CONVEX C^1 EXTENSIONS OF 1-JETS FROM COMPACT SUBSETS OF HILBERT SPACES

PROLONGEMENTS CONVEXES ET DIFFERENTIABLES DE CHAMPS TAYLORIENS D'ORDRE 1 DANS L'ESPACE DE HILBERT

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. **ABSTRACT**

Let X denote a Hilbert space. Given a compact subset K of X and two continuous functions $f : K \to \mathbb{R}$, $G : K \to X$, we show that a necessary and sufficient condition for the existence of a convex function $F \in C^1(X)$ such that $F = f$ on K and $\nabla F = G$ on K is that the 1-jet (f, G) satisfies:

(1) $f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in K$, and
(2) if $x, y \in K$ and $f(x) = f(y) + \langle G(y), x - y\rangle$ then $G(x) = G(y)$.

We also solve a similar problem for K replaced with an arbitrary bounded subset of X, and for $C^1(X)$ replaced with the class $C^{1,u}_b(X)$ of differentiable functions with uniformly continuous derivatives on bounded subsets of X.

RÉSUMÉ

Soit X un espace de Hilbert. Nous montrons que, étant donné un sous-ensemble compact K de X et deux fonctions continues $f : K \to \mathbb{R}$, $G : K \to X$, pour qu'il existe une fonction convexe $F \in C^1(X)$ tel que $(F, \nabla F) = (f, G)$ dans K, il faut et il suffit que

(1) $f(x) \geq f(y) + \langle G(y), x - y\rangle$ pour tout $x, y \in K$, et que
(2) si $x, y \in K$ et $f(x) = f(y) + \langle G(y), x - y\rangle$, $G(x) = G(y)$.

Nous résolvons également un problème similaire pour K remplacé par un sous-ensemble borné arbitraire de X, et pour $C^1(X)$ remplacé par la classe $C^{1,u}_b(X)$ de fonctions différentiables avec des dérivées uniformément continues sur les sous-ensembles bornés de X.

In [2], among other results, we showed the following.

Theorem 1. *If K is a compact subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$, $G : K \to \mathbb{R}^n$ are continuous functions, then a necessary and sufficient condition for the existence of a convex function $F \in C^1(\mathbb{R}^n)$ such that $F = f$ on K and $\nabla F = G$ on K is that the 1-jet (f, G) satisfies:*

(C) *$f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in K$, and*
(CW^1) *if $x, y \in K$ and $f(x) = f(y) + \langle G(y), x - y\rangle$ then $G(x) = G(y)$.*

Gilles Godefroy asked whether this statement should remain true if we replace $\mathbb{R}^n$ with a Hilbert space X. The purpose of this note is to give an affirmative answer to this question.

We refer to the introductions and the bibliography of [2, 1, 3] for motivation, insight and general reference about this kind of problems. Let us only mention that if one wants to replace K with a closed set in Theorem 1 then it is necessary to introduce more sophisticated conditions, see [3, Theorems 1.8 and 1.13]. Taking into account the difficulties that infinite dimensions add (such as the lack of local compactness and the existence of continuous convex functions which are not bounded on

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bounded sets), one can expect that even much more complicated conditions would be required to deal with the general case of a 1-jet (f, G) defined on a noncompact closed set E of a Hilbert space X. However, for a compact $E \subset X$, the result is as easy as in $\mathbb{R}^n$.

Theorem 2. *Let X denote a Hilbert space. Given a compact subset K of X and two continuous functions $f : K \to \mathbb{R}$, $G : K \to X$, a necessary and sufficient condition for the existence of a convex function $F \in C^1(X)$ such that $(F, \nabla F) = (f, G)$ on K is that the 1-jet (f, G) satisfies:*

- (C) *$f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in K$, and*
- (CW^1) *if $x, y \in K$ and $f(x) = f(y) + \langle G(y), x - y\rangle$ then $G(x) = G(y)$.*

Furthermore, whenever these conditions are satisfied, the extension F can be taken to be Lipschitz, with $Lip(F) \leq 5 \max_{z \in K} |G(z)|$.

