GLOBAL GEOMETRY AND C^1 CONVEX EXTENSIONS OF 1-JETS

DANIEL AZAGRA AND CARLOS MUDARRA

Abstract. Let E be an arbitrary subset of $\mathbb{R}^n$ (not necessarily bounded), and $f: E \to \mathbb{R}$, $G: E \to \mathbb{R}^n$ be functions. We provide necessary and sufficient conditions for the 1-jet (f, G) to have an extension $(F, \nabla F)$ with $F: \mathbb{R}^n \to \mathbb{R}$ convex and C^1. As an application we also solve a similar problem about finding convex hypersurfaces of class C^1 with prescribed normals at the points of an arbitrary subset of $\mathbb{R}^n$.

1. Introduction and main results

This paper concerns the following problem.

Problem 1.1. *Given $\mathcal{C}$ a differentiability class in $\mathbb{R}^n$, E a subset of $\mathbb{R}^n$, and functions $f: E \to \mathbb{R}$ and $G: E \to \mathbb{R}^n$, how can we decide whether there is a* convex *function $F \in \mathcal{C}$ such that $F(x) = f(x)$ and $\nabla F(x) = G(x)$ for all $x \in E$?*

This is a natural question which we could solve in [3] in the case that $\mathcal{C} = C^{1,\omega}(\mathbb{R}^n)$, where $\omega: [0, \infty) \to [0, \infty)$ is a (strictly increasing and concave) modulus of continuity. A necessary and sufficient condition is that there exists a constant $M > 0$ such that

$$f(x) \geq f(y) + \langle G(y), x - y \rangle + \|G(x) - G(y)\| \omega^{-1} \left(\frac{1}{2M} \|G(x) - G(y)\| \right) \text{ for all } x, y \in E. \quad (CW^{1,\omega})$$

Very recently, some explicit formulas for such extensions have been found in [8] for the $C^{1,1}$ case, and more in general in [4] for the $C^{1,\omega}$ case when ω is a modulus of continuity ω with the additional property that $\omega(\infty) = \infty$; in particular this includes all the Hölder differentiability classes $C^{1,\alpha}$ with $\alpha \in (0, 1]$. Moreover, it can be arranged that

$$\sup_{x,y \in \mathbb{R}^n, \, x \neq y} \frac{\|\nabla F(x) - \nabla F(y)\|}{\omega(\|x - y\|)} \leq 8M$$

(or even $\text{Lip}(\nabla F) \leq M$ in the $C^{1,1}$ case, that is to say when $\omega(t) = t$).
Besides the very basic character of Problem 1.1, there are other reasons for wanting to solve this kind of problems, as extension techniques for convex functions have natural applications in Analysis, Differential Geometry, PDE theory (in particular Monge-Ampère equations), Economics, and Quantum Computing. See the introductions of [3, 14, 26] for background about convex extensions problems, and see [6, 9, 10, 11, 13, 12, 17, 19, 21] and the references therein for information about general Whitney extension problems.

Date: June 29, 2017.
2010 *Mathematics Subject Classification.* 54C20, 52A41, 26B05, 53A99, 53C45, 52A20, 58C25, 35J96.
Key words and phrases. convex function, C^1 function, Whitney extension theorem, global differential geometry.
D. Azagra was partially supported by Ministerio de Educación, Cultura y Deporte, Programa Estatal de Promoción del Talento y su Empleabilidad en I+D+i, Subprograma Estatal de Movilidad. C. Mudarra was supported by Programa Internacional de Doctorado Fundación La Caixa–Severo Ochoa. Both authors partially suported by grant MTM2015-65825-P.

In [3], and for the class $\mathcal{C} = C^1(\mathbb{R}^n)$, we could only obtain a solution to Problem 1.1 in the particular case that E is a compact set. In this especial situation the three necessary and sufficient conditions on (f, G) that we obtained for C^1_{conv} extendibility are:

$$G \text{ is continuous, and } \lim_{|z-y|\to 0^+} \frac{f(z)-f(y)-\langle G(y), z-y\rangle}{|z-y|} = 0 \text{ uniformly on } E \tag{W^1}$$

(which is equivalent to Whitney's classical condition for C^1 extendibility),

$$f(x) - f(y) \geq \langle G(y), x-y\rangle \text{ for all } x, y \in E \tag{C}$$

(which ensures convexity), and

$$f(x) - f(y) = \langle G(y), x-y\rangle \implies G(x) = G(y), \text{ for all } x, y \in E \tag{CW^1}$$

(which tells us that if two points of the graph of f lie on a line segment contained in a hyperplane which we want to be tangent to the graph of an extension at one of the points, then our putative tangent hyperplanes at both points must be the same).

In [3] we also gave examples showing that the above conditions are no longer sufficient when E is not compact (even if E is an unbounded convex body). The reasons for this insufficiency can be mainly classified into two kinds of difficulties that only arise if the set E is unbounded and G is not uniformly continuous on E:

(1) There may be no convex extension of f to the whole of $\mathbb{R}^n$.
(2) Even when there are convex extensions of f defined on all of $\mathbb{R}^n$, and even when some of these extensions are differentiable in some neighborhood of E, there may be no $C^1(\mathbb{R}^n)$ convex extension of f.

The aim of this paper is to show how one can overcome these difficulties by adding new necessary conditions to $(W^1), (C), (CW^{1,1})$ in order to obtain a complete solution to Problem 1.1 for the case that $\mathcal{C} = C^1(\mathbb{R}^n)$.

The first kind of complication is well understood thanks to [23], and is not difficult to deal with: the requirement that

$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty \text{ for every sequence } (x_k)_k \subset E \text{ with } \lim_{k\to\infty} |G(x_k)| = +\infty \tag{EX}$$

guarantees that there exist convex functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi_{|_E} = f$.

The second kind of difficulty, however, is of a subtler geometrical character, and is related to the fact that a differentiable (or even real-analytic) convex function on $\mathbb{R}^n$ may have *corners at infinity*. In this introduction we will not attempt to rigourously define such corners at infinity; we will instead provide some examples that will hopefully show what the difficulties are (namely that in order to succeed one must take into account not only the global behavior of the differential data, but also the *differential behavior at infinity* of those data), and indicate a possible strategy for a solution of Problem 1.1 which we will then formulate in an equivalent way that we think is more suited to the analytical treatment of the problem.

Consider for instance the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \sqrt{x^2 + e^{-2y}}. \tag{1.1}$$

This function is real-analytic and convex, and its graph approaches from above, as $y \to \infty$, the graph of $\varphi(x, y) = |x|$. One can thus say that f has a 2-*dimensional corner at infinity* defined by the graph of φ, and directed by the line $y = 0$. Note also that the function f is *essentially coercive* (meaning that it is coercive up to a linear perturbation), while the function φ (the corner itself) is not.

In two dimensions there can only be 2-dimensional corners at infinity, but in $\mathbb{R}^n$ there are C^1 convex functions $f : \mathbb{R}^n \to \mathbb{R}$ which have corners at infinity of dimensions $k = 2, 3, \ldots, n-1$. For instance,

$$f(x_1, x_2, \ldots, x_n) = \sqrt{\sum_{j=1}^{k-1} x_j^2 + \sum_{j=k}^{n} e^{-2x_j}} \tag{1.2}$$

has a *corner at infinity of dimension* k, and is essentially coercive. On the other hand, in dimensions greater than or equal to 3 there are also C^1 convex functions with corners at infinity which are not coercive: for instance, if $2 \leq k < n$ then the function

$$f(x_1, \ldots, x_n) = \sqrt{x_1^2 + \sum_{j=2}^{k} e^{-2x_j}} \tag{1.3}$$

has a corner at infinity, and is not essentially coercive. Nevertheless f is *essentially k-coercive* (meaning that, up to summing a linear function, f can be written as $f = c \circ P$, where P is the orthogonal projection onto a k-dimensional subspace of X of $\mathbb{R}^n$ and $c : X \to \mathbb{R}$ is coercive).

In general, the presence of a corner at infinity in the graph of a differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$ always indicates some kind of essential k-coercivity for some $k \geq 2$ in the direction of the corner. This must be so, for otherwise the corner at infinity would touch the graph of f at some finite point, and this would be in contradiction with the convexity and differentiability of f (similarly to the reason why condition (CW^1) above is necessary for the existence of an extension $(F, \nabla F)$ of the 1-jet (f, G) with F convex and C^1).

Thus one could envisage a strategy for a solution to Problem 1.1 consisting in defining rigourously what a corner at infinity is for a 1-jet (f, G), adding to conditions (W^1), (C), (CW^1) and (EX), the requirement that *either there are coercive data in the directions of the corners at infinity, and they lie strictly above those corners, or else there is enough room to add new differential data which are coercive in those directions, compatible with the old data, and lying strictly above the corners*, and then follow the lines of the proofs of the results in [3] in order to show that these new conditions are necessary and sufficient for the existence of extensions $(F, \nabla F)$ of (f, G) with $F \in C^1_{\text{conv}}(\mathbb{R}^n)$ and F coercive in the directions of the corners.

Of course this is all very vague, and we will not pursue this line of proof (among other reasons, because it leads to greater complications than the equivalent reformulation we have chosen to present, and because in practice corners at infinity may be difficult to spot and deal with, as there may be infinitely many of them), but we hope that this heuristic approach will give a glimpse of the main ideas of the solution to Problem 1.1 that we next provide.

We will start by introducing some definitions and notation.

Definition 1.2. Let Z be a real vector space, and $P : Z \to X$ be the orthogonal projection onto a subspace $X \subseteq Z$. We will say that a function f defined on a subset E of Z is *essentially P-coercive* provided that there exists a linear function $\ell : Z \to \mathbb{R}$ such that for every sequence $(x_k)_k \subset E$ with $\lim_{k\to\infty} |P(x_k)| = \infty$ one has

$$\lim_{k\to\infty} (f - \ell)(x_k) = \infty.$$

We will say that f is *essentially coercive* whenever f is essentially I-coercive, where $I : Z \to Z$ is the identity mapping.

If X is a linear subspace of $\mathbb{R}^n$, we will denote by $P_X : \mathbb{R}^n \to X$ the orthogonal projection, and we will say that $f : E \to \mathbb{R}$ is *coercive in the direction of* X whenever f is P_X-coercive.

We will also denote by $X^\perp$ the orthogonal complement of X in $\mathbb{R}^n$. For a subset V of $\mathbb{R}^n$, span(V) will stand for the linear subspace spanned by the vectors of V. Finally, we define $C^1_{\text{conv}}(\mathbb{R}^n)$ as the set of all functions $f : \mathbb{R}^n \to \mathbb{R}$ which are convex and of class C^1.

