DELETING DIFFEOMORPHISMS WITH PRESCRIBED SUPPORTS IN BANACH SPACES

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Abstract. We show that, for every infinite-dimensional Banach space X with a Schauder basis, the following are equivalent: (1) X has a C^p smooth bump function; (2) for every compact subset K and every open subset U of X with $K \subset U$, there exists a C^p diffeomorphism $h : X \to X \setminus K$ such that h is the identity on $X \setminus U$.

A subset K of X is said to be topologically negligible provided there exists a homeomorphism $h : X \to X \setminus K$. The homeomorphism h is usually required to be the identity outside a given neighborhood U of K. Here X can be a Banach space, a manifold, or just a topological space, but we will only consider the case when X is an infinite-dimensional Banach space and h is a diffeomorphism (recall that points are *not* topologically negligible in finite-dimensional spaces). Such h will be called a deleting diffeomorphism, and we will say that h has its support on U.

Deleting diffeomorphisms are very powerful tools in infinite-dimensional global analysis and nonlinear analysis. We do not intend to make a history of the development of topological negligibility and its applications, and we refer the reader to the introductions of the papers [5, 10, 20] and the references therein for a better insight and a glimpse of the many important applications of smooth negligibility. We will only mention two facts here. First, in the case when X is the separable Hilbert space or even a Hilbert manifold, the most powerful result on smooth negligibility is that of West's [28]: for every locally compact set K, every open set $U \supset K$, and every open covering G of X, there exists a C^∞ diffeomorphism $h : X \to X \setminus K$ such that h is the identity off U and is limited by G (this means that h can be made to be arbitrarily close to the identity mapping). Second, when X is any infinite-dimensional Banach space with a (not necessarily equivalent) C^p smooth norm, for every compact set K and every ball B containing K there exists a C^p smooth diffeomorphism $h : X \to X \setminus K$ such that h is the identity off B. That is, the supports of those deleting diffeomorphisms are balls; see [5].

This paper could be regarded both as an addendum to [5] and as a bridge to possible generalizations of West's theorem [28]. Here we are concerned with the

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supports of diffeomorphisms deleting compacta from a Banach space: when can one get a diffeomorphism h deleting a compact set K from a Banach space X such that h is the identity outside a prescribed open neighborhood U of K?

Some readers might have the impression that the question about the supports of deleting diffeomorphisms is too technical, but the truth is that, for some important applications of negligibility, this is a crucial issue. For instance, Manuel Cepedello Boiso and the first-named author have recently shown that the C^∞ smooth functions with no critical points are dense in the set of continuous functions on the separable Hilbert space (or a Hilbert manifold); this is a sort of very strong approximate Morse-Sard theorem [3]. A key part of the proof of this result can only be made either by applying West's theorem or by using the fact that every compact set K of ℓ_2 can be removed by a diffeomorphism h which remains the identity outside a prescribed open neighborhood of K. If one is to establish strong approximate Morse-Sard-like theorems for infinite-dimensional Banach spaces other than ℓ_2, the first step should be to try to extend West's theorem to those spaces, or at least to show the existence of diffeomorphism deleting compacta with prescribed supports (as a matter of fact, the proof of West's theorem already uses the existence of such deleting diffeomorphisms with given supports, so this seems to be the first question one should look at).

Of course, if one wants to construct diffeomorphisms deleting compacta with supports that are much tighter than balls, it is reasonable to demand that the space X has some structure richer than merely possessing a smooth norm. In this respect, the assumption that X is separable and has a Schauder basis does not seem too restrictive: for instance, all the classical spaces meet this demand. Our main result says that diffeomorphisms deleting compacta with prescribed supports do exist in such spaces, provided they have a smooth bump function (a condition which is necessary as well).

Theorem 1. *Let X be an infinite-dimensional Banach space with a Schauder basis, and $p \in \mathbb{N} \cup \{\infty\}$. The following statements are equivalent:*

(1) *X has a C^p smooth bump function.*
(2) *For every compact subset K and every open subset U of X with $K \subset U$, there exists a C^p diffeomorphism $h : X \to X \setminus K$ such that h is the identity on $X \setminus U$.*

In the case when X is the Hilbert space, Theorem 1 is a particular instance of the above mentioned theorem of West's [28]. A simple proof of Theorem 1 for the Hilbert space was obtained by M. Cepedello Boiso and the first-named author and was included in the first version of the paper [3] (that version was then improved and the mentioned proof discarded because it was no longer needed). This proof of the result in the Hilbert case inspired the one we present here for any Banach space with a Schauder basis.

The proof of Theorem 1 is done mainly in two steps, which we next explain. The first one uses the noncomplete (asymmetric) norm technique of deleting compact sets introduced in [5, 19]. The main result of [5], Theorem 2.1, shows that a mapping of the form $\phi(x) = x + p(f(x))$, $x \in X \setminus K$, for a certain function $f : X \to [0, +\infty)$ with $f^{-1}(0) = K$ and a path $p : (0, +\infty) \to X$, establishes a C^∞ diffeomorphism between $X \setminus K$ and X. The map ϕ can be viewed as a *small* perturbation of the identity. In order that the perturbation $p \circ f$ be small, p and f must satisfy some Lipschitzian-type conditions with respect to a certain distance d_ω induced by a smooth *noncomplete (asymmetric) norm* ω. The function $f(x)$ can be viewed as a smooth substitute for the ω-distance function from x to the set K, and has the additional property that $f(x) = 1$ whenever $d_\omega(x, K) \geq \varepsilon$, where ε is a given fixed positive number. The path p is constructed in such a way that $p(t)$ asymptotically avoids compact sets by getting lost in the infinitely many dimensions of X as t goes to 0; moreover, $p(t) = 0$ for all $t \geq 1$, so that $\phi(x) = x$ whenever $d_\omega(x, K) \geq \varepsilon$. By pushing away ω-neighborhoods of K along the path p, the mapping ϕ^{-1} makes K disappear. Besides, from the formula defining ϕ and the properties of p and f it is clear that ϕ is the identity on the set $\{x \in X : d_\omega(x, K) \geq \varepsilon\}$. This last property is shown in the proof of [5, Theorem 2.1], but is not explicitly stated in that Theorem.

