SMOOTH CONVEX EXTENSIONS OF CONVEX FUNCTIONS

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ABSTRACT. Let C be a compact convex subset of $\mathbb{R}^n$, $f : C \to \mathbb{R}$ be a convex function, and $m \in \{1, 2, ..., \infty\}$. Assume that, along with f, we are given a family of polynomials satisfying Whitney's extension condition for C^m, and thus that there exists $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C. It is natural to ask for further (necessary and sufficient) conditions on this family of polynomials which ensure that F can be taken to be convex as well. We give a satisfactory solution to this problem in the case $m = \infty$, and make some remarks about the case of finite $m \geq 2$. For a solution to a similar problem in the case $m = 1$ (even for C not necessarily convex), see our preprint arXiv:1507.03931 [math.CA].

1. INTRODUCTION AND MAIN RESULTS

Let C be a closed subset of $\mathbb{R}^n$, and $m \in \mathbb{N}$. The famous Whitney Extension Theorem [27] provides a necessary and sufficient condition (W^m) for a function $f : C \to \mathbb{R}$ and a family of polynomials $P_y : \mathbb{R}^n \to \mathbb{R}$ of degree$(P_y) \leq m$ and such $P_y(y) = f(y)$ for every $y \in C$ (what we might call the would-be Taylor polynomial of f of order m at y) to admit a C^m extension F to all of $\mathbb{R}^n$ such that $J^m_y F = P_y$ for each $y \in C$, where $J^m f(x)$ denotes the Taylor polynomial of order m of F at y. Whitney's condition (W^m) can be reformulated by saying that

$$\lim_{\delta \to 0^+} \rho_m(K, \delta) = 0 \text{ for each compact subset } K \text{ of } C, \tag{1.1}$$

where we denote

$$\rho_m(K, \delta) = \sup\{\frac{\|D^j P_y(z) - D^j P_z(z)\|}{|y - z|^{m-j}} \ : \ j = 0, ..., m, \ y, z \in K, 0 < |y-z| \leq \delta\}.$$

If this condition is met, then Whitney's theorem provides us with a function $F \in C^m(\mathbb{R}^n)$ such that $D^j F(y) = D^j P_y(y)$ for every $j = 0, ..., m$ and $y \in C$; see [9, Theorem 3.1.14, p. 225] for instance. The converse is trivially true.

In the case $m = \infty$, Whitney's theorem states that if we are given a family of polynomials $\{P^m_y\}_{y \in C, m \in \mathbb{N} \cup \{0\}}$ such that $P^m_y(y) = f(y)$ and for

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every $k > j$ the polynomial P_y^j is the Taylor polynomial of order j at y of the polynomial P_y^k (let us call such a family *a compatible family of polynomials for C^∞ extension* of a function f defined on C), and if for each $m \in \mathbb{N}$ the subfamily $\{P_y^m\}_{y\in C}$ satisfies Whitney's condition (1.1), then there is a function $F \in C^\infty(\mathbb{R}^n)$ such that $P_y^m = J_y^m F$), for every $y \in C$ and $m \in \mathbb{N}$. Again, the converse is obviously true.

In recent years there has been great interest in solving Whitney-type extension theorems for functions rather than jets, in constructing continuous linear extension operators with nearly optimal norms, and in extending these results to other spaces of functions such as Sobolev spaces, see [17, 6, 3, 10, 11, 4, 12, 22, 19, 13, 23] and the references therein.

Returning to Whitney's theorem, it is natural to wonder what further conditions (if any) on those families of polynomials would be necessary and sufficient to ensure that F can be taken to be convex whenever f is convex. Besides its basic character, one should expect that a solution to this problem would find interesting applications in problems of differential geometry (see [14] and the references therein, and also [1, Theorem 1.8]), and of partial differential equations (such as the Monge-Ampère equations).

Let us begin by making a couple of general observations concerning solvability of our extension problem. Firstly, if C is not assumed to be compact, it is known that our problem has a negative solution. Indeed, there exists an unbounded closed convex subset C of $\mathbb{R}^2$ and a C^∞ convex function $f : C \to \mathbb{R}$ which has no *continuous* convex extension to all of $\mathbb{R}^2$, see [24, Example 4]. A modification of this example, which we will present in Section 4 below, shows that the obstruction persists even if we require that f have a strictly positive Hessian on a neighbourhood of C (such strongly convex functions f have smooth convex extensions to small open neighborhoods of C, but no convex extensions to $\mathbb{R}^n$). See also [5, 26], which show that there are infinite-dimensional Banach spaces X, closed *subspaces* $E \subset X$ and continuous convex functions $f : E \to \mathbb{R}$ which have no continuous convex extensions to X.

Secondly, if we do not require that C be convex, then the problem gets geometrically complicated, for the following reason. There are several possible, nonequivalent, definitions of convex functions defined on non-convex domains (see [28] for a study of three of them) but, no matter how one defines convexity of such functions, the problem cannot be solved just by adding further analytical conditions on the would-be Taylor polynomials of f and disregarding the global geometry of the graph of f. To see why this is so, let us consider the following example: take any four numbers $a, b, c, d \in \mathbb{R}$ with $a < b < 0 < c < d$, and define $C = \{a, b, 0, c, d\}$ and $f(x) = |x|$ for $x \in C$. Since C is a five-point set it is clear that, no matter what polynomials of degree up to $k \geq 1$ are chosen to be the differential data of f on C, the function f will satisfy Whitney's extension condition (W^k) for every $k \in \mathbb{N}$. Hence there are many C^1 (even infinitely many C^∞) functions F with $F = f$ on C. But none of these F can be convex on $\mathbb{R}$, because, as is easily checked, any convex extension g of f to $\mathbb{R}$ must satisfy $g(x) = |x|$ for every $x \in [a, d]$, and therefore g cannot be differentiable at 0. This example also shows that the most general forms of the extension problem for smooth convex functions are different in nature from the classical Whitney extension theorem [27] (for jets) and from the Whitney extension problems (for functions) dealt with in the mentioned papers [17, 6, 3, 10, 11, 4, 12, 22, 19, 13, 23], which are all of a local character.

Fortunately, there is evidence that the geometrical obstructions shown by these examples no longer exist when C is assumed to be compact and convex. In particular, it is clear that if f is convex on a compact convex set C and is C^1 on a neighbourhood of C then

$$m(f)(x) := \max_{y \in C}\{f(y) + \langle \nabla f(y), x - y\rangle\} \tag{1.2}$$

defines a Lipschitz, convex function on all of $\mathbb{R}^n$ which coincides with f on C (and that, in the case when C has nonempty interior, $m(f)$ happens to be the minimal convex extension of f to $\mathbb{R}^n$).

Therefore, at least in a first approach to the problem, it seems reasonable to assume that C is convex and compact, which we will do in the rest of this paper[1], and ask ourselves if our extension problem can always be solved in this relatively simple case. Extension problems related to the one we are dealing with have been considered by M. Ghomi [15] and by M. Yan [28]. A consequence of their results is that, under the assumptions that $m \geq 2$ and that f has a strictly positive Hessian on the boundary ∂C, there always exists an $F \in C^m(\mathbb{R}^n)$ such that F is convex and $F = f$ on C. See also [14, 16, 7] for related problems. Of course, strict positiveness of the Hessian is a very strong condition which is far from being necessary, and it would be desirable to get rid of this requirement altogether, if possible. However, some other assumptions must be made in its place, at least when $m \geq 3$, as already in one dimension there are examples of C^3 convex functions g defined on compact intervals I which cannot be extended to $C^3(J)$ convex functions for any open interval J containing I. Such an example is $g(x) = x^2 - x^3$ defined for $x \in I := [0, \frac{1}{3}]$. This example obviously generalizes to arbitrary dimension n by considering for instance

$$f(x_1, ..., x_n) = x_1^2 + \cdots + x_n^2 - x_1^3, \ \ (x_1, ..., x_n) \in B(0, 1/3). \tag{1.3}$$

In particular these examples show that the condition $D^2 f \geq 0$ on C is not sufficient to ensure the existence of a convex function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C for any $m \geq 3$. Therefore, we should look for conditions on the derivatives of f on C (beyond $D^2 f \geq 0$ on C) that are necessary and sufficient to guarantee that f has a C^m convex extension F to all of $\mathbb{R}^n$.

Now, observe that any such function F will satisfy that $D^2F(x)(v^2) \geq 0$ for every $x \in \mathbb{R}^n$, $v \in \mathbb{S}^{n-1}$, and therefore, if $m \geq 2$ is finite, the Taylor

[1] Nonetheless, in the special case $m = 1$, even for not necessarily convex C, we have found in [1] two global geometrical conditions which, along with (W^1), are necessary and sufficient for the existence of convex functions $F \in C^1(\mathbb{R}^n)$ such that $F = f$ on C.

polynomial of the second derivative D^2F at points $y \in C$ will also satisfy

$0 \leq D^2F(y+tw)(v^2) =$

$$D^2F(y)(v^2) + t\, D^3F(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m F(y)(w^{m-2}, v^2) + R_m(t, y, v, w),$$

where

$$\lim_{t \to 0^+} \frac{R_m(t, y, v, w)}{t^{m-2}} = 0 \text{ uniformly on } y \in C, w, v \in \mathbb{S}^{n-1}.$$

Then we will also have

$$\liminf_{t \to 0^+} \frac{1}{t^{m-2}} \left(D^2F(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m F(y)(w^{m-2}, v^2) \right) \geq 0$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$. This of course means that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$D^2F(y)(v^2) + t\, D^3F(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m F(y)(w^{m-2}, v^2) \geq -\varepsilon t^{m-2}$$

for all $y \in C,\ v, w \in \mathbb{S}^{n-1},\ 0 < t \leq t_\varepsilon$. We will abbreviate this by saying that

$$F \text{ satisfies condition } (CW^m) \text{ on } C.$$

Therefore we obtain the following necessary condition for the solution of the convex C^m extension problem.

Definition 1.1. *Let $m \in \mathbb{N}$, $m \geq 2$. We will say that f, together with a family of polynomials $\{P_y^m\}_{y \in C}$ of degree up to m such that $P_y^m(y) = f(y)$, satisfy the condition (CW^m) provided that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that*

$$D^2P_y^m(y)(v^2) + tD^3P_y^m(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m P_y^m(y)(w^{m-2}, v^2) \geq -\varepsilon t^{m-2}$$

for all $y \in C,\ v, w \in \mathbb{S}^{n-1},\ 0 < t \leq t_\varepsilon$.

We will also say that f and $\{P_y^m\}_{y \in C}$ satisfy (CW^m) with a strict inequality *if there are some $\eta > 0$ and $t_0 > 0$ such that*

$$D^2P_y^m(y)(v^2) + tD^3P_y^m(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m P_y^m(y)(w^{m-2}, v^2) \geq \eta t^{m-2}$$

for all $y \in C,\ v, w \in \mathbb{S}^{n-1},\ 0 < t \leq t_0$.

In the case that C has nonempty interior, the polynomials P_y^m are uniquely determined (even at the boundary points of C) by the values of f on C, and the above condition may be reformulated as follows

$$\liminf_{t \to 0^+} \frac{1}{t^{m-2}} \left(D^2f(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2) \right) \geq 0 \tag{CW^m}$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$, understanding that D^jf denotes the derivative of order j at y of any C^m extension of f to $\mathbb{R}^n$.

One might then think that for our convex extension problem, by considering the relative interior of the convex compact set C, there would be no loss of generality in assuming that C has nonempty interior (and thereforem considering that (CW^m) holds only for v, w in the linear span of the directions $y-y'$ with $y, y' \in C$). However, since we are looking for convex analogues of the classical Whitney's extension theorem (which deals with prescribing differential data as well as extending functions) such an approach would make us lose some valuable insight about the question as to what extent one can prescribe values and derivatives of convex functions on a given compact convex set with empty interior. Indeed, for a convex compact set C with empty interior and a convex function $f : C \to \mathbb{R}$, there are infinitely many convex functions $F : C \to \mathbb{R}$ with very different derivatives on C and such that $F = f$ on C. Let us look, for instance, at the extreme situation in which C is a singleton, say $C = \{0\}$. One of our main results in this paper (see Theorem 1.3 below) implies that, for any given family of polynomials P^m of degree up to m, $m \in \mathbb{N}$, such that $J_0^k P^m = P^k$ whenever $k \leq m$ and, for each $m \geq 2$,

$$\liminf_{t \to 0^+} \frac{D^2 P^m(0)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m P^m(0)(w^{m-2}, v^2)}{t^{m-2}} \geq 0$$

uniformly on $|v| = |w| = 1$, there exists a convex function F of class C^∞ such that the Taylor polynomial of F at 0 is P^m. Consequently, there are infinitely many degrees of freedom in prescribing derivatives of convex functions at a given point.

