# KIRSZBRAUN'S THEOREM VIA AN EXPLICIT FORMULA

D. AZAGRA, E. LE GRUYER, AND C. MUDARRA

Abstract. Let $X, Y$ be two Hilbert spaces, $E$ a subset of $X$ and $G : E \to Y$ a Lipschitz mapping. A famous theorem of Kirszbraun's states that there exists $\widetilde{G} : X \to Y$ with $\widetilde{G} = G$ on $E$ and $\operatorname{Lip}(\widetilde{G}) = \operatorname{Lip}(G)$. In this note we show that in fact the function

$$\widetilde{G} := \nabla_Y(\operatorname{conv}(g))(\cdot, 0), \qquad \text{where}$$

$$g(x,y) = \inf_{z\in E}\left\{ \langle G(z), y\rangle + \frac{\operatorname{Lip}(G)}{2}\|(x-z,y)\|^2 \right\} + \frac{\operatorname{Lip}(G)}{2}\|(x,y)\|^2,$$

defines such an extension. We apply this formula to get an extension result for *strongly biLipschitz homeomorphisms.* Related to the latter, we also consider extensions of $C^{1,1}$ strongly convex functions.

## 1. An explicit formula for Kirszbraun's theorem

In 1934 M.D. Kirszbraun [17] proved that, for every subset $E$ of $\mathbb{R}^n$ and every Lipschitz function $f : E \to \mathbb{R}^m$, there exists a Lipschitz extension $F : \mathbb{R}^n \to \mathbb{R}^m$ of $f$ such that $\operatorname{Lip}(F) = \operatorname{Lip}(f)$. Here $\operatorname{Lip}(\varphi)$ denotes the Lipschitz constant of $\varphi$, that is,

$$\operatorname{Lip}(\varphi) = \sup_{x\neq y} \frac{\|\varphi(x) - \varphi(y)\|}{\|x-y\|}.$$

This theorem was generalized for Hilbert spaces $X, Y$ in place of $\mathbb{R}^n$ and $\mathbb{R}^m$ by F.A. Valentine [23] in 1945, and the result is often referred to as the Kirszbraun-Valentine theorem. The proof is rather nonconstructive, in the sense that it requires to use Zorn's lemma or transfinite induction at least in the nonseparable case. In the separable case the proof can be made by induction, considering a dense sequence $\{x_k\}$ in $X$ and at each step managing to extend $f$ from $E \cup \{x_1, \dots, x_m\}$ to $E \cup \{x_1, \dots, x_{m+1}\}$ while preserving the Lipschitz constant of the extension by using Helly's theorem or intersection properties of families of balls, but still it is not clear what the extension looks like. Several other proofs and generalizations which are not constructive either have appeared in the literature; see [20, 16, 12, 8, 21, 4]; apart from Zorn's lemma or induction these proofs are based on intersection properties of arbitrary families of balls, or on maximal extensions of non-expansive operators and Fitpatrick functions. In 2008 H.H. Bauschke and X. Wang [7] gave the first constructive proof of the Kirszbraun-Valentine theorem of which we are aware; they relied on their previous work [6] on extension and representation of monotone operators and the Fitzpatrick function. See also [1], where some of these techniques are used to construct definable versions of Helly's and Kirszbraun theorems in arbitrary definably complete expansions of ordered fields. Finally, in 2015 E. Le Gruyer and T-V. Phan provided sup-inf explicit extension formulas for Lipschitz mappings between finite dimensional spaces by relying on Le Gruyer's solution to the minimal $C^{1,1}$ extension problem for 1-jets; see [19, Theorem 32 and 33] and [18].

In this note we present a short proof of the Kirszbraun-Valentine theorem in which the extension is given by an explicit formula. This proof is based on our previous work concerning $C^{1,1}$ extensions of

*Date*: Oct 01, 2018.
2010 *Mathematics Subject Classification.* 47H09, 52A41, 54C20.
*Key words and phrases.* Lipschitz function, Kirszbraun Theorem.

1-jets with optimal Lipschitz constants of the gradients [3]. See [11] for an alternative construction of such $C^{1,1}$ extensions on the Hilbert space, and [9, 13, 14, 15] for the much more difficult question of extending functions (as opposed to jets) to $C^{1,1}$ or $C^{m,1}$ functions on $\mathbb{R}^n$.