Let us make a precise definition of this ω-distance, which plays an important role in the proof of Theorem 1.

Definition 2. Let $(X, \|\cdot\|)$ be a Banach space. We say that a functional $\omega : X \to [0, \infty)$ is a noncomplete (asymmetric) norm provided ω is the Minkowski functional of a radially bounded convex body which is not bounded (and which is not necessarily symmetric either). That is, provided ω satisfies the following properties:

(1) $\omega(x + y) \leq \omega(x) + \omega(y)$ for all $x, y \in X$;
(2) $\omega(rx) = r\omega(x)$ for all $r \geq 0$, $x \in X$;
(3) $\omega(x) = 0$ if and only if $x = 0$.

We will say that ω is C^p smooth if it is so away from the origin. For each $x \in X$, $A \subset X$ and $r > 0$, we define the ω-body of center x and radius r as

$$B_\omega(x, r) = B(x, r; \omega) = \{y \in X : \omega(y - x) \leq r\},$$

and we define the ω-distance from x to the set A by

$$d_\omega(x, A) = d(x, A; \omega) = \inf\{\omega(x - z) : z \in A\}.$$

Finally, a subset V of X will be said to be an ω-neighborhood of $A \subset X$ provided that, for each $x \in A$, there exists $r > 0$ so that $B(x, r; \omega) \subset V$.

It is proved in [5] that every infinite-dimensional Banach space X with a C^p smooth equivalent norm has a C^p smooth noncomplete asymmetric norm ω as well.

With these notations, we can now state what the proof of [5, Theorem 2.1] really shows.

Theorem 3 (Azagra and Dobrowolski)**.** *Let $(X, \|\cdot\|)$ be a Banach space with a C^p smooth noncomplete (asymmetric) norm ω, where $p \in \mathbb{N} \cup \{\infty\}$. Then, for every $\varepsilon > 0$ and every compact set $K \subset X$, there exists a C^p diffeomorphism $\phi_\varepsilon : X \to X \setminus K$ such that ϕ_ε is the identity on $\{x \in X : d_\omega(x, K) \geq \varepsilon\}$.*

The second step in the proof of Theorem 1 is to construct a C^p smooth noncomplete norm ω and C^p diffeomorphism $F : X \to X$ such that $F(K) = K$ and $F(U)$ is a ω-neighborhood of K. Then, in order to obtain the desired deleting diffeomorphism $h : X \to X \setminus K$ with support on U, it will be enough to compose F with a diffeomorphism deleting K from X and being the identity off $F(U)$. The next Corollary makes sure that we can combine those two steps to obtain a proof of Theorem 1, but it also tells us under what more general conditions we can expect to obtain diffeomorphisms deleting compacta and with support on a prescribed open set: it turns out that we only need a diffeomorphism $F : X \to X$ such that $F(U)$ is an ω-neighborhood of $F(K)$ for some noncomplete (asymmetric) C^p smooth norm ω on X.

Corollary 4. *Let $(X, \|\cdot\|)$ be a Banach space. Assume that, for a compact set $K \subset X$ and an open set $U \supset K$, there are:*

(1) *a C^p smooth noncomplete (asymmetric) norm ω and*
(2) *a C^p diffeomorphism $F : X \to X$ such that $F(U)$ is an ω-neighborhood of $F(K)$.*

Then there exists a C^p diffeomorphism $h : X \to X \setminus K$ such that h is the identity on $X \setminus U$.

Proof. Since $F(U)$ is a ω-neighborhood of $F(K)$, for each $y \in F(K)$ there exists $r_y > 0$ so that $B_\omega(y, 2r_y) \subset F(U)$. We have

$$F(K) \subset \bigcup_{y \in F(K)} B_\omega(y, r_y) \subset \bigcup_{y \in F(K)} B_\omega(y, 2r_y) \subset F(U).$$

Since $F(K)$ is a compact set and the $B_\omega(y, r_y)$ are open in $(X, \|\cdot\|)$, we can get a finite number of points $y_1, ..., y_n \in F(K)$ so that

$$F(K) \subset \bigcup_{j=1}^{n} B_\omega(y_j, r_j) \subset \bigcup_{j=1}^{n} B_\omega(y, 2r_j) \subset F(U), \tag{1}$$

where we write $r_j = r_{y_j}$ for short. It follows that, for $\varepsilon := \min\{r_j : j = 1, ..., n\}$,

$$\{x \in X : d_\omega(x, F(K)) < \varepsilon\} \subset F(U) \tag{2}$$

(indeed, if $d_\omega(x, F(K)) < \varepsilon$ then, by definition, there exists $y \in F(K)$ so that $\omega(x - y) < \varepsilon$; now, since $y \in F(K) \subset \bigcup_{j=1}^n B_\omega(y_j, r_j)$, there exists $j_0 \in \{1, ..., n\}$ so that $\omega(y - y_{j_0}) < r_{j_0}$, and therefore $\omega(x - y_{j_0}) \leq \omega(x - y) + \omega(y - y_{j_0}) < \varepsilon + r_{j_0} \leq 2r_{j_0}$, that is, $x \in B_\omega(y_{j_0}, 2r_{j_0}) \subset F(U)$).

