Inner and outer smooth approximation of convex hypersurfaces. When is it possible?

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ABSTRACT

Let S be a convex hypersurface (the boundary of a closed convex set V with nonempty interior) in $\mathbb{R}^n$. We prove that S contains no lines if and only if for every open set $U \supset S$ there exists a real-analytic convex hypersurface $S_U \subset U \cap \text{int}(V)$. We also show that S contains no rays if and only if for every open set $U \supset S$ there exists a real-analytic convex hypersurface $S_U \subset U \setminus V$. Moreover, in both cases, S_U can be taken strongly convex. We also establish similar results for convex functions defined on open convex subsets of $\mathbb{R}^n$, completely characterizing the class of convex functions that can be approximated in the C^0-fine topology by smooth convex functions from above or from below. We also provide similar results for C^1-fine approximations.

1. Main results

The main purpose of this paper is to establish the following two results. The phrases 'S contains a ray' and 'S contains a line' in the two theorems below mean set-theoretic inclusion.

Theorem 1. *Let $S = \partial V$, where $V \subset \mathbb{R}^n$ is a closed convex set with nonempty interior. The following statements are equivalent.*

(1) S contains no rays.

(2) For every open set $U \supset S$ there exists a real-analytic convex hypersurface $S_U \subset U \setminus V$.

(3) For every open set $U \supset S$ there exists a real-analytic strongly convex hypersurface $S_U \subset U \setminus V$.

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Theorem 2. *Let $S = \partial V$, where V is a closed convex set with nonempty interior in $\mathbb{R}^n$. The following statements are equivalent.*

(1) S contains no lines.
(2) For every open set $U \supset S$ there exists a real-analytic convex hypersurface $S_U \subset U \cap \mathrm{int}(V)$.
(3) For every open set $U \supset S$ there exists a real-analytic strongly convex hypersurface $S_U \subset U \cap \mathrm{int}(V)$.

Corollary 3. *If S is a strictly convex hypersurface in $\mathbb{R}^n$, then for every open set $U \supset S$ there exist real-analytic strongly convex hypersurfaces S_I and S_O such that $S_I \subset U \cap \mathrm{int}(V)$ and $S_O \subset U \setminus V$.*

Results of this kind are important for extending locally concave functions, which are commonly used as Bellman functions of certain extremal problems in harmonic analysis. Results similar to Corollary 3 may be found along the lines of Section 4 in [5]. Our initial motivation comes from the need of Corollary 3 in higher dimensional generalizations of the work of that paper.

The preceding theorems are relatively easy consequences of the following results, which, as we believe, are of independent interest in themselves.

Theorem 4. *Let $U \subseteq \mathbb{R}^n$ be a non-empty open convex set, let $f : U \to \mathbb{R}$ be convex. The following statements are equivalent.*

(1) The graph of f does not contain any ray.
(2) For every continuous function $\varepsilon : U \to (0, \infty)$ there exists a real-analytic strongly convex function $g : U \to \mathbb{R}$ such that $f - \varepsilon < g < f$.
(3) For every continuous function $\varepsilon : U \to (0, \infty)$ there exists a convex function $g : U \to \mathbb{R}$ such that $f - \varepsilon < g < f$.

Theorem 5. *Let $U \subseteq \mathbb{R}^n$ be a non-empty open convex set, let $f : U \to \mathbb{R}$ be convex. The following statements are equivalent.*

(1) The graph of f does not contain any line.
(2) For every continuous function $\varepsilon : U \to (0, \infty)$ there exists a real-analytic strongly convex function $g : U \to \mathbb{R}$ such that $f < g < f + \varepsilon$.
(3) For every continuous function $\varepsilon : U \to (0, \infty)$ there exists a convex function $g : U \to \mathbb{R}$ such that $f < g < f + \varepsilon$.

In order to avoid any possible ambiguity in the preceding statements, let us fix some definitions. A *convex hypersurface S* is the boundary of a convex set with nonempty interior. Such a set S will be called *strictly convex* provided that S contains no line segments. Similarly, a convex function is strictly convex if its graph does not contain any line segment. If U is a nonempty convex subset of $\mathbb{R}^n$, we say that a C^2 function $f : U \to \mathbb{R}$ is *strongly convex* whenever $D^2 f(x)$ is strictly positive definite for every $x \in U$. A (not necessarily C^2) function $\varphi : U \to \mathbb{R}$ will be said to be strongly convex if for every $x \in U$ there exist $r_x > 0$ and a C^2 strongly convex function $\psi_x : B(x, r_x) \to \mathbb{R}$ such that $\varphi - \psi_x$ is convex on $B(x, r_x)$. A convex hypersurface $S \subset \mathbb{R}^n$ will be called real-analytic (resp. a strongly convex real-analytic surface) provided that there exists a real-analytic convex (resp. strongly convex) function $g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ such that $S = g^{-1}(r)$ for some $r > \inf_{x \in \mathbb{R}^n} g(x)$ (which implies that $Dg(x) \neq 0$ for all $x \in S$).

Let us now explain what we mean, in Theorems 4 and 5, by a ray in the case that $U \neq \mathbb{R}^n$. The phrase *the graph of f contains a ray* will mean, in this paper, that there exists $x \in U$ and $e \in \mathbb{R}^d$ such that the restriction of the function f to the set $\{x + te : t \in [0, \infty)\} \cap U$ is affine. Line segments of the form $[x, z] := \{(1 - t)x + tz : t \in [0, 1)\}$, where $x \in U$ and $z \in \partial U$, are rays for us. Similarly, in the above results

and what follows, in the case $U \neq \mathbb{R}^n$, a line in U will be a nonempty intersection of U with a line in $\mathbb{R}^n$. The phrase *the graph of f contains a line* will mean that the restriction of the function f to a line in U is affine. It is worth noting that, when $U = \mathbb{R}^n$, saying that the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ does not contain any line is equivalent to asserting that f is *essentially coercive* (which means coercive up to a linear perturbation). This is a consequence of [1, Lemma 4.2] or [3, Theorem 1.11]. However, in the statements of Theorems 1 and 2, a ray means a ray in $\mathbb{R}^n$ (not necessarily the intersection of a ray with S), and a line means a line in $\mathbb{R}^n$ (which again does not need to be the intersection of a line with S).

For background about this kind of problems, see [1] and the references therein. In [1] it was proved that every convex function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon \in (0, \infty)$ there exists a real-analytic convex function $g : U \to \mathbb{R}$ such that $|f - g| \leq \varepsilon$. This result is no longer valid in general when the number ε is replaced with a strictly positive continuous function, although in [1] it was also shown that if f is *properly convex*, then the result is still true.[1] However, unless $U = \mathbb{R}^n$, proper convexity is not a necessary condition for this kind of approximation. The following result enlarges the class of functions known to admit such approximations, providing a simple geometrical characterization of the class of convex functions (defined on an arbitrary convex and open subset of $\mathbb{R}^n$) that can be approximated in the C^0-fine topology by real-analytic strictly convex functions.

Corollary 6. *Let $U \subseteq \mathbb{R}^n$ be a non-empty open convex set, let $f : U \to \mathbb{R}$ be convex. The following statements are equivalent.*

(1) The graph of f does not contain any line.
(2) For every continuous function $\varepsilon : U \to (0, \infty)$ there exists a real-analytic strongly convex function $g : U \to \mathbb{R}$ such that $|f - g| < \varepsilon$.
(3) For every continuous function $\varepsilon : U \to (0, \infty)$ there exists a strictly convex function $g : U \to \mathbb{R}$ such that $|f - g| < \varepsilon$.

Our methods can be tuned to obtain C^1-fine approximation of C^1 convex functions by real-analytic convex functions. The following result improves [1, Theorem 1.10].

Theorem 7. *Let $U \subseteq \mathbb{R}^n$ be a non-empty open convex set and let $f : U \to \mathbb{R}$ be convex and of class C^1. The following statements are equivalent.*

(1) The graph of f does not contain any line.
(2) For every continuous function $\varepsilon : U \to (0, \infty)$ there exists a real-analytic strongly convex function $g : U \to \mathbb{R}$ such that $|f - g| < \varepsilon$ and $\|Df - Dg\| < \varepsilon$.

Section 2 contains results on approximation of convex functions by other convex functions from below and above; here we do not care about the smoothness of functions. Section 3 derives Theorems 4, 5, and Corollary 6 from the results of Section 2 and techniques of [1]. Section 4 contains the proofs Theorems 1 and 2. The last Section 5 is devoted to the proof of Theorem 7.

2. Approximation by rough functions

In the proofs of Theorems 4 and 5 we will use the following two theorems (Theorems 8 and 10 below), which we believe to be novel and of independent interest.

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[1] A function $f : U \to \mathbb{R}$ is properly convex provided that $f = \varphi + \ell$, with ℓ linear and $\varphi : U \to [a, b)$ convex and proper (meaning that $\varphi^{-1}[a, c]$ is compact for every $c \in [a, b)$); here $b \in \mathbb{R} \cup \{\infty\}$.