This theorem can be viewed as a particular case of the following result. We let $C_b^{1,u}(X)$ stand for the class of all differentiable functions $f : X \to \mathbb{R}$ such that their gradients $\nabla f : X \to X$ are uniformly continuous on each bounded subset of X. If ω is a modulus of continuity, we also define $C^{1,\omega}(X)$ as the set of all differentiable functions $f : X \to \mathbb{R}$ such that for some $M > 0$ we have $|\nabla f(x) - \nabla f(y)| \leq M\omega(|x - y|)$ for all $x, y \in X$.

Theorem 3. *Given a Hilbert space X, a bounded subset B of X, and two functions $f : B \to \mathbb{R}$, $G : B \to X$ such that G is bounded, a necessary and sufficient condition for the existence of a convex function $F \in C_b^{1,u}(X)$ such that $(F, \nabla F) = (f, G)$ on B is that the 1-jet (f, G) satisfies:*

- (C) *$f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in B$;*
- (SCW^1) *if (x_n), (y_n) are sequences in B and $\lim_{n\to\infty} (f(x_n) - f(y_n) - \langle G(y_n), x_n - y_n\rangle) = 0$ then $\lim_{n\to\infty} (G(x_n) - G(y_n)) = 0$.*

Furthermore, whenever these conditions are satisfied, the extension F can be taken to be Lipschitz, with $Lip(F) \leq 5 \sup_{z \in B} |G(z)|$.

Obviously one can take $x_n = x$ and $y_n = y$ in condition (SCW^1), so it is clear that this condition is generally stronger than (CW^1). But in the case of a compact set K, these conditions are equivalent (under the continuity assumption on f and G). Indeed, suppose that (CW^1) holds and we are given two sequences (x_n), $(y_n) \subseteq K$ such that $\lim_{n\to\infty} (f(x_n) - f(y_n) - \langle G(y_n), x_n - y_n\rangle) = 0$. If we do not have $\lim_{n\to\infty} (G(x_n) - G(y_n)) = 0$ then we can take subsequences converging to points $x, y \in K$ respectively such that $f(x) - f(y) - \langle G(y), x - y\rangle = 0$ and $|G(x) - G(y)| > 0$, and so condition (CW^1) fails. Thus Theorem 3 generalizes Theorem 2 (which in turn implies Theorem 1).

Proof of Theorem 3. We start by proving that (SCW^1) is a necessary condition. Let $F \in C_b^{1,u}(X)$ be a convex function and assume, for the sake of contradiction, that there are two sequences (x_n), $(y_n) \subset B$ and some $\varepsilon > 0$ for which

$$\alpha_n := f(x_n) - f(y_n) - \langle \nabla F(y_n), x_n - y_n\rangle \to 0, \quad \text{and} \quad |\nabla F(x_n) - \nabla F(y_n)| \geq \varepsilon \quad \text{for all} \quad n.$$

By convexity and the necessity of condition (CW^1) in Theorem 1 we must have $\alpha_n > 0$ for all $n \in \mathbb{N}$. Let us set, for every n,

$$v_n := \frac{\nabla F(y_n) - \nabla F(x_n)}{|\nabla F(y_n) - \nabla F(x_n)|}.$$

By convexity of F we obtain

$$\begin{aligned}\sqrt{\alpha_n}\langle \nabla F(x_n + \sqrt{\alpha_n} v_n), v_n\rangle &\geq F(x_n + \sqrt{\alpha_n} v_n) - F(x_n) \\ &\geq F(y_n) + \langle \nabla F(y_n), x_n + \sqrt{\alpha_n} v_n - y_n\rangle - F(x_n) \\ &= -\alpha_n + \sqrt{\alpha_n}\langle \nabla F(y_n), v_n\rangle\end{aligned}$$

for all n. Hence we deduce

$$\langle \nabla F(x_n + \sqrt{\alpha_n} v_n) - \nabla F(x_n), v_n \rangle \geq -\sqrt{\alpha_n} + |\nabla F(y_n) - \nabla F(x_n)| \geq -\sqrt{\alpha_n} + \varepsilon.$$

Since $\lim_n \alpha_n = 0$, the above inequality contradicts the fact that ∇F is uniformly continuous on bounded sets. Thus condition (SCW^1) is necessary. The necessity of condition (C) is obvious.