Now we can apply Theorem 3 above with the noncomplete (asymmetric) norm ω, the compact set $F(K)$ and the positive number $\varepsilon = \min\{r_j : j = 1, ..., n\}$ to obtain a C^p diffeomorphism $\phi_\varepsilon : X \to X \setminus F(K)$ so that ϕ_ε is the identity on the set $\{x \in X : d_\omega(x, K) \geq \varepsilon\}$, and in particular (thanks to (2) above) ϕ_ε is the identity outside $F(U)$.

Define then $h := F^{-1} \circ \phi_\varepsilon \circ F$. It is clear that h is a C^p diffeomorphism from X onto $X \setminus K$. Finally, if $x \notin U$, then $F(x) \notin F(U)$, hence $\phi_\varepsilon(F(x)) = F(x)$, and $h(x) = x$. $\square$

In the case when X has a Schauder basis, for instance, we are able to construct a diffeomorphism F and a noncomplete norm ω with such properties. The construction is rather technical and will be split into several lemmas. The reader is advised to skip a few pages and begin reading the proof of Proposition 14, which goes directly into the construction of ω and F, and only later go back to the most technical details.

We will also need to make use of smooth starlike bodies and their Minkowski functionals. A closed subset A of a Banach space X is said to be a starlike body if there exists a point a_0 in the interior of A such that every ray emanating from a_0 meets ∂A, the boundary of A, at most once. We will say that a_0 is a center of A. There can obviously exist many centers for a given starlike body. Up to a suitable translation, we can always assume that $a_0 = 0$ is the origin of X, and we will often do so, unless otherwise stated. For a starlike body A with center a_0, we define the characteristic cone of A as

$$ccA = \{x \in X | a_0 + r(x - a_0) \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A with respect to the center a_0 as

$$\mu_{A,a_0}(x) = \mu_A(x) = \inf\{t > 0 \mid x - a_0 \in t(-a_0 + A)\} \text{ for all } x \in X.$$

Note that $\mu_A(x) = \mu_{-a_0+A}(x - a_0)$ for all $x \in X$. It is easily seen that μ_A is a continuous function which satisfies $\mu_A(a_0 + rx) = r\mu_A(a_0 + x)$ for every $r \geq 0$ and $x \in X$, and $\mu_A^{-1}(0) = ccA$. Moreover, $A = \{x \in X | \mu_A(x) \leq 1\}$, and $\partial A = \{x \in X \mid \mu_A(x) = 1\}$. Conversely, if $\psi : X \to [0, \infty)$ is continuous and satisfies $\psi(a_0 + \lambda x) = \lambda\psi(a_0 + x)$ for all $\lambda \geq 0$, then $A_\psi = \{x \in X \mid \psi(x) \leq 1\}$ is a starlike body. Of every such function we will say that ψ is a positively homogeneous functional.

We will say that A is a C^p smooth starlike body provided its Minkowski functional μ_A is C^p smooth on the set $X \setminus ccA = X \setminus \mu_A^{-1}(0)$. This is equivalent to saying that ∂A is a C^p smooth one-codimensional submanifold of X such that no affine hyperplane tangent to ∂A contains a ray emanating from the center a_0. Throughout this paper, $p = 0, 1, 2, ..., \infty$, and C^0 smooth means just continuous.

All the starlike bodies that we will deal with in this paper are *radially bounded.* A starlike body A is said to be radially bounded provided that, for every ray emanating from the center a_0 of A, the intersection of this ray with A is a bounded set. This amounts to saying that $ccA = \{a_0\}$.

For every *bounded* starlike body A in a Banach space $(X, \|\cdot\|)$ there are constants $M, m > 0$ such that

$$m\|x\| \leq \mu_A(x) \leq M\|x\| \text{ for all } x \in X.$$

If A is just *radially bounded* then we can only ensure that

$$\mu_A(x) \leq M\|x\| \text{ for all } x \in X,$$

for some $M > 0$. As is shown implicitly in [18, Proposition II.5.1], a Banach space X has a C^p smooth bump function if and only if there is a bounded C^p smooth starlike body in X. The reader might want to consult the references [2, 4, 7, 8, 9] for other properties of starlike bodies.

Notation 5. Let $(X, \|\cdot\|)$ be a Banach space, and ρ be the Minkowski functional of a radially bounded C^p smooth starlike body. For each $x \in X$, $A \subset X$, $r > 0$, we define the ρ-pseudoball of center x and radius r as

$$B(x, r; \rho) = \{y \in X : \rho(y - x) \leq r\},$$

and the ρ-pseudodistance of x to the set A by

$$d(x, A; \rho) = \inf\{\rho(x - z) : z \in A\}.$$

Lemma 6. *Let X be a Banach space, A be a C^p smooth bounded starlike body in X (with Minkowski functional μ_A), and Y and Z be linear subspaces of X satisfying the following properties:*

(i) *$X = Y \oplus Z$; and*

(ii) *the function $t \in (0, \infty) \longrightarrow \mu_A(y + tz)$ is nondecreasing for each $(y, z) \in Y \times Z$.*

Take Υ, any finite subset of Y; δ and Δ, numbers with $0 < \delta < \Delta$; and $\{r_\upsilon : \upsilon \in \Upsilon\}$ and $\{R_\upsilon : \upsilon \in \Upsilon\}$, families of numbers such that $0 < r_\upsilon < R_\upsilon$ for each $\upsilon \in \Upsilon$. Then, there exists a function $g : X = Y \oplus Z \longrightarrow [0, +\infty)$ satisfying:

(1) *g is C^p smooth on X;*

(2) *$g = 0$ on $\bigcup_{\upsilon \in \Upsilon} B(\upsilon, r_\upsilon; \mu_A) \cup \{(y, z) \in X : \mu_A(z) \leq \delta\}$;*

(3) *$g = 1$ on $[X \setminus \bigcup_{\upsilon \in \Upsilon} B(\upsilon, R_\upsilon; \mu_A)] \cap \{(y, z) \in X : \mu_A(z) \geq \Delta\}$; and*