In [1] essentially coercive convex functions were called *properly convex*, and some approximation results, which fail for general convex functions, were shown to be true for this class of functions. The following result was also implicitly proved in [1, Lemma 4.2]. Since this will be a very important tool in the statements and proofs of all the results of the present paper, and because we have introduced new terminology and added conclusions, we will provide a self-contained proof in Section 2 for the readers' convenience.

Theorem 1.3. *For every convex function $f : \mathbb{R}^n \to \mathbb{R}$, there exists a unique linear subspace X_f of $\mathbb{R}^n$, a unique vector $v_f \in X_f^{\perp}$, and a unique essentially coercive function $c_f : X_f \to \mathbb{R}$ such that f can be written in the form*

$$f(x) = c_f(P_{X_f}(x)) + \langle v_f, x\rangle \text{ for all } x \in \mathbb{R}^n.$$

Moreover, if Y is a linear subspace of $\mathbb{R}^n$ such that f is coercive in the direction of Y, then $Y \subseteq X_f$.

The following Proposition shows that the directions X_f given by these decompositions are stable by approximation.

Proposition 1.4. *With the notation of the preceding theorem, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are convex functions and A is a positive number such that $f(x) \leq g(x) + A$, for all $x \in \mathbb{R}^n$, then $X_f \subseteq X_g$.*
In particular, if $|f - g| \leq A$ then $X_f = X_g$.

Proof. The inequality $f(x) \leq g(x)+A$ and the esential coercivity of f in the direction X_f implies that g is essentially coercive in the direction X_f. Then $X_f \subseteq X_g$ by the last part of Theorem 1.3. □

Our solution to Problem 1.1 is as follows.

Theorem 1.5. *Given an arbitrary subset E of $\mathbb{R}^n$, a linear subspace $X \subset \mathbb{R}^n$, the orthogonal projection $P = P_X : \mathbb{R}^n \to X$, and two mappings $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, the following is true. There exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and $X_F = X$, if and only if the following conditions are satisfied.*

(*i*) *G is continuous and $f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in E$.*

(*ii*) *If $(x_k)_k \subset E$ is a sequence for which $\lim_{k\to\infty} |G(x_k)| = +\infty$, then*

$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty.$$

(*iii*) *$Y := span\left(\{G(x) - G(y) : x, y \in E\}\right) \subseteq X$.*

(*iv*) *If $Y \neq X$ and we denote $k = \dim Y$ and $d = \dim X$, there exist vectors $w_1, \ldots, w_{d-k} \in \mathbb{R}^n$, points $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$ and numbers $\beta_1, \ldots, \beta_{d-k} \in \mathbb{R}$ such that*
   - (a) $X = span\left(\{u - v \ : \ u, v \in G(E) \cup \{w_1, \ldots, w_{d-k}\}\}\right)$.
   - (b) $\beta_j > \max_{1\leq i\neq j\leq d-k}\{\beta_i + \langle w_i, p_j - p_i\rangle\}$ *for all* $1 \leq j \leq d - k$.
   - (c) $\beta_j > \sup_{z\in E,\, |G(z)|\leq N}\{f(z) + \langle G(z), p_j - z\rangle\}$ *for all* $1 \leq j \leq d - k$ *and* $N \in \mathbb{N}$.
   - (d) $\inf_{x\in E,\, |P(x)|\leq N} f(x) > \max_{1\leq j\leq d-k}\{\beta_j + \langle w_j, x - p_j\rangle\}$ *for all* $N \in \mathbb{N}$.

(*v*) *If $(x_k)_k, (z_k)_k$ are sequences in E such that $(P(x_k))_k$ and $(G(z_k))_k$ are bounded and*

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0,$$

then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$.

The reader may wonder why in the $C^{1,\omega}$ case just one simple inequality such as $(CW^{1,\omega})$ is necessary and sufficient for a solution of Problem 1.1, while in the C^1 case so many, rather complicated conditions are required. This is of course due to the issue of the *corners at infinity* that we mentioned above. The reader can check that if G is uniformly continuous on E then no extension $(F, \nabla F)$ of (f, G) with $F \in C^{1,\omega}(\mathbb{R}^n)$ and convex can have corners at infinity. This is also related to an important

step in the proof of the above result: while coercivity of a function g is not necessary in order that its convex envelope be of class $C^{1,\omega}$ (see [7, Theorem 7] or [4, Theorem 2.3]), it does matter in the C^1 case (see [20] and [5, Example 4.1]). Besides, in the case $\mathcal{C} = C^1$, Problem 1.1 may be *geometrically underdetermined*, in the following sense. In order to solve this problem one must gain some information about the geometry of the possible corners at infinity determined by the given 1-jet (f, G). This information will only indicate, in general, the least possible subspace X for which there are functions $F \in C^1_{\text{conv}}(\mathbb{R}^n)$ such that $(F, \nabla F)$ extends (f, G) and $X_F = X$. But there may be larger subspaces X with this property, and in practice it may be useful to be able to find the largest such X (for instance, because one needs to find a coercive extension). For these reasons, and as long as we want to have necessary and sufficient conditions that ask for geometrical information of the behavior of the jet with respect to a subspace X and in return provide extensions F such that $X_F = X$, we need conditions (iii) and (iv) in the above theorem.
Let us consider some examples that will hopefully clarify these comments.

Example 1.6. Consider the following 1-jets (f_j, G_j) defined on subsets E_j of $\mathbb{R}^n$:

(1) $E_1 = \{(x, y) \in \mathbb{R}^2 : y = \log|x|, x \in \mathbb{N} \cup \{\pm\frac{1}{n} : n \in \mathbb{N}\}\}$, $f_1(x, y) = |x|$, $G_1(x, y) = (-1, 0)$ if $x < 0$, $G_1(x, y) = (1, 0)$ if $x > 0$.
(2) $E_2 = \{(x, y) \in \mathbb{R}^2 : y = \log|x|, x \in \mathbb{N} \cup \{\pm\frac{1}{n} : n \in \mathbb{N}\}\}$, $f_2 = \varphi$, $G_2 = \nabla\varphi$, where $\varphi(x, y) = \sqrt{x^2 + e^{-2y}}$.
(3) $E_3 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, y = \log|x|, x \in \mathbb{N} \cup \{\pm\frac{1}{n} : n \in \mathbb{N}\}\}$, $f_3 = \varphi$, $G_3 = \nabla\varphi$, where $\varphi(x, y, z) = \sqrt{x^2 + e^{-2y}}$.
(4) $E_4 = E_1 \cup \{(x, y) \in \mathbb{R}^2 : |x| \geq 1\}$, $f_4(x, y) = |x|$, $G_4(x, y) = (-1, 0)$ if $x < 0$, $G_4(x, y) = (1, 0)$ if $x > 0$.

Then one can check that:

(i) For the jet (f_1, G_1) and with the notation of Theorem 1.5, we have $Y = \mathbb{R} \times \{0\}$, but the least possible X we can take is $X = \mathbb{R}^2$ (all possible extensions F must be essentially coercive on $\mathbb{R}^2$).
(ii) For the jet (f_2, G_2) we have $Y = \mathbb{R}^2$, and all possible extensions F must be essentially coercive on $\mathbb{R}^2$.
(iii) For the jet (f_3, G_3) we have $Y = \mathbb{R}^2 \times \{0\}$, and we can take either $X = Y$ or $X = \mathbb{R}^3$.
(iv) For the jet (f_4, G_4) we have $Y = \mathbb{R} \times \{0\}$, but one cannot apply Theorem 1.5 with any X (because as $y \to \infty$ there is a lot of data that are not compatible with the fact that this jet has a corner at infinity directed by the line $x = 0$, and therefore any differentiable convex extension F should be essentially coercive in the direction $\{0\} \times \mathbb{R}$). Thus there exists no $F \in C^1_{\text{conv}}(\mathbb{R}^2)$ such that $(F, \nabla F)$ extends (f_4, G_4).

Of course, as the dimension n grows larger, things get more and more complicated. The reader is invited to verify this assertion by constructing higher dimensional variants of these examples and using the functions of (1.2) and (1.3).

In practice, if $Y \neq X$ and we are able to calculate (or at least appropriately estimate) the minimal extension of the jet (f, G), defined by

$$m(x) = m(f, G)(x) = \sup_{y \in E}\{f(y) + \langle G(y), x - y\rangle\},$$

then a natural way to check condition (iv) is as follows.
Define, for each $u \in X \setminus \{0\}$, $p \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, the sets

$$S(m, u, p, \alpha, \beta) = \{x \in \mathbb{R}^n : m(x) \leq \beta + \alpha\langle u, x - p\rangle\},$$

and consider an orthonormal basis $\{u_1, \ldots, u_{d-k}\}$ of the orthogonal complement of Y in X. Find $p_1 \in \mathbb{R}^n \setminus \overline{E}$, $\alpha_1, \beta_1 \in \mathbb{R}$ such that

$$\text{int}\,(S(m, u_1, p_1, \alpha_1, \beta_1)) \neq \emptyset \text{ and } \text{dist}\,(E, S(m, u_1, p_1, \alpha_1, \beta_1)) > 0.$$

Thus, we can take $r > 0$ such that

$$m(x) \geq \beta_1 + \langle \alpha_1 u_1, x - p_1 \rangle + r \quad \text{for all} \quad x \in E.$$

Also, find $q_1 \in \text{int}\,(S(m, u_1, p_1, \alpha_1, \beta_1))$ sufficiently close to p_1 such that

$$m(q_1) \leq \beta_1 + \langle \alpha_1 u_1, q_1 - p_1 \rangle - r' \quad \text{and} \quad |\langle \alpha_1 u_1, p_1 - q_1 \rangle| \leq \frac{r'}{2},$$

for some $r' > 0$ with $r' \leq r$.
Then set $E_1^* = E \cup \{q_1\}$, and define $f_1^* := E_1^* \to \mathbb{R}$, $G_1^* : E_1^* \to \mathbb{R}^n$ by

$$f_1^*(q_1) = \beta_1, \; f_1^*(x) = f(x) \text{ if } x \in E; \;\; G_1^*(q_1) = \alpha_1 u_1, \; G_1^*(x) = G(x) \text{ if } x \in E.$$

Notice that the new putative tangent hyperplane $h(x) = \beta_1 + \langle G_1^*(q_1), x - q_1 \rangle$ that we have added to our problem lies strictly below the graph of the old function f. Indeed, for all $x \in E$:

$$\begin{aligned} f(x) - f_1^*(q_1) - \langle G_1^*(q_1), x - q_1 \rangle &= m(x) - \beta_1 - \langle \alpha_1 u_1, x - p_1 \rangle + \langle \alpha_1 u_1, q_1 - p_1 \rangle \\ &\geq r + \langle \alpha_1 u_1, q_1 - p_1 \rangle \geq \frac{r}{2}. \end{aligned}$$