Now assume that G is bounded on B and the pair $(f, G) : B \to \mathbb{R} \times X$ satisfies conditions (C) and (SCW^1) on B. Using condition (C) we have that

$$\langle G(y), x - y \rangle \leq f(x) - f(y) \leq \langle G(x), x - y \rangle \quad x, y \in B,$$

and this implies that f is Lipschitz on B. In particular, f is bounded on B. For each $y \in B$ let us define $\psi_y : X \to \mathbb{R}$ by

$$\psi_y(x) = \sup_{z \in B} \{ f(z) + \langle G(z), x - z \rangle - f(y) - \langle G(y), x - y \rangle \}.$$

Since f and G are bounded it is clear that ψ_y is everywhere finite. Also, because ψ_y is the supremum of a family of convex C-Lipschitz functions, where $C := 2\|G\|_\infty = 2\sup_{z \in B} |G(z)|$, we have that ψ_y is convex and C-Lipschitz for every $y \in B$. In particular, also using condition (C), we obtain

$$\psi_y(y) = 0 \leq \psi_y(x) \leq C|x - y| \text{ for all } x \in X, \, y \in B. \tag{1}$$

Now let us consider the function $\omega_0 : (0, \infty) \to [0, \infty)$ defined by

$$\omega_0(t) = \sup \left\{ \frac{\psi_y(x)}{|x - y|} \, : \, 0 < |x - y| \leq t, \, x \in X, \, y \in B \right\}.$$

It is obvious that $\omega_0(s) \leq \omega_0(t)$ for all $0 < s < t$, and $\omega_0(t) \leq C$ for all $t \in [0, \infty)$. We also have the following.

Lemma 4. $\lim_{t \to 0^+} \omega_0(t) = 0$.

Proof. Suppose $\limsup_{t \to 0^+} \omega_0(t) > 0$. Then there exist $\varepsilon > 0$, a sequence of numbers $(t_n) \searrow 0$, and two sequences of points $(y_n) \subset B$ and $(x_n) \subset X$ such that $x_n \in B(y_n, t_n)$ and

$$\frac{\psi_{y_n}(x_n)}{|x_n - y_n|} \geq \varepsilon$$

for all $n \in \mathbb{N}$. By approximating the supremum defining $\psi_{y_n}(x_n)$ we may also find sequences $(z_n) \subset B$ and $(\delta_n) \subset [0, 1]$ such that $\lim_{n \to \infty} \delta_n = 0$ and

$$\psi_{y_n}(x_n) = f(z_n) + \langle G(z_n), x_n - z_n \rangle - f(y_n) - \langle G(y_n), x_n - y_n \rangle + \delta_n |x_n - y_n|. \tag{2}$$

Then, using condition (C), we deduce that

$$\begin{aligned}
0 < \varepsilon \leq \frac{\psi_{y_n}(x_n)}{|x_n - y_n|} &= \frac{f(z_n) + \langle G(z_n), x_n - z_n \rangle - f(y_n) - \langle G(y_n), x_n - y_n \rangle}{|x_n - y_n|} + \delta_n \\
&= \frac{f(z_n) + \langle G(z_n), y_n - z_n \rangle - f(y_n) + \langle G(z_n) - G(y_n), x_n - y_n \rangle}{|x_n - y_n|} + \delta_n \\
&\leq \frac{\langle G(z_n) - G(y_n), x_n - y_n \rangle}{|x_n - y_n|} + \delta_n \leq |G(z_n) - G(y_n)| + \delta_n,
\end{aligned}$$

which implies

$$0 < \varepsilon \leq \liminf_{n \to \infty} |G(z_n) - G(y_n)|. \tag{3}$$

But on the other hand, since $|x_n - y_n| \to 0$ and G is bounded, using (1) and (2) we also obtain

$$0 = \lim_{n\to\infty} \psi_{y_n}(x_n) = \lim_{n\to\infty} \left(f(z_n) + \langle G(z_n), x_n - z_n\rangle - f(y_n) - \langle G(y_n), x_n - y_n\rangle\right)$$
$$= \lim_{n\to\infty} \left(f(z_n) + \langle G(z_n), y_n - z_n\rangle - f(y_n)\right),$$

which by (SCW^1) implies $\lim_{n\to\infty}(G(z_n) - G(y_n)) = 0$, in contradiction with (3). □