Theorem 8. *Let $U \subseteq \mathbb{R}^n$ be a non-empty open convex set, let $f: U \to \mathbb{R}$ be convex. The graph of f does not contain any ray if and only if for every compact subset K of U there exists a compact subset C of U such that $K \subset \mathrm{int}(C)$ and, for*

$$\varphi_C(x) := \sup\{f(y) + \xi(x - y) : y \in U \setminus C, \xi \in \partial f(y)\}, \quad x \in U,$$

we have that

$$\inf\{f(x) - \varphi_C(x) : x \in K\} > 0. \tag{1}$$

Here $\partial f(x)$ stands for the subdifferential of f at the point x:

$$\partial f(x) = \Big\{L \text{ is a linear function} : \forall y \in U \quad f(y) \geq f(x) + L(y - x)\Big\}, \quad x \in U. \tag{2}$$

Recall that the set $\partial f(x)$ is non-empty and

$$f(x) = \sup\Big\{f(y) + L(x - y) : y \in U, \, L \in \partial f(y)\Big\}, \quad x \in U. \tag{3}$$

Proof of Theorem 8. The 'only if' implication is evident, let us prove the 'if' part. First, we notice that $f(x) \geq \varphi_C(x)$ for any $x \in U$. If the statement is not true, then, for any compact set C such that $K \subset \mathrm{int}(C)$, the infimum in (1) equals zero. Therefore, there exist sequences (x_0^k) in K, (y_k) in U such that $\lim_{k \to \infty} \|y_k\| = \infty$ or $\lim_{k \to \infty} d(y_k, \partial U) = 0$ (if $U \neq \mathbb{R}^n$), and (η_k), $\eta_k \in \partial f(y_k)$ such that

$$f(x_0^k) - f(y_k) - \eta_k(x_0^k - y_k) \to 0, \qquad k \to \infty. \tag{4}$$

Since K is a compact set, we may assume, without loss of generality, that $x_0^k \to x_0 \in K$. Thus, by the continuity of f,

$$f(x_0) = \lim_{k \to \infty} \Big(f(y_k) + \eta_k(x_0 - y_k)\Big). \tag{5}$$

Denoting

$$v_k := \frac{y_k - x_0}{\|y_k - x_0\|},$$

up to passing to some subsequence, we may assume that (v_k) converges to some $v_0 \in \mathbb{S}^{n-1}$. Here and in what follows we use the standard Euclidean norm on $\mathbb{R}^n$ and denote the unit sphere by $\mathbb{S}^{n-1}$. Since the graph of f does not contain any ray and f is convex, there exist two points $z_0, w_0 \in U \cap \{x_0 + tv_0 : t > 0\}$ such that $\|w_0 - x_0\| > \|z_0 - x_0\|$ and

$$\frac{f(z_0) - f(x_0)}{\|z_0 - x_0\|} < \frac{f(w_0) - f(z_0)}{\|w_0 - z_0\|} \leq L(v_0)$$

for every $L \in \partial f(w_0)$.

Let us define

$$w_k := x_0 + \|w_0 - x_0\| v_k, \quad z_k := x_0 + \|z_0 - x_0\| v_k.$$

The points w_k and z_k may fall out of U for some k, but for all k large enough we have that $w_k, z_k \in U$. Up to extracting a subsequence, we may, thus, assume that $w_k, z_k \in U$ for all $k \in \mathbb{N}$. Let us also set

$$r_k := \frac{f(w_k) - f(z_k)}{\|w_k - z_k\|} - \frac{f(z_k) - f(x_0)}{\|z_k - x_0\|},$$

and choose $\xi_k \in \partial f(w_k)$ for each $k \in \mathbb{N}$. Note that

$$\frac{f(w_k) - f(z_k)}{\|w_k - z_k\|} \leq \xi_k(v_k). \tag{6}$$

Since $\lim_{k\to\infty} w_k = w_0$, $\lim_{k\to\infty} z_k = z_0$, and f is continuous, we have that

$$\lim_{k\to\infty} r_k = r := \frac{f(w_0) - f(z_0)}{\|w_0 - z_0\|} - \frac{f(z_0) - f(x_0)}{\|z_0 - x_0\|} > 0.$$

For sufficiently large $k \in \mathbb{N}$, we have $\|y_k - x_0\| > \|w_k - x_0\|$, and by convexity,

$$\xi_k(v_k) \le \eta_k(v_k),$$

and

$$\eta_k(v_k) \ge \frac{f(y_k) - f(w_k)}{\|y_k - w_k\|}. \tag{7}$$

Therefore we have

$$f(y_k) + \eta_k(x_0 - y_k) = f(y_k) - \|y_k - x_0\|\eta_k(v_k) =$$
$$f(y_k) - \|y_k - w_k\|\eta_k(v_k) - \|w_k - x_0\|\eta_k(v_k).$$

By (7), an upper bound of this expression is

$$f(y_k) - f(y_k) + f(w_k) - \|w_k - x_0\|\eta_k(v_k) \le$$
$$f(w_k) - \|w_k - x_0\|\xi_k(v_k) =$$
$$f(w_k) - \|w_k - z_k\|\xi_k(v_k) - \|z_k - x_0\|\xi_k(v_k).$$

Due to (6), this is estimated with

$$f(w_k) + f(z_k) - f(w_k) + \|z_k - x_0\|\frac{f(z_k) - f(w_k)}{\|w_k - z_k\|} =$$
$$f(z_k) + \|z_k - x_0\|\left(-r_k - \frac{f(z_k) - f(x_0)}{\|z_k - x_0\|}\right) =$$
$$f(x_0) - r_k\|z_k - x_0\|,$$

which implies

$$\limsup_{k\to\infty}\Big(f(y_k) + \eta_k(x_0 - y_k)\Big) \le f(x_0) - \|z_0 - x_0\|r < f(x_0),$$

in contradiction to (5). $\square$

Corollary 9.   *Let $U \subseteq \mathbb{R}^n$ be a non-empty open convex set, let $f : U \to \mathbb{R}$ be convex. Let $\varepsilon : U \to \mathbb{R}_+$ be a strictly positive continuous function. Assume that the graph of f does not contain rays. There exists a convex function $g : U \to \mathbb{R}$ such that*

$$f(x) - \varepsilon(x) < g(x) < f(x) \tag{8}$$

for all $x \in U$.

The strict sign in the second inequality is important.

Proof. Let us construct g with the formula

$$g(y) = \sup\left\{f(x) + L(y - x) - \frac{1}{2}\varepsilon(x) : x \in U, \quad L_x \in \partial f(x)\right\}. \tag{9}$$

The function g is clearly convex. Plugging $x := y$ into this formula, we get $g(y) \ge f(y) - \frac{1}{2}\varepsilon(y) > f(y) - \varepsilon(y)$. The inequality $g(y) < f(y)$ follows from Theorem 8 and the continuity of ε. $\square$
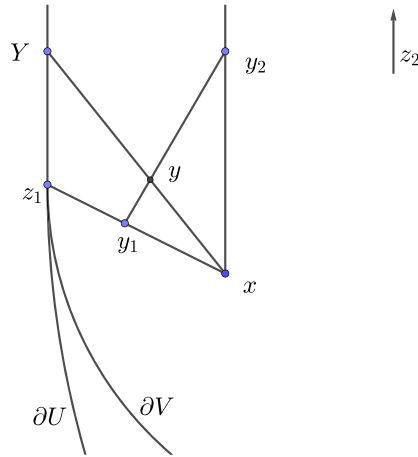
Fig. 1. Illustration to the proof of Lemma 11.

Theorem 10. *The graph of f does not contain lines if and only if for any $x \in U$ there exists a compact set $C_x \subset U$ and an affine function A_x such that $f(x) < A_x(x)$, however, $f(y) > A_x(y)$ provided $y \in U \setminus C_x$.*

The proof of the theorem will take some time.

In what follows, we fix $x \in U$ and a linear function L in the subdifferential of f at x. Consider the set

$$V = \{y \in U : f(y) = f(x) + L(y - x)\}. \tag{10}$$

Then V is a relatively closed convex subset of U (of course, V is not necessarily closed as a subset of $\mathbb{R}^n$). Let V_∞ be another set,

$$V_\infty = \left\{ y \in U : \exists \text{ a ray } [x, z) \subset V \text{ such that } y \in [x, z) \right\}. \tag{11}$$

As usual, by a ray we understand either a classical ray (then z is an infinite point) or the segment $[x, z)$ with $z \in \partial U$. The set V_∞ is relatively closed in U. However, in general situation, it might be non-convex.

Lemma 11. *Assume V does not contain lines. Then V_∞ is convex.*

Proof. Let y_1 and y_2 be two points in V_∞ lying on the rays $[x, z_1)$ and $[x, z_2)$ correspondingly. The reasoning depends on whether z_1 and z_2 are finite or infinite. Let us consider the case where z_1 is a finite point and z_2 is infinite (this case is the most 'representative'), see Fig. 1 for a visualization. Consider the ray (z_1, z_2) (which means a ray with the vertex z_1 and collinear with $[x, z_2)$). It $(z_1, z_2) \cap U \neq \varnothing$, then this intersection is contained in V (by convexity and closedness of V); this cannot happen since in such a case $(z_1, z_2) \cap U$ is a line.

Let $y \in (y_1, y_2)$, we wish to show that $y \in V_\infty$. Let $Y = (z_1, z_2) \cap \{x + t(y - x) : t > 0\}$. By the above, $Y \in \partial U$ and $[x, Y) \subset V$. Thus, $y \in V_\infty$.