On the other hand the old hyperplanes $x \mapsto f(y) + \langle G(y), x - y \rangle$, $y \in E$, lie strictly below the point $(q_1, f_1^*(q_1))$, as for all $y \in E$ we have

$$f_1^*(q_1) - f(y) - \langle G(y), q_1 - y \rangle \geq \beta_1 - m(q_1) \geq r' + \langle \alpha_1 u_1, p_1 - q_1 \rangle \geq \frac{r'}{2}.$$

Next, for the jet (f_1^*, G_1^*) defined on E_1^* we consider the analogous C^1_{conv} extension problem. Now we have that

$$Y_1 := \text{span}\{G_1^*(x) - G_1^*(y) : x, y \in E_1^*\}$$

has dimension $d - k + 1$ and contains Y. Proceeding as before we consider the minimal function

$$m_1(x) = m(f_1^*, G_1^*)(x)$$

and find $p_2, q_2 \in \mathbb{R}^n$, $\alpha_2, \beta_2 \in \mathbb{R}$ with the same properties as $p_1, q_1, \alpha_1, \beta_1$ with respect to E_1^* instead of E. Then we set $E_2^* = E_1^* \cup \{q_2\}$ and define $f_2^* := E_2^* \to \mathbb{R}$, $G_2^* : E_2^* \to \mathbb{R}^n$ by

$$f_2^*(q_2) = \beta_2, \; f_1^*(x) = f_1^*(x) \;\text{ if } x \in E_1^*; \;\; G_2^*(q_2) = \alpha_2 u_2, \; G_2^*(x) = G_1^*(x) \;\text{ if } x \in E_1^*.$$

It is not difficult to see that by continuing the process in this manner we will obtain, in $d - k$ steps, points q_j, vectors $w_j = \alpha_j u_j$ and numbers β_j, $j = 1, \ldots, d - k$, satisfying condition (iv) of Theorem 1.5.
In the special case that G is bounded, this process is geometrically much simpler, because the function $m(f, G)$ is Lipschitz, hence the sets $S(m, u, p, \alpha, \beta)$ can be checked to be equivalent to (in the sense that they contain and are contained in) cones of the form

$$\{x \in \mathbb{R}^n : |P_Y(x - q)| \leq \varepsilon \langle w, x - p \rangle\}.$$

Therefore, in this case, our task boils down to finding cones of this form which are free of points in E. Of course, when there is enough differential data in the initial problem so as to have $Y = X$, all of the preceding considerations are unnecessary, and Theorem 1.5 takes on a much more user-friendly look.

Corollary 1.7. *Given an arbitrary subset E of $\mathbb{R}^n$, a linear subspace $X \subset \mathbb{R}^n$, the orthogonal projection $P = P_X : \mathbb{R}^n \to X$, and two mappings $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ such that $X = \text{span}\left(\{G(x) - G(y) \, : \, x, y \in E\}\right)$, the following is true. There exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and $X_F = X$, if and only if the following conditions are satisfied:*

(i) *G is continuous and $f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in E$.*

(ii) *If $(x_k)_k \subset E$ is a sequence for which $\lim_{k\to\infty} |G(x_k)| = +\infty$, then*
$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty.$$

(iii) *If $(x_k)_k, (z_k)_k$ are sequences in E such that $(P(x_k))_k$ and $(G(z_k))_k$ are bounded and*
$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0,$$
then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$.

A especially useful instance[1] is the particular case that $X = \mathbb{R}^n$, where we obtain essentially coercive convex extensions of class C^1.

Corollary 1.8. *Given an arbitrary subset E of $\mathbb{R}^n$ and two mappings $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$, assume that the following conditions are satisfied.*

(i) *G is continuous and $f(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x, y \in E$.*

(ii) *If $(x_k)_k \subset E$ is a sequence for which $\lim_{k\to\infty} |G(x_k)| = +\infty$, then*
$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = +\infty.$$

(iii) *$\text{span}\left(\{G(x) - G(y) : x, y \in E\}\right) = \mathbb{R}^n$.*

(iv) *If $(x_k)_k, (z_k)_k$ are sequences in E such that $(x_k)_k$ and $(G(z_k))_k$ are bounded and*
$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0,$$
then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$.

Then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $F_{|_E} = f$, $(\nabla F)_{|_E} = G$, and F is essentially coercive.
In particular, if for some $x_0 \in E$ we have $G(x_0) = 0$, then F is coercive.

Let us mention that the above corollary is applied in [2] to show that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ has a Lusin property of type $C^1_{\text{conv}}(\mathbb{R}^n)$ (meaning that for every $\varepsilon > 0$ there exists a function $g \in C^1_{\text{conv}}(\mathbb{R}^n)$ such that $\mathcal{L}^n\left(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}\right) < \varepsilon$, where $\mathcal{L}^n$ denotes Lebesgue's measure) if and only if either f is essentially coercive or else f is already C^1 (in which case taking $g = f$ is the only possible option).

Finally, let us turn our attention to a geometrical problem which is closely related to our results.

Problem 1.9. *Given an arbitrary subset E of $\mathbb{R}^n$ and a unitary vector field $N : E \to \mathbb{R}^n$, what conditions will be necessary and sufficient in order to guarantee the existence of a convex hypersurface M of class C^1 with the properties that $E \subset M$ and $N(x)$ is normal to M at each $x \in E$?.*

Our solution to this problem is as follows. We say that a subset W of $\mathbb{R}^n$ is a (possibly unbounded) convex body provided that W is closed and convex, with nonempty interior. We will say that W is of class C^1 provided that its Minkowski functional
$$\mu_W(x) = \inf\{\lambda > 0 : \tfrac{1}{\lambda}x \in W\}$$

[1]Coercitivity of a convex function may well be relevant or even essential to a number of possible applications, e.g. in PDE theory.

is of class C^1 on the open set $\mathbb{R}^n \setminus \mu_W^{-1}(0)$. This is equivalent to saying that W can be locally parametrized as a graph $(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1}))$ (coordinates taken with respect to an appropriate permutation of the canonical basis of $\mathbb{R}^n$), where g is of class C^1. We will denote

$$n_W(x) = \frac{\nabla \mu_W(x)}{|\nabla \mu_W(x)|}, \quad x \in \partial W,$$

the outer normal to ∂W.

Theorem 1.10. *Let E be an arbitrary subset of $\mathbb{R}^n$, $N : E \to \mathbb{S}^{n-1}$ a continuous mapping, X a linear subspace of $\mathbb{R}^n$, and $P : \mathbb{R}^n \to X$ the orthogonal projection. Then there exists a (possibly unbounded) convex body W of class C^1 such that $0 \in int(W)$, $N(x) = n_W(x)$ for all $x \in E$, and $X = span\,(n_W(\partial W))$, if and only if the following conditions are satisfied:*

(1) $\langle N(y), x - y \rangle \leq 0$ *for all* $x, y \in E$.
(2) *For all sequences $(x_k)_k$, $(z_k)_k$ contained in E with $(P(x_k))_k$ bounded, we have that*
$$\lim_{k\to\infty} \langle N(z_k), x_k - z_k \rangle = 0 \implies \lim_{k\to\infty} |N(z_k) - N(x_k)| = 0.$$
(3) $0 < \inf_{y\in E} \langle N(y), y \rangle$.
(4) *Denoting $d = dim(X)$, $Y = span(N(E))$, $\ell = dim(Y)$, we have that $Y \subseteq X$, and, if $Y \neq X$ then there exist vectors $w_1, \ldots, w_{d-\ell}$ and points $x_1, \ldots, x_{d-\ell}$ such that:*
   (a) $X = span\,(N(E) \cup \{w_1, \ldots, w_{d-\ell}\})$.
   (b) $\langle w_j, x_i - x_j \rangle < 0$ *for* $j \neq i$.
   (c) $\sup_{z\in E, |P(z)|\leq k} \langle N(z), x_j - z \rangle < 0$ *for each* $k \in \mathbb{N}$, $j = 1, \ldots, d-\ell$.
   (d) $\langle w_j, x_j \rangle > 1$ *for all* $j = 1, \ldots, d - \ell$.
   (e) $\sup_{x\in E, |P(x)|\leq k} \langle w_j, x \rangle < 1 + \langle w_j, x_j \rangle$ *for each* $k \in \mathbb{N}$, $j = 1, \ldots, d-\ell$.

As before, in the case that $X = \mathrm{span}(N(E))$, the above result is much easier to use.

Corollary 1.11. *Let E be an arbitrary subset of $\mathbb{R}^n$, $N : E \to \mathbb{S}^{n-1}$ a continuous mapping, X a linear subspace of $\mathbb{R}^n$ such that $X = span(N(E))$, and $P : \mathbb{R}^n \to X$ the orthogonal projection. Then there exists a (possibly unbounded) convex body W of class C^1 such that $0 \in int(W)$, $N(x) = n_W(x)$ for all $x \in E$, and $X = span\,(n_W(\partial W))$, if and only if the following conditions are satisfied:*

(1) $\langle N(y), x - y \rangle \leq 0$ *for all* $x, y \in E$.
(2) *For all sequences $(x_k)_k$, $(z_k)_k$ contained in E with $(P(x_k))_k$ bounded, we have that*
$$\lim_{k\to\infty} \langle N(z_k), x_k - z_k \rangle = 0 \implies \lim_{k\to\infty} |N(z_k) - N(x_k)| = 0.$$
(3) $0 < \inf_{y\in E} \langle N(y), y \rangle$.

2. Proof of Theorem 1.3.

Let us first recall some terminology from [1]. We say that a function $C : \mathbb{R}^n \to \mathbb{R}$ is a k-dimensional *corner function* on $\mathbb{R}^n$ if it is of the form

$$C(x) = \max\{\, \ell_1 + b_1,\, \ell_2 + b_2,\, \ldots,\, \ell_k + b_k \,\},$$

where the $\ell_j : \mathbb{R}^n \to \mathbb{R}$ are linear functions such that the functions $L_j : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ defined by $L_j(x, x_{n+1}) = x_{n+1} - \ell_j(x)$, $1 \leq j \leq k$, are linearly independent in $\mathbb{R}^{n+1}$, and the $b_j \in \mathbb{R}$. This is equivalent to saying that the functions $\{\ell_2 - \ell_1, \ldots, \ell_k - \ell_1\}$ are linearly independent in $\mathbb{R}^n$.
We also say that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is supported by C at a point $x \in \mathbb{R}^n$ provided we have $C \leq f$ and $C(x) = f(x)$.
Now let us prove Theorem 1.3.