Now let us set $\omega_0(0) = 0$. If $\omega_0 : [0,\infty) \to [0,\infty)$ is constantly 0 then G is constant, and for any $y_0 \in B$ the function $F(x) = f(y_0) + \langle G(y_0), x - y_0\rangle$ has the property that $(F, \nabla F) = (f, G)$ on B. Therefore we can assume that ω_0 is not constant, and define $\omega_1 : [0,\infty) \to [0,\infty)$ by

$$\omega_1(t) = \inf\{g(t) \,|\, g : [0,\infty) \to \mathbb{R} \text{ is concave and } g \geq \omega_0\}$$

(the concave envelope of ω_0). Then ω_1 is a nondecreasing continuous concave modulus of continuity such that $\omega_1 \leq C$. Let us also set

$$\varphi_1(t) = \int_0^t \omega_1(s)ds, \quad t \in [0,\infty).$$

The function φ_1 is convex and C^1, with a uniformly continuous derivative, and satisfies $\varphi_1(0) = 0$. For each $y \in B$, let us define the function

$$X \ni x \mapsto \varphi_y(x) := \varphi_1\left(|x - y|\right).$$

Lemma 5. The functions $\varphi_y : X \to [0,\infty)$ are of class $C^{1,\omega_1}(X)$, with

$$|\nabla\varphi_y(x) - \nabla\varphi_y(z)| \leq M\omega_1\left(|x - z|\right)$$

for all $x, z \in X$, where M is a constant independent of $y \in B$.

Proof. Since

$$\nabla\varphi_y(x) = \omega_1(|x - y|)\frac{x - y}{|x - y|},$$

it is clearly enough to show that the function $X \ni x \mapsto \varphi(x) := \varphi_1(|x|)$ is of class $C^{1,\omega_1}(X)$. Recall that ω_1 is a concave, nondecreasing, modulus of continuity. In particular the function $(0,\infty) \ni t \mapsto \omega_1(t)/t$ is nonincreasing. Fix $x, z \in X \setminus \{0\}$, and let us estimate $|\nabla\varphi(x) - \nabla\varphi(z)|$. Assume that $|x| \geq |z|$ for instance. Then

$$|\nabla\varphi(x) - \nabla\varphi(z)| = \left|\omega_1(|x|)\frac{x}{|x|} - \omega_1(|z|)\frac{z}{|z|}\right| \leq |\omega_1(|x|) - \omega_1(|z|)|\left|\frac{x}{|x|}\right| + \omega_1(|x|)\left|\frac{x}{|x|} - \frac{z}{|z|}\right|$$
$$\leq \omega_1(|x - z|) + \omega_1(|x|)\frac{||z|\,x - |x|\,z|}{|x||z|} \leq \omega_1(|x - z|) + 2\,\omega_1(|x|)\frac{|x - z|}{|x|}.$$

Now observe that $|x| \geq \frac{1}{2}|x| + \frac{1}{2}|z| \geq \frac{1}{2}|x - z|$, and therefore $\omega_1(|x|)/|x| \leq \omega_1(\frac{1}{2}|x - z|)/(\frac{1}{2}|x - z|)$. We obtain

$$|\nabla\varphi(x) - \nabla\varphi(z)| \leq \omega_1(|x - z|) + 2\,\omega_1\left(\tfrac{1}{2}|x - z|\right)\frac{|x - z|}{\frac{1}{2}|x - z|} \leq 5\,\omega_1(|x - z|).$$