The case where z_1 and z_2 are finite points is similar. The only difference is that now (z_1, z_2) is a classical segment. The case where z_1 and z_2 are both infinite is a little bit different (in fact, it simplifies). In this case, we do not need the assumption that V does not contain lines. We consider the ray $\{x + t(y - x) : t > 0\}$ and prove directly that it belongs to V_∞ (this follows from the closedness and convexity of V). $\square$

We will also need a version of the hyperplane separation theorem. We provide the proof since the construction will be used in Section 4 below.

Theorem 12. *Let K be a convex closed cone in $\mathbb{R}^n$ with the vertex at the origin. Assume K does not contain lines. There exists a linear hyperplane H such that $H \cap K = \{0\}$.*

Proof. First, we note that the origin is an extreme point of K. Second, consider the set $\tilde{K} = \operatorname{conv}(K \cap \mathbb{S}^{n-1})$. This is a compact convex set that does not contain the origin (since $K \cap \mathbb{S}^{n-1} \subset K \setminus \{0\}$, $K \cap \mathbb{S}^{n-1}$ is compact, and 0 is an extreme point of K). Thus, by the classical hyperplane separation theorem, there exists a hyperplane $\tilde{H}$ that strongly separates $\tilde{K}$ and 0. For example, one may consider the point $\zeta \in \tilde{K}$ that has the smallest possible Euclidean norm and set $\tilde{H}$ to be the midperpendicular of ζ and the origin. Let H be the translate of $\tilde{H}$ passing through 0. Then $H \cap \tilde{K} = \varnothing$. Therefore, $H \cap K = \{0\}$. $\quad\square$

Proof of Theorem 10. If C_x and A_x as in the second condition of the theorem exist for any x, then the graph of f does not contain lines. The reverse implication is less trivial.

Set $x = 0$ for convenience. Pick some $L \in \partial f(0)$, consider the set V defined in (10) and note that it does not contain lines (since the graph of f does not). Then, by Lemma 11, V_∞ given by (11) is a closed convex set that does not contain lines. Consider the minimal convex cone with the vertex 0 that contains V_∞ and call it V^* (note that V^* is not necessarily a subset of U). This cone is also convex, closed, and does not contain lines. By Theorem 12, there exists a hyperplane H that intersects V^* (and therefore, V_∞) at the origin only. Without loss of generality, we may assume $H = \{y \in \mathbb{R}^n : y_n = 0\}$ and that V_∞ lies in the hyperspace where $y_n \geq 0$. We also assume $f(0) = 0$ and $L = 0$ (we may subtract an affine function from f and A_x).

Since V^* meets H only at the origin, by compactness of the intersection of V^* with the unit sphere, there exists $\delta > 0$ such that

$$V_\infty \setminus \{0\} \subset \left\{ y \in \mathbb{R}^n : y_n > \delta \|y\| \right\}. \tag{12}$$

Let us call the latter open set K_δ. Note that $W = \operatorname{conv}(V \setminus K_\delta)$ is a compact set lying inside U. We set $A(y) = -\varepsilon y_n$, $A_x(y) := A(y) + \varepsilon_1$, where ε and ε_1 are sufficiently small parameters to be specified later. Let C_x be a star-shaped (not necessarily convex) closed bounded set such that $V \setminus K_\delta \subset \operatorname{int}(C_x)$ and $(C_x \cap K_\delta) \subset U$. We may construct the set C_x in the following way. Since U is an open set, there exists $\nu > 0$ such that U contains the closed Euclidean ball of radius ν centered at the origin. Let ρ be the distance between W and ∂U. Define the function $s : \mathbb{S}^{n-1} \to \mathbb{R}_+$ by the formula

$$s(z) = \max\big(\nu, (\mu_W(z))^{-1} + \rho/2\big), \quad z \in \mathbb{S}^{n-1}; \tag{13}$$

here μ_W denotes the Minkowski functional of W. We set $C_x = \cup_{z \in \mathbb{S}^{n-1}}[0, s(z)z]$.

Let

$$M = \max_{z \in C_x} \|z_n\|; \quad m = \inf_{z \in \partial C_x \setminus K_\delta} \big(f(z)\big); \quad \varepsilon = \frac{m}{2M}. \tag{14}$$

Note that $m > 0$ since $\partial C_x \setminus K_\delta$ is a compact set, which does not intersect V. Let us prove that with this choice of ε, $A(y) > f(y)$ when $y \in U \setminus C_x$. Since $y \notin C_x$, we get $\|y\| \geq \nu$. If $y \in K_\delta$, then

$$A(y) \leq -\varepsilon \delta \nu < 0 \leq f(y). \tag{15}$$

In the case $y \notin K_\delta$, we also have $y \notin C_x$. Let y^* be the point on the intersection of the segment $[0, y]$ with the boundary of C_x. Then,

$$A(y) = -\varepsilon y_n = -\varepsilon \frac{\|y\|}{\|y^*\|} y_n^* \leq \frac{m}{2} \frac{\|y\|}{\|y^*\|} < \frac{\|y\|}{\|y^*\|} f(y^*) \leq f(y), \tag{16}$$

the last inequality in the chain follows from the convexity of f (since $y^* \in [0, y)$). Thus, if

$$0 < \varepsilon_1 < \min\Big\{\varepsilon \delta \nu, \inf_{y \in U \setminus (C_x \cup K_\delta)} \frac{\|y\|}{\|y^*\|} \big(f(y^*) - m/2\big)\Big\}, \tag{17}$$

the function $A_0(y) := A(y) + \varepsilon_1$ satisfies the required properties. $\quad\square$

Corollary 13. *Let $\varepsilon: U \to \mathbb{R}_+$ be a positive continuous function. Assume the graph of f does not contain lines. There exists a convex function $g: U \to \mathbb{R}$ such that*

$$f(x) < g(x) \leq f(x) + \varepsilon(x), \quad x \in U. \tag{18}$$

Proof. Let the graph of g coincide with the convex hull of the graph of $f + \varepsilon$. The inequality $g \leq f + \varepsilon$ is evident. To prove the inequality $f < g$, we need to modify the function A_x provided by [Theorem 10](#). Given any $x \in U$, we will construct an affine function $\tilde{A}_x$ such that $f(x) < \tilde{A}_x(x)$ and $\tilde{A}_x(y) < f(y) + \varepsilon(y)$ for all $y \in U$. This will prove the desired inequality $f < g$.

Let C_x be the compact set delivered by [Theorem 10](#) together with A_x. We pick some number $\theta \in (0,1)$ such that

$$\theta < \frac{\inf\{\varepsilon(y) : y \in C_x\}}{\sup\{A_x(y) - f(y) : y \in C_x\}} \tag{19}$$

and define

$$\tilde{A}_x(y) = \theta A_x(y) + (1-\theta)\big(f(x) + L(y-x)\big), \tag{20}$$

here $L \in \partial f(x)$ is an arbitrary function. Then,

$$\tilde{A}_x(y) - f(y) = \theta\big(A_x(y) - f(y)\big) + (1-\theta)\big(f(x) + L(y-x) - f(y)\big) < \varepsilon(y), \qquad y \in C_x. \tag{21}$$

In the case $y \notin C_x$ we simply have $\tilde{A}_x(y) < f(y)$; the inequality $f(x) < \tilde{A}_x(x)$ is also true. $\square$

3. Proofs of [Theorems 4](#) and [5](#), and of [Corollary 6](#)

We need to gather some facts and techniques from [1]. For instance we will be using *smooth maxima*: for any number $\delta > 0$, denote

$$M_\delta(x,y) = \frac{x + y + \theta(x-y)}{2}, \quad (x,y) \in \mathbb{R}^2,$$

where $\theta: \mathbb{R} \to (0, \infty)$ is a C^∞ function such that:

(1) $\theta(t) = |t|$ if and only if $|t| \geq \delta$;
(2) θ is convex and symmetric;
(3) $\mathrm{Lip}(\theta) = 1$.

If $f, g: U \subseteq \mathbb{R}^n \to \mathbb{R}$, define the function $M_\delta(f,g): U \to \mathbb{R}$ by

$$M_\delta(f,g)(x) = M_\delta(f(x), g(x)).$$

By $Lip(f)$ we mean the Lipschitz constant of f.

Proposition 14. *Let $f, g: U \subseteq \mathbb{R}^n \to \mathbb{R}$ be convex functions. For every $\delta > 0$, the function $M_\delta(f,g): U \to \mathbb{R}$ has the following properties:*

(1) *$M_\delta(f,g)$ is convex.*
(2) *If f is C^k on $\{x : f(x) \geq g(x) - \delta\}$ and g is C^k on $\{x : g(x) \geq f(x) - \delta\}$ then $M_\delta(f,g)$ is C^k on U. In particular, if f,g are C^k, then so is $M_\delta(f,g)$.*
(3) *$M_\delta(f,g)(x) = f(x)$ if $f(x) \geq g(x) + \delta$.*
(4) *$M_\delta(f,g)(x) = g(x)$ if $g(x) \geq f(x) + \delta$.*
(5) *$\max\{f,g\} \leq M_\delta(f,g) \leq \max\{f,g\} + \delta/2$.*
(6) *$M_\delta(f,g) = M_\delta(g,f)$.*
(7) *$Lip(M_\delta(f,g)|_B) \leq \max\{Lip(f|_B), Lip(g|_B)\}$ for every ball $B \subset U$.*

(8) If f, g are strictly convex on a set $B \subseteq U$, then so is $M_\delta(f, g)$.