On the other hand, if C is a convex compact set with nonempty interior (what is usually called a convex body) and $f : C \to \mathbb{R}$ is a convex function which has a (not necessarily convex) C^m extension to an open neighbourhood of C, then it is clear that f automatically satisfies (CW^m) on the interior of C. Conversely, if f satisfies (CW^m) on the interior of C then it immediately follows, using Taylor's theorem, that $D^2 f(x) \geq 0$ for all x in the open convex set $\text{int}(C)$, hence f is convex on $\text{int}(C)$, and by continuity we infer that f is also convex on C. These observations show that if C is a convex compact subset of $\mathbb{R}^n$, $m \in \mathbb{N}, m \geq 2$, and $f : C \to \mathbb{R}$ is a function then:

(1) A necessary condition for f to have a C^m convex extension to all of $\mathbb{R}^n$ is that f satisfies both (W^m) and (CW^m) on C.
(2) In the event that one can show that this requirement is sufficient as well, then one also has that a necessary and sufficient condition for a convex function $f : C \to \mathbb{R}$ to have a C^m convex extension to all of $\mathbb{R}^n$ is that f satisfy (W^m) on C, and (CW^m) on the boundary ∂C.

One can also easily show the following.

Remark 1.2. *Let f and $\{P_y^{m+1}\}_{y \in C}$ satisfy (CW^{m+1}) for some $m \geq 2$. Then f and $\{P_y^m\}_{y \in C}$ satisfy (CW^m) too, where each P_y^m is obtained from P_y^{m+1} by discarding its $(m+1)$-homogeneous terms.*

Our first main result is as follows.

Theorem 1.3. *Let C be a compact convex subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a function, and let $\{P_y^m\}_{y\in C, m\in\mathbb{N}}$ be a compatible family of polynomials for C^∞ extension of f. Then f has a convex, C^∞ extension F to all of $\mathbb{R}^n$, with $J_y^m F = P_y^m$ for every $y \in C$ and $m \in \mathbb{N}$, if and only if $\{P_y^m\}_{y\in C}$ satisfies (W^m) and (CW^m) on C, for every $m \in \mathbb{N}$, $m \geq 2$.*

Moreover, if C has nonempty interior and $f : C \to \mathbb{R}$ is convex, then f has a convex, C^∞ extension F to all of $\mathbb{R}^n$, with $J_y^m F = P_y^m$ for every $y \in C$ and $m \in \mathbb{N}$, if and only if $\{P_y^m\}_{y\in C}$ satisfies (W^m) on C, and (CW^m) on ∂C, for every $m \in \mathbb{N}$, $m \geq 2$.

If $m \geq 2$ finite and $n \geq 2$, if C has empty interior then conditions (CW^m) and (W^m) are not sufficient for a convex function $f : C \to \mathbb{R}$ to have a C^m convex extension to $\mathbb{R}^n$; see Example 4.2 in section 4 below. However, it is conceivable that these conditions might be sufficient in the case when C has nonempty interior. As of now, we only know that in dimension $n = 1$ this is indeed so, and moreover, since in this case the boundary of C has only two points and there are only two directions in which to differentiate, the condition (CW^m) can be very much simplified.

Proposition 1.4. *Let I be a closed interval in $\mathbb{R}$, and $m \in \mathbb{N}$ with $m \geq 2$. Let $f : I \to \mathbb{R}$ be a convex function of class C^m in the interior of I, and assume that f has one-sided derivatives of order up to m, denoted by $f^{(k)}(a^+)$ or $f^{(k)}(b^-)$, at the extreme points of I. Then f has a convex extension of class $C^m(\mathbb{R})$ if and only if the first (if any) non-zero derivative which occurs in the finite sequence $\{f^{(2)}(b^-), f^{(3)}(b^-), ..., f^{(m)}(b^-)\}$ is positive and of even order, and similarly for $\{f^{(2)}(a^+), f^{(3)}(a^+), ..., f^{(m)}(a^+)\}$.*

The easy proof is left to the reader's care.

In the special case when condition (CW^k) is satisfied with a strict inequality for some k, the problem also becomes much easier to solve, because in this situation f must be convex on a neighbourhood of C, and then we may use the following by-product of our proof of Theorem 1.3

Proposition 1.5. *Let $m \in \mathbb{N}$. If $C \subset \mathbb{R}^n$ is compact, and if there exists an open convex neighbourhood U of C such that $f : U \to \mathbb{R}$ is C^m and convex, then there exists a convex function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C.*

See subsection 2.9 below for more information. Let us also note that in this case f automatically satisfies (CW^p) for all the rest of p's.

Proposition 1.6. *Let $m \in \mathbb{N}\cup\{\infty\}$, $m \geq 2$. If $f \in C^m(\mathbb{R}^n)$ satisfies (CW^k) with a strict inequality on ∂C for some $k \geq 2$, then f satisfies (CW^p) on ∂C for every $p \in \{2, ..., m\}$, if m is finite, and for every $p \in \mathbb{N}$ with $p \geq 2$, if $m = \infty$.*

We leave the easy verification to the reader's care. As a straightforward consequence of Proposition 1.5 we will obtain the following.

Corollary 1.7. *Let $m \in \mathbb{N} \cup \{\infty\}$, $m \geq 2$. Let C be a convex compact subset of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}$ be a convex function having a (not necessarily convex) C^m extension to an open neighbourhood of C. If f and its derivatives satisfy (CW^k) with a strict inequality on C for some $2 \leq k \leq m$, then there exists a convex function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C.*

The easiest instance of application of this Corollary is of course when f has a strictly positive Hessian on ∂C, in which case we recover the aforementioned consequence of the results of M. Ghomi's [15] and M. Yan's [28].

In the case $m \geq 2$ with m finite, the method of proof of Theorem 1.3 does not allow us to obtain sufficiency of the conditions (W^m) and (CW^m) for a function f to have a convex C^m extension. The best we can obtain with this method is the following.

Theorem 1.8. *Let C be a compact convex subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a function, $m \in \mathbb{N}$ with $m \geq n+3$, and let $\{P_y^m\}_{y \in C}$ be a family of polynomials of degree less than or equal to m and $P_y^m(y) = f(y)$ for every $y \in C$. Assume that $\{P_y^m\}_{y \in C}$ satisfies (W^m) and (CW^m). Then f has a convex extension $F \in C^{m-n-1}(\mathbb{R}^n)$ such that $J_y^{m-n-1}F = P_y^{m-n-1}$ for every $y \in C$.*

The above result is probably not optimal, at least in the case when C has nonempty interior. However, Example 4.2 below will show that if C has empty interior then one cannot expect to find smooth convex extensions of jets satisfying (W^m) and (CW^m) on C without losing at least two orders of smoothness. On the positive side, there is a class of relatively nice convex bodies for which Theorem 1.8 can be very much improved.

Definition 1.9 (FIO bodies of class m)**.** *Given an integer $m \geq 2$, we will say that a subset C of $\mathbb{R}^n$ is an ovaloid of class C^m if there exist $M > 0$ and a function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *ψ is of class $C^m(\mathbb{R}^n)$.*
(ii) *$D^2\psi(x)(v^2) \geq M$ for all $x \in \mathbb{R}^n$ and for all $v \in \mathbb{S}^{n-1}$.*
(iii) *$C = \psi^{-1}(-\infty, 1]$.*

We will also say that a set C is (FIO^m), or an FIO body of class C^m, if C is the intersection of a finite family of ovaloids of class C^m.

By restricting our attention to the class of FIO bodies, we can find convex extensions of functions satisfying (W^m) and (CW^m) with a loss of just one order of smoothness.

Theorem 1.10. *Let C be a convex subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a function, $m \in \mathbb{N}$ with $m \geq 3$, and let $\{P_y^m\}_{y \in C}$ be a family of polynomials of degree less than or equal to m and $P_y^m(y) = f(y)$ for every $y \in C$. Assume that $\{P_y^m\}_{y \in C}$ satisfies (W^m) and (CW^m), and that C is (FIO^{m-1}). Then f has a convex extension $F \in C^{m-1}(\mathbb{R}^n)$ such that $J_y^{m-1}F = P_y^{m-1}$ for every $y \in C$.*

Let us conclude this introduction with an important remark: one might wonder whether the conditions (CW^m) could be deduced from the condition $D^2 f \geq 0$ on C, at least in the case that C has nonempty interior. The answer is negative: in view of Theorem 1.10 and the example given in equation (1.3) above, the condition $D^2 f \geq 0$ on a convex body C does not imply condition (CW^m) on C for any $m \geq 4$. Furthermore, by making some straightforward calculations on can show that the function f defined in (1.3) does not satisfy condition (CW^3) either.

The rest of the paper is organized as follows. In Section 2 we will prove Theorem 1.3 and Corollary 1.7. In Section 3 we will prove Theorem 1.10. Finally, in Section 4 we will make some remarks and present some counterexamples related to extension problems for convex functions.

2. C^∞ CONVEX EXTENSIONS

In this section we will prove Theorem 1.3. By using Whitney's extension theorem we may and do assume that $f \in C^\infty(\mathbb{R}^n)$, with $J^m_y f = P^m_y$ for all $m \in \mathbb{N}$ and all $y \in C$, and that f satisfies condition (CW^m) on C for every $m \in \mathbb{N}$. We may also assume that f has a compact support contained in $C + B(0, 2)$.

2.1. **Idea of the proof.** Let us give a rough sketch of the proof so as to guide the reader through the inevitable technicalities. We warn the reader, however, that what we now say we are going to do is not exactly what we will actually do. Our proof could be rewritten to match this sketch exactly, but at the cost of adding further technicalities, which we do not feel would be pertinent. This proof has two main parts. In the first part we will estimate the possible lack of convexity of f outside C: by using the conditions (CW^m), a Whitney partition of unity, and some ideas from the proof of the Whitney extension theorem in the C^∞ case, we will construct a function $\eta \in C^\infty(\mathbb{R})$ such that $\eta \geq 0$, $\eta^{-1}(0) = (-\infty, 0]$, and $\min_{|v|=1} D^2 f(x)(v^2) \geq -\eta\,(d(x, C))$ for every $x \in \mathbb{R}^n$. In the second part of the proof we will compensate the lack of convexity of f outside C with the construction of a function $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi \geq 0$, $\psi^{-1}(0) = C$, and $\min_{|v|=1} D^2\psi(x)(v^2) \geq 2\eta\,(d(x, C))$. Then, by setting $F := f + \psi$ we will conclude the proof of Theorem 1.3.

There are many ways to construct such a function ψ. The essential point is to write C as an intersection of a family of half-spaces, and then to make a weighted sum, or an integral, of suitable convex functions composed with the linear forms that provide those half-spaces. If the sequence of linear forms is equi-distributed, in the weighted sum approach, or if one uses a measure equivalent to the standard measure on $\mathbb{S}^{n-1}$, in the integral approach, then the different functions ψ produced by these methods will have equivalent convexity properties. See [2] for an instance of the weighted sum approach, and [16, Proposition 2.1] for the integral approach. Of course our situation is more complicated than that of these references, as we need to find quantitative estimations of the convexity of ψ outside C which are good enough to outweigh our previous estimations of the lack of convexity of f outside C. It turns out that, in the present C^∞ case, this goal can be achieved with either method of construction of ψ. Here we will follow the integral approach of Ghomi's in [16, Proposition 2.1], as it will lead us to easier calculations.

2.2. **First lower estimates for the Hessian of f: the numbers $\{r_m\}_m$.** We next show how the assumption of conditions (CW^m) for every $m \geq 2$ implies a lower bound for the Hessian of f in terms of the distance to C.