(4) *the function $t \in (0, \infty) \longrightarrow g(y + tz)$ is nondecreasing, for every $(y, z) \in Y \times Z$.*

Proof. For each $\upsilon \in \Upsilon$, pick a function $\theta_\upsilon \in C^\infty(\mathbb{R}, [0, 1])$ which is nondecreasing and satisfies $\theta_\upsilon^{-1}(0) = (-\infty, r_\upsilon]$ and $\theta_\upsilon^{-1}(1) = [R_\upsilon, +\infty)$. Take also a nondecreasing function $\theta \in C^\infty(\mathbb{R}, [0, 1])$ such that $\theta^{-1}(0) = (-\infty, \delta]$ and $\theta^{-1}(1) = [\Delta, +\infty)$. Then we can define $g : X \longrightarrow [0, 1]$ as

$$g(x) = g(y, z) = \theta(\mu_A(z)) \prod_{\upsilon \in \Upsilon} \theta_\upsilon(\mu_A(y - \upsilon + z)).$$

It is obvious that g satisfies conditions (1), (2) and (3). Condition (4) follows from (ii), and the facts that $\Upsilon \subset Y$ and the functions $\{\theta_\upsilon : \upsilon \in \Upsilon\}$ and θ are nonnegative and nondecreasing. $\square$

Lemma 7. *Let X be a Banach space; D be a C^p smooth bounded starlike body, with Minkowski functional ρ; Y and Z be linear subspaces of X; and $g, \omega : X \longrightarrow [0, +\infty)$ be functions satisfying the following properties:*

(i) *$X = Y \oplus Z$;*

(ii) *g is C^p smooth on X; $g = 0$ on the set $\{(y, z) \in X : \rho(z) \leq \delta\}$, for some $\delta > 0$; and the function $t \in (0, \infty) \longrightarrow g(y + tz)$ is nondecreasing for each $(y, z) \in Y \times Z$;*

(iii) *ω is the Minkowski functional of a C^p smooth radially bounded convex body W such that $W \cap Z$ contains $D \cap Z$ (that is $\omega \leq \rho$ on Z).*

Then, the mapping $F : X = Y \oplus Z \longrightarrow X$ *defined by*

$$F(x) = F(y,z) = \Big(y, \big[g(x)\frac{\rho(z)}{\omega(z)} + 1 - g(x)\big]z\Big) \ \textit{if } z \neq 0, \textit{ and } F(y,z) = (y,z) \textit{ if } z = 0$$

is a C^p *diffeomorphism.*

Proof. It is clear that F is of class C^p on the set $\{(y,z) \in X : z \neq 0\}$. Bearing in mind that $g = 0$ on $\{(y,z) \in X : \rho(z) \leq \delta\}$, we immediately see that F is the identity on this set, and therefore, F is C^p on X.

Let us check that F is a bijection. Since F is the identity on $\{(y,z) \in X : z = 0\}$, it will suffice to show that for every $(y,z) \in X$ with $z \neq 0$, there exists a unique $t(y,z) > 0$ with $F(y, t(y,z)z) = (y,z)$. Take $x_0 = (y_0, z_0) \in X$ such that $z_0 \neq 0$. Consider the function $G_{x_0} = G_{(y_0,z_0)} : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$G_{x_0}(t) = [g(y_0 + tz_0)\frac{\rho(z_0)}{\omega(z_0)} + 1 - g(y_0 + tz_0)]t,$$

which is of class C^p. We have that $G_{x_0}(0) = 0$ and $\lim_{t\to+\infty} G_{x_0}(t) = +\infty$, and the derivative of G_{x_0} is given by

$$\frac{dG_{x_0}}{dt} = [g(y_0 + tz_0)\frac{\rho(z_0)}{\omega(z_0)} + 1 - g(y_0 + tz_0)] + tDg(y_0 + tz_0)(z_0)[\frac{\rho(z_0)}{\omega(z_0)} - 1].$$

Now, for $t \geq 0$, condition (ii) implies that $Dg(y_0 + tz_0)(z_0) \geq 0$. Hence,

$$\frac{dG_{x_0}}{dt} \geq [g(y_0 + tz_0)\frac{\rho(z_0)}{\omega(z_0)} + 1 - g(y_0 + tz_0)] \geq 1,$$

and G_{x_0} is strictly increasing. Therefore, there is a unique $t(y_0, z_0) > 0$ such that $G_{(y_0,z_0)}(t(y_0,z_0)) = 1$, which means $F(y_0, t(y_0,z_0)z_0) = (y_0, z_0)$.

Finally, from the fact that $\frac{dG_{x_0}}{dt} \geq 1$, and by using the implicit function theorem, it is immediate that the function $X \setminus Y \ni (y,z) \longrightarrow t(y,z)$ is of class C^p, and therefore F is a C^p diffeomorphism. $\square$

Lemma 8. *Let* $(X, \|\cdot\|)$ *be a normed space, and* Y *and* Z *be linear subspaces of* X *with* $X = Y \oplus Z$. *Assume that the norm* $\|\cdot\|$ *has the following property of symmetry:*

$$\|(y,-z)\| = \|(y,z)\| \textit{ for every } (y,z) \in Y \times Z.$$

Then, for any given $(y,z) \in Y \times Z$, *the function*

$$[0,\infty) \ni t \longrightarrow g(t) = \|(y,tz)\|$$

is nondecreasing.