(9) If $f, g \in C^2(U)$ are strongly convex on a set $B \subseteq U$, then so is $M_\delta(f, g)$.

(10) If $f_1 \le f_2$ and $g_1 \le g_2$ then $M_\delta(f_1, g_1) \le M_\delta(f_2, g_2)$.

Proof. See [1, Section 2]. □

The result below follows from the proof of [1, Theorem 1.1], although it was not explicitly mentioned there.

Theorem 15. *Let U be a nonempty convex open subset of $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ be convex. Assume that f cannot be written as $f = c \circ P + \ell$, where $P : \mathbb{R}^n \to \mathbb{R}^k$ is linear and surjective, $k < n$, $c : P(U) \to \mathbb{R}$, and ℓ is linear. Then f can be uniformly approximated on U by real-analytic strongly convex functions.*

For the sake of completeness, let us review the main points of the proof of [1, Theorem 1.1] and make some remarks as to why the approximations can be taken strongly convex if f is not of the form $f = c \circ P + \ell$. We will use some terminology from [1].

Definition 16. We will say that a function $C : \mathbb{R}^n \to \mathbb{R}$ is a k-dimensional *corner function* on $\mathbb{R}^n$ if it is of the form
$$C(x) = \max\{\, \ell_1 + b_1, \, \ell_2 + b_2, \, \ldots, \, \ell_k + b_k \,\},$$
where the $\ell_j : \mathbb{R}^n \to \mathbb{R}$ are linear functions such that the functions $L_j : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $L_j(x, x_{n+1}) = x_{n+1} - \ell_j(x)$, $1 \le j \le k$, are linearly independent, and the $b_j \in \mathbb{R}$.

We will also say that a convex function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is supported by C at a point $x \in U$ provided we have $C \le f$ on U and $C(x) = f(x)$.

The following lemma is a refinement of [1, Lemma 4.2].

Lemma 17. *Let $U \subseteq \mathbb{R}^n$ be open and convex, let $f : U \to \mathbb{R}$ be a convex function and $x_0 \in U$. Assume that f is not supported at x_0 by any $(n+1)$-dimensional corner function. Then there exist $k < n$, a linear projection $P : \mathbb{R}^n \to \mathbb{R}^k$, a convex function $c : P(U) \subseteq \mathbb{R}^k \to \mathbb{R}$, and a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $f = c \circ P + \ell$.*

In the statement of [1, Lemma 4.2], f was assumed to be C^1, but this was just for convenience; the same proof can be used to show the result for an arbitrary convex function (using the fact that if the range of the subdifferential of a convex function is contained in $\{0\}$ then the function is constant, and applying this to the function $(t_1, \ldots, t_{n-k}) \mapsto (f - \ell_1)(y + \sum_{j=1}^{n-k} t_j w_j)$).

Proof of Theorem 15. In order to show Theorem 15, one can argue as follows. Let us consider a compact convex subset K of U. Given $\varepsilon > 0$, since f is convex and Lipschitz on K we can find finitely many points $x_1, \ldots, x_m \in K$ and affine functions $h_1, \ldots, h_m : \mathbb{R}^n \to \mathbb{R}$ such that f is differentiable at each x_j, each h_j supports $f - \varepsilon$ at x_j, and $f - 2\varepsilon \le \max\{h_1, \ldots, h_m\}$ on K. By convexity we also have $\max\{h_1, \ldots, h_m\} \le f - \varepsilon$ on all of U. By the preceding lemma, for each x_j we may find a $(n+1)$-dimensional corner function C_j that supports $f - \varepsilon$ at x_j. Note that these corner functions are always defined on all of $\mathbb{R}^n$ (even when f is not). Since f is convex and differentiable at x_j, we have $h_j = C_j$ on a neighborhood of x_j and, by convexity, also $h_j \le C_j \le f - \varepsilon$ and $\max\{C_1, \ldots, C_m\} \le f - \varepsilon$ on U. We also have $f - 2\varepsilon \le \max\{h_1, \ldots, h_m\} \le \max\{C_1, \ldots, C_m\} \le f - \varepsilon$ on K. Now apply [1, Lemma 4.1] to the functions

$C_j + \varepsilon'/2$ in order to find C^∞ strongly convex functions $g_1, \ldots, g_m \colon \mathbb{R}^n \to \mathbb{R}$ such that $C_j \le g_j \le C_j + \varepsilon'$, where $\varepsilon' := \varepsilon/2m$, and define $g \colon \mathbb{R}^n \to \mathbb{R}$ by

$$g = M_{\varepsilon'}(g_1, M_{\varepsilon'}(g_2, M_{\varepsilon'}(g_3, \ldots, M_{\varepsilon'}(g_{m-1}, g_m))...))$$

(for instance, if $m = 3$, then $g = M_{\varepsilon'}(g_1, M_{\varepsilon'}(g_2, g_3))$). By Proposition 14, we have that $g \in C^\infty(\mathbb{R}^d)$ is strongly convex,

$$\max\{C_1, \ldots, C_m\} \le g \le \max\{C_1, \ldots, C_m\} + m\varepsilon' \le f - \frac{\varepsilon}{2} \quad \text{on} \quad U,$$

and

$$f - 2\varepsilon \le \max\{C_1, \ldots, C_m\} \le g \quad \text{on} \quad K.$$

Therefore, $f \colon U \subseteq \mathbb{R} \to \mathbb{R}$ can be approximated from below by C^∞ strongly convex functions, uniformly on each compact convex subset of U. By [1, Theorem 1.2] and Remark 1 in Section 2 of the same paper, we conclude that, given $\varepsilon > 0$ we may find a C^∞ strongly convex function h such that $f - 2\varepsilon \le h \le f - \varepsilon$ on U.

Finally, set

$$\eta(x) = \frac{1}{2} \min\{\varepsilon, \min\{D^2 h(x)(v)^2 : v \in \mathbb{R}^n, \|v\| = 1\}\}, \quad x \in U.$$

The function $\eta \colon U \to (0, \infty)$ is continuous, so we can apply Whitney's theorem (Lemma 6 in [6]) on C^2-fine approximation of C^2 functions by real-analytic functions to find a real analytic function $g \colon U \to \mathbb{R}$ such that

$$\max\{|h - g|, \|Dh - Dg\|, \|D^2 h - D^2 g\|\} \le \eta.$$

This implies that $f - 3\varepsilon \le g \le f$ and $D^2 g \ge \frac{1}{2} D^2 h > 0$, so g is strongly convex as well. $\square$

We will also make use of the following simple fact.

Lemma 18. *Let $I = (a, b)$, where $-\infty \le a < b \le +\infty$, let $\varphi \colon I \to \mathbb{R}$ be convex and let $\psi \colon \mathbb{R} \to [0, 1]$ be differentiable and such that $\lim_{|t| \to \infty} \psi(t) = 0$. If $I \ne \mathbb{R}$, also assume that $\psi^{-1}(0) = \mathbb{R} \setminus I$. If $|\varphi(t)| \le \psi(t)$ for all $t \in I$, then $\varphi(t) = 0$ for all $t \in I$.*

Proof. If $\varphi(s) > 0$ for some $s \in I$, then φ attains a maximum in I, and since φ is convex and $\lim_{t \to a^+} \varphi(t) = 0$, φ it must be constantly 0. Hence $\varphi \le 0$. Let us see that $\varphi(t) = 0$ for all $t \in I$. Take $t_0 \in I$. If $a \in \mathbb{R}$, by convexity we have

$$-\frac{\psi(t)}{t - a} \le \frac{\varphi(t)}{t - a} \le \frac{\varphi(t_0)}{t_0 - a} \quad \text{for all } t \in [a, t_0],$$

but

$$\lim_{t \to a^+} \frac{-\psi(t)}{t - a} = -\psi'(a) = 0,$$

so $0 \le \varphi(t_0)$. If $b \in \mathbb{R}$, similarly we get $\varphi(t_0) \ge 0$. Finally, if $I = \mathbb{R}$, since $\varphi \le 0$ is convex, φ must be constant, and the assumptions that $\lim_{|t| \to \infty} \psi(t) = 0$ and $|\varphi| \le \psi$ imply that this constant must be 0. $\square$

Proof of Theorem 5. (2) $\implies$ (3) is trivial.
(3) $\implies$ (1) is a consequence of Lemma 18: if the restriction of f to $U \cap \{x + tv : t \in \mathbb{R}\}$ is affine, we may consider a function $\varepsilon \colon \mathbb{R}^d \to [0, 1]$ of class C^1 such that $\lim_{|x| \to \infty} \varepsilon(x) = 0$ and $\mathbb{R}^n \setminus U = \varepsilon^{-1}(0)$ (if $U \ne \mathbb{R}^n$). By assumption there exists a convex function $g \colon U \to \mathbb{R}$ such that $f < g < f + \varepsilon$. Then we may apply Lemma 18 with $\varphi(t) := g(x + tv) - f(x + tv)$ and $\psi(t) = \varepsilon(x + tv)$ to find that $g(x + tv) = f(x + tv)$ for all t, contradicting that $f < g$.