Lemma 2.1. *Given $m \in \mathbb{N}$ if $f \in C^m(\mathbb{R}^n)$ and f satisfies (CW^m) then there is a number $r_m > 0$ such that, whenever $d(x, C) \leq r_m$, we have*

$$D^2 f(x)(v^2) \geq -d(x,C)^{m-2}, \quad \textit{for all} \quad v \in \mathbb{S}^{n-1}.$$

Proof. Given $x \in \mathbb{R}^n \setminus C$, $|v| = 1$, $t := d(x, C)$, let y be the unique point of C with the property that $d(x, C) = |x - y|$. Take $w = (x - y)/|x - y|$. We have $y = x + tw$. By Taylor's Theorem, we can write

$$\begin{aligned} D^2 f(x)(v^2) =& D^2 f(y)(v^2) + t\, D^3 f(y)(w, v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2) \\ &+ \frac{t^{m-2}}{(m-2)!} \left[D^m f(y + sw)(w^{m-2}, v^2) - D^m f(y)(w^{m-2}, v^2) \right], \end{aligned}$$

for some $s \in [0, t]$. Since f satisfies the condition $(CW)^m$, there exists a positive number r_m, independent of y, v and w, for which

$$\inf_{0<r\leq r_m} \left\{ \frac{D^2 f(y)(v^2) + r D^3 f(y)(w, v^2) + \cdots + \frac{r^{m-2}}{(m-2)!} D^m f(y)(w^{m-2}, v^2)}{r^{m-2}} \right\} \geq -\frac{1}{2}.$$

Thus, for $0 < t \leq r_m$,

$$D^2 f(x)(v^2) \geq -\frac{t^{m-2}}{2} + \frac{t^{m-2}}{(m-2)!} \left[D^m f(y + sw)(w^{m-2}, v^2) - D^m f(y)(w^{m-2}, v^2) \right].$$

On the other hand, if $s \in [0, t]$, we can write

$$D^m f(y + sw)(w^{m-2}, v^2) - D^m f(y)(w^{m-2}, v^2) \leq \|D^m f(x + sw) - D^m f(y)\|,$$

where we denote $\|A\| := \sup_{u_i \in \mathbb{S}^{n-1}} |A(u_1, \ldots, u_m)|$, for every m-linear form A on $\mathbb{R}^n$. Moreover, the above expression is smaller than or equal to

$$\varepsilon_m(t) := \sup_{\{z \in \mathbb{R}^n,\, z' \in \partial C,\, |z - z'| \leq t\}} \|D^m f(z) - D^m f(z')\|.$$

Since $D^m f$ is uniformly continuous, there is $r'_m > 0$ such that if $0 < r \leq r'_m$, then $\varepsilon_m(r) \leq \frac{1}{2}$ (in fact we have $\lim_{r \to 0^+} \varepsilon_m(r) = 0$). Therefore, if we suppose $0 < t \leq \min\{r_m, r'_m\}$, we obtain

$$D^2 f(x)(v)^2 \geq -\frac{t^{m-2}}{2} - \frac{t^{m-2}}{(m-2)!} \varepsilon_m(t) \geq -t^{m-2}.$$

□

2.3. **A Whitney partition of unity on $(0,+\infty)$.** For all $k\in\mathbb{Z}$, we define the closed intervals

$$I_k=[2^k,2^{k+1}],\quad I_k^*=\left[\frac{3}{4}2^k,\frac{9}{8}2^{k+1}\right].$$

Obviously $(0,+\infty)=\bigcup_{k\in\mathbb{Z}}I_k$. We note that I_k and I_k^* have the same midpoint and $\ell(I_k^*)=\frac{3}{2}\ell(I_k)$, where $\ell(I_k)=2^k$ denotes the length of I_k. In other words, the interval I_k^* is I_k expanded by the factor $3/2$.

Proposition 2.2. *The intervals I_k, I_k^* satisfy:*

1. *If $t\in I_k^*$, then*

$$\frac{3}{4}\ell(I_k)\leq t\leq\frac{9}{4}\ell(I_k).$$

2. *If I_k^* and I_j^* are not disjoint, then*

$$\frac{1}{2}\ell(I_k)\leq\ell(I_j)\leq 2\ell(I_k).$$

3. *Given any $t>0$, there exists an open neighbourhood $U_t\subset(0,+\infty)$ of t such that U_t intersects at most 2 intervals of the collection $\{I_k^*\}_{k\in\mathbb{Z}}$.*

This is a special case of the decomposition of an open set in Whitney's cubes, see [25, Chapter VI] for instance. In the one dimensional case things are much simpler and, for instance, it is easy to see that one may replace the number $N=12$ in [25, Proposition VI.1.2, p. 169] with the number 2. Anyhow, dealing with the number 12 instead of 2 would have no harmful effect in our proof.

We now relabel the families $\{I_k\}_k$ and $\{I_k^*\}_k$, $k\in\mathbb{Z}$, as sequences indexed by $k\in\mathbb{N}$, so we will write $\{I_k\}_{k\geq 1}$ and $\{I_k^*\}_{k\geq 1}$. For every $k\geq 1$, we will denote by t_k and ℓ_k the midpoint and the length of I_k, respectively.

Next we recall how to define a Whitney partition of unity subordinated to the intervals I_k^*. Let us take a bump function $\theta_0\in C^\infty(\mathbb{R})$ with $0\leq\theta_0\leq 1$, $\theta_0=1$ on $[-1/2,1/2]$; and $\theta_0=0$ on $\mathbb{R}\setminus(-\frac{3}{4},\frac{3}{4})$. For every k, we define the function θ_k by

$$\theta_k(t)=\theta_0\left(\frac{t-t_k}{\ell_k}\right),\quad t\in\mathbb{R}.$$

It is clear that $\theta_k\in C^\infty(\mathbb{R})$, that $0\leq\theta_k\leq 1$, that $\theta_k=1$ on I_k, and that $\theta_k=0$ outside $\operatorname{int}(I_k^*)$.

Now we consider the function $\Phi=\sum_{k\geq 1}\theta_k$ defined on $(0,+\infty)$. Using Proposition 2.2, every point $t>0$ has an open neighbourhood which is contained in $(0,+\infty)$ and intersects at most two of the intervals $\{I_k^*\}_k$. Since $\operatorname{supp}(\theta_k)\subset I_k^*$, the sum defining Φ has only two terms and therefore Φ is of class C^∞. For the same reason, $\Phi(t)=\sum_{I_k^*\ni t}\theta_k(t)\leq 2$, for $t>0$. On the other hand, every $t>0$ must be contained in some I_k, where the function θ_k takes the constant value 1, so we have $1\leq\Phi\leq 2$. These properties allow us to define, on $(0,\infty)$, the functions $\theta_k^*=\frac{\theta_k}{\Phi}$. These are C^∞ functions satisfying $\sum_k \theta_k^* = 1$, $0 \leq \theta_k^* \leq 1$, and $\mathrm{supp}(\theta_k^*) \subseteq I_k^*$. Less elementary, but crucial, is the following property; see [27, 25] for a proof.

Proposition 2.3. *For every $j \in \mathbb{N} \cup \{0\}$, there is a positive constant A_j such that*

$$|(\theta_k^*)^{(j)}(t)| \leq A_j \ell_k^{-j} \quad \textit{for every} \quad t > 0,\ k \in \mathbb{N}.$$

2.4. **The sequence $\{\delta_p\}_p$ and the function ε.** Let us consider the numbers r_m of Lemma 2.1. We can easily construct a sequence $\{\delta_p\}_p$ of positive numbers satisfying

$$\delta_p \leq \min\{r_{p+2}, \frac{1}{(p+2)!}\}, \text{ for } p \geq 1$$

$$\delta_p < \frac{\delta_{p-1}}{2}, \text{ for } p \geq 2.$$

Of course the sequence $\{\delta_p\}_p$ is strictly decreasing to 0. Now, for every k we define a positive integer γ_k as follows. In the case that $\ell_k \geq \delta_1$, we set $\gamma_k = 1$. In the opposite case, $\ell_k < \delta_1$, we take γ_k as the unique positive integer for which

$$\delta_{\gamma_k+1} \leq \ell_k < \delta_{\gamma_k}.$$

Finally let us define:

$$\varepsilon(t) = \begin{cases} \sum_{k\geq 1} t^{\gamma_k} \theta_k^*(t) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

In the following Lemma we show that ε is of class C^∞ on $\mathbb{R}$ and satisfies an additional property which will be important in Subsection 2.7.

Lemma 2.4. *The function ε satisfies the following properties.*

1. *ε is of class $C^\infty(\mathbb{R})$ and satisfies $\varepsilon^{(j)}(0) = 0$ for every $j \in \mathbb{N} \cup \{0\}$.*
2. *If $0 < t \leq \delta_4$ and $q \in \mathbb{N}$ are such that $\delta_{q+1} \leq t < \delta_q$ and $\frac{t}{2} \leq s \leq t$, then $\varepsilon(2s) \geq t^{q+2}$.*

Proof. For the first statement, we inmediately see that $\varepsilon^{-1}(0) = (-\infty, 0]$, that $\varepsilon > 0$ on $(0, +\infty)$ and that $\varepsilon \in C^\infty(\mathbb{R} \setminus \{0\})$. In order to prove the differentiability of ε at $t = 0$ and that all the derivatives of ε at $t = 0$ are 0, it is sufficient to prove that for all $j \in \mathbb{N} \cup \{0\}$,

$$\lim_{t \to 0^+} \frac{|\varepsilon^{(j)}(t)|}{t} = 0.$$

To check this, fix $j \in \mathbb{N} \cup \{0\}$ and $\eta > 0$ and take

$$\widetilde{t_j} := \min\{\frac{\eta}{2B_j 4^j (j+1)!}, \delta_{j+5}\}, \quad \text{where} \quad B_j = \max\{A_l \ : \ 0 \leq l \leq j\}.$$

Recall that the numbers A_l are those given by Proposition 2.3. Let $0 < t \leq \widetilde{t_j}$. Due to the fact that $\{\delta_p\}_p$ is strictly decreasing, we can find a unique positive integer q such that $\delta_{q+1} \leq t < \delta_q$, and because $t \leq \delta_{j+5} < \delta_1$, we

must have $q \geq j+4$. Now, if k is such that $t \in I_k^*$, Proposition 2.2 tells us that
$$\ell_k \leq \frac{4}{3}t < 2t \leq 2\delta_{j+5} < \delta_1,$$
and using the definition of γ_k, we have
$$\delta_{\gamma_k+1} \leq \ell_k \leq \frac{4}{3}t < 2t < 2\delta_q < \delta_{q-1}.$$
The above inequalities imply that $\gamma_k + 1 > q-1$, that is $\gamma_k \geq q-1$. In particular $\gamma_k \geq j+3$. On the other hand, using Proposition 2.2 again, we obtain:
$$\delta_{\gamma_k} > \ell_k \geq \frac{4t}{9} \geq \frac{t}{4} \geq \frac{\delta_{q+1}}{4} > \delta_{q+3},$$
so $\gamma_k \leq q+2$.
If we use Leibnitz's Rule, we obtain
$$\varepsilon^{(j)}(t) = \sum_{k\geq 1}\sum_{l=0}^{j}\binom{j}{l}\frac{d^l}{dt^l}(t^{\gamma_k})(\theta_k^*)^{(j-l)}(t),$$
and since $\gamma_k \geq j+3$ for those k such that $t \in I_k^*$, we can write
$$\frac{|\varepsilon^{(j)}(t)|}{t} = \Big|\sum_{I_k^*\ni t}\sum_{l=0}^{j}\binom{j}{l}\frac{\gamma_k!}{(\gamma_k-l)!}t^{\gamma_k-l-1}(\theta_k^*)^{(j-l)}(t)\Big| \leq \sum_{I_k^*\ni t}\sum_{l=0}^{j} j!\gamma_k!t^{\gamma_k-l-1}A_{j-l}\ell_k^{l-j}.$$
Now, by Proposition 2.2 we know that $\ell_k \geq \frac{4}{9}t \geq \frac{1}{4}t$. Moreover, because $\gamma_k \leq q+2$, we have $\gamma_k! \leq (q+2)!$ and the last sum is smaller than or equal to
$$\sum_{I_k^*\ni t}\sum_{l=0}^{j} j!\,(q+2)!\,t^{\gamma_k-l-1}\,A_{j-l}\,\frac{t^{l-j}}{4^{l-j}}.$$
Writing $t^{\gamma_k-l-1} = t^2t^{\gamma_k-l-3} \leq t\,\delta_q\,t^{\gamma_k-l-3}$, this sum is smaller than or equal to
$$\sum_{I_k^*\ni t}\sum_{l=0}^{j} j!\,(q{+}2)!\,t\delta_q\,t^{\gamma_k-l-3}\,A_{j-l}\,\frac{t^{l-j}}{4^{l-j}} \leq \left(4^j j!B_j\sum_{I_k^*\ni t}\sum_{l=0}^{j}(q+2)!\delta_q\,t^{\gamma_k-j-3}\right)t.$$
Noting that $t \leq \delta_{j+5} < 1$ and $\gamma_k \geq j+3$, we must have $t^{\gamma_k-j-3} \leq 1$. By construction of the sequence $\{\delta_p\}_p$ we have that $(q+2)!\,\delta_q \leq 1$, and using that the sum $\sum_{I_k^*\ni t}$ has at most 2 terms, we obtain
$$\frac{|\varepsilon^{(j)}(t)|}{t} \leq 4^j(j+1)j!\,2B_jt \leq 4^j(j+1)!\,2B_j\widetilde{t_j} \leq \eta.$$
This completes the proof of statement **1**.
Now we prove the second statement. First of all, we note that $\delta_{q+1} \leq t \leq 2s \leq 2t < 2\delta_q < \delta_{q-1}$, and in particular $q \geq 3$. Let us suppose that $2s \in I_k^*$. Using Proposition 2.2,
$$\delta_{\gamma_k+1} \leq \ell_k \leq \frac{4}{3}(2s) < 2(2s) < 2\delta_{q-1} < \delta_{q-2},$$