Case 1. We will first assume that f is differentiable (and therefore of class C^1, since f is convex). If f is affine, say $f(x) = a\langle u, x\rangle + b$, then the result is trivially true with $X = \{0\}$, $c(0) = b$, and $v = au$. On the other hand, if f is essentially coercive then the result also holds obviously with $X = \mathbb{R}^n$, $v = 0$, and $c = f$. So we may assume that f is neither affine nor essentially coercive. In particular there exist $x_0, y_0 \in \mathbb{R}^n$ with $Df(x_0) \neq Df(y_0)$. It is clear that $L_1(x, x_{n+1}) = x_{n+1} - Df(x_0)(x)$ and $L_2(x, x_{n+1}) = x_{n+1} - Df(y_0)(x)$ are two linearly independent linear functions on $\mathbb{R}^{n+1}$, hence f is supported at x_0 by the two-dimensional corner $x \mapsto \max\{f(x_0) + Df(x_0)(x - x_0), f(y_0) + Df(y_0)(x - y_0)\}$.
Let us then define k as the greatest integer number so that f is supported at x_0 by a $(k+1)$-dimensional corner. By assumption we have $1 \leq k < n$. Then we also have that there exist $\ell_1, \ldots, \ell_{k+1} \in (\mathbb{R}^n)^*$ with $L_j(x, x_{n+1}) = x_{n+1} - \ell_j(x)$, $j = 1, \ldots, k+1$, linearly independent in $(\mathbb{R}^{n+1})^*$, and $b_1, \ldots, b_{k+1} \in \mathbb{R}$, so that $C = \max_{1\leq j\leq k+1}\{\ell_j + b_j\}$ supports f at x_0.
Observe that the $\{L_j - L_1\}_{j=2}^{k+1}$ are linearly independent in $(\mathbb{R}^{n+1})^*$, hence so are the $\{\ell_j - \ell_1\}_{j=2}^{k+1}$ in $(\mathbb{R}^n)^*$, and therefore $\bigcap_{j=2}^{k+1} \mathrm{Ker}\,(\ell_j - \ell_1)$ has dimension $n - k$. Then we can find linearly independent vectors $w_1, \ldots, w_{n-k}$ such that $\bigcap_{j=2}^{k+1} \mathrm{Ker}\,(\ell_j - \ell_1) = \mathrm{span}\{w_1, \ldots, w_{n-k}\}$.
Now, given any $y \in \mathbb{R}^n$, if $\frac{d}{dt}(f - \ell_1)(y + tw_q)|_{t=t_0} \neq 0$ for some t_0 then $Df(y + t_0 w_q) - \ell_1$ is linearly independent with $\{\ell_j - \ell_1\}_{j=2}^{k+1}$, which implies that $(x, x_{n+1}) \mapsto x_{n+1} - Df(y + t_0 w_q)$ is linearly independent with $L_1, \ldots, L_{k+1}$, and therefore the function

$$x \mapsto \max\{\ell_1(x) + b_1, \ldots, \ell_{k+1}(x) + b_{k+1}, Df(y + t_0 w_q)(x - y - t_0 w_q) + f(y + t_0 w_q)\}$$

is a $(k + 2)$-dimensional corner supporting f at x_0, which contradicts the choice of k. Thus we must have

$$\frac{d}{dt}(f - \ell_1)(y + tw_q) = 0 \quad \text{for all } \; y \in \mathbb{R}^n, t \in \mathbb{R} \; \text{ with } \; y + tw_q \in \mathbb{R}^n, \; q = 1, ..., n-k. \tag{2.1}$$

This implies that

$$(f - \ell_1)(y + \sum_{j=1}^{n-k} t_j w_j) = (f - \ell_1)(y) \tag{2.2}$$

if $y \in \mathbb{R}^n$ and $y + \sum_{j=1}^{n-k} t_j w_j \in \mathbb{R}^n$. Let P be the orthogonal projection of $\mathbb{R}^n$ onto the subspace $X := \mathrm{span}\{w_1, \ldots, w_{n-k}\}^\perp$. For each $z \in X$ we may define

$$\widetilde{c}(z) = (f - \ell_1)(z + \sum_{j=1}^{n-k} t_j w_j)$$

if $z + \sum_{j=1}^{n-k} t_j w_j \in \mathbb{R}^n$ for some $t_1, \ldots, t_{n-k}$. It is clear that $\widetilde{c} : X \to \mathbb{R}$ is well defined and convex, and satisfies

$$f - \ell_1 = \widetilde{c} \circ P.$$

Now let us write

$$\ell_1(x) = \langle u, x\rangle + \langle v, x\rangle,$$

where $u \in X$ and $v \in X^\perp$. We then have

$$f(x) = c(P(x)) + \langle v, x\rangle,$$

where $c : X \to \mathbb{R}$ is defined by

$$c(x) = \widetilde{c}(x) + \langle u, x\rangle.$$

Moreover, since $\bigcap_{j=2}^{k+1} \mathrm{Ker}\,(\ell_j - \ell_1) = X^\perp$, it is clear that the restriction of the corner function $C = \max_{1\leq j\leq k+1}\{\ell_j + b_j\}$ to X is a $(k + 1)$ dimensional corner function on X, which has dimension k, and

it is obvious that $(k+1)$-dimensional corner functions on k-dimensional spaces are essentially coercive; therefore, because $c(x) \geq C(x)$ for all $x \in X$, we deduce that c is essentially coercive.
Now let us see that X is the only linear subspace of $\mathbb{R}^n$ for which f admits a decomposition of the form

$$f(x) = c(P_X(x)) + \langle v, x\rangle, \tag{2.3}$$

with c essentially coercive and $v \in X^{\perp}$. Assume that we have two subspaces Z_1, Z_2 for which (2.3) holds, say

$$f(x) = \varphi_1(P_{Z_1}(x)) + \langle \xi_1, x\rangle, \tag{2.4}$$

and

$$f(x) = \varphi_2(P_{Z_2}(x)) + \langle \xi_2, x\rangle, \tag{2.5}$$

with φ_j essentially coercive and $\xi_j \in X_j^{\perp}$. In order to show that $Z_1 = Z_2$ is enough to check that $Z_1^{\perp} = Z_2^{\perp}$. Suppose this equality does not hold; then, either $\in Z_1^{\perp} \setminus Z_2^{\perp} \neq \emptyset$ or $\in Z_2^{\perp} \setminus Z_1^{\perp} \neq \emptyset$. Assume for instance that there exists $\xi_0 \in Z_1^{\perp} \setminus Z_2^{\perp}$. Then, on the one hand (2.4) implies that the function $t \mapsto f(t\xi_0) = \varphi_1(0) + t\langle \xi_1, \xi_0\rangle$ is linear, and on the other hand (2.5) implies that the same function $t \mapsto f(t\xi_0) = \varphi_2(P_{Z_2}(t\xi_0)) + t\langle \xi_2, \xi_0\rangle$ is essentially coercive (indeed, we have $\lim_{|t|\to\infty} |P_{Z_2}(t\xi_0)| = \infty$ because $\xi_0 \notin Z_2^{\perp}$). This is absurd, so we must have $Z_1^{\perp} \subset Z_2^{\perp}$. By a similar argument, just changing the roles of Z_1 and Z_2, we also obtain that $Z_2^{\perp} \subset Z_1^{\perp}$. Therefore $Z_1^{\perp} = Z_2^{\perp}$, as we wanted to check. [2]
Next, let us see that $\xi_1 = \xi_2$. For every $v \in Z_1^{\perp}$ we have

$$\varphi_1(0) + \langle \xi_1, v\rangle = f(v) = \varphi_2(0) + \langle \xi_2, v\rangle.$$

Since the equality of two affine function imply the equality of their linear parts, we have that

$$\langle \xi_1, v\rangle = \langle \xi_2, v\rangle$$

for all $v \in Z_1^{\perp}$, and because $\xi_1, \xi_2 \in Z_1^{\perp}$ this shows that $\xi_1 = \xi_2$.
Once we know that $X_1 = X_2$ and $\xi_1 = \xi_2$, it immediately follows from (2.4) and (2.5) that $\varphi_1 = \varphi_2$. This shows that the decomposition is unique.
Finally let us prove that if f is essentially coercive in the direction of a subspace Y (say that there exists a linear form ℓ on $\mathbb{R}^n$ such that $|f(x) - \ell(x)| \to \infty$ as $|P_Y(x)| \to \infty$), then $Y \subseteq X_f$. Indeed, otherwise there would exist a vector $\xi \in X^{\perp} \setminus Y^{\perp}$, and the function

$$\mathbb{R} \ni t \mapsto f(t\xi) = c(P_X(t\xi)) + t\langle v, \xi\rangle = c(0) + t\langle v, \xi\rangle$$

would be affine, hence so would be the function

$$\mathbb{R} \ni t \mapsto f(t\xi) - \ell(t\xi).$$

But this function cannot be affine, because $\xi \notin Y^{\perp}$ implies that $|P_Y(t\xi)| \to \infty$ as $|t| \to \infty$, and we have $|f(x) - \ell(x)| \to \infty$ as $|P_Y(x)| \to \infty$. This completes the proof of Theorem 1.3 in the case that f is everywhere differentiable.
Case 2. In the case that $f : \mathbb{R}^n \to \mathbb{R}$ is convex but not everywhere differentiable, we can use [1, Theorem 1.1] in order to find a C^1 (or even real analytic) convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f - 1 \leq g \leq f$. Then we may apply Case 1 in order to find a unique subspace $X \subseteq \mathbb{R}^n$, an essentially coercive convex function $C : X \to \mathbb{R}$ and a vector $v \in X^{\perp}$ such that

$$g(z) = c(P(z)) + \langle v, z\rangle$$

[2] It is worth noting that the preceding argument also shows that the dimension of X_f is k, the largest integer such that f is supported at some point by a $(k+1)$-dimensional corner function. In particular, it follows that a function is essentially coercive in $\mathbb{R}^n$ if and only if it is supported by an $(n+1)$-dimensional corner function.

for all $z \in \mathbb{R}^n$. Now take $x \in X$ and $\xi \in X^\perp$. The function $\mathbb{R} \ni t \mapsto g(t\xi)$, is affine, and because $f \leq g+1$ and f is convex, so must be the function $\mathbb{R} \ni t \mapsto f(t\xi)$, and with the same linear part (this immediately follows form the fact that the only convex functions which are bounded above on $\mathbb{R}$ are constants). This shows that

$$f(x+t\xi) = f(x) + t\langle v, \xi\rangle$$

for every $x \in X$, $\xi \in X^\perp$, $t \in \mathbb{R}$. Equivalently, we can write

$$f(z) = \varphi(P(z)) + \langle v, z\rangle \text{ for all } z \in \mathbb{R}^n,$$

where $\varphi : X \to \mathbb{R}$ is defined by $\varphi(x) = f(x)$ for all $x \in X$. Moreover, φ is essentially coercive because so is $g_{|X}$ and we have $|f-g| \leq 1$. This shows the existence of the decomposition in the statement. The uniqueness of the decomposition, as well as the last part of the statement of Theorem 1.3, follows by the same arguments as in Case 1, because that part of the proof does not need to use the differentiability of f. The proof of Theorem 1.3 is thus complete. □

3. Necessity of Theorem 1.5

Let F be a convex function of class $C^1(\mathbb{R}^n)$ such that $(F, \nabla F)$ extends (f, G) from E, and $X_F = X$.