If $X$ is a Hilbert space, $E \subset X$ is an arbitrary subset and $(f, G) : E \to \mathbb{R} \times X$ is a 1-jet on $E$, we will say that $(f, G)$ satisfies condition $(W^{1,1})$ with constant $M > 0$ on $E$ provided that

$$f(y) \leq f(x) + \tfrac{1}{2}\langle G(x) + G(y), y - x\rangle + \tfrac{M}{4}\|x - y\|^2 - \tfrac{1}{4M}\|G(x) - G(y)\|^2, \quad \text{for all} \quad x, y \in E. \tag{1}$$

In [24, 18] it was proved that condition $(W^{1,1})$ with constant $M > 0$ is a necessary and sufficient condition on $f : E \to \mathbb{R}$, $G : E \to X$ for the existence of a function $F \in C^{1,1}(X)$ with $\mathrm{Lip}(\nabla F) \leq M$ and such that $F = f$ and $\nabla F = G$ on $E$. More recently, as a consequence of a similar extension theorem for $C^{1,1}$ convex functions, we have found an explicit formula for such an extension $F$.

**Theorem 1.** [3, Theorem 3.4] *Let $E$ be a subset of a Hilbert space $X$. Given a 1-jet $(f, G)$ satisfying condition $(W^{1,1})$ with constant $M$ on $E$, the formula*

$$F = \mathrm{conv}(g) - \tfrac{M}{2}\|\cdot\|^2,$$
$$g(x) = \inf_{y \in E}\{f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x - y\|^2\} + \tfrac{M}{2}\|x\|^2, \quad x \in X,$$

*defines a $C^{1,1}(X)$ function with $F_{|E} = f$, $(\nabla F)_{|E} = G$, and* $\mathrm{Lip}(\nabla F) \leq M$.

Here $\mathrm{conv}(g)$ denotes the convex envelope of $g$, defined by

$$\mathrm{conv}(g)(x) = \sup\{h(x) \, : \, h \text{ is convex, proper and lower semicontinuous, } h \leq g\}. \tag{2}$$

Another expression for $\mathrm{conv}(g)$ is given by

$$\mathrm{conv}(g)(x) = \inf\left\{\sum_{j=1}^{k}\lambda_j g(x_j) \, : \, \lambda_j \geq 0, \sum_{j=1}^{k}\lambda_j = 1, \, x = \sum_{j=1}^{k}\lambda_j x_j, \, k \in \mathbb{N}\right\}, \tag{3}$$

and also by the Fenchel biconjugate of $g$, that is,

$$\mathrm{conv}(g) = g^{**}, \tag{4}$$

where

$$h^*(x) := \sup_{v \in X}\{\langle v, x\rangle - h(v)\}; \tag{5}$$

see [10, Proposition 4.4.3] for instance. In the case that $X$ is finite dimensional, say $X = \mathbb{R}^n$, the expression (3) can be made simpler: by using Carathéodory's Theorem one can show that it is enough to consider convex combinations of at most $n+1$ points. That is to say, if $g : \mathbb{R}^n \to \mathbb{R}$ then

$$\mathrm{conv}(g)(x) = \inf\left\{\sum_{j=1}^{n+1}\lambda_j g(x_j) \, : \, \lambda_j \geq 0, \sum_{j=1}^{n+1}\lambda_j = 1, \, x = \sum_{j=1}^{n+1}\lambda_j x_j\right\}; \tag{6}$$

see [22, Corollary 17.1.5] for instance.