that is $\gamma_k \geq q-2$. If we use Proposition 2.2 again,

$$\delta_{\gamma_k} > \ell_k \geq \frac{4(2s)}{9} \geq \frac{(2s)}{4} \geq \frac{\delta_{q+1}}{4} > \delta_{q+3},$$

and then $\gamma_k \leq q+2$.

Finally, note that $2s \leq 2t < \delta_{q-1} < \delta_1 < 1$, and due to the fact that $\gamma_k \leq q+2$ for those k such that $2s \in I_k^*$, we have that $(2s)^{q+2} \leq (2s)^{\gamma_k}$. Therefore we easily obtain the desired inequality:

$$t^{q+2} \leq (2s)^{q+2} = \sum_{I_k^* \ni 2s} (2s)^{q+2} \theta_k^*(2s) \leq \sum_{I_k^* \ni 2s} (2s)^{\gamma_k} \theta_k^*(2s) = \varepsilon(2s).$$

□

2.5. **The function φ.** Next, we will first adapt the function constructed by Ghomi in [16, Proposition 2.1] to suit our purposes, and then we will find quantitative estimates for its Hessian. We begin by defining

$$\tilde{\varepsilon}(t) = \begin{cases} \dfrac{\varepsilon(2t)}{t^{n+3}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Since $\varepsilon \in C^\infty(\mathbb{R})$, with $\varepsilon^{(j)}(0) = 0$ for all $j \in \mathbb{N} \cup \{0\}$, we have that $\tilde{\varepsilon} \in C^\infty(\mathbb{R})$ and $\tilde{\varepsilon}^{(j)}(0) = 0$ for all $j \in \mathbb{N} \cup \{0\}$ as well. Now, let us consider the function

$$g(t) = \begin{cases} \int_0^t \int_0^s \tilde{\varepsilon}(r) dr\, ds & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

It is clear that $g \in C^\infty(\mathbb{R})$ and $g^{(j)}(0) = 0$ for all $j \in \mathbb{N} \cup \{0\}$. In addition, $g^{-1}(0) = (-\infty, 0]$ and $g''(t) = \tilde{\varepsilon}(t) > 0$ for all $t > 0$. In particular, g is convex on $\mathbb{R}$ and positive, with a strictly positive second derivative, on $(0, +\infty)$.

We may assume that $0 \in C$. Now, for every vector $w \in \mathbb{S}^{n-1}$, define $h(w) = \max_{z \in C} \langle z, w \rangle$, the support function of C (for information about support functions of convex sets, see [21] for instance). We also define the function

$$\begin{array}{rccl} \phi: & \mathbb{S}^{n-1} \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (w, x) & \longmapsto & \phi(w,x) = g(\langle x, w \rangle - h(w)). \end{array}$$

It is easy to see that, for every $w \in \mathbb{S}^{n-1}$ and every multi-index α, we have

$$\frac{\partial^\alpha}{\partial x^\alpha} \phi(w,x) = g^{(|\alpha|)}(\langle x, w \rangle - h(w)) w^\alpha,$$

where $w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n}$. In addition, we note that when $x \in C$, we have $\langle x, w \rangle \leq h(w)$ for every $w \in \mathbb{S}^{n-1}$. Therefore, the properties of g and its derivatives imply that $\phi(w, \cdot)$ is a function of class $C^\infty(\mathbb{R}^n)$ whose derivatives of every order and itself vanish on C for every $w \in \mathbb{S}^{n-1}$. It is also easy to check that the function $\phi(w, \cdot)$, being a composition of a convex function with a non-decreasing convex function, is convex as well.

Finally, we define the function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ as follows:

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} \phi(w,x)\, dw \quad \text{for every} \quad x \in \mathbb{R}^n.$$

Again it is easy to check that $\varphi^{-1}(0)=C$ and φ is convex. Because $\phi(w,\cdot)$ is in $C^\infty(\mathbb{R}^n)$, the derivatives $(w,x)\mapsto \frac{\partial^\alpha}{\partial x^\alpha}\phi(w,x)$ are continuous for every multi-index α, and $\mathbb{S}^{n-1}$ is compact, it follows from standard results on differentiation under the integral sign that the function φ is of class $C^\infty(\mathbb{R}^n)$ as well and that $D^\alpha\varphi(x)=0$ for every $x\in C$ and every multi-index α. In other words, $J^m_x\varphi=0$ for all $m\in\mathbb{N}\cup\{0\}$ and all $x\in C$. One can also check easily that

$$D^2\varphi(x)(v^2)=\int_{\mathbb{S}^{n-1}} g''(\langle x,w\rangle-h(w))\langle w,v\rangle^2\,dw.$$

2.6. **Selection of angles and directions.** For given $x\in\mathbb{R}^n\setminus C$ and $v\in\mathbb{S}^{n-1}$ we will now find a region $W=W(x,v)$ of $\mathbb{S}^{n-1}$ of sufficient volume (depending only, and conveniently, on $d(x,C)$) on which we have good lower estimates for $g''(\langle x,w\rangle-h(w))\langle w,v\rangle^2$. This will involve a careful selection of angles and directions.

Fix a point $x\in\mathbb{R}^n\setminus C$, let x_C be the metric projection of x onto the compact convex C, and set

$$u_x=\frac{1}{|x-x_C|}(x-x_C),$$

and

$$\alpha_x=\frac{d(x,C)}{d(x,C)+\operatorname{diam}(C)}.$$

Lemma 2.5. *With the above notation we have $\langle x,u_x\rangle-h(u_x)=d(x,C)$ and*

$$d(x,C)\geq\langle x,w\rangle-h(w)\geq\frac{1}{2}d(x,C)$$

for all $w\in\mathbb{S}^{n-1}$ such that $\widehat{w\,u_x}\in\left[\frac{\alpha_x}{3},\frac{\alpha_x}{2}\right]$.

Here $\widehat{w\,u_x}$ denotes the length of the shortest geodesic (or angle) between w and u_x in $\mathbb{S}^{n-1}$.

Proof. The fact that $\langle x,u_x\rangle-h(u_x)=d(x,C)$ is a straightforward consequence of the definition of h and u_x. For the second part, given $w\in\mathbb{S}^{n-1}$ with $\widehat{w\,u_x}\in\left[\frac{\alpha_x}{3},\frac{\alpha_x}{2}\right]$, let us denote $\theta=\widehat{w\,u_x}$. Since C is compact, we can find $\xi\in C$ such that $h(w)=\langle\xi,w\rangle$. Using that $\langle x,u_x\rangle-h(u_x)=|x-x_C|$ and $|w-u_x|\leq\theta$, we have

$$\begin{aligned}
\langle x,w\rangle-h(w)&=\langle x,w-u_x\rangle+|x-x_C|+h(u_x)-h(w)\\
&\geq\langle x,w-u_x\rangle+|x-x_C|+\langle\xi,u_x-w\rangle\\
&=\langle x-\xi,w-u_x\rangle+|x-x_C|\\
&\geq-\left(\operatorname{diam}(C)+|x-x_C|\right)\theta+|x-x_C|\\
&\geq-\left(\operatorname{diam}(C)+|x-x_C|\right)\frac{\alpha_x}{2}+|x-x_C|\\
&=\frac{1}{2}|x-x_C|.
\end{aligned}$$

The other inequality, $d(x,C) \geq \langle x, w\rangle - h(w)$, is straightforward. $\square$

Next we find the region W we need.

Lemma 2.6. *Given any $v \in \mathbb{S}^{n-1}$ with $\langle u_x, v\rangle \geq 0$, there exists a vector $w_0 = w_0(x,v) \in \mathbb{S}^{n-1}$ such that if we define*

$$W = \{w \in \mathbb{S}^{n-1} \, : \, \widehat{w\, w_0} \in [0, \frac{\alpha_x}{12}]\},$$

then:

1. *For every $w \in W$, we have $\widehat{u_x\, w} \in \left[\frac{\alpha_x}{3}, \frac{\alpha_x}{2}\right]$.*
2. *For every $w \in W$, we have $\langle w, v\rangle \geq \sin(\frac{\alpha_x}{3})$.*
3. *$\mathrm{vol}_{\mathbb{S}^{n-1}}(W) \geq V(n)\alpha_x^{n-1}$, where $V(n) > 0$ is a constant depending only on the dimension n.*

Proof. We prove **1** and **2** at the same time by studying two cases separately.
Case (i): $u_x \neq v$. Take an w_0 in the unit circle of the plane spanned by the vectors u_x and v, in such a way that $\widehat{w_0\, u_x} = \frac{5\alpha_x}{12}$, and that the arc in that circle joining u_x with w_0 has the same orientation as the arc joining u_x with v. Set $W = \{w \in \mathbb{S}^{n-1} \, : \, \widehat{w\, w_0} \in [0, \frac{\alpha_x}{12}]\}$ and let $w \in W$.
First, recalling that the angles shorter than π give the usual distance between points of $\mathbb{S}^{n-1}$, we may use the triangle inequality to estimate

$$\widehat{u_x\, w} \leq \widehat{u_x\, w_0} + \widehat{w_0\, w} \leq \frac{5\alpha_x}{12} + \frac{\alpha_x}{12} = \frac{\alpha_x}{2}$$

and

$$\widehat{u_x\, w} \geq \widehat{u_x\, w_0} - \widehat{w_0\, w} \geq \frac{5\alpha_x}{12} - \frac{\alpha_x}{12} = \frac{\alpha_x}{3},$$

that is $\widehat{u_x\, w} \in \left[\frac{\alpha_x}{3}, \frac{\alpha_x}{2}\right]$. It only remains to see that $\langle w, v\rangle \geq \sin(\alpha_x/3)$ for all $w \in W$. First, we easily check that $\widehat{v\, w_0} \leq \frac{\pi}{2} - \frac{5\alpha_x}{12}$. Now, for an arbitrary $w \in W$, we have

$$\widehat{v\, w} \leq \widehat{v\, w_0} + \widehat{w_0\, w} \leq \frac{\pi}{2} - \frac{5\alpha_x}{12} + \frac{\alpha_x}{12} = \frac{\pi}{2} - \frac{\alpha_x}{3}.$$