3.1. **Condition** (i)**.** The inequality $f(x) - f(y) - \langle G(y), x-y\rangle \geq 0$ for all $x, y \in E$ follows from the fact that F is convex and differentiable with $(F, \nabla F) = (f, G)$ on E.

3.2. **Condition** (ii)**.** Assume that $(|\nabla F(x_k)|)_k$ tends to $+\infty$ for a sequence $(x_k)_k \subset \mathbb{R}^n$ but

$$\frac{\langle \nabla F(x_k), x_k\rangle - F(x_k)}{|\nabla F(x_k)|}$$

does not go to $+\infty$. Then, passing to a subsequence, we may assume that there exists $M > 0$ such that $\langle \nabla F(x_k), x_k\rangle - F(x_k) \leq M|\nabla F(x_k)|$ for all k. We denote $z_k = 2M\frac{\nabla F(x_k)}{|\nabla F(x_k)|}$. By convexity, we have, for all k, that

$$0 \leq F(z_k) - F(x_k) - \langle \nabla F(x_k), z_k - x_k\rangle \leq F(z_k) - M|\nabla F(x_k)|,$$

which contradicts the assumption that $|\nabla F(x_k)| \to \infty$.

3.3. **Condition** (iii)**.** Making use of Theorem 1.3 and bearing in mind that $X_F = X$, we can write $F = c \circ P_X + \langle v, \cdot\rangle$, where $P_X : \mathbb{R}^n \to X$ is the orthogonal projection onto the subspace X, the function $c : X \to \mathbb{R}$ is convex and essentially coercive on X, and $v \perp X$. It is easy to see that c is differentiable on X and that $\nabla F(x) = \nabla c(P_X(x)) + v$ for all $x \in \mathbb{R}^n$. Since $F = G$ on E, we easily get $G(x) - G(y) \in X$ for all $x, y \in E$.

3.4. **Condition** (v)**.** Let us consider sequences $(x_k)_k, (z_k)_k$ on E such that $(P_X(x_k))_k$ and $(\nabla F(z_k))_k$ are bounded and

$$\lim_{k\to\infty} \left(F(x_k) - F(z_k) - \langle \nabla F(z_k), x_k - z_k\rangle\right) = 0. \tag{3.1}$$

Suppose that $|\nabla F(x_k) - \nabla F(z_k)|$ does not converge to 0. Then, using that $(P_X(x_k))_k$ is bounded, there exist some $x_0 \in X$ and $\varepsilon > 0$ for which, possibly after passing to a subsequence, $P_X(x_k)$ converges to x_0 and $|\nabla F(x_k) - \nabla F(z_k)| \geq \varepsilon$ for every k. Using the decomposition $F = c \circ P_X + \langle v, \cdot\rangle$ and elementary properties of orthogonal projections on (3.1) we obtain

$$\lim_{k\to\infty} \left(c(P_X(x_k)) - c(P_X(z_k)) - \langle \nabla c(P_X(z_k)), P_X(x_k) - P_X(z_k)\rangle\right) = 0.$$

Since $\nabla F(y)-v = \nabla c(P_X(y))$ for all $y \in \mathbb{R}^n$ we have that $(\nabla c(P_X(z_k)))_k$ is bounded and $|\nabla c(P_X(x_k)) - \nabla c(P_X(z_k))| \geq \varepsilon$ for every k. This yields

$$\lim_{k\to\infty} \left(c(x_0) - c(P_X(z_k)) - \langle \nabla c(P_X(z_k)), x_0 - P_X(z_k) \rangle \right) = 0.$$

The contradiction follows from the following Lemma.

Lemma 3.1. *Let $h : X \to \mathbb{R}$ be a differentiable convex function and let x_0 and $(y_k)_k$ be a point and a sequence in X such that $(\nabla h(y_k))_k$ is bounded and*

$$\lim_{k\to\infty} \left(h(x_0) - h(y_k) - \langle \nabla h(y_k), x_0 - y_k \rangle \right) = 0.$$

Then $\lim_{k\to\infty} |\nabla h(x_0) - \nabla h(y_k)| = 0$.

Proof. Suppose that the conclusion is not satisfied. Then, up to extracting a subsequence, we would have $|\nabla h(x_0) - \nabla h(y_k)| \geq \varepsilon$, for some positive ε and for every k. Now, for every k, we set

$$\alpha_k := h(x_0) - h(y_k) - \langle \nabla h(y_k), x_0 - y_k \rangle, \quad v_k := \frac{\nabla h(y_k) - \nabla h(x_0)}{|\nabla h(y_k) - \nabla h(x_0)|}.$$

In Lemma 2.1 of [3] it is proved that $\alpha_k = 0$ implies $|\nabla h(x_0) - \nabla h(y_k)| = 0$, which is absurd. Thus we must have $\alpha_k > 0$ for every k. By convexity we have

$$\begin{aligned} \sqrt{\alpha_k} \langle \nabla h(x_0 + \sqrt{\alpha_k} v_k), v_k \rangle &\geq h(x_0 + \sqrt{\alpha_k} v_k) - h(x_0) \\ &\geq h(y_k) + \langle \nabla h(y_k), x_0 + \sqrt{\alpha_k} v_k - y_k \rangle - h(x_0) \\ &= -\alpha_k + \sqrt{\alpha_k} \langle \nabla h(y_k), v_k \rangle \quad \text{for all} \quad k. \end{aligned}$$

Hence, we obtain

$$\langle \nabla h(x_0 + \sqrt{\alpha_k} v_k) - \nabla h(x_0), v_k \rangle \geq -\sqrt{\alpha_k} + |\nabla h(y_k) - \nabla h(x_0)| \geq -\sqrt{\alpha_k} + \varepsilon.$$

The above inequality is impossible, as ∇h is continuous and $\alpha_k \to 0$. □

3.5. **Condition** (iv)**.** By applying Theorem 1.3 we may write

$$F(x) = c(P_X(x)) + \langle v, x \rangle,$$

with $c : X \to \mathbb{R}$ convex and essentially coercive, and $v \perp X$. This implies that

$$X = \operatorname{span}\{\nabla c(x) - \nabla c(y) : x, y \in X\},$$

and because $\nabla F = \nabla(c \circ P_X) + v$, also that

$$X = \operatorname{span}\{\nabla F(x) - \nabla F(y) : x, y \in \mathbb{R}^n\}.$$

Since $Y := \operatorname{span}\{\nabla F(x) - \nabla F(y) : x, y \in E\} \subset X$, if $Y \neq X$ and k and d denote the dimensions of Y and X respectively, we can find points $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus E$ such that

$$X = \{u - w : u, w \in \nabla F(E) \cup \{\nabla F(p_1), \ldots, \nabla F(p_{d-k}\}\}.$$

This shows the necessity of $(iv)(a)$. Obviously we have $\nabla F(p_j) \in X \setminus Y$ for all $j = 1, \ldots, d-k$, and because Y is closed and ∇F is continuous this implies that

$$p_j \in \mathbb{R}^n \setminus \overline{E} \text{ for all } j = 1, \ldots, d-k.$$

By the (already shown) necessity of condition (v), applied with $E^* = E \cup \{p_1, \ldots, p_{d-k}\}$ in place of E, we have that

$$\lim_{\ell} |G(x_\ell) - G(z_\ell)| = 0 \tag{3.2}$$

whenever $(x_\ell)_\ell, (z_\ell)_\ell$ are sequences in E^* such that $(P_X(x_\ell))_\ell$ and $(G(z_\ell))_\ell$ are bounded and

$$\lim_{\ell} \left(f(x_\ell) - f(z_\ell) - \langle G(z_\ell), x_\ell - z_\ell \rangle \right) = 0.$$

But the fact that $\text{dist}(\nabla F(p_j), Y) > 0$ for each $j = 1, \ldots, d-k$ prevents the limiting condition (3.2) from holding true with $(z_\ell)_\ell \subset \{p_1, \ldots, p_{d-k}\}$ and $(x_\ell)_\ell \subset E$. This implies that the inequalities

$$\begin{aligned} & F(p_j) \geq F(p_i) + \langle \nabla F(p_i), p_j - p_i \rangle, \quad 1 \leq i, j \leq d-k, \; i \neq j, \\ & F(p_j) \geq \sup_{z \in E, |G(z)| \leq N} \{ F(z) + \langle \nabla F(z), p_j - z \rangle \}, \quad 1 \leq j \leq d-k, \; N \in \mathbb{N}, \text{ and} \\ & \inf_{x \in E, |P_X(x)| \leq N} F(x) \geq \max_{1 \leq j \leq d-k} \{ F(p_j) + \langle \nabla F(p_j), x - p_j \rangle \}, \quad N \in \mathbb{N}, \end{aligned}$$

which generally hold by convexity of F, must all be strict. This shows the necessity of $(iv)(b) - (d)$.