(1) $\implies$ (2): We may assume that $\lim_{|x| \to \infty} \varepsilon(x) = 0$, and if $U \neq \mathbb{R}^n$, we may also assume that ε has C^1 extension to all of $\mathbb{R}^n$, denoted still by ε, such that $\varepsilon^{-1}(0) = \mathbb{R}^n \setminus U$. Let us fix a sequence of compact sets (K_j) such that

$$U = \bigcup_{j=1}^{\infty} K_j \quad \text{and} \quad K_j \subset \text{int}(K_{j+1}) \text{ for every } j \in \mathbb{N}.$$

By Corollary 13 there exists a convex function $h_1 : U \to \mathbb{R}$ such that

$$f < h_1 < f + \varepsilon.$$

Since the graph of f contains no lines, using the preceding lemma it is easy to see that the graph of h_1 contains no lines either (if the restriction of f to $U \cap \{x + te : t \in \mathbb{R}\}$ is affine, we may apply the lemma with the functions $\varphi(t) := h_1(x + te) - f(x + te)$ and $\psi(t) := \varepsilon(x + te)$). In particular h_1 is not of the form $h_1 = c \circ P + \ell$ for any linear projection $P : \mathbb{R}^n \to \mathbb{R}^k$ with $k < n$ and ℓ linear. Then, according to Theorem 15, we may find a strongly convex C^∞ function $g_1 : U \to \mathbb{R}$ such that

$$h_1 - \frac{\varepsilon_1}{3} < g_1 < h_1 - \frac{\varepsilon_1}{6} \text{ on } U,$$

where

$$\varepsilon_1 := \inf_{x \in K_1} \{h_1(x) - f(x)\}.$$

For future notational consistency, we also write $\varphi_1 = g_1$. By continuity of ε and compactness of K_1 there exists $m_1 \in \mathbb{N}$ such that

$$f + \frac{\varepsilon}{m_1} < h_1 - \frac{2}{3}\varepsilon_1 \text{ on } K_1,$$

and applying again Corollary 13 we can take a convex function $h_2 : U \to \mathbb{R}$ such that

$$f < h_2 < f + \frac{\varepsilon}{m_1} \text{ on } U.$$

Observe that the graph of h_2 cannot contain any line. Now let us set

$$\varepsilon_2 := \inf_{x \in K_2} \{h_2(x) - f(x)\} > 0,$$

and use Theorem 15 to obtain a strongly convex C^∞ function $\varphi_2 : U \to \mathbb{R}$ such that

$$h_2 - \frac{\varepsilon_2}{3} < \varphi_2 < h_2 - \frac{\varepsilon_2}{6} \text{ on } U.$$

Let us define

$$g_2 := M_{\delta_2}(g_1, \varphi_2),$$

where $\delta_2 = \varepsilon_2 / 12$. By using Proposition 14 we see that g_2 is a strongly convex C^∞ function satisfying

$$\max\{g_1, \varphi_2\} \leq g_2 \leq \max\{g_1, \varphi_2\} + \delta_2/2.$$

Also, since $\varphi_2 < h_2 - \varepsilon_2/6 < h_1 - 2\varepsilon_1/3 < g_1 - \varepsilon_1/3$ on K_1, and $\varepsilon_1/3 > \delta_2$, we obtain

$$g_2 = g_1 \text{ on } K_1.$$

Moreover, we have

$$f < g_2 < f + \varepsilon \text{ on } K_2,$$

because

$$g_2 \geq \varphi_2 > h_2 - \varepsilon_2/3 > h_2 - \varepsilon_2 \geq f \text{ on } K_2$$

and

$$g_2 \leq \max\{g_1, \varphi_2\} + \delta_2/2 \leq \max\{h_1 - \varepsilon_1/6, h_2 - \varepsilon_2/6\} + \delta_2/2 \leq \max\{h_1, h_2\} - \varepsilon_2/12 < f + \varepsilon$$

on U.

We continue this process by induction: suppose that, for $N \geq 2$, we have defined convex functions $h_1, \ldots, h_N : U \to \mathbb{R}$, strongly convex functions $g_1, \ldots, g_N$, and $\varphi_1, \ldots, \varphi_N \in C^\infty(U)$ (with $\varphi_1 = g_1$), numbers $1 = m_0 < m_1 < m_2 < \cdots < m_{N-1} \in \mathbb{N}$ such that, for every $j = 1, \ldots, N$,

$$f + \frac{\varepsilon}{m_j} < h_j - \frac{2}{3}\varepsilon_j \text{ on } K_j,$$

and

$$h_j - \frac{\varepsilon_j}{3} < \varphi_j < h_j - \frac{\varepsilon_j}{6} \text{ on } U,$$

where

$$\varepsilon_j := \inf_{x \in K_j}\{h_j(x) - f(x)\},$$

$$g_j = M_{\delta_j}(g_{j-1}, \varphi_j),$$

with

$$\delta_j = \frac{\varepsilon_j}{3 \cdot 2^j},$$

and also

$$g_j = g_{j-1} \text{ on } K_{j-1},$$

$$g_j \leq \max\{h_1, \ldots, h_j\} - \delta_j < f + \varepsilon \text{ on } U,$$

$$g_j \geq h_j - \varepsilon_j \geq f \text{ on } K_j.$$

Then we can find $m_N \in \mathbb{N}$ such that $m_N > m_{N-1}$ and

$$f + \frac{\varepsilon}{m_N} < h_N - \frac{2}{3}\varepsilon_N \text{ on } K_N,$$

and using Corollary 13, we obtain a convex function $h_{N+1} : U \to \mathbb{R}$ such that

$$f < h_{N+1} < f + \frac{\varepsilon}{m_N} \text{ on } U.$$

According to Lemma 18 the graph of h_N cannot contain any line, so by Theorem 15, for

$$\varepsilon_{N+1} := \inf_{x \in K_{N+1}}\{h_{N+1}(x) - f(x)\} > 0,$$

there exists a strongly convex function $\varphi_{N+1} \in C^\infty(U)$ such that

$$h_{N+1} - \frac{\varepsilon_{N+1}}{3} < \varphi_{N+1} < h_{N+1} - \frac{\varepsilon_{N+1}}{6} \text{ on } U.$$

We define

$$\delta_{N+1} = \frac{\varepsilon_{N+1}}{3 \cdot 2^{N+1}},$$

and

$$g_{N+1} = M_{\delta_{N+1}}(g_N, \varphi_{N+1}),$$

which is a strongly convex C^∞ function satisfying

$$\max\{g_N, \varphi_{N+1}\} \leq g_{N+1} \leq \max\{g_N, \varphi_{N+1}\} + \delta_{N+1}/2.$$

Since $\varphi_{N+1} < h_{N+1} - \varepsilon_{N+1}/6 < h_{N+1} - 2\varepsilon_N/3 < g_N - \varepsilon_N/3$ on K_N, and $\varepsilon_N/3 > \delta_{N+1}$, Proposition 14 implies

$$g_{N+1} = g_N \text{ on } K_N.$$

On the other hand,

$$g_{N+1} \geq \varphi_{N+1} > h_{N+1} - \varepsilon_{N+1}/3 > h_{N+1} - \varepsilon_{N+1} \geq f \text{ on } K_{N+1}$$

and

$$g_{N+1} \leq \max\{g_N, \varphi_{N+1}\} + \frac{\delta_{N+1}}{2} \leq \max\{\max\{h_1, \ldots, h_N\} - \delta_N, h_{N+1} - \frac{\varepsilon_{N+1}}{6}\} + \delta_{N+1}$$
$$\leq \max\{h_1, \ldots, h_{N+1}\} - \delta_{N+1} < f + \varepsilon$$

on U.

Therefore, by induction there exist sequences of functions (g_j), (φ_j), (h_j) satisfying the properties listed above for every $j \in \mathbb{N}$.

Let us finally define

$$g(x) = \lim_{j \to \infty} g_j(x), \quad x \in U.$$

Since the $g_j \in C^\infty(U)$ are strongly convex and satisfy $g_{j+1} = g_j$ on K_j, $K_j \subset \text{int } K_{j+1}$ for every j, and $U = \bigcup_{j \in \mathbb{N}} K_j$, it is clear that g is well defined, strongly convex, and of class $C^\infty(U)$. We also have $g_j > f$ on K_j for every j, and $g_j < f + \varepsilon$ on U for every j, which imply $f < g < f + \varepsilon$ on U.

In order to obtain a real-analytic function ψ with these properties, let

$$\eta(x) := \frac{1}{2} \min\left\{ g(x) - f(x), \ f(x) + \varepsilon(x) - g(x), \ \min\{D^2 g(x)(v)^2 : v \in \mathbb{R}^n, \|v\| = 1\} \right\}, x \in U,$$

which defines a strictly positive continuous function on U. We can apply Whitney's theorem (Lemma 6 in [6]) on C^2-fine approximation of C^2 functions to find a real-analytic function $\psi : U \to \mathbb{R}$ such that

$$\max\{|\psi - g|, \|D\psi - Dg\|, \|D^2\psi - D^2g\|\} \leq \eta.$$

This implies that $f < \psi < f + \varepsilon$ and $D^2\psi \geq \frac{1}{2}D^2g > 0$, so g is strongly convex too. $\quad\square$

Proof of Theorem 4. $(1) \implies (2)$: Since the graph of f does not contain any ray, it does not contain any line either. Then, according to Theorem 8, there exists a convex function $h : U \to \mathbb{R}$ such that $f - \varepsilon < h < f$. Setting $\delta(x) = f(x) - h(x)$, $x \in U$, we may apply Theorem 5 to h to find a real-analytic strongly convex function $g : U \to \mathbb{R}$ such that $h < g < h + \delta$, which implies $f - \varepsilon < g < f$.