Therefore $\langle v, w\rangle = \cos(\widehat{v\, w}) \geq \cos(\frac{\pi}{2} - \frac{\alpha_x}{3}) = \sin\frac{\alpha_x}{3}$.
Case (ii): $u_x = v$. Take w_0 in the sphere $\mathbb{S}^{n-1}$ such that $\widehat{w_0\, u_x} = \frac{5\alpha_x}{12}$. If we define $W = \{w \in \mathbb{S}^{n-1} \, : \, \widehat{w\, w_0} \in [0, \frac{\alpha_x}{12}]\}$, following the same estimations as in Case (i) we obtain $\widehat{u_x\, w} \in [\frac{\alpha_x}{3}, \frac{\alpha_x}{2}]$ for every $w \in W$. And we easily have $\langle w, v\rangle = \langle w, u_x\rangle \geq \sin\frac{\alpha_x}{3}$.
We now prove **3.** Since the standard measure on $\mathbb{S}^{n-1}$ is invariant under isometries we may assume that $W = \{w \in \mathbb{S}^{n-1} \, : \, \widehat{w\, e_1} \in [0, \frac{\alpha_x}{12}]\}$, where $e_1 = (1, 0, ..., 0)$. The set W is a hyperspherical cap, and its volume is given by

$$\mathrm{vol}_{\mathbb{S}^{n-1}}(W) = \mathrm{vol}(\mathbb{S}^{n-2}) \int_0^{\alpha_x/12} \sin^{n-2}(\beta) d\beta,$$

where $\mathrm{vol}(\mathbb{S}^{n-2}) = 1$ in the special case $n = 2$. But for angles β such that $0 \leq \beta \leq \frac{\alpha_x}{12} \leq \frac{\pi}{3}$, it is clear that $\sin\beta \geq \frac{1}{2}\beta$, and therefore

$$\mathrm{vol}_{\mathbb{S}^{n-1}}(W) \geq \mathrm{vol}(\mathbb{S}^{n-2}) \int_0^{\alpha_x/12} (\frac{1}{2}\beta)^{n-2} d\beta = V(n)\alpha_x^{n-1},$$

where

$$V(n) = \frac{\mathrm{vol}(\mathbb{S}^{n-2})}{12(n-1)(24)^{n-2}}$$

for $n \geq 2$. $\square$

2.7. **Convexity of $f + \psi$ on a neighbourhood of C.** Now, using the constant $V(n)$ obtained in Lemma 2.6, define

$$C(n) = \frac{V(n)}{36(1 + \mathrm{diam}(C))^{n+1}}.$$

Lemma 2.7. *With the notation of Subsection 2.4, consider the function $H = f + \frac{2}{C(n)}\varphi$ defined on $\mathbb{R}^n$, and take $r = \delta_4$. Then, for every $x \in \mathbb{R}^n \setminus C$ such that $t := d(x, C) \leq r$, and for every $v \in \mathbb{S}^{n-1}$, we have*

$$D^2H(x)(v^2) \geq t^q,$$

where q is the unique positive integer such that $\delta_{q+1} \leq t < \delta_q$.

Proof. Fix x, t, v, q as in the statement. Since $D^2H(x)(v^2) = D^2H(x)((-v)^2)$, we may suppose that $\langle v, u_x\rangle \geq 0$, where $u_x = (x - x_C)/|x - x_C|$ and x_C is the metric projection of x onto C. Take the angle α_x and the set $W = W(x, v)$ as in Lemmas 2.5 and 2.6 respectively. By the construction of φ, we have

$$\begin{aligned}(2.1)\quad D^2\varphi(x)(v^2) &= \int_{\mathbb{S}^{n-1}} \tilde{\varepsilon}(\langle x, w\rangle - h(w))\langle w, v\rangle^2\, dw \\ &\geq \int_W \tilde{\varepsilon}(\langle x, w\rangle - h(w))\langle w, v\rangle^2\, dw > 0,\end{aligned}$$

and for $w \in W$, Lemma 2.6 gives us that $\widehat{w\, u_x} \in \left[\frac{\alpha_x}{3}, \frac{\alpha_x}{2}\right]$; on the other hand Lemma 2.5 says that, in this case,

$$\frac{t}{2} \leq \langle x, w\rangle - h(w) \leq t \leq \delta_4.$$

Using the second property of Lemma 2.4 we obtain

$$\tilde{\varepsilon}(\langle x, w\rangle - h(w)) = \frac{\varepsilon(2(\langle x, w\rangle - h(w)))}{(\langle x, w\rangle - h(w))^{n+3}} \geq \frac{t^{q+2}}{(\langle x, w\rangle - h(w))^{n+3}} \geq \frac{t^{q+2}}{t^{n+3}} = \frac{t^q}{t^{n+1}}.$$

On the other hand, due to Lemma 2.6 the product $\langle v, w\rangle$ is greater than or equal to $\sin(\frac{\alpha_x}{3})$ for all $w \in W$. By combining the preceding inequalities, we get

$$D^2\varphi(x)(v^2) \geq \frac{t^q}{t^{n+1}} \sin^2(\frac{\alpha_x}{3})\, \mathrm{vol}_{\mathbb{S}^{n-1}}(W).$$

By the third part of Lemma 2.6, the last term is greater or equal than

$$\frac{t^q}{t^{n+1}}\sin^2\left(\frac{\alpha_x}{3}\right)V(n)\alpha_x^{n-1}.$$

Since $\alpha_x \leq 1$, we have that $\sin(\frac{\alpha_x}{3}) \geq \frac{1}{2}\frac{\alpha_x}{3}$, so we obtain

$$D^2\varphi(x)(v^2) \geq \frac{t^q}{t^{n+1}}\frac{\alpha_x^2}{36}V(n)\alpha_x^{n-1} = \frac{t^q}{t^{n+1}}\frac{\alpha_x^{n+1}}{36}V(n).$$

Moreover, we have

$$\alpha_x = \frac{t}{t+\operatorname{diam}(C)} \geq \frac{t}{1+\operatorname{diam}(C)},$$

because $t \leq r = \delta_4 < 1$. Gathering these inequalities, we get

$$D^2\varphi(x)(v^2) \geq \frac{t^q}{t^{n+1}}\frac{t^{n+1}}{36(1+\operatorname{diam}(C))^{n+1}}V(n) = C(n)t^q.$$

Finally, due to the construction of the sequence $\{\delta_p\}_p$, (see Subsection 2.4) we have $d(x,C) = t < \delta_q \leq r_{q+2}$, hence Lemma 2.1 ensures that

$$D^2f(x)(v^2) \geq -t^q.$$

Therefore

$$D^2H(x)(v^2) = D^2f(x)(v^2) + \frac{2}{C(n)}D^2\varphi(x)(v^2) \geq -t^q + 2t^q = t^q.$$

□

Since $J^m_y\varphi = 0$ for $y \in C$ and each $m \in \mathbb{N}\cup\{0\}$, we have proved that H is of class $C^\infty(\mathbb{R}^n)$, $H = f$ on C, $J^m_yH = J^m_yf = P^m_y$ for every $y \in C$ and every $m \in \mathbb{N}$, and H has a strictly positive Hessian on the set $\{x \in \mathbb{R}^n : 0 < d(x,C) \leq r\}$.

2.8. **Conclusion of the proof: convexity of $f+\psi$ on $\mathbb{R}^n$.** To complete the proof of Theorem 1.3 we only have to change the funcion H slightly.

Lemma 2.8. *There exists a number $a > 0$ such that the function $F := f + a\varphi$ is of class $C^\infty(\mathbb{R}^n)$, concides with f on C, satisfies $J^m_yF = P^m_y$ for every $y \in C$, $m \in \mathbb{N}$, is convex on $\mathbb{R}^n$, and has a strictly positive Hessian on $\mathbb{R}^n \setminus C$.*

Proof. Let us denote $\psi = \frac{2}{C(n)}\varphi$. We recall that $f = 0$ outside $C + B(0,2)$. Take $r > 0$ as in Lemma 2.7. Since $C_r := \{x : r \leq d(x,C) \leq 2\}$ is a compact subset where ψ has a strictly positive Hessian (cf. (2.1)), and using again that f has compact support, we can find $M \geq 1$ such that

$$\sup_{x\in\mathbb{R}^n,\, v\in\mathbb{S}^{n-1}} |D^2f(x)(v^2)| \leq M \quad \text{and} \quad \inf_{x\in C_r,\, v\in\mathbb{S}^{n-1}} D^2\psi(x)(v^2) \geq \frac{1}{M}.$$

Let us take $A = 2M^2$ and $F = f + A\psi$. If $d(x,C) \leq r$ and $v \in \mathbb{S}^{n-1}$ we have, by Lemma 2.7, that

$$D^2F(x)(v^2) = 2M^2D^2\psi(x)(v^2) + D^2f(x)(v^2) > D^2\psi(x)(v^2) + D^2f(x)(v^2) > 0.$$

In the case when $d(x,C)\in[r,2]$, given any $|v|=1$, we easily see that

$$D^2F(x)(v^2)=2M^2D^2\psi(x)(v^2)+D^2f(x)(v^2)\geq 2M-M=M>0.$$

Finally, in the region $\{x\,:\,d(x,C)>2\}$, we have that $f=0$. Hence

$$D^2F(x)(v^2)=2M^2D^2\psi(x)(v^2)>0.$$

Therefore, in any case, by setting $a=2A/C(n)$, we get that $F=f+A\psi=f+a\varphi$ is of class $C^\infty(\mathbb{R}^n)$ satisfies $F(y)=f(y)$ and $J^m_yF=P^m_y$ for every $y\in C$, $m\in\mathbb{N}$, and has a positive Hessian on $\mathbb{R}^n\setminus C$. Since f is convex on C and F is differentiable, this is easily seen to imply that F is convex on all of $\mathbb{R}^n$. $\square$

2.9. **Proof of Corollary 1.7.** An obvious variation of the proof of the above Lemma shows Proposition 1.5. On the other hand Proposition 1.5 can easily be used to show Corollary 1.7. Indeed, we have

$$D^2f(y)(v^2)+t\,D^3f(y)(w,v^2)+\cdots+\frac{t^{k-2}}{(k-2)!}D^kf(y)(w^{k-2},v^2)\geq\eta t^{k-2}$$

for all $y\in C$, $w,v\in\mathbb{S}^{n-1}$, $0<t\leq t_0$ and, on the other hand, by Taylor's theorem and uniform continuity of D^mf,

$$\begin{aligned}&D^2f(y+tw)(v^2)=\\&D^2f(y)(v^2)+t\,D^3f(y)(w,v^2)+\cdots+\frac{t^{m-2}}{(m-2)!}D^mf(y)(w^{m-2},v^2)+R_m(t,y,v,w),\end{aligned}$$

where

$$\lim_{t\to0^+}\frac{R_m(t,y,v,w)}{t^{m-2}}=0\text{ uniformly on }y\in C,w,v\in\mathbb{S}^{n-1}.$$

We may assume $t_0\leq 1$. Then we may also find $t_0'\in(0,t_0)$ such that $R_m(t,y,v,w)\geq-\frac{\eta}{2}t^{m-2}$ for all $y\in C$, $w,v\in\mathbb{S}^{n-1}$, $0<t\leq t_0'$, and it follows that

$$D^2f(y+tw)(v^2)\geq\frac{\eta}{2}t^{m-2}$$

for all $y\in C$, $w,v\in\mathbb{S}^{n-1}$, $0<t\leq t_0'$. This implies that $D^2f(x)\geq 0$ whenever $d(x,C)\leq t_0'$, and therefore that f is convex on $U:=\{x\in\mathbb{R}^n : d(x,C)<t_0'\}$. Corollary 1.7 then follows from Proposition 1.5.

3. C^m convex extensions for $m\geq 2$ finite

We start this section with the proof of Theorem 1.8. We may assume that C is a compact convex subset of $\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}$ is of class $C^m(\mathbb{R}^n), m\geq n+3$, with support contained on $C+B(0,2)$, and such that f satisfies condition (CW^m) on C. We will split the proof into several subsections.