Proof. The function g is obviously convex, and $\lim_{t\to\infty} g(t) = \infty$. A standard convexity argument shows that g is nondecreasing if and only if g has a global minimum at the point $t = 0$, that is, if and only if $\|(y,0)\| \leq \|(y,tz)\|$ for all $t > 0$. Therefore it suffices to check that $\|(y,0)\| \leq \|(y,v)\|$ for all $v \in Z$. The verification is trivial thanks to the symmetry condition:

$$\|(y,0)\| = \|\frac{1}{2}(y,v) + \frac{1}{2}(y,-v)\| \leq \frac{1}{2}\|(y,v)\| + \frac{1}{2}\|(y,-v)\| = \|(y,v)\|.$$

□

Definition 9. We say that a subset V of the plane $\mathbb{R}^2$ is a *smooth square* provided

(i) V is a C^∞ smooth bounded convex body with the origin as an interior point;
(ii) $(y,z) \in \partial V \Leftrightarrow (\epsilon_1 y, \epsilon_2 z) \in \partial V$ for every $(\epsilon_1, \epsilon_2) \in \{-1,1\}^2$.
(iii) $[-\frac{1}{2}, \frac{1}{2}] \times \{-1,1\} \cup \{-1,1\} \times [-\frac{1}{2}, \frac{1}{2}] \subset \partial V$;
(iv) $V \subset [-1,1] \times [-1,1]$.

It is obvious that there are plenty of smooth squares in $\mathbb{R}^2$. As we will see, whenever we have a decomposition $X = Y \oplus Z$, smooth squares V are very useful to combine two norms ρ_Y, ρ_Z (or two Minkowski functionals of starlike bodies) defined on Y and Z into a norm (or a Minkowski functional of a starlike body) defined on X by

$$\rho(x) = \rho(y,z) = \mu_V(\rho_Y(y), \rho_Z(z)),$$

without losing any differentiability property of ρ_Y and ρ_Z, and keeping the equivalence with the functional $\max\{\rho_Y(y), \rho_Z(z)\}$. The next lemma (whose proof is easy and therefore omitted) says this is indeed so. The rather strange statement of property (iv) tells us that the unit sphere of μ_V is locally flat and orthogonal to the axis (in a neighborhood of the intersection of ∂V with the lines $x = 0$ and $y = 0$). This property (iv) accounts for the fact that such ρ enjoys the same degree of differentiability as ρ_Y and ρ_Z do.

Lemma 10. *Let $V \subset \mathbb{R}^2$ be a smooth square. Then, its Minkowski functional $\mu_V : \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ smooth norm in $\mathbb{R}^2$ such that*

(1) $\mu_V \leq \|\cdot\|_1 \leq 2\mu_V$;
(2) $\|\cdot\|_\infty \leq \mu_V \leq 2\|\cdot\|_\infty$;
(3) $\mu_V(0,z) = |z|, \quad \mu_V(y,0) = |y|$;
(4) *for every $(y,z) \in \mathbb{R}^2 \setminus \{(0,0)\}$, there exist $\sigma > 0$ and $(\kappa_1, \kappa_2) \in \mathbb{R}^2$ such that $\mu_V(y',z') = \kappa_1 \cdot y'$ if $\|(y'-y, z')\|_\infty \leq \sigma$, and $\mu_V(y',z') = \kappa_2 \cdot z'$ if $\|(y', z'-z)\|_\infty \leq \sigma$.*
(5) *the functions $t \in (0,\infty) \to \mu_V(y,tz)$ and $t \in (0,\infty) \to \mu_V(ty,z)$ are both nondecreasing.*

Here, as is customary, $\|(x,y)\|_1 = |x| + |y|$, and $\|(x,y)\|_\infty = \sup\{|x|,|y|\}$.

Notation 11. Let $(X, \|\cdot\|)$ be a Banach space with Schauder basis $(e_i)_{i=1}^\infty$. For each $n \in \mathbb{N}$, let us consider

$$Y_n := span\{e_i : 1 \leq i \leq n\}, \quad Z_n := \overline{\text{span}}\{e_i : i > n\}.$$

The uniformly bounded family of canonical projections $X \to Y_n$ associated to the basis $(e_i)_{=1}^\infty$ will be denoted by $(P_n)_{n=1}^\infty$.

Lemma 12. *Let $(X, \|\cdot\|)$ be a Banach space with Schauder basis $(e_i)_{i=1}^\infty$, and let $A \subset X$ be a C^p smooth bounded starlike body. Consider a smooth square $V \subset \mathbb{R}^2$. Then, for every $n \in \mathbb{N}$, the function $\rho_n : X = Y_n \times Z_n \to \mathbb{R}$ defined by*

$$\rho_n(y,z) = \mu_V(\mu_A(y), \mu_A(z))$$

is the Minkowski functional of a C^p smooth bounded starlike body A_n such that the functions

$$(0,\infty) \ni t \to \rho_n(y,tz) \quad \text{and} \quad (0,\infty) \ni t \to \rho_n(ty,z)$$

are both nondecreasing. Moreover, there exists $\alpha > 0$ such that

$$\alpha^{-1}\rho_n \le \|\cdot\| \le \alpha\rho_n$$

for every $n \in \mathbb{N}$.

Proof. Thanks to Lemma 10 and Lemma 8 it is straightforward to check that ρ_n has the above properties. We only show the existence of an $\alpha > 0$ satisfying the last part of the statement. Choose $M > 0$ such that $M^{-1}\|\cdot\| \le \mu_A \le M\|\cdot\|$ and

$$\sup\{\|P_n\| : n \in \mathbb{N}\} \le M,$$

where $(P_n)_{n\ge1}$ are the projections associated to the Schauder basis of X. Fix an $n \in \mathbb{N}$.

Let $x = (y+z) \equiv (y,z) \in Y_n \times Z_n$. If $\rho_n(y,z) \le 1$ then, by Lemma 10, we have that $\|(\mu_A(y),\mu_A(z))\|_\infty \le 1$, hence $\|(\|y\|,\|z\|)\|_\infty \le M$. This implies that $\|y+z\| \le 2M$ and $\|\cdot\| \le 2M\rho_n$.