$(2) \implies (3)$ is trivial.

$(3) \implies (1)$ can be proved by using the following variant of Lemma 18 with the function $\varphi(t) = g(x + tv) - f(x + tv)$, $t \in (a, t_0]$, assuming that the graph of f is affine on some ray $\{x + tv : t \in (a, t_0]\}$ of U.

Lemma 19. *Let $I = (a, b)$, where $-\infty \leq a < b \leq +\infty$, let $\varphi : I \to \mathbb{R}$ be convex and let $\psi : \mathbb{R} \to [0, 1]$ be differentiable and such that $\lim_{|t| \to \infty} \psi(t) = 0$. If $I \neq \mathbb{R}$, also assume that $\psi^{-1}(0) = \mathbb{R} \setminus I$. Let $t_0 \in I$. If $-\psi(t) \leq \varphi(t)$ for all $t \in I \cap (a, t_0]$, then $\varphi(t) \geq 0$ for all $t \in I$ sufficiently close to a.*

The proof of this lemma is similar to that of Lemma 18 and is left to the reader. $\quad\square$

Proof of Corollary 6. $(1) \implies (2)$ is an obvious consequence of Theorem 5, and $(2) \implies (3)$ is trivial. Let us see that $(3) \implies (1)$: assume (1) is false; then there exists a line $\{x + tv : t \in \mathbb{R}\} \cap U = \{x + tv : t \in (a, b)\}$, on which f is affine. Let $\varepsilon : \mathbb{R}^n \to [0, 1]$ be of class C^1 and such that $\lim_{|x| \to \infty} \varepsilon(x) = 0$ and (if $U \neq \mathbb{R}^n$) also $\mathbb{R}^n \setminus U = \varepsilon^{-1}(0)$. By the assumption there exists a strictly convex function $g : U \to \mathbb{R}$ such that $|f - g| < \varepsilon$. Then, by applying Lemma 18 with $\varphi(t) := g(x + tv) - f(x + tv)$ and $\psi(t) = \varepsilon(x + tv)$, we deduce that $g(x + tv) = f(x + tv)$ for all t. This contradicts that g is strictly convex. $\quad\square$

4. Proofs of Theorems 1 and 2

Besides Theorems 4 and 5, in the proofs of Theorems 1 and 2 we will use the following lemmas.

Lemma 20 (*See [2], Lemma 3.2*). *Let $W \subset \mathbb{R}^n$ be a closed convex set such that $0 \in \mathrm{int}(W)$, and let $\mu = \mu_W$ denote the Minkowski functional of W. The following assertions are equivalent:*

(a) W does not contain any line.
(b) ∂W does not contain any line.
(c) $\mu^{-1}(0)$ does not contain any line
(d) μ is essentially coercive.

Lemma 21. *Let $S = \partial V$, where V is a closed convex set V with nonempty interior in $\mathbb{R}^n$. If S does not contain any rays and is unbounded, then S can be regarded (up to a suitable rotation and translation) as the graph of a convex function $f : U \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\lim_{y \in U, |y| \to \infty} f(y) = \infty$ (if U is unbounded) and $\lim_{y \to x} f(y) = \infty$ for every $x \in \partial U$ (if $U \neq \mathbb{R}^{n-1}$). In particular f is properly convex and its graph contains no ray.*

Proof. Since S does not contain any line, nor does V (according to the preceding lemma). And since S is unbounded, so is V, hence V contains a ray. Consider the maximal inscribed cone of V:

$$K = \left\{ e \in \mathbb{R}^n : \exists x \in U \text{ such that } \{x + te : t > 0\} \subset V \right\}. \tag{22}$$

The cone K is non-empty, closed, convex, and does not contain lines. Consider the hyperplane H constructed in the proof of Theorem 12 (we need the explicit construction with the closest point ζ presented in the proof). Let us introduce the orthogonal coordinates such that $H = \{x \in \mathbb{R}^n : x_n = 0\}$ and $x_n > 0$ on K. Note that in such a case ζ lies on the Ox_n axis, which yields the ray $(0, 0, \dots, 0, t)$, where $t \in \mathbb{R}_+$, belongs to K. We will call this ray the positive half of the Ox_d-axis.

Let P_H be the orthogonal projection of $\mathbb{R}^n$ onto H. Set $U = P_H(V)$ and $f(y) = \inf\{t \in \mathbb{R} : (y, t) \in V\}$, here $y \in U$. Let us prove that this choice indeed fulfills the requirements. First, since the positive half of the Ox_d axis lies in K, any ray of the form $\{x + (0, 0, \dots, 0, t) : t > 0\}$ lies in V, provided $x \in V$. Thus, V is indeed the epigraph of f.

Second, let us check two limit assertions. Similar to (12),

$$K \subset \left\{ y \in \mathbb{R}^n : y_n > \delta \|y\| \right\} \tag{23}$$

for a sufficiently small number $\delta > 0$. This, in particular, leads to the bound $f(y) \geq \delta \|y\| - C$ for a sufficiently large constant C. Therefore, $f(x)$ tends to infinity as x tends to infinity inside U, supporting the first limit assertion. On the other hand, if $U \neq \mathbb{R}^{n-1}$ and $x \in \partial U$, the limit $\beta := \lim_{y \to x} f(y)$ exists in $(0, +\infty]$. If β were finite then S would contain the ray $\{(x, t) : t \geq \beta\}$, contradicting the assumption that S contain no rays. Therefore β is infinite, and the second limit assertion is also verified. This also shows S is the graph of f. $\quad\square$

Proof of Theorem 1. $(1) \implies (3)$: If S is unbounded then this implication is an immediate consequence of Theorem 4 and Lemma 21. On the other hand, if S is compact, the result is well known. Nevertheless, for completeness, let us provide a short proof of this case based on the preceding results. We may assume that $0 \in \mathrm{int}(V)$ and U is of the form $\varphi^{-1}(1 - 2\varepsilon, 1 + 2\varepsilon)$, where $\varphi = \mu^2$, μ denoting the Minkowski functional of V, and ε is a positive constant. The function φ is convex and coercive, and its graph does not contain any ray. By Theorem 4 there exists a real-analytic strongly convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi - \varepsilon < g < \varphi$.

Let us define $W = g^{-1}(-\infty, 1]$. Then $S_U := \partial W = g^{-1}(1)$ is a real-analytic strongly convex hypersurface with $S_U \subset U \setminus V$.

(3) $\implies$ (2) is obvious.

(2) $\implies$ (1): this can be proved similarly to (3) $\implies$ (1) of Theorem 4. The details are left to the reader. $\quad\square$

Proof of Theorem 2. (1) $\implies$ (3): By Lemma 20, the Minkowski functional of V, which we will denote μ, is essentially coercive (and in particular its graph does not contain any line). Given an open set $U \supset S$, by using partitions of unity for instance, it is not difficult to construct a continuous function $\varepsilon : \mathbb{R}^n \to (0, 1]$ such that

$$\mu(x) + \varepsilon(x) < 1 \text{ for all } x \in V \setminus U.$$

Then we may apply Theorem 5 to find a real-analytic strongly convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\mu < g < \mu + \varepsilon$. Define $S_U = g^{-1}(1)$. It is clear that S_U is a real-analytic strongly convex hypersurface (the boundary of the convex body $g^{-1}(-\infty, 1]$). If $x \in V \setminus U$, then we have $g(x) < \mu(x) + \varepsilon(x) < 1$, and if $x \in \mathbb{R}^n \setminus \text{int}(V)$, then $g(x) > \mu(x) \geq 1$. Therefore, $S_U = g^{-1}(1) \subset U \cap \text{int}(V)$.

(3) $\implies$ (2) is trivial.

(2) $\implies$ (1) is similar to (3) $\implies$ (1) of Theorem 5. The details are left to the reader. $\quad\square$

5. Proof of Theorem 7

Let us gather some preliminary results that we will be using in the proof. The following theorem is well known; see, for instance, [4, Theorem 25.7].

Theorem 22. *Let U be a nonempty open convex subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}$ be a differentiable convex function, and (f_k) be a sequence of differentiable convex functions such that $f(x) = \lim_{k \to \infty} f_k(x)$ for every $x \in U$. Then Df_k converges to Df, uniformly on each compact subset of U.*

The following fact about smooth maxima is shown in [1, Lemma 7.1].

Lemma 23. *Let M_δ the smooth maximum of Proposition 14, and let $V \subseteq \mathbb{R}^n$ be an open set. If $\varphi, \psi \in C^1(V)$, then*

$$\|DM_\delta(\varphi, \psi) - \frac{D\varphi + D\psi}{2}\| \leq \frac{1}{2}\|D\varphi - D\psi\|.$$

We will also use the following consequence of Theorems 10, 4 and 22.