3.1. **The function ω.**

Lemma 3.1. *Let us denote*

$$Q_m(t,y,v,w)=\frac{D^2f(y)(v^2)+\cdots+\frac{t^{m-2}}{(m-2)!}D^mf(y)(w^{m-2},v^2)}{t^{m-2}}$$

for all $t>0, y\in C,\ v,w\in\mathbb{S}^{n-1}$. There exists a non decreasing continuous function $\omega:[0,+\infty)\to[0,+\infty)$ with $\omega(0)=0$ such that

$$D^2f(x)(v^2)\geq -\omega(d(x,C))d(x,C)^{m-2}\quad \textit{for all}\quad x\in\mathbb{R}^n,\ v\in\mathbb{S}^{n-1}.$$

Proof. We start with the following easy remark. If Q_m is as above and

$$\varepsilon_m(t)=\sup_{\{z\in\mathbb{R}^n,\, z'\in\partial C,\, |z-z'|\leq t\}}\|D^mf(z)-D^mf(z')\|,$$

by using (CW^m) and uniform continuity of D^mf, given a positive integer p, there exists $r_p>0$ such that

$$Q_m(t,y,v,w)\geq -\frac{1}{2p}\quad\text{and}\quad \varepsilon_m(t)\leq\frac{1}{2p} \tag{3.1}$$

for every $y\in\partial C,\ v,w\in\mathbb{S}^{n-1}$ and $0<t\leq r_p$. We may suppose that this sequence $\{r_p\}_{p\geq 1}$ is strictly decreasing to 0. Since the derivatives of f up to order m are bounded on $\mathbb{R}^n$ we can find a constant $M>1$ such that

$$\varepsilon_m(t)-Q_m(t,y,v,w)\leq M\quad\text{for all}\quad y\in\partial C, v,w\in\mathbb{S}^{n-1}, t\geq r_1. \tag{3.2}$$

Now, given $x\in\mathbb{R}^n\setminus C$ and $v\in\mathbb{S}^{n-1}$, we denote by y the metric projection of x onto C, $w=(x-y)/|x-y|$ and $t=d(x,C)$. By Taylor's theorem and the definition of Q_m and ε_m, we have

$$D^2f(x)(v^2)\geq t^{m-2}Q_m(t,y,v,w)-t^{m-2}\varepsilon_m(t)=-t^{m-2}\left(\varepsilon_m(t)-Q_m(t,y,v,w)\right).$$

We define $\omega:[0,+\infty)\to[0,+\infty)$ by setting

$$\begin{aligned}&\omega(0)=0,\ \omega(r_p)=\frac{1}{p-1}\quad p\geq 2,\quad \omega(r_1)=M,\\ &\omega \text{ affine on each } [r_{p+1},r_p]\quad p\geq 1,\quad \omega(t)=M\quad t\geq r_1.\end{aligned}$$

It is easy to check that ω is a non decreasing continuous function such that $\omega(t)\geq\frac{1}{p}$ for every $t\geq r_{p+1}$ and every $p\geq 2$, and that $\omega(t)\geq 1$ for every $t\geq r_2$. Using inequalities (3.1) and (3.2) we deduce that

$$\begin{aligned}&D^2f(x)(v^2)\geq -Mt^{m-2}\quad\text{for}\quad t\geq r_1\\ &D^2f(x)(v^2)\geq -\frac{1}{p}t^{m-2}\quad\text{for}\quad t\leq r_p,\quad p\in\mathbb{N}\end{aligned}$$

and by the properties of ω we conclude

$$D^2f(x)(v^2)\geq -\omega(d(x,C))d(x,C)^{m-2}\quad\text{for every}\quad x\in\mathbb{R}^n,\ v\in\mathbb{S}^{n-1}.$$

$\square$

3.2. **The function φ.** Using the function ω defined in Lemma 3.1, we introduce two new functions

$$g(t) = \begin{cases} \int_0^t \int_0^{t_2} \cdots \int_0^{t_{m-n-1}} \omega(2^{m-n-2}s)ds\, dt_{m-n-1} \cdots dt_2 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} g(\langle x, w \rangle - h(w))\, dw, \quad x \in \mathbb{R}^n.$$

Since ω is continuous, the function g is of class $C^{m-n-1}(\mathbb{R})$ with $g^{(k)}(0) = 0$ for every $1 \leq k \leq m-n-1$. The same arguments and calculations of Section 2.5 allow us to deduce that φ is of class $C^{m-n-1}(\mathbb{R}^n)$ with $\varphi^{-1}(0) = C$ and $J_x^{m-n-1}\varphi = 0$ for all $x \in C$. It is also easy to see

$$D^2\varphi(x)(v^2) = \int_{\mathbb{S}^{n-1}} g''(\langle x, w \rangle - h(w))\langle v, w \rangle^2 dw,$$

for all $x \in \mathbb{R}^n$ and all $v \in \mathbb{S}^{n-1}$.

3.3. **Conclusion of the proof of Theorem 1.8.** Suppose that $x \in \mathbb{R}^n \setminus C$ with $d(x, C) \leq 1$ and denote $t := d(x, C)$. Fix also a direction $v \in \mathbb{S}^{n-1}$. If we consider the angle $\alpha = \alpha_x$ and the subset $W = W_{x,v}$ of $\mathbb{S}^{n-1}$ as in Section 2.6 we obtain

$$D^2\varphi(x)(v^2) \geq \int_W g''(\langle x, w \rangle - h(w)) \sin^2\left(\frac{\alpha}{3}\right) dw.$$

Recall that, since $t \leq 1$, the angle α satisfies

$$\alpha = \frac{t}{t + \operatorname{diam}(C)} \geq \frac{t}{1 + \operatorname{diam}(C)}.$$

Combining Lemmas 2.5 and 2.6, we deduce that $\frac{t}{2} \leq \langle x, w \rangle - h(w) \leq t$ for every $w \in W$. Because g'' is non decreasing, we have that

$$g''(\langle x, w \rangle - h(w)) \geq g''(t/2) \quad \text{for all} \quad w \in W.$$

These estimations lead us to

$$D^2\varphi(x)(v^2) \geq \operatorname{vol}_{\mathbb{S}^{n-1}}(W) g''(t/2) \sin^2(\alpha/3).$$

Note that Lemma 2.6 also shows that there exists a positive constant $V(n)$ only depending on n such that $\operatorname{vol}_{\mathbb{S}^{n-1}}(W) = V(n)\alpha^{n-1}$. Since $\sin^2\left(\frac{\alpha}{3}\right) \geq \frac{\alpha^2}{36}$, the Hessian of φ at x on the direction v satisfies

$$D^2\varphi(x)(v^2) \geq \frac{V(n)}{36}\left(\frac{t}{1 + \operatorname{diam}(C)}\right)^{n+1} g''(t/2). \tag{3.3}$$

Now we give a lower bound for $g''(t/2)$. By the construction of g we have

$$g''(t/2) = \int_0^{t/2} \int_0^{t_2} \cdots \int_0^{t_{m-n-3}} \omega(2^{m-n-2}s)ds\, dt_{m-n-3} \cdots dt_2,$$

where, in the special case $m = n+3$, the above expression means $g''(t/2) = \omega(t)$. Using that ω is nonnegative and nondecreasing we may estimate:

$$\begin{aligned} g''(t/2) &\geq \int_{t/4}^{t/2} \int_0^{t_2} \cdots \int_0^{t_{m-n-3}} \omega(2^{m-n-2}s)ds\, dt_{m-n-3} \cdots dt_2 \\ &\geq \frac{t}{4} \int_0^{t/4} \int_0^{t_2} \cdots \int_0^{t_{m-n-4}} \omega(2^{m-n-2}s)ds\, dt_{m-n-4} \cdots dt_2 \\ &\geq \frac{t}{4} \cdot \frac{t}{8} \int_0^{t/8} \int_0^{t_2} \cdots \int_0^{t_{m-n-5}} \omega(2^{m-n-2}s)ds\, dt_{m-n-5} \cdots dt_2 \\ &\geq \frac{t}{4} \cdot \frac{t}{8} \cdots \frac{t}{2^{m-n-3}} \cdot \frac{t}{2^{m-n-2}} \omega(t) = \frac{t^{m-n-3}}{2^{2+3+\cdots+(m-n-2)}} \omega(t). \end{aligned}$$

By plugging this estimation in (3.3), we obtain that

$$D^2\varphi(x)(v^2) \geq k(n,m,C) t^{m-2} \omega(t),$$

where

$$k(n,m,C) = \frac{V(n)}{36 \cdot 2^{2+3+\cdots+(m-n-2)} (1+\operatorname{diam}(C))^{n+1}}.$$

On the other hand, Lemma 3.1 implies that

$$D^2 f(x)(v^2) \geq -\omega(t) t^{m-2}.$$

Therefore, the function $\psi = f + \frac{2}{k(n,m,C)} \varphi$ satisfies $D^2\psi(x)(v^2) \geq 0$ on the neighbourhood $\{x \in \mathbb{R}^n \,:\, d(x,C) \leq 1\}$ of C with strict inequality whenever $0 < d(x,C) \leq 1$. We also have that the function ψ is of class $C^{m-n-1}(\mathbb{R}^n)$ with $f = \psi$ on C and $J_y^{m-n-1}\psi = J_y^{m-n-1} f$ for all $y \in C$. Finally, using the same argument of Section 2.8 we can construct a convex function $F \in C^{m-n-1}(\mathbb{R}^n)$ with $F = f$ on C and $J_y^{m-n-1} F = J_y^{m-n-1} f$ for all $y \in C$. The proof of Theorem 1.8 is complete.

In the rest of this section we will give the proof of Theorem 1.10.

3.4. **Sublevel sets of strongly convex functions.** Here we gather some elementary properties of ovaloids and Minkowski functionals that we will need in the proof of Theorem 1.10.

Proposition 3.2. *Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a convex function of class $C^m(\mathbb{R}^n)$, with $m \geq 2$, such that there exists a constant $M > 0$ with $D^2\psi(x)(v^2) \geq M$ for all $x \in \mathbb{R}^n$ and for all $v \in \mathbb{S}^{n-1}$. If we denote $C = \psi^{-1}(-\infty, 1]$, then the following is true.*

(i) *C is a convex compact set.*
(ii) *$\partial C = \{x \in \mathbb{R}^n \,:\, \psi(x) = 1\}$.*
(iii) *If $\operatorname{int}(C) = \emptyset$, then C is a singleton.*

If we further assume that $\operatorname{int}(C) \neq \emptyset$ then we also have:

(iv) *ψ attains a unique minimum in $\operatorname{int}(C)$.*
(v) *∂C is a one-codimensional manifold of class C^m.*

(vi) *If $x \notin C$ and $x_C \in \partial C$ is such that $|x - x_C| = d(x, C)$, then $\nabla\psi(x_C)$ and $x - x_C$ are paralell and outwardly normal to ∂C at the point x_C.*

(vii) *There is a constant $\beta > 0$ such that*

$$\psi(x) - 1 \geq \beta d(x, C) \quad \textit{for every} \quad x \in \mathbb{R}^n \setminus C.$$

Proof. Properties (i)-(vi) are well known facts about strongly convex functions of class C^m. Perhaps only property (vii) requires an explanation. Due to the compactness of ∂C and the continuity of $\nabla\psi$, we can find $\beta > 0$ such that $|\nabla\psi(x)| \geq \beta$ for all $x \in \partial C$. If $x \notin C$, by taking $x_C \in \partial C$ with $|x - x_C| = d(x, C)$, by convexity of ψ we have

$$\psi(x) - 1 = \psi(x) - \psi(x_C) \geq \langle \nabla\psi(x_C), x - x_C \rangle,$$

and by (vi), the last product coincides with $|\nabla\psi(x_C)||x - x_C| \geq \beta d(x, C)$. $\square$

Given any subset C of a normed space $(X, |\cdot|)$ the Minkowski functional of C is defined by

$$\mu_C(x) = \inf\{t \geq 0 \, : \, x \in tC\}, \quad x \in X.$$

The following Proposition sums up some well known properties of gauges asociated to convex bodies and also shows other properties that are crucial to our purposes and are not so well known.

Proposition 3.3. *If $C \subseteq X$ is convex with $0 \in \mathrm{int}(C)$ we have:*

(i) $0 \leq \mu_C(x) < +\infty$ *for all $x \in X$, and $\mu_C(0) = 0$.*

(ii) $\mu_C = \mu_{\mathrm{int}(C)} = \mu_{\overline{C}}$.

(iii) *If $0 < t < \infty$, then $\mu_C(x) < t$ if and only if $x \in t\,\mathrm{int}(C) = \mathrm{int}(tC)$.*

(iv) *μ_C is positively homogeneous subadditive functional.*

(v) $\{x \in X \, : \, \mu_C(x) < 1\} = \mathrm{int}(C) \subset C \subset \overline{C} = \{x \in X \, : \, \mu_C(x) \leq 1\}$,

(vi) *If $r > 0$ is such that $B(0, r) \subset C$, then $\mu_C(x) \leq r^{-1}|x|$ for all $x \in X$. Also,*

(vii) *μ_C is $\frac{1}{r}$-Lipschitz, and*

(viii) $\mu_C(x) - 1 \leq r^{-1} d(x, C)$ *for all $x \in X$.*

(ix) *If $C = \bigcap_{k=1}^N C_k$, where each C_k is a convex subset with $0 \in \mathrm{int}\, C$ then $\mu_C = \max_{1 \leq k \leq N} \mu_{C_k}$.*

Suppose in addition that $C \subset X$ is bounded.