4. Sufficiency of Theorem 1.5

First of all, with the notation of condition (iv), we define $E^* = E \cup \{p_1, ..., p_{d-k}\}$ and extend the functions f and G to E^* by setting

$$f(x_j) := \beta_j, \quad G(p_j) := w_j \quad \text{for} \quad j = 1, \ldots, d-k. \tag{4.1}$$

Lemma 4.1. *Let us denote $E^* := E \cup \{p_1, \ldots, p_{d-k}\}$. The following holds:*
(a) *$X = span\left(\{ G(x) - G(y) \, : \, x, y \in E^* \} \right)$.*
(b) *There exists $r > 0$ such that $f(p_i) - f(p_j) - \langle G(p_j), p_i - p_j \rangle \geq r$ for all $1 \leq i \neq j \leq d-k$.*
(c) *For every $N \in \mathbb{N}$, there exists $r_N > 0$ with $f(p_i) - f(z) - \langle G(z), p_i - z \rangle \geq r_N$ for all $z \in E$ with $|G(z)| \leq N$ and all $1 \leq i \leq d-k$.*
(d) *For every $N \in \mathbb{N}$, there exists $r_N > 0$ with $f(x) - f(p_i) - \langle G(p_i), x - p_i \rangle \geq r_N$ for all $x \in E$ with $|P_X(x)| \leq N$ and all $1 \leq i \leq d-k$.*

Proof. This follows immediately from condition (iv) and the definitions of (4.1). □

Lemma 4.2. *The jet (f, G) defined on $E^* = E \cup \{p_1, \ldots, p_{d-k}\}$ satisfies the inequalities of the assumption (i) on E^*. Moreover, if $(x_k)_k, (z_k)_k$ are sequences in E^* such that $(P_X(x_k))_k$ and $(G(z_k))_k$ are bounded, then*

$$\lim_{k \to \infty} \left((x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \right) = 0 \implies \lim_{k \to \infty} |G(x_k) - G(z_k)| = 0.$$

Proof. Suppose that$(x_k)_k, (z_k)_k$ are sequences in E^* such that $(P_X(x_k))_k$ and $(G(z_k))_k$ are bounded and $\lim_{k \to \infty} \left((x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \right) = 0$. In view of Lemma 4.1 (b), (c) and (d), it is immediate that there exists k_0 such that either there is some $1 \leq i \leq d-k$ with $x_k = z_k = p_i$ for all $k \geq k_0$ or else $x_k, z_k \in E$ for all $k \geq k_0$. In the first one of these cases, the conclusion is trivial. In the second one, $\lim_{k \to \infty} |G(x_k) - G(z_k)| = 0$ follows from condition (v) of Theorem 1.5. □

We now consider the minimal convex extension of the jet (f, G) from E^*, defined

$$m(x) = m(f, G, E^*)(x) := \sup_{y \in E^*} \{ f(y) + \langle G(y), x - y \rangle \}, \quad x \in \mathbb{R}^n.$$

It is clear that m, being the supremum of a family of affine functions, is a convex function on $\mathbb{R}^n$. In fact, we have the following.

Lemma 4.3. *$m(x)$ is finite for every $x \in \mathbb{R}^n$. In addition, $m = f$ on E^* and $G(x) \in \partial m(x)$ for all $x \in E^*$.*

Proof. Fix a point $z_0 \in E^*$. For any given point $x \in \mathbb{R}^n$ it is clear that there exists a sequence $(y_k)_k$ (maybe stationary) in E^* such that

$$f(z_0) + \langle G(z_0), x - z_0 \rangle \leq f(y_k) + \langle G(y_k), x - y_k \rangle \quad \text{for all} \quad k,$$

and $f(y_k) + \langle G(y_k), x - y_k \rangle \uparrow m(x)$ as $k \to \infty$. On the other hand, by the first statement of Lemma 4.2, we have

$$f(y_k) + \langle G(y_k), x - y_k \rangle \leq f(z_0) + \langle G(y_k), x - z_0 \rangle.$$

Then it is clear that $m(x) < +\infty$ when $(G(y_k))_k$ is a bounded sequence. We next show that this sequence cannot be unbounded. Indeed, in such case, by the condition (ii) in Theorem 1.5 (which obviously holds with E^* instead of E), we would have a subsequence for which $\lim_{k \to \infty} |G(y_k)| = +\infty$ which in turn implies

$$\lim_{k \to \infty} \frac{\langle G(y_k), y_k \rangle - f(y_k)}{|G(y_k)|} = +\infty.$$

Hence, by the assumption on $(y_k)_k$ it would be

$$\frac{f(y_k) - \langle G(y_k), y_k \rangle}{|G(y_k)|} \geq \frac{f(z_0) + \langle G(z_0), x - z_0 \rangle}{|G(y_k)|} - \Big\langle \frac{G(y_k)}{|G(y_k)|}, x \Big\rangle.$$

Since $\lim_{k \to \infty} |G(y_k)| = +\infty$, the right-hand term is bounded below and this leads to a contradiction. Therefore $m(x) < +\infty$ for all $x \in \mathbb{R}^n$. In addition, by using the definition of m and the first statement of Lemma 4.2 for the jet (f, G), we easily obtain that $m = f$ on E^* and that $G(x)$ belongs to $\partial m(x)$ for all $x \in E^*$ (where $\partial m(x)$ denotes the subdifferential of m at x). □

Lemma 4.4. *The function m is essentially coercive in the direction X, and in fact, with the notation of Theorem 1.3 we have that*

$$X_m = X.$$

Proof. By Lemma 4.1 (a), we have $X = \text{span}\left(\{G(x) - G(y) \,:\, x, y \in E^*\}\right)$. Let us first see that m is essentially coercive in the direction of X. If $X = \{0\}$ then m is affine and the result is obvious. Therefore we can assume $\dim(X) \geq 1$ and take points $x_0, x_1, \ldots, x_k \in E$ such that $\{v_1, \ldots, v_k\}$ is a basis of X, where

$$v_j = G(x_j) - G(x_0), \;\; j = 1, \ldots, k.$$

Then

$$C(x) = \max\{f(x_0) + \langle G(x_0), x - x_0 \rangle, \, f(x_1) + \langle G(x_1), x - x_1 \rangle, \ldots, \, f(x_k) + \langle G(x_k), x - x_k \rangle\}$$

defines a k-dimensional corner function such that

$$C(x) \leq m(x) \text{ for all } x \in \mathbb{R}^n,$$

and it is not difficult to see that C is essentially coercive in the direction of X, hence so is m.
In particular, by Theorem 1.3, it follows that $X \subseteq X_m$.
Now, if $X_m \neq X$, we can take a vector $w \in X_m \setminus \{0\}$ such that $w \perp X$, and then we obtain, for all $t \in \mathbb{R}$, that

$$\begin{aligned}
&m(x_0 + tw) - f(x_0) - \langle G(x_0), tw \rangle = \\
&\sup_{z \in E} \{f(z) - f(x_0) + \langle G(z) - G(x_0), tw \rangle + \langle G(z), x_0 - z\} = \\
&\sup_{z \in E} \{f(z) - f(x_0) + \langle G(z), x_0 - z\} \leq 0.
\end{aligned}$$

By convexity, this implies that

$$m(x_0 + tw) = f(x_0) + \langle G(x_0), tw \rangle$$

for all $t \in \mathbb{R}$, and in particular the function $\mathbb{R} \ni t \mapsto m(x_0 + tw)$ cannot be essentially coercive, contradicting the assumption that $w \in X_m$. Therefore we must have $X_m = X$. □

Making use of Theorem 1.3 in combination with Lemma 4.4, we can write

$$m = c \circ P_X + \langle v, \cdot \rangle \quad \text{on} \quad \mathbb{R}^n, \tag{4.2}$$

where $c : X \to \mathbb{R}$ is convex and essentially coercive on X and $v \perp X$. In addition, the subdifferential mappings of m and c satisfy the following.

Claim 4.5. *Given $x \in \mathbb{R}^n$ and $\eta \in \partial m(x)$, then $\eta - v \in X$ and $\eta - v \in \partial c(P_X(x))$.*

Proof. Suppose that $x \in \mathbb{R}^n$ and $\eta \in \partial m(x)$ but $\eta - v \notin X$. Then we can find $w \in X^\perp$ with $\langle \eta - v, w \rangle = 1$. Using (4.2) we get that

$$\langle \eta, w \rangle \leq m(x+w) - m(x) = c(P_X(x+w)) + \langle v, x+w \rangle - c(P_X(x)) - \langle v, x \rangle = \langle v, w \rangle.$$

This implies that $\langle \eta - v, w \rangle \leq 0$, a contradiction. This shows that $\eta - v \in X$. Now, let $z \in X$ and $x \in \mathbb{R}^n$. We have

$$c(z) - c(P_X(x)) = m(z) - \langle v, z \rangle - m(x) + \langle v, x \rangle \geq \langle \eta - v, z - x \rangle = \langle \eta - v, z - P_X(x) \rangle.$$

Therefore, $\eta - v \in \partial c(P_X(x))$. □

By combining the previous Claim with the second part of Lemma 4.3 we obtain that

$$G(x) - v \in \partial c(P_X(x)) \subset X \quad \text{for all} \quad x \in E^*. \tag{4.3}$$

Lemma 4.6. *c is differentiable on $\overline{P_X(E^*)}$ and if $y \in P_X(E^*)$, then $\nabla c(y) = G(x) - v$, where $x \in E^*$ is such that $P_X(x) = y$.*

Proof. Let us suppose that c is not differentiable at some $y_0 \in \overline{P_X(E^*)}$. Then, by the convexity of c on X, we may suppose that there exist a sequence $(h_k)_k \subset X$ with $|h_k| \downarrow 0$ and $\varepsilon > 0$ such that

$$\varepsilon \leq \frac{c(y_0 + h_k) + c(y_0 - h_k) - 2c(y_0)}{|h_k|} \quad \text{for all} \quad k.$$

We consider sequences $(y_k)_k \subset P_X(E^*)$ and $(x_k)_k \subset E^*$ with

$$P_X(x_k) = y_k \quad \text{and} \quad y_k \to y_0.$$

In particular, the sequence $(P_X(x_k))_k$ is bounded. Since each h_k belongs to X, we can use (4.2) to rewrite the last inequality as

$$\varepsilon \leq \frac{m(y_0 + h_k) + m(y_0 - h_k) - 2m(y_0)}{|h_k|} \quad \text{for all} \quad k. \tag{4.4}$$

By the definition of m we can pick two sequences $(z_k)_k, (\widetilde{z_k})_k \subset E^*$ with the following property:

$$m(y_0 + h_k) \geq f(z_k) + \langle G(z_k), y_0 + h_k - z_k \rangle \geq m(y_0 + h_k) - \frac{|h_k|}{2^k},$$
$$m(y_0 - h_k) \geq f(\widetilde{z_k}) + \langle G(\widetilde{z_k}), y_0 - h_k - \widetilde{z_k} \rangle \geq m(y_0 - h_k) - \frac{|h_k|}{2^k}$$

for every k. We claim that $(G(z_k))_k$ must be bounded. Indeed, otherwise, after passing to a subsequence and using the condition (ii) of Theorem 1.5, we would obtain that

$$\lim_{k\to\infty} G(z_k) = \lim_{k\to\infty} \frac{\langle G(z_k), z_k \rangle - f(z_k)}{|G(z_k)|} = +\infty.$$