Lemma 24. *Let U be a nonempty open convex subset of $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ be convex and C^1. Assume that the graph of f contains no lines. Then, for every continuous function $\varepsilon : U \to (0, 1)$ and every compact set $K \subset U$ there exist a compact set C and a convex C^1 function $\psi : U \to \mathbb{R}$ such that:*

(1) $f \leq \psi < f + \varepsilon$ on U;
(2) $f < \psi$ on K;
(3) $K \subset \text{int}(C) \subset C \subset U$;
(4) $f = \psi$ on $U \setminus C$;
(5) ψ is strongly convex on $\text{int}(C)$, and
(6) $\|D\psi - Df\| < \varepsilon$ on U.

Proof. For every $x \in K$, by Theorem 10 there exist an affine function $A_x : \mathbb{R}^n \to \mathbb{R}$ and a compact set $C_x \subset U$ such that $f(x) - A_x(x) < 0$ and $f(y) - A_x(y) > 0$ for all $y \in U \setminus C_x$. In particular

$D_x := \{y \in U : f(y) - A_x(y) \leq 0\}$ is a compact convex neighborhood of x. Since K is compact, we may find finitely many points $x_1, \ldots, x_m \in K$ such that

$$K \subset \bigcup_{j=1}^{m} \text{int}(D_{x_j}).$$

Observe that the graph of the restriction of f to $\text{int}(D_{x_j})$ cannot contain any line for any $j = 1, \ldots, m$. For every $j = 1, \ldots, m$, let $\varepsilon_j : U \to [0, 1]$ be a C^1 function such that $\varepsilon_j^{-1}(0) = U \setminus \text{int}(D_{x_j})$ and $\varepsilon_j \leq \varepsilon$. According to Theorem 4, for each j there exists a strongly convex C^∞ function $\varphi_j : \text{int}(D_{x_j}) \to \mathbb{R}$ such that $f < \varphi_j < \varepsilon_j$ on $\text{int}(D_{x_j})$. For each j, we can extend φ_j to all of U by setting $\varphi_j = f$ on $U \setminus \text{int}(D_{x_j})$, and since ε_j is of class C^1 and satisfies $\varepsilon_j = 0$ on $U \setminus \text{int}(D_{x_j})$, we have that φ_j is differentiable on U, which (because φ_j is convex) amounts to saying that $\varphi_j \in C^1(U)$.

Let us call $C = \bigcup_{j=1}^{m} D_{x_j}$ and

$$\varphi = \frac{1}{m} \sum_{j=1}^{m} \varphi_j.$$

It is easy to check that C and φ satisfy properties $(1) - (5)$ of the statement (with φ in place of ψ).

Now, for each $k \in \mathbb{N}$, we may apply what we have just proved with ε/k replacing ε, and we obtain a sequence (ψ_k) of C^1 convex functions satisfying properties $(1) - (5)$ (with the same C) and also

$$f \leq \psi_k \leq f + \frac{\varepsilon}{k} \text{ on } U$$

for every k. Then, by Theorem 22, $D\psi_k$ converges to Df uniformly on C, and therefore we can find k large enough so that

$$\|D\psi_k(x) - Df(x)\| \leq \min_{y \in C} \varepsilon(y) \text{ for all } x \in C.$$

Since $\psi_k = f$ on $U \setminus C$, we also have $D\psi_k = Df$ on $U \setminus C$, so by setting $\psi = \psi_k$ we get a C^1 convex function satisfying properties $(1) - (6)$. □

We are ready to establish a C^1-fine version of Corollary 13.

Proposition 25. *Let U be an nonempty open convex subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}$ a C^1 convex function, and $\varepsilon : U \to (0, 1)$ a continuous function. Assume that the graph of f contains no lines. Then there exists a C^1 strongly convex function $g : U \to \mathbb{R}$ such that*

$$f < g < f + \varepsilon \text{ and } \|Dg - Df\| < \varepsilon \text{ on } U.$$

Proof. Let us fix a sequence of compact sets (K_j) such that

$$U = \bigcup_{j=1}^{\infty} K_j \quad \text{and} \quad K_j \subset \text{int}(K_{j+1}) \text{ for every } j \in \mathbb{N}.$$

For every $j \in \mathbb{N}$, by Lemma 24 there exist a compact set C_j and a convex C^1 function $g_j : U \to \mathbb{R}$ such that:

(1) $f \leq g_j \leq f + \varepsilon/2$ on U;
(2) $f < g_j$ on K_j;
(3) $K_j \subset \text{int}(C_j) \subset C_j \subset U$;
(4) $f = g_j$ on $U \setminus C_j$;
(5) g_j is strongly convex on $\text{int}(C_j)$, and
(6) $\|Dg_j - Df\| \leq \varepsilon/2$ on U.

Let us define

$$g = \sum_{j=1}^{\infty} \frac{1}{2^j} g_j.$$

It is routine to check that $g : U \to \mathbb{R}$ is of class C^1, strongly convex, and satisfies $f < g < f + \varepsilon$ and $\|Dg - Df\| < \varepsilon$ on U. $\square$

Now let us proceed with the proof of Theorem 7. We only need to show that $(1) \implies (2)$ (the converse is easily shown as in the proof of Corollary 6).

As in the proof of Theorem 5, we may assume that $\lim_{|x| \to \infty} \varepsilon(x) = 0$ and, if $U \neq \mathbb{R}^d$, that ε has C^1 extension to all of $\mathbb{R}^d$, denoted still by ε, such that $\varepsilon^{-1}(0) = \mathbb{R}^d \setminus U$. Let us fix a sequence of compact sets (K_j) such that

$$U = \bigcup_{j=1}^{\infty} K_j \quad \text{and} \quad K_j \subset \text{int}(K_{j+1}) \text{ for every } j \in \mathbb{N}.$$

By the preceding proposition there exist a strongly convex C^1 function $h_1 : U \to \mathbb{R}$ such that

$$f < h_1 < f + \varepsilon \text{ and } \|Dh_1 - Df\| < \varepsilon \text{ on } U.$$

Let us set

$$\varepsilon_1 := \inf_{x \in K_1} \{h_1(x) - f(x)\}.$$

By continuity of ε and compactness of K_1 there exists $m_1 \in \mathbb{N}$ such that

$$f + \frac{\varepsilon}{m_1} < h_1 - \frac{2}{3}\varepsilon_1 \text{ on } K_1,$$

and applying again Proposition 25 we can take a C^1 strongly convex function $h_2 : U \to \mathbb{R}$ such that

$$f < h_2 < f + \frac{\varepsilon}{m_1} \text{ and } \|Df - Dh_2\| \leq \frac{1}{4}\varepsilon \text{ on } U.$$

Using the limiting properties of ε, the inequalities $f < h_1 < f + \varepsilon$ and $f < h_2 < f + \varepsilon/m_1$, and the fact that $U = \bigcup_{j=1}^{\infty} K_j$, we may find a number $n_2 \in \mathbb{N}$ such that

$$h_2 > h_1 - \frac{\varepsilon_1}{12} \text{ on } U \setminus K_{n_2}.$$

We set $n_1 = 1$, $\varepsilon_2 = \inf_{y \in K_{n_2}} \{h_2(y) - f(y)\}$, and find $n_3 > n_2$ so that

$$h_3 > h_2 - \frac{\varepsilon_2}{96} \text{ on } U.$$

By continuing this process by induction, we obtain sequences $m_0 = 1 < m_1 < m_2 < \cdots$ and $n_1 = 1 < n_2 < n_3 < \cdots$ of positive integers, and C^1 strongly convex functions $h_j : U \to \mathbb{R}$, $j \in \mathbb{N}$, satisfying

$$f + \frac{\varepsilon}{m_j} < h_j - \frac{2}{3}\varepsilon_j \text{ on } K_{n_j},$$

$$f < h_j < f + \frac{\varepsilon}{m_{j-1}} \text{ and } \|Dh_j - Df\| < \frac{1}{4}\varepsilon \text{ on } U,$$

and

$$h_{j+1} > h_j - \frac{\varepsilon_j}{2^{3j-1} \cdot 3} \text{ on } U \setminus K_{n_{j+1}},$$

where

$$\varepsilon_j := \inf_{x \in K_{n_j}} \{h_j(x) - f(x)\}.$$

Next, for every $j \in \mathbb{N}$, we may combine Theorems 5 and 22 in order to find a C^∞ strongly convex function $\varphi_j : U \to \mathbb{R}$ such that

$$h_j - \frac{\varepsilon_j}{2^{3j-3} \cdot 3} < \varphi_j < h_j - \frac{\varepsilon_j}{2^{3j-2} \cdot 3} \text{ on } U,$$

and

$$\|D\varphi_j - Dh_j\| < \frac{1}{4} \inf\{\varepsilon(y) : y \in K_{n_{j+1}}\} \text{ on } K_{n_{j+1}}.$$

Now let us call $\varphi_1 = g_1$, and for every $j \geq 2$, define

$$g_j = M_{\delta_j}(g_{j-1}, \varphi_j),$$

where

$$\delta_j = \frac{\varepsilon_j}{2^{3j-2} \cdot 3}.$$

Claim 26. *For every $j \in \mathbb{N}$, g_j is a C^∞ strongly convex function satisfying:*

(1) $g_{j+1} = g_j$ on K_{n_j}
(2) $g_{j+1} = \varphi_{j+1}$ on $U \setminus K_{n_{j+1}}$
(3) $g_j \leq \max\{h_1, \ldots, h_j\} - \delta_j/2 < f + \varepsilon$ on U
(4) $g_j \geq h_j - \varepsilon_j \geq f$ on K_{n_j}.