(x) *There is $R > 0$ such that $\mu_C(x) \geq R^{-1}|x|$ for all $x \in X$.*

(xi) *For all $x \in X$, we have $d(x, \partial C) \leq R|\mu_C(x) - 1|$. In particular $d(x, C) \leq R(\mu_C(x) - 1)$ if $x \in X \setminus C$.*

(xii) *If $C = \bigcap_{k=1}^N C_k$, where each C_k is convex and bounded with $0 \in \mathrm{int}(C)$, we have*

$$\max_{1 \leq k \leq N} d(x, C_k) \leq d(x, C) \leq \frac{R}{r} \max_{1 \leq k \leq N} d(x, C_k) \quad \textit{for all} \quad x \in X,$$

whenever $r, R > 0$ are such that $\overline{B}(0, r) \subseteq C \subseteq \overline{B}(0, R)$.

(xiii) *If* $C = \bigcap_{k=1}^N C_k$, *where each* C_k *is convex and bounded with* $\mathrm{int}(C) \neq \emptyset$, *but not necessarily* $0 \in \mathrm{int}(C)$, *we have*

$$\max_{1 \leq k \leq N} d(x, C_k) \leq d(x, C) \leq \frac{R}{r} \max_{1 \leq k \leq N} d(x, C_k) \quad \textit{for all} \quad x \in X,$$

where $r, R > 0$ *are such that* $\overline{B}(x_0, r) \subseteq C \subseteq \overline{B}(x_0, R)$ *and* $x_0 \in \mathrm{int}(C)$.

Proof. Properties (i)-(viii),(x),(xi) are all well known properties about Minkowski functional. See [8, Chapter (II)] for details. Now we check the rest of the properties.

(ix) It is an easy consequence of (iii).

(xii) When $x \in C$ there is nothing to prove. If $x \notin C$, using (ix) and (xi) we obtain

$$d(x, C) \leq R[\mu_C(x) - 1] = R[\max_{1 \leq k \leq N} \mu_{C_k}(x) - 1] = R[\max_{1 \leq k \leq N} (\mu_{C_k}(x) - 1)].$$

By (viii), the last term is less than or equal to $\frac{R}{r} \max_{1 \leq k \leq N} d(x, C_k)$.

(xiii) After a translation, the same proof as in (xii) holds. □

3.5. **Proof of Theorem 1.10.** First of all, let us make an small remark: If a set C is (FIO^m), (see Definition 1.9), then either C has nonempty interior or C is a single point. We may thus suppose that C has nonempty interior, as the result follows immediately from Theorem 1.3 in the case that C is a singleton.

Fix $m \in \mathbb{N}$ with $m \geq 3$. Suppose that C is (FIO^{m-1}) with nonempty interior and $f \in C^m(\mathbb{R}^n)$ a function satisfying (CW^m) on C. According to Definition 1.9, we can write $C = \bigcap_{j=1}^N C_j$, where for each $1 \leq j \leq N$ there are $M_j > 0$ and a function $\psi_j : \mathbb{R}^n \to \mathbb{R}$ of class $C^{m-1}(\mathbb{R}^n)$ such that $C_j = \psi_j^{-1}(-\infty, 1]$ and $D^2\psi_j(x)(v^2) \geq M_j$ for all $x \in \mathbb{R}^n$ and $v \in \mathbb{S}^{n-1}$. Let us denote $M = \min\{M_j \ : \ 1 \leq j \leq N\}$. By Proposition 3.2, for each $j \in \{1, \ldots, N\}$, the set C_j is a convex compactum and there is a constant $\beta_j > 0$ with $\psi_j(x) - 1 \geq \beta_j d(x, C_j)$ whenever $x \notin C_j$. Set $\beta = \min\{\beta_j \ : \ j = 1, \ldots, N\}$. Using Proposition 3.3 **(xiii)**, we obtain $L > 0$ with $d(x, C) \leq L \max_{1 \leq j \leq N} d(x, C_j)$ for all $x \in \mathbb{R}^n$. To sum up, we have found L, β, M are positive constants satisfying

$$d(x, C) \leq L \max_{1 \leq j \leq N} d(x, C_j) \quad \text{for all} \quad x \in \mathbb{R}^n; \tag{3.4}$$

$$\psi_j(x) - 1 \geq \beta d(x, C_j) \quad \text{for all} \quad x \notin C_j, \quad 1 \leq j \leq N; \tag{3.5}$$

$$D^2\psi_j(x)(v^2) \geq M \quad \text{for all} \quad x \in \mathbb{R}^n, \ v \in \mathbb{S}^{n-1}, \quad 1 \leq j \leq N. \tag{3.6}$$

Since $f : \mathbb{R}^n \to \mathbb{R}$ satisfies (CW^m) on C, the estimation given in Lemma 3.1 holds for f. For these positive constants $L, \beta > 0$, we define the following functions

$$g(t) = \begin{cases} \int_0^t \int_0^{t_2} \cdots \int_0^{t_{m-1}} \omega\left(2^{m-2}s\right) ds\, dt_{m-1} \cdots dt_2 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

$$h(t) = g(L\beta^{-1}t), \quad t \in \mathbb{R},$$

and

$$\varphi(x) = \sum_{j=1}^{N} h(\psi_j(x) - 1), \quad x \in \mathbb{R}^n.$$

It is clear that $g \in C^{m-1}(\mathbb{R})$ with $g^{(k)}(0) = 0$ for all $0 \leq k \leq m-1$. By the definition of the ψ's and g, we have that $\varphi^{-1}(0) = C$ and $\varphi \in C^{m-1}(\mathbb{R}^n)$. It is routine to check that $\partial^\alpha \varphi(x) = 0$ for all $x \in C$ and $|\alpha| \leq m-1$, that is, $J_x^{m-1}\varphi = 0$ for all $x \in C$. A simple calculation and the fact that $g'' \geq 0$ lead us to

$$\begin{aligned} D^2\varphi(x)(v^2) &= \sum_{j=1}^{N} h''(\psi_j(x) - 1)\langle \nabla\psi_j(x), v\rangle^2 + \sum_{j=1}^{N} h'(\psi_j(x) - 1) D^2\psi_j(x)(v^2) \\ &\geq \sum_{j=1}^{N} h'(\psi_j(x) - 1) D^2\psi_j(x)(v^2), \end{aligned}$$

for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$. Now, we study the convexity of φ outside of C. Fix $x \in \mathbb{R}^n \setminus C$ and $v \in \mathbb{S}^{n-1}$. From (3.6) we deduce

$$D^2\varphi(x)(v^2) \geq \sum_{j=1}^{N} h'(\psi_j(x) - 1) D^2\psi_j(x)(v^2) \geq M \sum_{j=1}^{N} h'(\psi_j(x) - 1).$$

But the above sum is greater than or equal to $Mh'(\psi_j(x) - 1)$, where we consider an index $j := j_x$ with $d(x, C_j) = \max_{1 \leq i \leq N} d(x, C_i)$. Of course, for this index j, we have that $x \notin C_j$. This implies $\psi_j(x) > 1$ and therefore

$$D^2\varphi(x)(v^2) \geq Mh'(\psi_j(x) - 1) = ML\beta^{-1} g'(L\beta^{-1}(\psi_j(x) - 1)).$$

Using inequalities (3.5) and (3.4) and the choice of j, we obtain

$$\psi_j(x) - 1 \geq \beta d(x, C_j) \geq \beta L^{-1} d(x, C).$$

The above inequality and the fact that g' is non decreasing imply that

$$g'(L\beta^{-1}(\psi_j(x) - 1)) \geq g'(d(x, C)) = g'(t),$$

where $t := d(x, C)$. Recalling that ω is nonnegative and nondecreasing, we obtain

$$\begin{aligned} g'(t) &= \int_0^t \int_0^{t_2} \cdots \int_0^{t_{m-2}} \omega\left(2^{m-2}s\right) ds\, dt_{m-2} \cdots dt_2 \\ &\geq \int_{t/2}^t \int_0^{t_2} \cdots \int_0^{t_{m-2}} \omega(2^{m-2}s) ds\, dt_{m-2} \cdots dt_2 \\ &\geq \frac{t}{2} \int_0^{t/2} \int_0^{t_2} \cdots \int_0^{t_{m-3}} \omega(2^{m-2}s) ds\, dt_{m-3} \cdots dt_2 \\ &\geq \frac{t}{2} \cdot \frac{t}{4} \int_0^{t/4} \int_0^{t_2} \cdots \int_0^{t_{m-4}} \omega(2^{m-2}s) ds\, dt_{m-4} \cdots dt_2 \\ &\geq \frac{t}{2} \cdot \frac{t}{4} \cdots \frac{t}{2^{m-3}} \int_0^{t/2^{m-3}} \omega(2^{m-2}s) ds \\ &\geq \frac{t}{2} \cdot \frac{t}{4} \cdots \frac{t}{2^{m-3}} \cdot \frac{t}{2^{m-2}} \omega(t) = \frac{t^{m-2}}{2^{1+2+3+\cdots+(m-2)}} \omega(t). \end{aligned}$$

Therefore

$$D^2\varphi(x)(v^2) \geq ML\beta^{-1} g'(t) = k(n, m, C) t^{m-2} \omega(t),$$

where

$$k(n, m, C) = \frac{ML\beta^{-1}}{2^{1+2+3+\cdots+(m-2)}}.$$

On the other hand, Lemma 3.1 gives us the following inequality:

$$D^2 f(x)(v^2) \geq -\omega(t) t^{m-2}.$$

Hence $F := f + \frac{2}{k(n,m,C)}\varphi$ has a strictly positive Hessian on $\mathbb{R}^n \setminus C$, is of class $C^{m-1}(\mathbb{R}^n)$ and coincides with f on C. Since $J_x^{m-1}\varphi = 0$ for all $x \in C$, we have that $J_x^{m-1}F = J_x^{m-1}f$ for all $x \in C$. Because f is convex on C and the extension F is differentiable, we have that F is convex in $\mathbb{R}^n$. The proof of Theorem 1.10 is complete.

4. Remarks and Counterexamples

The following example is a variation of [24, Example 4] and shows that our main result fails if we drop the assumption that C be compact, even in the presence of strictly positive Hessians.

Example 4.1. Let $C = \{(x, y) \in \mathbb{R}^2 : x > 0, xy \geq 1\}$, and define

$$f(x, y) = -2\sqrt{xy} + \frac{1}{x+1} + \frac{1}{y+1}$$

for every $(x, y) \in C$. The set C is convex and closed, with a nonempty interior, and it is routine to verify that f has a strictly positive Hessian on C. We also have

$$\nabla f(x, y) = \left(-x^{-\frac{1}{2}} y^{\frac{1}{2}} - \frac{1}{(x+1)^2},\ -x^{\frac{1}{2}} y^{-\frac{1}{2}} - \frac{1}{(y+1)^2} \right).$$

We claim that f does not have any convex extension to all of $\mathbb{R}^2$. In order to prove this it is sufficient to see that, for instance, $m(f)(-1,-1)=\infty$, where $m(f)$ is the minimal convex extension of f defined in Section 2. Considering the curve $\gamma(t)=(t,\frac{1}{t})$, $t>0$, which parameterizes the boundary of C, we have

$$m(f)(-1,-1)\geq f(t,\frac{1}{t})+\langle\nabla f(t,\frac{1}{t}),(-1-t,-1-\frac{1}{t})\rangle=2+t+\frac{1}{t},$$

so by letting either $t\to\infty$ or $t\to 0^+$ we obtain $m(f)(-1,1)=\infty$. As a matter of fact, it is not difficult to see that $m(f)(x,y)=\infty$ for every $(x,y)\in\mathbb{R}^2$ such that $x<0$ or $y<0$.

The following example shows that if C has empty interior then one cannot expect to find smooth convex extensions (of functions satisfying (W^m) and (CW^m) on C) without experiencing a certain loss of differentiability. The example also shows that in $\mathbb{R}^2$ this loss amounts to at least two orders of smoothness, and that the situation does not improve as m grows large (unless $m=\infty$, of course).