Due to the choice of $(z_k)_k$ we must have

$$\begin{aligned} m(y_0) &= \lim_{k\to\infty} f(z_k) + \langle G(z_k), x_0 + h_k - z_k\rangle \\ &= \lim_{k\to\infty} |G(z_k)| \left(\frac{f(z_k) - \langle G(z_k), z_k\rangle}{|G(z_k)|} + \left\langle \frac{G(z_k)}{|G(z_k)|}, x_0 + h_k \right\rangle \right) = -\infty, \end{aligned}$$

which is absurd. Similarly we show that $(G(\tilde{z_k}))_k$ is bounded. Now we write

$$\begin{aligned} f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle \\ &= f(x_k) - \langle v, x_k\rangle - (m(y_0 + k_k) - \langle v, y_0 + h_k\rangle) \\ &\quad + m(y_0 + h_k) - f(z_k) - \langle G(z_k), y_0 + h_k - z_k\rangle \\ &\quad + \langle G(z_k) - v, y_0 + h_k - x_k\rangle. \end{aligned}$$

By (4.2), the first term in the sum equals $c(P_X(x_k)) - c(y_0 + h_k)$, which converges to 0 because $P_X(x_k) \to y_0$ and c is continuous. Thanks to the choice of the sequence $(z_k)_k$, the second term also converges to 0. From (4.3), we have $G(z_k) - v \in X$ for all k, and then the third term in the sum is actually $\langle G(z_k) - v, y_0 - P_X(x_k) + h_k\rangle$, which converges to 0 as $(G(z_k))_k$ is bounded and $P_X(x_k) \to y_0$. We then have

$$\lim_{k\to\infty} (f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle) = 0,$$

where $(P_X(x_k))_k$ and $(G(z_k))_k$ are bounded sequences. We obtain from Lemma 4.2 that $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$ and similarly we show that $\lim_{k\to\infty} |G(x_k) - G(\widetilde{z_k})| = 0$. This obviously implies

$$\lim_{k\to\infty} |G(z_k) - G(\widetilde{z_k})| = 0. \tag{4.5}$$

By the choice of the sequence $(z_k)_k$, $(\widetilde{z_k})_k$ and by inequality (4.4) we have, for every k,

$$\begin{aligned} \varepsilon &\le \frac{f(z_k) + \langle G(z_k), y_0 + h_k - z_k\rangle}{|h_k|} + \frac{f(\widetilde{z_k}) + \langle G(\widetilde{z_k}), y_0 - h_k - \widetilde{z_k}\rangle}{|h_k|} \\ &\quad - \frac{f(z_k) + \langle G(z_k), y_0 - z_k\rangle + f(\widetilde{z_k}) + \langle G(\widetilde{z_k}), y_0 - \widetilde{z_k}\rangle}{|h_k|} \\ &= \left\langle G(z_k) - G(\widetilde{z_k}), \frac{h_k}{|h_k|} \right\rangle + \frac{1}{2^{k-1}} \le |G(z_k) - G(\widetilde{z_k})| + \frac{1}{2^{k-1}}. \end{aligned}$$

Then (4.5) leads us to a contradiction. We conclude that c is differentiable on $\overline{P_X(E^*)}$.
We now prove the second part of the Lemma. Consider a point $y \in P_X(E^*)$ and $x \in E^*$ with $P_X(x) = y$. Using (4.3), $G(x) - v \in \partial c(y)$. Because c is differentiable at y, we further have that $G(x) - v = \nabla c(y)$. □

In order to complete the proof of Theorem 1.5, we will need the following Lemma.

Lemma 4.7. *Let $h : X \to \mathbb{R}$ be a convex and coercive function such that h is differentiable on a closed subset A of X. There exists $H \in C^1(X)$ convex and coercive such that $H = h$ and $\nabla H = \nabla h$ on A.*

Proof. Since h is convex, its gradient ∇h is continuous on A. Then, for all $x, y \in A$, we have

$$0 \le \frac{h(x) - h(y) - \langle \nabla h(y), x - y\rangle}{|x - y|} \le \left\langle \nabla h(x) - \nabla h(y), \frac{x - y}{|x - y|} \right\rangle \le |\nabla h(x) - \nabla h(y)|,$$

where the last term tends to 0 as $|x - y| \to 0$ uniformly on $x, y \in K$ for every compact subset K of A. This shows that the pair $(h, \nabla h)$ defined on A satisfies the conditions of the classical Whitney Extension Theorem for C^1 functions. Therefore, there exists a function $\widetilde{h} \in C^1(X)$ such that $\widetilde{h} = h$ and $\nabla \widetilde{h} = \nabla h$ on A. We now define

$$\phi(x) := |h(x) - \widetilde{h}(x)| + 2d(x,A)^2, \quad x \in X. \tag{4.6}$$

Claim 4.8. *ϕ is differentiable on A, with $\nabla\phi(x_0) = 0$ for every $x_0 \in A$.*

Proof. The function $d(\cdot, A)^2$ is obviously differentiable, with a null gradient, at x_0, hence we only have to see that $|h - \widetilde{h}|$ is differentiable, with a null gradient, at x_0. Since $\nabla\widetilde{h}(x_0) = \nabla h(x_0)$, the Claim boils down to the following easy exercise: if two functions h_1, h_2 are differentiable at x_0, with $\nabla h_1(x_0) = \nabla h_2(x_0)$, then $|h_1 - h_2|$ is differentiable, with a null gradient, at x_0. □

Now, because $d(\cdot, A)^2$ is continuous and positive on $X \setminus A$, according to Whitney's approximation theorem [25] we can find a function $\varphi \in C^\infty(X \setminus A)$ such that

$$|\varphi(x) - \phi(x)| \leq d(x,A)^2 \quad \text{for every } x \in X \setminus A, \tag{4.7}$$

Let us define $\widetilde{\varphi} : X \to \mathbb{R}$ by $\widetilde{\varphi} = \varphi$ on $X \setminus A$ and $\widetilde{\varphi} = 0$ on A.

Claim 4.9. *The function $\widetilde{\varphi}$ is differentiable on X.*

Proof. It is obvious that $\widetilde{\varphi}$ is differentiable on $\text{int}(A) \cup (X \setminus A)$. We only have to check that $\widetilde{\varphi}$ is differentiable on ∂A. If $x_0 \in \partial A$ we have

$$\frac{|\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0)|}{|x - x_0|} = \frac{|\widetilde{\varphi}(x)|}{|x - x_0|} \leq \frac{|\phi(x)| + d(x,A)^2}{|x - x_0|} \to 0$$

as $|x - x_0| \to 0^+$, because both H and $d(\cdot, A)^2$ vanish at x_0 and are differentiable, with null gradients, at x_0. Therefore $\widetilde{\varphi}$ is differentiable at x_0, with $\nabla\widetilde{\varphi}(x_0) = 0$. □

Now we set

$$g := \tilde{h} + \tilde{\varphi}$$

on X. It is clear that $g = h$ on A. Also, by Claim 4.9, g is differentiable on X with $\nabla g = \nabla h$ on A. By combining (4.6) and (4.7) we easily obtain that

$$g(x) \geq \tilde{h}(x) + \phi(x) - d(x,A)^2 \geq h(x) \quad x \in X \setminus A.$$

Therefore $g \geq h$ on X and in particular g is coercive on X, because so is h, by assumption.
We next consider the *convex envelope* of g. Recall that, for a function $\psi : X \to \mathbb{R}$, the convex envelope of ψ is defined by

$$\text{conv}(\psi)(x) = \sup\{\Phi(x) \,:\, \Phi \text{ is convex }, \Phi \leq \psi\}$$

(another expression for $\text{conv}(\psi)$, which follows from Carathéodory's Theorem, is

$$\text{conv}(\psi)(x) = \inf\left\{\sum_{j=1}^{n+1} \lambda_j \psi(x_j) \,:\, \lambda_j \geq 0, \sum_{j=1}^{n+1} \lambda_j = 1, x = \sum_{j=1}^{n+1} \lambda_j x_j\right\},$$

see [22, Corollary 17.1.5] for instance). The following result is a restatement of a particular case of the main theorem in [20]; see also [18].

Theorem 4.10 (Kirchheim-Kristensen)**.** *If $\psi : X \to \mathbb{R}$ is differentiable and $\lim_{|x|\to\infty} \psi(x) = \infty$, then $conv(\psi) \in C^1(X)$.*

If we define

$$H = \text{conv}(g)$$

we immediately get that H is convex on X and $H \in C^1(X)$. By definition of H we have that $h \leq H \leq g$ on X, which implies that H is coercive. Also, because $g = h$ on A, we have that $H = h$ on A. In order to shows that $\nabla H = \nabla h$ on A, we use the following well known criterion for differentiability of convex functions, whose proof is straightforward.

Lemma 4.11. *If ψ is convex, Φ is differentiable at x, $\psi \leq \Phi$, and $\phi(x) = \Phi(x)$, then ψ is differentiable at x, with $\nabla\psi(x) = \nabla\Phi(x)$.*

(This fact can also be phrased as: a convex function ψ is differentiable at x if and only if ψ is superdifferentiable at x.)
Since h is convex and H is differentiable on X with $h = H$ on A and $h \leq H$ on X, the preceding Lemma shows that $\nabla H = \nabla h$ on A. This completes the proof of Lemma 4.7. □

Now we are able to finish the proof of Theorem 1.5. Setting $A := \overline{P_X(E^*)}$, we see from Lemma 4.6 that c is differentiable on A. Moreover, since $c : X \to \mathbb{R}$ is convex and essentially coercive on X, there exists $\eta \in X$ such that $h := c - \langle \eta, \cdot \rangle$ is convex, differentiable on A and coercive on X. Applying Lemma 4.7 to h, we obtain $H \in C^1(X)$ convex and coercive on X with $(H, \nabla H) = (h, \nabla h)$ on A. Thus, the function $\varphi := H + \langle \eta, \cdot \rangle$ is convex, essentially coercive on X and of class $C^1(X)$ with $(\varphi, \nabla\varphi) = (c, \nabla c)$ on A. We next show that $F := \varphi \circ P_X + \langle v, \cdot \rangle$ is the desired extension of (f, G). Since φ is $C^1(X)$ and convex, it is clear that F is $C^1(\mathbb{R}^n)$ and convex as well. Bearing in mind Theorem 1.3 and the fact that φ is essentially coercive, it follows that $X_F = X$. Also, since $\varphi(y) = c(y)$ for $y \in P_X(E)$, we obtain from (4.2) and Lemma 4.3 that

$$F(x) = \varphi(P_X(x)) + \langle v, x \rangle = c(P_X(x)) + \langle v, x \rangle = m(x) = f(x).$$

Finally, from the second part of Lemma 4.6, we have for all $x \in E$ that

$$\nabla F(x) = \nabla\varphi(P_X(x)) + v = G(x) - v + v = G(x)$$

The proof of Theorem 1.5 is complete. □

5. Proof of Theorem 1.10

Let us assume first that there exists such a convex body W, and let us check that N satisfies conditions $(1) - (4)$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x) = \mu_W(x)^2.$$