On the other hand, for each $(y,z) \in Y_n \times Z_n$ we have

$$\begin{aligned}\|y\| &= \|P_n(y+z)\| \le M\|y+z\| \le (1+M)\|y+z\|,\\ \|z\| &= \|y+z-y\| \le \|y+z\| + \|y\| \le (1+M)\|y+z\|.\end{aligned}$$

Therefore, again by Lemma 10, we can estimate

$$\rho_n(y,z) \le \|(\mu_A(y),\mu_A(z))\|_1 \le M(\|y\|+\|z\|) \le 2M(1+M)\|y+z\|.$$

Now we can deduce that

$$[2M(1+M)]^{-1}\rho_n \le \|\cdot\| \le 2M\rho_n$$

for each $n \in \mathbb{N}$. Hence it is enough to choose $\alpha > 0$ with $\alpha \ge 2M(1+M)$. □

Lemma 13. *Let $(X,\|\cdot\|)$ be a Banach space with Schauder basis$(e_i)_{i\ge1}$ and $A \subset X$ be a C^p smooth bounded starlike body. Consider a smooth square $V \subset \mathbb{R}^2$. Let $K \subset X$ be a compact subset of X and $U \supset K$ an open set. Then, for every $\gamma > 1$, there exist a number $N \in \mathbb{N}$, a finite subset Υ of $Y_N := sp\{e_i : 1 \le i \le N\}$, and positive numbers r and R satisfying the following properties:*

1. $\frac{R}{r} \ge \gamma$;
2. $K \subset \bigcup_{\upsilon\in\Upsilon} B(\upsilon,r;\rho_N) \subset \bigcup_{\upsilon\in\Upsilon} B(\upsilon,R;\rho_N) \subset U$,

where ρ_N is defined as in the statement of Lemma 12.

Proof. By Lemma 12, there exists $\alpha \ge 1$ such that $\alpha^{-1}\rho_n \le \|\cdot\| \le \alpha\rho_n$ for every $n \in \mathbb{N}$. Let us define $L := 2\big[\alpha^2 + \gamma^{-1}\big]$. Set $R' = \frac{1}{2}\text{dist}(K, X\setminus U) > 0$, and $r' = (L\gamma)^{-1}R'$. For each point $\xi \in K$ we have that $B(\xi,r';\|\cdot\|) \subset B(\xi,R';\|\cdot\|) \subset U$. Since K is compact, there is a finite subset I of K such that

$$K \subset \bigcup_{\xi\in I} B(\xi,r';\|\cdot\|) \subset \bigcup_{\xi\in I} B(\xi,R';\|\cdot\|) \subset U.$$

Taking into account that $\frac{R'}{r'} = L\gamma = 2(\gamma\alpha^2 + 1)$, we may find an $\varepsilon > 0$ such that

$$\frac{R' - \varepsilon}{r' + \varepsilon} \geq \gamma\alpha^2 > 1.$$

Now let $N \in \mathbb{N}$ be such that $I \subset Y_N + \varepsilon B_X$ (such N exists because $(e_i)_{i=1}^{\infty}$ is a Schauder basis of X). For every $\xi \in I$, choose $\upsilon_\xi \in Y_N$ satisfying $\|\xi - \upsilon_\xi\| \leq \varepsilon$. The set $\Upsilon := \{\upsilon_\xi : \xi \in I\}$ is a finite subset of Y_N. Moreover,

$$K \subset \bigcup_{\xi \in I} B(\xi, r'; \|\cdot\|) \subset \bigcup_{\xi \in I} B(\upsilon_\xi, r' + \varepsilon; \|\cdot\|) \subset \bigcup_{\xi \in I} B(\upsilon_\xi, \alpha(r' + \varepsilon); \rho_N)$$
$$\subset \bigcup_{\xi \in I} B(\upsilon_\xi, \frac{R' - \varepsilon}{\alpha}; \rho_N) \subset \bigcup_{\xi \in I} B(\upsilon_\xi, R' - \varepsilon; \|\cdot\|) \subset \bigcup_{\xi \in I} B(\xi, R'; \|\cdot\|) \subset U.$$

Therefore we can finish the proof by setting $r := \alpha(r' + \varepsilon)$ and $R := \alpha^{-1}(R' - \varepsilon)$. □

Now we have all the tools we need to construct the noncomplete norm ω and the diffeomorphism F we need.

Proposition 14. *Let $(X, \|\cdot\|)$ be a Banach space with Schauder basis and with a C^p smooth bump function. Then, for every compact set $K \subset X$ and every open neighborhood $U \supset K$, there are a C^∞ smooth noncomplete norm ω on X, and a C^p diffeomorphism $F : X \to X$ such that $F(K) = K$ and $F(U)$ is a ω-neighborhood of K.*

Proof. Let $(e_i)_{i=1}^{\infty}$ be a normalized Schauder basis of $(X, \|\cdot\|)$, $(P_i)_{i=1}^{\infty}$ be the family of associated projections, and M_1 the basic constant. Since X has a C^p smooth bump function, there exists a C^p smooth bounded symmetric starlike body $A \subset X$. Choose a smooth square $V \subset \mathbb{R}^2$. Define the family $(\rho_n)_{n \geq 1}$, as in the statement of Lemma 12, that is

$$\rho_n(y, z) = \mu_V(\mu_A(y), \mu_A(z)).$$

By Lemma 12, there exists some $\alpha \geq 1$ satisfying $\alpha^{-1}\rho_n \leq \|\cdot\| \leq \alpha\rho_n$ for all $n \in \mathbb{N}$. Take $M_2 \geq 1$ such that $M_2^{-1}\|\cdot\| \leq \mu_A \leq M_2\|\cdot\|$. Define $M = M_1 + M_2 + \alpha$. Then we have

$$\begin{aligned} &\sup\{\|P_i\| : i \in \mathbb{N}\} \leq M; \\ &M^{-1}\rho_n \leq \|\cdot\| \leq M\rho_n, \quad \text{for each } n \in \mathbb{N}; \\ &M^{-1}\|\cdot\| \leq \mu_A \leq M\|\cdot\|. \end{aligned}$$