Example 4.2. Consider the function $\theta(y)=\frac{1-\cos(2\pi y)}{2\pi}$, $y\in\mathbb{R}$. Clearly, $\theta\in C^\infty(\mathbb{R})$, with $\theta(0)=\theta(1)=0$, $\theta(1/2)=\frac{1}{\pi}$ and $\theta'(y)=\sin(2\pi y)$. Define $h(x,y)=\theta(y)x^m$, $(x,y)\in\mathbb{R}^2$. Let $C:=\{0\}\times[0,1]$. We have $D^k h=0$ on C for all $k\in\{0,\ldots,m-1\}$, and

$$D^m h(x,y)=m!\theta(y)\,\overbrace{e_1^*\otimes\cdots\otimes e_1^*}^{m}\ \text{ for } (x,y)\in C$$

(here e_1^* denotes the linear function $(x_1,x_2)\mapsto x_1$). Therefore $D^m h(0,0)=D^m h(0,1)=0$, and $D^m h(0,\frac{1}{2})=\frac{m!}{\pi}e_1^*\otimes\cdots\otimes e_1^*$. We claim that if $m\geq 2$ is even then there is no convex function $F\in C^m(\mathbb{R}^2)$ such that $D^kF=D^kh$ on C for $k\in\{0,\ldots,m\}$. We also claim that h satisfies conditions (W^∞) and (CW^{m+1}) (and in particular (CW^m) too) on C.

The first claim immediately follows from the following.

Remark 4.3. *If $m\geq 2$, there exists no convex function $f\in C^m(\mathbb{R}^2)$ such that $D^kf(0,y)=0$ for all $k\in\{0,\ldots,m-1\}$, $y\in[0,1]$, and such that $D^mf(0,0)=D^mf(0,1)=0$ and $D^mf(0,\frac{1}{2})=Ae_1^*\otimes\cdots\otimes e_1^*$, where $A>0$ is a constant.*

Proof. For the sake of contradiction, suppose there is such an f. Using Taylor's theorem we have

$$f(x,y)=\frac{1}{m!}D^mf(0,y_0)(x,y-y_0)^m+R(x,y,y_0)\quad (x,y)\in\mathbb{R}^2,\ y_0\in[0,1],$$

where $\dfrac{R(x,y)}{|(x,y-y_0)|^m}\to 0$ as $(x,y)\to(0,y_0)$, uniformly on $y_0\in[0,1]$. Fix $0<\varepsilon<\frac{A}{2m!}$, and take $\delta=\delta(\varepsilon)>0$ such that if $y_0\in[0,1]$ and $(x,y)\in\mathbb{R}^2$

satisfy $(x^2+(y-y_0)^2)^{1/2}\leq\delta$ then

$$\left|f(x,y)-\frac{1}{m!}D^m f(0,y_0)(x,y-y_0)^m\right| = |R(x,y)| \leq \varepsilon(x^2+(y-y_0)^2)^{\frac{m}{2}}.$$

Evaluating for $y=y_0=1/2$ we obtain

$$\left|f(x,\tfrac{1}{2})-A\frac{x^m}{m!}\right|\leq\varepsilon|x|^m, \text{ if } |x|\leq\delta.$$

For $y=y_0\in\{0,1\}$ and $|x|\leq\delta$ we get

$$\max\{|f(x,0)|,|f(x,1)|\}\leq\varepsilon|x|^m.$$

Fix $x_0>0$ with $x_0\leq\delta$. We then have

$$f(x_0,\tfrac{1}{2})\geq A\frac{x_0^m}{m!}-\varepsilon x_0^m > 2\varepsilon x_0^m-\varepsilon x_0^m=\varepsilon x_0^m\geq\max\{f(x_0,0),f(x_0,1)\}.$$

This implies that $[0,1]\ni t\mapsto\varphi(t)=f(x_0,t)$ satisfies $\varphi(\frac{1}{2})>\frac{1}{2}\varphi(0)+\frac{1}{2}\varphi(1)$, and in particular f cannot be convex. □

Let us now prove our second claim. It is obvious that h satisfies (W^k) for every k. We only have to check that h satisfies (CW^{m+1}) on C. We must see that, given $\varepsilon>0$ there exists $t_\varepsilon>0$ such that

$$Q_{m+1}(y,t,v,w)=\frac{\frac{1}{(m-2)!}D^m h(0,y)(v^2,w^{m-2})+\frac{t}{(m-1)!}D^{m+1}h(0,y)(v^2,w^{m-1})}{t}\geq-\varepsilon,$$

for every $y\in[0,1],\ v,w\in\mathbb{S}^1,\ 0<t\leq t_\varepsilon$. It is not difficult to check that

$$\begin{aligned}D^{m+1}h(0,y)(v^2,w^{m-1}) &= \frac{\partial^{m+1}h}{\partial x^m\partial y}(0,y)\left[(m-1)v_1^2w_1^{m-2}w_2+2v_1v_2w_1^{m-1}\right]\\ &= m!\,\theta'(y)\left[(m-1)v_1^2w_1^{m-2}v_2+2v_1v_2w_1^{m-1}\right].\end{aligned}$$

On the other hand $D^m h(0,y)(v^2,w^{m-2})=m!\,\theta(y)v_1^2w_1^{m-2}$. For our given $\varepsilon>0$, let us fix t_ε such that

$$0<t_\varepsilon\leq\min\left(1,\ \frac{\varepsilon}{4\pi(2m+3)(m+1)m(m-1)}\right).$$

Take $y\in[0,1],\ v,w\in\mathbb{S}^1$ and $0<t\leq t_\varepsilon$. We have

$$Q_{m+1}(y,t,v,w)=\frac{1}{t}\left[\frac{m!}{(m-2)!}\theta(y)v_1^2w_1^{m-2}+\frac{m!}{(m-1)!}t\theta'(y)\left((m-1)v_1^2w_1^{m-2}w_2+2v_1v_2w_1^{m-1}\right)\right].$$

Since m is even, we have $w_1^{m-2}\geq 0$, and it is not difficult to estimate

(4.1)
$$Q_{m+1}(y,t,v,w)\geq\frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left(\theta(y)|v_1|-(m+1)t|\theta'(y)|\right).$$

Let us now distinguish the following cases.

Case 1: Assume $y \in [\frac{1}{4}, \frac{3}{4}]$. Then $2\pi y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Therefore $\cos(2\pi y) \leq 0$, which implies $\theta(y) \geq \frac{1}{2\pi}$. Since we always have $|\theta'(y)| = |\sin(2\pi y)| \leq 1$, it follows from (4.1) that

$$Q_{m+1}(y,t,v,w) \geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t} \left(\frac{|v_1|}{2\pi} - (m+1)t \right).$$

Subcase 1.1: Assume $|v_1| \geq 2\pi(m+1)t$. Then it is clear that $Q_{m+1}(y,t,v,w) \geq 0 \geq -\varepsilon$.

Subcase 1.2: Assume $2\pi(m+1)t^2 \leq |v_1| \leq 2\pi(m+1)t$. Then, since $|w_1|, t, 1-t \leq 1$, we obtain

$$\begin{aligned}
&Q_{m+1}(y,t,v,w) \geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t} \left((m+1)t^2 - (m+1)t\right) \\
&= (m+1)m(m-1)|v_1||w_1|^{m-2}(t-1) \geq -2\pi(m+1)^2 m(m-1)t|w_1|^{m-2}(1-t) \\
&\geq -2\pi t(m+1)^2 m(m-1) \geq -2\pi t_\varepsilon (m+1)^2 m(m-1) \geq -\varepsilon.
\end{aligned}$$

Subcase 1.3: Assume $|v_1| \leq 2\pi(m+1)t^2$. We have

$$\begin{aligned}
&Q_{m+1}(y,t,v,w) \geq -\frac{m(m-1)|v_1||w_1|^{m-2}}{t} \left((m+1)t - \frac{|v_1|}{2\pi}\right) \\
&\geq -\frac{2\pi m(m-1)(m+1)t^2|w_1|^{m-2}}{t} \left(m+1+\frac{1}{2\pi}\right) \geq -2\pi(m+1)m(m-1)(m+2)\, t \\
&\geq -2\pi(m+1)m(m-1)(m+2)\, t_\varepsilon \geq -\varepsilon.
\end{aligned}$$

Case 2: Assume $y \in [0, \frac{1}{4})$. Then $\pi y \in [0, \frac{\pi}{4})$ and $2\pi y \in [0, \frac{\pi}{2})$. We have

$$\theta(y) = \frac{1-\cos(2\pi y)}{2\pi} = \frac{\sin^2(\pi y)}{\pi}.$$

On the other hand,

$$|\theta'(y)| = |\sin(2\pi y)| = \sin(2\pi y) = 2\sin(\pi y)\cos(\pi y).$$

By substituting in (4.1), we get

$$\begin{aligned}
Q_{m+1}(y,t,v,w) &\geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t} \left(\frac{\sin^2(\pi y)|v_1|}{\pi} - (m+1)\sin(2\pi y)t\right) \\
&= \frac{m(m-1)|v_1||w_1|^{m-2}\sin(\pi y)}{t} \left(\frac{\sin(\pi y)|v_1|}{\pi} - 2(m+1)\cos(\pi y)t\right)
\end{aligned}$$

Subcase 2.1: Assume $\sin(\pi y)|v_1| \geq 2\pi(m+1)\cos(\pi y)t$. Then obviously $Q_{m+1}(y,t,v,w) \geq 0 \geq -\varepsilon$.

Subcase 2.2: Assume $2\pi(m+1)\cos(\pi y)t^2 \leq \sin(\pi y)|v_1| \leq 2\pi(m+1)\cos(\pi y)t$. We have

$$Q_{m+1}(y,t,v,w) \geq m(m-1)|v_1||w_1|^{m-2}\sin(\pi y)2(m+1)\cos(\pi y)(t-1),$$

whose modulus is less than or equal to

$$\begin{aligned}
&2(m+1)m(m-1)|v_1|\sin(\pi y) \leq 4\pi(m+1)^2 m(m-1)\cos(\pi y)t \\
&\leq 4\pi(m+1)^2 m(m-1)t \leq 4\pi(m+1)^2 m(m-1)t_\varepsilon \leq \varepsilon.
\end{aligned}$$

This shows that $Q_{m+1}(y,t,v,w) \geq -\varepsilon$.
Subcase 2.3: Assume $\sin(\pi y)|v_1| \leq 2\pi(m+1)\cos(\pi y)t^2$. Recall that

$$Q_{m+1}(y,t,v,w) \geq \frac{m(m-1)|v_1||w_1|^{m-2}\sin(\pi y)}{t}\left(\frac{\sin(\pi y)|v_1|}{\pi} - 2(m+1)\cos(\pi y)t\right).$$

The modulus of the last term is less than or equal to

$$\frac{m(m-1)|v_1|\sin(\pi y)\left(\frac{1}{\pi}+2(m+1)\right)}{t} \leq \frac{m(m-1)2\pi(m+1)\cos(\pi y)t^2(1+2(m+1))}{t}$$
$$\leq 2\pi(m+1)m(m-1)(2m+3)t \leq 2\pi(2m+3)(m+1)m(m-1)t_\varepsilon \leq \varepsilon.$$

Hence $Q_{m+1}(y,t,v,w) \geq -\varepsilon$.
Case 3: Assume finally that $y \in (\frac{3}{4}, 1]$. Take $z = 1-y$. Clearly $\cos(2\pi z) = \cos(2\pi y)$, and $\sin(2\pi z) = -\sin(2\pi y)$. Therefore $\theta(z) = \theta(y)$ and $|\theta'(y)| = |\theta'(z)|$, hence

$$\begin{aligned} Q_{m+1}(y,t,v,w) &\geq \frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left(\theta(y)|v_1| - (m+1)t|\theta'(y)|\right) \\ &= \frac{m(m-1)|v_1||w_1|^{m-2}}{t}\left(\theta(z)|v_1| - (m+1)t|\theta'(z)|\right), \end{aligned}$$

and since $z \in [0, \frac{1}{4})$, we can apply Case 2 with z instead of y to obtain $Q_{m+1}(y,t,v,w) \geq -\varepsilon$.

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