We have that $\partial W = F^{-1}(1)$, and in particular $F = 1$ on E; besides

$$N(x) = \frac{\nabla F(x)}{|\nabla F(x)|} \text{ for all } x \in E.$$

It is clear that $F \in C^1_{\text{conv}}(\mathbb{R}^n)$, and therefore $(F, \nabla F)$ satisfies conditions $(i) - (iv)$ of Theorem 1.5 on the set $E^* := E \cup \{0\}$. Then condition (1) follows directly from (i) (or from the fact that W is convex and N is normal to ∂W). In order to check (2), take two sequences $(x_k)_k$, $(z_k)_k$ contained in E with $(P(x_k))_k$ bounded. Note that, because F is bounded on $\{x \in \mathbb{R}^n : F(x) \leq 2\}$ and F is convex, F is Lipschitz on the set $\{x \in \mathbb{R}^n : F(x) \leq 1\}$, and in particular ∇F is bounded on E, so $(\nabla F(z_k))_k$ is bounded as well. Now suppose that

$$\lim_{k\to\infty} \langle N(z_k), x_k - z_k \rangle = 0.$$

Then we also have, using that $F(x_k) = 1 = F(z_k)$ and the fact that $(|\nabla F(z_k)|)_k$ is bounded, that

$$\lim_{k\to\infty} \left(F(x_k) - F(z_k) - \langle \nabla F(z_k), x_k - z_k \rangle \right) = 0,$$

and according to (i) of Theorem 1.5 we obtain

$$\lim_{k\to\infty} |\nabla F(x_k) - \nabla F(z_k)| = 0. \tag{5.1}$$

Suppose, seeking a contradiction that we do not have $\lim_{k\to\infty} |N(x_k) - N(z_k)| = 0$. Then, after possibly passing to subsequences, we may assume that there exists some $\varepsilon > 0$ such that

$$|N(x_k) - N(z_k)| \geq \varepsilon \text{ for all } k \in \mathbb{N}.$$

Since $F(x_k) = 1 = F(z_k)$, $F(0) = 0$, and F is convex we have that

$$|\nabla F(x_k)| \geq 1, \; |\nabla F(z_k)| \geq 1,$$

and these sequences are also bounded above, so we may assume, possibly after extracting subsequences again, that $F(x_k)$ and $F(z_k)$ converge, respectively, to vectors $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$. By (5.1) we then get $\xi = \eta$, hence also

$$\varepsilon \leq |N(x_k) - N(z_k)| = \left| \frac{\nabla F(x_k)}{|\nabla F(x_k)|} - \frac{\nabla F(z_k)}{|\nabla F(z_k)|} \right| \to \left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| = 0,$$

a contradiction.

Let us now check (3). Since $0 \in \text{int}(W)$, we can find $r > 0$ such that $B(0, 2r) \subset W$. Let $y \in \partial W$. If $y \in \partial W$ is parallel to $N(y)$, then $\langle N(y), y \rangle = |y| \geq 2r$. Otherwise, by convexity of W, the triangle of vertices 0, $rN(y)$ and y, with angles α, β, γ at those vertices, is contained in W. So is the triangle of vertices 0, $N(y)$, p, where p is the intersection of the line segment $[0, y]$ with the line $L = \{rN(y) + tv : t \in \mathbb{R}\}$, where v is perpendicular to $N(y)$ in the plane $\text{span}\{y, N(y)\}$. Then we have that $|p| < |y|$, and $|p| \cos \alpha = r$, hence

$$\langle N(y), y \rangle = |y| \cos \alpha > |p| \cos \alpha = r > 0.$$

Finally let us check (4). The fact that $Y \subseteq X$ is a straightforward consequence of (iii) of Theorem 1.5 applied with $E^* = E \cup \{0\}$ and of the fact that $\nabla F(x) = 2\mu_W(x)\nabla\mu_W(x)$ and $n_W(x)$ are linearly dependent. If $Y \neq X$, then it is immediately seen that conditions $(iv)(a) - (d)$ of Theorem 1.5, with E^* in place of E, imply $(4)(a) - (d)$.

Conversely, assume that $N : E \to \mathbb{S}^{n-1}$ satisfies $(1) - (4)$, and let us construct a suitable W with the help of Theorem 1.5. Choose r such that

$$0 < r < \inf_{y \in E} \langle N(y), y \rangle, \tag{5.2}$$

and define $E^* = E \cup \{0\}$, $f : E^* \to \mathbb{R}$, $G : E^* \to \mathbb{R}^n$ by

$$f(0) = 0, f(x) = 1 \text{ if } x \in E; \;\; G(0) = 0, G(x) = \frac{2}{r} N(x) \text{ if } x \in E.$$

It is clear that condition (3) implies that $\text{dist}(0, E) > 0$, hence the continuity of G on E^* is obvious. As for checking that

$$f(x) - f(y) - \langle G(y), x - y \rangle \geq 0 \;\text{ for all } x, y \in E^*,$$

the only interesting case is that of $x = 0$, $y \in E$, for which we have

$$f(0) - f(y) - \langle G(y), x - y \rangle = -1 + \frac{2}{r} \langle N(y), y \rangle \geq -1 + 2 = 1 > 0.$$

Therefore condition (i) of Theorem 1.5 is fulfilled. Condition (ii) is satisfied trivially, because G is bounded on E^*. Conditions (iii) and (iv) follow immediately from (4). It only remains for us to check (v). As before, an a priori less trivial situation consists in taking $x_k = 0$, $(z_k)_k \subseteq E$. Note that $(G(z_k))_k$ is always bounded. Assuming that

$$\lim_{k \to \infty} f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle = 0,$$

we get $\lim_{k \to \infty} \langle G(z_k), z_k \rangle = 1$, which implies

$$\lim_{k \to \infty} \langle N(z_k), z_k \rangle = \frac{r}{2},$$

contradicting (5.2). Therefore this situation cannot occur. The rest of cases are immediately dealt with.

Thus we may apply Theorem 1.5 in order to find a function $F \in C^1_{\text{conv}}(\mathbb{R}^n)$ such that $(F, \nabla F)$ extends the jet (f, G) from E. We then define $W = F^{-1}(-\infty, 1]$. It is easy to check that W is a (possibly unbounded) convex body of class C^1 such that $0 \in \text{int}(W)$, $N(x) = n_W(x)$ for all $x \in E$, and $X = \text{span}\,(n_W(\partial W))$. □

References

[1] D. Azagra, *Global and fine approximation of convex functions.* Proc. Lond. Math. Soc. (3) 107 (2013), no. 4, 799–824.

[2] D. Azagra and P. Hajłasz, *Lusin-type properties of convex functions*, preprint.

[3] D. Azagra and C. Mudarra, *Whitney Extension Theorems for convex functions of the classes C^1 and $C^{1,\omega}$*, Proc. London Math. Soc. 114 (2017), no.1, 133–158.

[4] D. Azagra and C. Mudarra, *Explicit formulas for $C^{1,1}$ and $C^{1,\omega}_{conv}$ extensions of 1-jets in Hilbert and superreflexive spaces*, preprint, arXiv:1706.02235

[5] J. Benoist and J.-B. Hiriart-Urruty, *What is the subdifferential of the closed convex hull of a function?*, SIAM J. Math. Anal. 27 (6) (1996) 1661–1679.

[6] Y. Brudnyi, P. Shvartsman, *Whitney's extension problem for multivariate $C^{1,\omega}$-functions.* Trans. Am. Math. Soc. 353 (2001), 2487–2512.

[7] M. Cepedello, *On regularization in superreflexive Banach spaces by infimal convolution formulas*, Studia Math. 129 (1998), 265–284.

[8] A. Daniilidis, M. Haddou, E. Le Gruyer, and O. Ley, *Explicit formulas for $C^{1,1}$ Glaeser-Whitney extensions of 1-fields in Hilbert spaces*, preprint, arXiv:1706.01721

[9] C. Fefferman, *A sharp form of Whitney's extension theorem.* Ann. of Math. (2) 161 (2005), no. 1, 509–577.

[10] C. Fefferman, *Whitney's extension problem for C^m.* Ann. of Math. (2) 164 (2006), no. 1, 313–359.

[11] C. Fefferman, *Whitney's extension problems and interpolation of data.* Bull. Amer. Math. Soc. (N.S.) 46 (2009), no. 2, 207–220.

[12] C. Fefferman, A. Israel, G.K Luli, *Interpolation of data by smooth nonnegative functions*, Rev. Mat. Iberoam. 33 (2017), no. 1, 305–324.

[13] C. Fefferman, A. Israel, G.K Luli, *Finiteness principles for smooth selection.* Geom. Funct. Anal. 26 (2016), no. 2, 422–477.

[14] M. Ghomi, *Strictly convex submanifolds and hypersurfaces of positive curvature.* J. Differential Geom. 57 (2001), 239–271.

[15] M. Ghomi, *The problem of optimal smoothing for convex functions.* Proc. Amer. Math. Soc. 130 (2002) no. 8, 2255–2259.

[16] M. Ghomi, *Optimal smoothing for convex polytopes.* Bull. London Math. Soc. 36 (2004), 483–492.

[17] G. Glaeser, *Etudes de quelques algèbres tayloriennes*, J. d'Analyse 6 (1958), 1-124.

[18] A. Griewank, P.J. Rabier, *On the smoothness of convex envelopes.* Trans. Amer. Math. Soc. 322 (1990) 691–709.

[19] M. Jiménez-Sevilla, L. Sánchez–González, *On smooth extensions of vector-valued functions defined on closed subsets of Banach spaces.* Math. Ann. 355 (2013), no. 4, 1201–1219.

[20] B. Kirchheim, J. Kristensen, *Differentiability of convex envelopes.* C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), no. 8, 725–728.

[21] E. Le Gruyer, *Minimal Lipschitz extensions to differentiable functions defined on a Hilbert space.* Geom. Funct. Anal 19(4) (2009), 1101-1118.

[22] T. Rockafellar, *Convex Analysis.* Princeton Univ. Press, Princeton, NJ, 1970.

[23] K. Schulz, B. Schwartz, *Finite extensions of convex functions.* Math. Operationsforsch. Statist. Ser. Optim. 10 (1979), no. 4, 501–509.

[24] E. Stein, *Singular integrals and differentiability properties of functions.* Princeton, University Press, 1970.

[25] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63–89.

[26] M. Yan, *Extension of Convex Function.* J. Convex Anal. 21 (2014) no. 4, 965–987.

ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain
E-mail address: azagra@mat.ucm.es

ICMAT (CSIC-UAM-UC3-UCM), Calle Nicolás Cabrera 13-15. 28049 Madrid, Spain
E-mail address: carlos.mudarra@icmat.es