Now, for our given sets $K \subset U$, by applying Lemma 13 with $\gamma = 12M^6$, we get an $N \in \mathbb{N}$, a finite subset Υ of $Y_N := \mathrm{sp}\{e_i : 1 \leq i \leq N\}$, and numbers $0 < r < R$ satisfying:

(i) $R/r \geq \gamma$;
(ii) $K \subset \bigcup_{\upsilon \in \Upsilon} B(\upsilon, r; \rho_N) \subset \bigcup_{\upsilon \in \Upsilon} B(\upsilon, R; \rho_N) \subset U$,

By Lemma 12 again, we know that ρ_N is the Minkowski functional of a C^p smooth bounded starlike body such that the functions

$$(0, \infty) \ni t \to \rho_N(y, tz) \quad \text{and} \quad (0, \infty) \ni t \to \rho_N(ty, z) \text{ are nondecreasing.}$$

Next, choose numbers numbers δ, Δ with $0 < \delta < \Delta < \frac{1}{2}R$, and apply Lemma 6 to find a C^p smooth function $g : X = Y_N \oplus Z_N \longrightarrow [0, +\infty)$ such that

(iii) $g = 0$ on $\bigcup_{\upsilon\in\Upsilon} B(\upsilon, r; \rho_N) \cup \{(y, z) \in X : \rho_N(z) \leq \delta\}$;
(iv) $g = 1$ on $[X \setminus \bigcup_{\upsilon\in\Upsilon}(\upsilon, R; \rho_N)] \cap \{(y, z) \in X : \rho_N(z) \geq \Delta\}$;
(v) $t \in (0, \infty) \longrightarrow g(y + tz)$ is nondecreasing, for each $(y, z) \in Y_N \times Z_N$.

To save notation we will denote $\rho := \rho_N$. We are going to construct a C^∞ smooth noncomplete norm ω on $(X, \|\cdot\|)$ such that $\omega \leq \rho$. It is well known that every bounded symmetric convex body in $\mathbb{R}^n = Y_n$ can be approximated in the Hausdorff distance by C^∞ smooth bounded symmetric convex bodies. In particular, for the body $\mathrm{conv(A)} \cap Y_N \subset \mathbb{R}^N$, there exists a C^∞ smooth bounded symmetric convex body W_N in Y_N such that

$$\mathrm{conv}(A) \cap Y_N \subset W_N \subset (MB_X \cap Y_N).$$

In particular, since $M^{-1}B_X \subset A$, we know that

$$\mathrm{conv}(A) \cap Y_N \subset W \subset (MB_X \cap Y_N) = M^2(\frac{1}{M}B_X \cap Y_N) \subset$$
$$M^2(A \cap Y_N) \subset M^2(\mathrm{conv}(A) \cap Y_N) \tag{1}$$

Let $(e'_i)_{i\geq 1}$ be an orthonormal basis in ℓ_2. Consider the mapping $T : X \to \ell_2$ defined by

$$T(\sum_{i=1}^{\infty} x_i e_i) = \sum_{i\geq 1} \frac{x_i}{2^i} e'_i$$

for each $x = \sum_{i=1}^{\infty} x_i e_i \in X$. For every $j \in \mathbb{N}$ and $x \in X$ we have that $|x_j| \leq 2M\|x\|$. It follows that $\|T(x)\|_{\ell_2} \leq 2M\|x\|$ for every $x \in X$. Then T is an injective continuous linear map. Define now $\omega : X = Y_N \oplus Z_N \longrightarrow \mathbb{R}$ by

$$\omega(x) = \omega(y, z) = \frac{1}{3M^2}\mu_V(\mu_W(y), \|T(z)\|_{\ell_2}).$$

Taking into account the properties of V stated in Lemma 10, it is clear that ω is a C^∞ smooth norm on X. To see that ω is not complete it is enough to consider the sequence $(2^{\frac{i}{2}}e_i)_{i>N}$, which is not bounded in $(X, \|\cdot\|)$ and yet

$$\omega\big(2^{\frac{i}{2}}e_i\big) = \omega\big(0, 2^{\frac{i}{2}}e_i\big) = \frac{1}{3M^2}\mu_V(0, 2^{-\frac{i}{2}}) = \frac{1}{3M^2}2^{-\frac{i}{2}}$$

for each $i > N$, so the sequence $(2^{\frac{i}{2}}e_i)_{i>N}$ is bounded in (X, ω). Therefore ω is not equivalent to $\|\cdot\|$, that is, ω is not complete. Finally, let us check that $\omega \leq \rho$. Let us first see that

$$\mu_W(y) \leq M^2\rho(y, z) \text{ for each } (y, z) \in Y_N \times Z_N.$$

Indeed, if $\rho(y, z) \leq 1$ then $\|y\| \leq M\mu_A(y) \leq M\mu_V(\mu_A(y), \mu_A(z)) \leq M$. Since $M^{-1}B_X \cap Y_N \subset W$, we get that $M^{-2}y \in W$, and $\mu_W(y) = M^2\mu_W(\frac{y}{M^2}) \leq M^2$.

Hence $\mu_W(y) \leq M^2\rho(y,z)$. Now, for any $(y,z)\times Y_N\times Z_N$ we have that

$$\begin{aligned}
&\mu_V(\mu_W(y), \|T(z)\|_{\ell_2}) \leq \mu_W(y) + \|T(z)\|_{\ell_2}\\
&\leq M^2\rho(y,z) + 2M\|z\| \leq M^2\rho(y,z) + 2M^2\mu_A(z)\\
&= M^2\rho(y,z) + 2M^2\mu_V(0,\mu_A(z)) \leq 3M^2\rho(y,z),
\end{aligned}$$

and therefore $\omega \leq \rho$.

