n = K$,
(iii) P_n is starlike with respect to c_{n+1} for all $n \in \mathbb{N}$,
(iv) $C_{n+1} \subset_d P_n \cap C_n$ for all $n \in \mathbb{N}$.

•*1st step.* Choose $\delta_1 > 0$ with $\delta_1 < \min\{\varepsilon/\Delta, 1\}$, and set $P_1 = \{x \in X : \mu_1(x) \leq 1 + \delta_1\}$. By Lemma 2.6, there is $\delta_1' > 0$ such that $P_1 \supset V_1 + \delta_1' B_X$.
•*2nd step.* Now choose $\delta_2 > 0$ such that $\delta_2 < \min\{\delta_1'/2\Delta, 1/2\}$. Then $P_2 = \{x \in X : \mu_2(x) \leq 1 + \delta_2\} \subset V_2 + (\delta_1'/2)B_X$, and therefore $\operatorname{dist}(P_2, X \setminus P_1) > 0$.
•*(n+1)-th step.* Assume δ_j and P_j are already defined for $j = 1, 2, ..., n$ in such a way that $P_{j+1} \subset_d P_j$ for $j \leq n-1$. By Lemma 2.6, there is $\delta_n' > 0$ such that $P_n \supset V_n + \delta_n' B_X$. Pick $\delta_{n+1} > 0$ so that $\delta_{n+1} < \min\{\delta_n'/2\Delta, 1/2^n\}$, and set $P_{n+1} = \{x \in X : \mu_{n+1}(x) \leq 1 + \delta_{n+1}\}$. Then we have that $P_{n+1} \subset V_n + (\delta_n'/2)B_X$, hence $\operatorname{dist}(P_{n+1}, X \setminus P_n) > 0$.

By induction the sequence (P_n) is well-defined and satisfies properties (i) and (iii) above. To see that $P_1 \subset_d (r/2)E$, just note that $P_1 \subset V_1 + \delta_1 \Delta B_X = [C_1, K] + \delta_1 \Delta B_X \subset [C_1, K] + 3\varepsilon B_X \subset (r/2)E$. On the other hand, since $P_n \cap C_n = C_n$, it is clear that $C_{n+1} \subset_d P_n \cap C_n$, that is, the sequence (P_n) satisfies property (iv).

Finally, let us check that condition (ii) is met as well. It is immediate that $K \subset \bigcap_{n=1}^{\infty} P_n$. Let us take $q \in \bigcap_{n=1}^{\infty} P_n$ and show that $q \in K$. For each $n \in \mathbb{N}$ we have

$q \in P_n \subset V_n + \delta_n \Delta B_X = [C_n, K] + \delta_n \Delta B_X$, so there are $x_n \in C_n, y_n \in K, t_n \in [0,1]$ with $\|q - (1-t_n)x_n - t_n y_n\| \le \delta_n \Delta$, and in particular $\lim_{n\to\infty}[(1-t_n)x_n - t_n y_n] = q$. Since K is weakly compact and $[0,1]$ is compact, we may assume (passing to a subsequence if necessary) that y_n converges to some $y_0 \in K$ weakly, and $t_n \to t_0 \in [0,1]$. Then $(1-t_n)x_n$ converges to $q - t_0 y_0$ weakly. If $t_0 \neq 1$ then we have that x_n converges weakly to $x_0 := (1-t_0)^{-1}(q - t_0 y_0)$; but, since each $C_n \supset (x_j)_{j \ge n}$ is closed and convex, hence weakly closed, we have $x_0 \in C_n$ for each n, and then $x_0 \in \bigcap_{n=1}^{\infty} C_n = \emptyset$, a contradiction. Therefore, $t_0 = 1$, and $q = y_0 \in K$.

Now we are going to define the bodies A_n, B_n, D_n, E_n, and Q_n. Fix $n \in \mathbb{N}$. Since C_{n+2} and C_{n+1} are bounded starlike bodies with respect to c_{n+2}, and $C_{n+2} \subset_d C_{n+1}$, we can apply Lemma 2.11 to obtain two C^p smooth starlike bodies D_n, E_n with respect to c_{n+2} such that

$$C_{n+2} \subset D_n \subset_\mu E_n \subset_d C_{n+1}.$$

Another application of Lemma 2.11 gives us a C^p smooth starlike body A_n with respect to c_{n+2} such that

$$E_n \subset_\mu A_n \subset C_{n+1} = C_{n+1} \cap P_{n+1}.$$

Besides, $P_{n+1} \subset_d P_n$, and P_{n+1} is starlike with respect to c_{n+1}. Then, applying Lemma 2.11 for the last time (now P_n acts as a mere set, it is not necessary that P_n be starlike with respect to c_{n+2}, only P_{n+1} has to meet this condition), we get B_n and Q_n, two C^p smooth starlike bodies with respect to c_{n+2}, satisfying

$$P_{n+1} \subset_\mu B_n \subset_\mu Q_n \subset P_n.$$

Moreover, we also have $E_n \subset C_{n+1} \subset P_{n+1} \subset_\mu B_n$. Summing up, we get that

$$C_{n+2} \subset D_n \subset_\mu E_n \subset_\mu A_n \subset C_{n+1} \subset P_{n+1} \subset_\mu B_n \subset_\mu Q_n \subset P_n,$$

and now it is clear that the sequences of bodies we have just constructed satisfy conditions $(1) - (4)$ of Proposition 2.3. Finally, B is the required body and satisfies condition (5). Indeed, notice that $K \subset (r/2)\mathrm{int}(E) \subset rB$, and $P_1 \subset (r/2)E \subset 4rB$.