On the other hand, if one of the points x, z is 0, for instance $z = 0$, then

$$|\nabla\varphi(x) - \nabla\varphi(0)| = \left|\omega_1(|x|)\frac{x}{|x|} - 0\right| = \omega_1(|x|),$$

so in either case we have what we need, with $M = 5$. □

Now consider the functions $g : X \to \mathbb{R}$ defined by

$$g(x) = \inf_{y \in B} \left\{ f(y) + \langle G(y), x - y \rangle + 2\varphi_y(x) \right\},$$

and

$$F = \mathrm{conv}(g)$$

(the convex envelope of g, that is to say, the largest convex function which is less than or equal to g). As in [1, Lemma 4.14] it is not difficult to check that

$$g(x+h) + g(x-h) - 2g(x) \leq 2\varphi_1\left(2|h|\right)$$

for all $x, h \in X$, which implies, as in [1, Theorem 2.3], that

$$F(x+h) + F(x-h) - 2F(x) \leq 2\varphi_1\left(2|h|\right)$$

for all $x, h \in X$. Since F is convex this inequality implies that $F \in C^{1,\omega_1}(X)$ (see [1, Proposition 4.5]), and in particular $F \in C_b^{1,u}(X)$.

Let us see that $(F, \nabla F) = (f, G)$ on B. We first observe that, by concavity of ω_1, we have

$$\frac{1}{2}\omega_1(t)t \leq \int_0^t \omega_1(s)ds = \varphi_1(t),$$

hence

$$t\omega_0(t) \leq t\omega_1(t) \leq 2\varphi_1(t).$$

Therefore, setting

$$m(x) := \sup_{z \in B} \left\{ f(z) + \langle G(z), x - z \rangle \right\}$$

(the minimal extension of the jet (f, G)) we have

$$\begin{aligned} & f(y) + \langle G(y), x - y \rangle + 2\varphi_y(x) = f(y) + \langle G(y), x - y \rangle + 2\varphi_1(|x - y|) \\ & \geq f(y) + \langle G(y), x - y \rangle + |x - y|\omega_0(|x - y|) \geq f(y) + \langle G(y), x - y \rangle + \psi_y(x) = m(x), \end{aligned}$$

hence

$$m(x) \leq g(x)$$

for all $x \in X$, and since m is convex this implies that

$$m \leq F \leq g \quad \text{on } X.$$

But we also have

$$f \leq m \leq g \leq f \quad \text{on } B.$$

Therefore $F = f$ on B. On the other hand, since $m \leq F$ on X and $F = m$ on B, where m is convex and F is differentiable on X, we deduce that m is differentiable on B with $\nabla m(x) = \nabla F(x)$ for all $x \in B$. But it is clear, by definition of m, that $G(x) \in \partial m(x)$ (the subdifferential of m at x) for every $x \in B$, so we must have $\nabla F(x) = G(x)$ for every $x \in B$.

Finally let us see that F is $5\|G\|_\infty$-Lipschitz. It is clear that φ_y is $2C$-Lipschitz for all $y \in B$, and this implies that g is $5\|G\|_\infty$-Lipschitz. Besides, we have that

$$F(x) = \mathrm{conv}(g)(x) = \inf \left\{ \sum_{j=1}^n \lambda_j g(x_j) \ : \ \lambda_j \geq 0, \sum_{j=1}^n \lambda_j = 1, x = \sum_{j=1}^n \lambda_j x_j, n \in \mathbb{N} \right\}.$$

Then, given $x, h \in X$ and $\varepsilon > 0$, we can pick $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n > 0$ such that

$$F(x) \geq \sum_{i=1}^n \lambda_i g(x_i) - \varepsilon, \quad \sum_{i=1}^n \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^n \lambda_i x_i = x.$$

Because $x+h=\sum_{i=1}^{n}\lambda_i(x_i+h)$, we have $F(x+h)\leq\sum_{i=1}^{n}\lambda_i g(x_i+h)$, which leads us to

$$F(x+h)-F(x)\leq\sum_{i=1}^{n}\lambda_i\left(g(x_i+h)-g(x_i)\right)+\varepsilon\leq 5\|G\|_\infty|h|+\varepsilon,$$

and since $\varepsilon>0$ is arbitrary, we get $F(x+h)-F(x)\leq 5\|G\|_\infty|h|$ for all $x,h\in X$, which means that $\text{Lip}(F)\leq 5\|G\|_\infty$. $\square$

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