From Lemma 10 and the definition of ω, it is obvious that the functions

$$t\in(0,\infty)\to\omega(ty,z) \text{ and } t\in(0,\infty)\to\omega(y,tz) \text{ are nondecreasing,} \tag{2}$$

for every $(y,z)\in Y_N\oplus Z_N$. By Lemma 7, and according to properties (iii)-(iv)-(v) above, the mapping $F: X = Y_N\oplus Z_N \longrightarrow X$ defined by

$$F(x) = F(y,z) = \begin{cases} (y, [g(x)\frac{\rho(z)}{\omega(z)} + 1 - g(x)]z) \text{ if } z\neq 0,\\ (y,z) \text{ if } z = 0\end{cases}$$

is a C^p diffeomorphisms.

It only remains to check that $F(K) = K$ and $F(U)$ is an ω-neighborhood of K. From property (iii) above it follows that $g = 0$ en $\bigcup_{\upsilon\in\Upsilon} B(\upsilon, r;\rho)\cup\{(y,z)\in X : \rho(z)\leq\delta\}\supset K$, hence F is the identity on K. Let us show that $F(U)$ is an ω-neighborhood of K. Take $x = (y,z)\in X$ and a vector $\upsilon\in\Upsilon$ so that $\rho(x-\upsilon) = \rho(y-\upsilon, z)\geq R$. Then

$$\|\big(\mu_A(y-\upsilon),\mu_A(z)\big)\|_\infty \geq \frac{1}{2}\mu_V\big(\mu_A(y-\upsilon),\mu_A(z)\big) = \frac{1}{2}\rho(y-\upsilon,z)\geq\frac{R}{2}. \tag{3}$$

We now have to consider three cases.

First case. Assume that $\rho(z)\leq\Delta$. Then we get

$$\mu_A(z) = \mu_V(0,\mu_A(z)) = \rho(z)\leq\Delta<\frac{R}{2}$$

and, bearing (3) in mind, we deduce that

$$\mu_A(y-\upsilon)\geq\frac{R}{2}. \tag{4}$$

Now, by combining (1), (2), (3), (4), and the inclusion $\mathrm{conv}(A)\subset M^2A$, we may estimate as follows

$$\begin{aligned}
&\omega(F(x)-\upsilon) = \omega(y-\upsilon, [g(x)\frac{\rho(z)}{\omega(z)} + 1 - g(x)]z) \geq \omega(y-\upsilon, 0)\\
&= \frac{1}{3M^2}\cdot\mu_V\big(\mu_W(y-\upsilon),0\big) = \frac{1}{3M^2}\cdot\mu_W(y-\upsilon)\\
&\geq\frac{1}{3M^4}\cdot\mu_{\mathrm{conv}(A)\cap Y_N}(y-\upsilon) = \frac{1}{3M^4}\cdot\mu_{\mathrm{conv}(A)}(y-\upsilon)\\
&\geq\frac{1}{3M^6}\cdot\mu_A(y-\upsilon)\geq\frac{R}{2}\cdot\frac{1}{3M^6} = \frac{R}{6M^6} > r.
\end{aligned}$$

Second case. If $\rho(z)\geq\Delta$ and $\mu_A(y-\upsilon)\geq\frac{1}{2}R$, we may copy the estimation just done to see that $\omega(F(x)-\upsilon) > r$.

Third case. Suppose that $\rho(z) \geq \Delta$ and $\mu_A(y-\upsilon) < \frac{1}{2}R$. Then, from (3) it follows that $\mu_A(z) \geq \frac{R}{2}$, and

$$\omega(F(x)-\upsilon) = \omega(y-\upsilon, \frac{\rho(z)}{\omega(z)}) \geq \omega(0, \frac{\rho(z)}{\omega(z)})$$
$$= \rho(z) = \mu_V(0, \mu_A(z)) = \mu_A(z) \geq \frac{R}{2} > r.$$

From this discussion we conclude $F\big(X \setminus \bigcup_{\upsilon\in\Upsilon} B(\upsilon, R; \rho)\big) \subset X \setminus \bigcup_{\upsilon\in\Upsilon} B(\upsilon, r; \omega)$, that is,

$$\bigcup_{\upsilon\in\Upsilon} B(\upsilon, r; \omega) \subset F\big(\bigcup_{\upsilon\in\Upsilon} B(\upsilon, R; \rho)\big).$$

We have thus shown that

$$K \subset \bigcup_{\upsilon\in\Upsilon} B(\upsilon, r; \rho) \subset \bigcup_{\upsilon\in\Upsilon} B(\upsilon, r; \omega) \subset F\big(\bigcup_{\upsilon\in\Upsilon} B(\upsilon, R; \rho)\big) \subset F(U),$$

and $F(U)$ is an ω-neighborhood of K. $\square$

Proof of Theorem 1

(1) $\Rightarrow$ (2). It is enough to combine Corollary 4 with Proposition 14.
(2) $\Rightarrow$ (1). Take U an open bounded set containing the origin. By the hypothesis there exists a C^p diffeomorphism $h : X \to X \setminus \{0\}$ such that h is the identity off U. Consider the mapping $g : X \to X$ defined as $g(x) = h(x) - x$ for each $x \in X$. The mapping g is C^p smooth, it vanishes outside U, and $g(0) = h(0) \neq 0$. Choose a continuous linear functional $x^* \in X^*$ satisfying $x^*(g(0)) \neq 0$, and define $b := x^* \circ g$. Then b is a C^p smooth bump function on X. $\square$

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