Case II. Assume now that X is reflexive.

In this case it is known that there exists a continuous linear injection $T : X \longrightarrow c_0(\Gamma)$ for some (infinite) set Γ (see [14], p.246, for instance). It is also well known that for an infinite set Γ, the space $c_0(\Gamma)$ is c_0-saturated, that is, every infinite-dimensional closed subspace of $c_0(\Gamma)$ has a closed subspace which is isomorphic to c_0. This clearly implies that $c_0(\Gamma)$ contains no closed infinite-dimensional reflexive subspaces. Therefore $Y := \overline{T(X)} \subset c_0(\Gamma)$ is nonreflexive, and $T(X)$ is not a closed subspace of $Y \subset c_0(\Gamma)$. On the other hand, the space $c_0(\Gamma)$ has a C^∞ smooth equivalent norm (see [14], chapter V, theorem 1.5), whose restriction to Y defines a C^∞ smooth equivalent norm $|\cdot|$. Finally, it is well known [14] that the space $c_0(\Gamma)$ has C^∞ smooth partitions of unity, hence so does Y.

Summing up, we have a continuous linear injection $T : X \to Y$, where $(Y, |\cdot|)$ is a nonreflexive Banach space with a C^∞ smooth norm and C^∞ smooth partitions of unity, and $T(X)$ is dense in Y.

Set $B' = \{y \in Y : |y| \leq 1\}$, which is a C^∞ smooth bounded convex body with $0 \in \operatorname{int}(B')$. Define $B = T^{-1}(B')$. It is clear that B is a radially bounded C^∞ smooth convex body.

Let K be a weakly compact subset of X and $r > 0$ with $K \subset rB$. Since T is continuous, $T(K)$ is weakly compact. Moreover $T(K) \subset T(rB) \subset rB' \subset_d (r/2)(4B')$. Now we may copy the above proof (nonreflexive case), with $4B' = E$ and $T(K)$ replacing K, in order to obtain sequences of C^∞ smooth starlike bodies, $(P'_n), (C'_n), (A'_n), (B'_n), (Q'_n), (D'_n), (E'_n)$, and a sequence of points (c'_n) of Y satisfying the conditions $(1) - (4)$ of the statement of Proposition 2.3 and $P'_1 \subset (r/2)(4B') = 2rB'$. Ensure further that $c'_n \in T(X) \cap \operatorname{int}(C'_n)$ for each $n \in \mathbb{N}$ (this is possible because $T(X)$ is dense in Y, hence $T(X) \cap \operatorname{int}(C'_n) \neq \emptyset$ for all n).

Then, for each $n \in \mathbb{N}$, define $c_n = T^{-1}(c'_n) \in X$, and

$$C_n = T^{-1}(C'_n),\ B_n = T^{-1}(B'_n),\ P_n = T^{-1}(P'_n),\ A_n = T^{-1}(A'_n)$$
$$Q_n = T^{-1}(Q'_n),\ D_n = T^{-1}(D'_n),\ E_n = T^{-1}(E'_n) \subset X.$$

By Lemma 2.8, these are radially bounded C^∞ smooth starlike bodies with respect to c_{n+2}. On the other hand, Lemma 2.9 guarantees that

$$C_{n+2} \subset D_n \subset_\mu E_n \subset_\mu A_n \subset C_{n+1} \subset P_{n+1} \subset B_n \subset_\mu Q_n \subset P_n.$$

Finally, it is immediately checked that $\bigcap_{n=1}^\infty C_n = \emptyset$, $\bigcap_{n=1}^\infty P_n = K$, $P_1 = T^{-1}(P'_1) \subset T^{-1}(2rB') = 2rB \subset 4rB$. □

Now we are in a position to finish the proof of the main result.

Proof of Theorem 1.1.

We may assume that A is a bounded starlike body with respect to the origin. Let B be the radially bounded C^p smooth starlike body provided by Proposition 2.3. Choose $r > 0$ such that $A \subset_\mu rB$. Bearing in mind that $K \subset_d A \subset_\mu rB$, it follows from Proposition 2.3 that there are sequences (P_n), (C_n), (A_n), (B_n), (Q_n), (D_n), (E_n) of subsets of X and a sequence (c_n) of points of X which satisfy the conditions of Proposition 2.1. Then we can apply this Proposition to find a C^p diffeomorphism $\Psi : X \to X \setminus K$ such that Ψ is the identity on $X \setminus P_1 \supset X \setminus 4rB$.

On the other hand, since $K \subset_d A$, Lemma 2.11 allows us to find two C^p smooth starlike bodies U_1, U_2 with respect to 0 such that $K \subset U_1 \subset_\mu U_2 \subset A$. Now, by the Four Bodies Lemma 2.2, there is a C^p diffeomorphism $g : X \to X$ such that $g(U_2) = 4rB$ and g is the identity on $U_1 \supset K$; notice in particular that $g(K) = K$.

Define then $h = g^{-1} \circ \Psi \circ g$. It is clear that h is a C^p diffeomorphism from X onto $X \setminus K$. Moreover, if $x \in X \setminus A$ then $x \notin U_2$, so $g(x) \notin 4rB$, which implies that $\Psi(g(x)) = g(x)$, hence $h(x) = x$; that is, h is the identity off A. □

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