Proof. By using Proposition 14 we see that g_{j+1} is a strongly convex C^∞ function satisfying

$$\max\{g_j, \varphi_{j+1}\} \leq g_j \leq \max\{g_j, \varphi_{j+1}\} + \delta_{j+1}/2.$$

Since

$$g_j \geq \varphi_j > h_j - \frac{\varepsilon_j}{2^{3j-3} \cdot 3} > h_{j+1} + \frac{2}{3}\varepsilon_j - \frac{\varepsilon_j}{2^{3j-3} \cdot 3}$$
$$> \varphi_{j+1} + \frac{\varepsilon_{j+1}}{2^{3(j+1)-2} \cdot 3} + \frac{2}{3}\varepsilon_j - \frac{\varepsilon_j}{2^{3j-3} \cdot 3} > \varphi_{j+1} + \delta_{j+1},$$

Proposition 14 also implies that

$$g_{j+1} = M_{\delta_{j+1}}(g_j, \varphi_{j+1}) = g_j \text{ on } K_{n_j},$$

which shows (1). As for (4), we have

$$g_j \geq \varphi_j > h_j - \frac{\varepsilon_j}{2^{3j-3} \cdot 3} > h_j - \varepsilon_j \geq f \text{ on } K_{n_j}.$$

We show the rest of these properties by induction on j. On $U \setminus K_{n_2}$, we have

$$\varphi_2 > h_2 - \varepsilon_2/24 > h_1 - \varepsilon_1/12 - \varepsilon_2/24 > g_1 + \varepsilon_1/6 - \varepsilon_1/12 - \varepsilon_2/24 > g_1 + \delta_2,$$

so we obtain that $g_2 = \varphi_2$ outside K_{n_2}. Assuming that (2) is true for $1 \leq j \leq \ell-1$, let us see that $g_{\ell+1} = \varphi_{\ell+1}$ on $U \setminus K_{n_{\ell+1}}$. On $U \setminus K_{n_{\ell+1}}$ we have

$$\varphi_{\ell+1} > h_{\ell+1} - \frac{\varepsilon_{\ell+1}}{2^{3(\ell+1)-3} \cdot 3} > h_\ell - \frac{\varepsilon_j}{2^{3\ell-1} \cdot 3} - \frac{\varepsilon_{\ell+1}}{2^{3(\ell+1)-3} \cdot 3}$$
$$> \varphi_\ell + \frac{\varepsilon_\ell}{2^{3\ell-2} \cdot 3} - \frac{\varepsilon_\ell}{2^{3\ell-1} \cdot 3} - \frac{\varepsilon_{\ell+1}}{2^{3(\ell+1)-3} \cdot 3}$$
$$= g_\ell + \frac{\varepsilon_\ell}{2^{3\ell-1} \cdot 3} - \frac{\varepsilon_{\ell+1}}{2^{3(\ell+1)-3} \cdot 3} \geq g_\ell + \frac{\varepsilon_{\ell+1}}{2^{3\ell-1} \cdot 3} - \frac{\varepsilon_{\ell+1}}{2^{3(\ell+1)-3} \cdot 3} > g_\ell + \delta_{\ell+1},$$

hence $g_{\ell+1} = M_{\delta_{\ell+1}}(g_\ell, \varphi_{\ell+1}) = \varphi_{\ell+1}$ outside $K_{n_{\ell+1}}$. This proves (2).

Finally, for (3), we have

$$g_2 \leq \max\{g_1, \varphi_2\} + \delta_2/2 \leq \max\{h_1 - \varepsilon_1/6, h_2 - \varepsilon_2/48\} + \delta_2/2 \leq \max\{h_1, h_2\} - \delta_2/2 < f + \varepsilon$$

on U. Assume now that property (3) is true for j, and let us see that it is also true for $j+1$. Indeed we have

$$g_{j+1} \leq \max\{g_j, \varphi_{j+1}\} + \frac{\delta_{j+1}}{2} \leq \max\{\max\{h_1, \ldots, h_j\} - \frac{\delta_j}{2}, h_{j+1} - \delta_{j+1}\} + \frac{\delta_{j+1}}{2}$$
$$\leq \max\{h_1, \ldots, h_{j+1}\} - \delta_{j+1} < f + \varepsilon$$

on U. This shows (3). $\square$

Claim 27. *For every $j \in \mathbb{N}$ we have $\|Dg_j - Df\| \leq \varepsilon$ on $K_{n_{j+1}}$.*

Proof. We will use induction in j. We start with the base case $j = 1$. On $K_{n_1} = K_1$, we have $g_2 = g_1 = \varphi_1$, hence

$$\|Dg_2 - Df\| \leq \|Dg_1 - Dh_1\| + \|Dh_1 - Df\| \leq \frac{1}{4} \min_{y \in K_{n_2}} \varepsilon(y) + \frac{1}{4}\varepsilon \leq \varepsilon.$$

What is more, for any j, we have that on $K_{n_{j+1}} \setminus K_{n_j}$ we know by the preceding claim that $g_j = \varphi_j$, so we also have

$$\|Dg_j - Df\| = \|D\varphi_j - Df\| \leq \|D\varphi_j - Dh_j\| + \|Dh_j - Df\| \leq \frac{1}{4} \min_{y \in K_{n_{j+1}}} \varepsilon(y) + \frac{1}{4} < \varepsilon.$$

This finishes the proof of the base. Now we pass to the induction step $(j-1) \rightarrow j$. We have already proved the desired inequality on $K_{n_{j+1}} \setminus K_{n_j}$.

On $K_{n_j} \setminus K_{n_{j-1}}$ we have, using Lemma 23 and the above claim,

$$\|Dg_j - Df\| \leq \frac{1}{2}\|Dg_{j-1} - D\varphi_j\| + \frac{1}{2}\|Dg_j + D\varphi_j - 2Df\|$$
$$\leq \frac{1}{2}\|Dg_{j-1} - D\varphi_j\| + \frac{1}{2}\|Dg_{j-1} - Df\| + \frac{1}{2}\|Df - D\varphi_j\|$$
$$\leq \frac{1}{2}\|D\varphi_{j-1} - Dh_{j-1}\| + \frac{1}{2}\|Dh_{j-1} - Df\| + \frac{1}{2}\|Df - D\varphi_j\| + \frac{1}{2}\|Dg_{j-1} - Df\| + \frac{1}{2}\|Df - D\varphi_j\|$$
$$\leq \|D\varphi_{j-1} - Dh_{j-1}\| + \|Df - Dh_{j-1}\| + \|D\varphi_j - Dh_j\| + \|Df - Dh_j\|$$
$$\leq \frac{1}{4} \min_{y \in K_{n_j}} \varepsilon(y) + \frac{1}{4}\varepsilon + \frac{1}{4} \min_{y \in K_{n_{j+1}}} \varepsilon(y) + \frac{1}{4}\varepsilon.$$

On $K_{n_{j-1}}$, the desired inequality is a consequence of property (1) of the preceding claim and the induction hypothesis. Thus, we have deduced that $\|Dg_j - Df\| \leq \varepsilon$ on $K_{n_{j+1}}$ for every j. $\square$

Let us finally define

$$g(x) = \lim_{j \to \infty} g_j(x), \quad x \in U.$$

Since the $g_j \in C^\infty(U)$ are strongly convex and satisfy $g_{j+1} = g_j$ on K_{n_j}, $K_{n_j} \subset \text{int}(K_{n_{j+1}})$ for every j, and $U = \bigcup_{j \in \mathbb{N}} K_{n_j}$, it is clear that g is well defined, strongly convex, and of class $C^\infty(U)$. From Claim 26 we see that

$$f < g < f + \varepsilon \text{ on } U,$$

and from Claim 27 we deduce that

$$\|Dg - Df\| \leq \varepsilon \text{ on } U.$$

In order to obtain a real-analytic function ψ with these properties, we consider

$$\eta(x) := \frac{1}{4} \min\{g(x) - f(x), \ f(x) + \varepsilon(x) - g(x), \ \min_{|v|=1} D^2 g(x)(v)^2\}, \quad x \in U,$$

which defines a strictly positive continuous function on U, and we apply Whitney's approximation theorem to find a real-analytic function $\psi : U \to \mathbb{R}$ such that

$$\max\{|\psi - g|, \|D\psi - Dg\|, \|D^2\psi - D^2g\|\} \leq \eta.$$

This implies that $f < \psi < f + \varepsilon$, $\|Df - D\psi\| < 2\varepsilon$ and $D^2\psi \geq \frac{1}{2}D^2g > 0$, so ψ is strongly convex too. $\quad\square$

Let us make one final remark. One can wonder if there are analogues of Theorem 7 for C^k fine approximation with $k \geq 2$. Our methods cannot be employed to answer this question, due to the following fact: if f, g are C^2 convex functions, then in general the second derivative of $M_\delta(f, g)$ blows up as δ goes to 0.

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