**Theorem 2** (Kirszbraun's theorem via an explicit formula)**.** *Let $X, Y$ be two Hilbert spaces, $E$ a subset of $X$ and $G : E \to Y$ a Lipschitz mapping. There exists $\widetilde{G} : X \to Y$ with $\widetilde{G} = G$ on $E$ and* $\mathrm{Lip}(\widetilde{G}) = \mathrm{Lip}(G)$. *In fact, if $M = \mathrm{Lip}(G)$, then the function*

$$\widetilde{G}(x) := \nabla_Y(\mathrm{conv}(g))(x, 0), \quad x \in X, \quad \textit{where}$$
$$g(x, y) = \inf_{z \in E}\left\{\langle G(z), y\rangle_Y + \tfrac{M}{2}\|x - z\|_X^2\right\} + \tfrac{M}{2}\|x\|_X^2 + M\|y\|_Y^2, \quad (x, y) \in X \times Y,$$

*defines such an extension.*

Here $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the norm on $X$ and $Y$ respectively. Also, the inner products in $X$ and $Y$ are denoted by $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ respectively. For any function $F$, $\nabla_Y F$ will stand for the $Y$-*partial derivatives* of $F$, that is the canonical projection from $X\times Y$ onto $Y$ composed with $\nabla F$.

*Proof.* We consider on $X\times Y$ the norm given by $\|(x,y)\| = \sqrt{\|x\|_X^2+\|y\|_Y^2}$ for every $(x,y)\in X\times Y$. Then $X\times Y$ is a Hilbert space whose inner product is $\langle (x,y),(x',y')\rangle = \langle x,x'\rangle_X + \langle y,y'\rangle_Y$ for every $(x,y),(x',y')\in X\times Y$. We define the 1-jet $(f^*,G^*)$ on $E\times\{0\}\subset X\times Y$ by $f^*(x,0)=0$ and $G^*(x,0)=(0,G(x))$. Using that $G$ is $M$-Lipschitz, it is easy to see that $(f^*,G^*)$ satisfies condition $(W^{1,1})$ on $E\times\{0\}$ with constant $M$, see inequality (1). Therefore, Theorem 1 asserts that the function $F$ defined by

$$F = \operatorname{conv}(g) - \tfrac{M}{2}\|\cdot\|^2, \quad \text{where}$$
$$g(x,y) = \inf_{z\in E}\{f^*(z,0) + \langle G^*(z,0),(x-z,y)\rangle + \tfrac{M}{2}\|(x-z,y)\|^2\} + \tfrac{M}{2}\|(x,y)\|^2,$$

is of class $C^{1,1}(X\times Y)$ with $(F,\nabla F) = (f^*,G^*)$ on $E\times\{0\}$ and $\mathrm{Lip}(\nabla F)\le M$. In particular, the mapping $X\ni x\mapsto \widetilde{G}(x) := \nabla_Y F(x,0)\in Y$ is $M$-Lipschitz and extends $G$ from $E$ to $X$. Finally, the expressions defining $\widetilde{G}$ and $g$ can be simplified as

$$\widetilde{G}(x) = \nabla_Y\left(\operatorname{conv}(g) - \tfrac{M}{2}\|\cdot\|^2\right)(x,0) = \nabla_Y(\operatorname{conv}(g))(x,0) - \nabla_Y\left(\tfrac{M}{2}\|\cdot\|^2\right)(x,0) = \nabla_Y(\operatorname{conv}(g))(x,0)$$

and

$$g(x,y) = \inf_{z\in E}\left\{\langle G(z),y\rangle_Y + \tfrac{M}{2}\|x-z\|_X^2\right\} + \tfrac{M}{2}\|x\|_X^2 + M\|y\|_Y^2.$$

$\square$

Let $X$ be a Hilbert space with inner product and associated norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. If $E\subset X$ is arbitrary and $G: E\to X$ is a mapping, we say that $G$ is firmly non-expansive if

$$\langle G(x)-G(y), x-y\rangle \ge \|G(x)-G(y)\|^2 \quad \text{for all} \quad x,y\in E.$$

It is well-known (and easy to prove) that a mapping $G: E\to X$ is firmly non-expansive if and only $2G-I: E\to X$ is 1-Lipschitz, where $I$ denotes the identity map. Using this characterization and Theorem 2 we obtain the following corollary.

**Corollary 3.** *Let $G: E\to X$ be a firmly non-expansive mapping defined on a subset $E$ of a Hilbert space $X$. Then $G$ can be extended to a firmly non-expansive mapping $\widetilde{G}: X\to X$ by means of the formula*

$$\widetilde{G}(x) := \tfrac{1}{2}\left(P_2\left(\nabla(\operatorname{conv}(g))(x,0)\right) + x\right) \quad x\in X, \quad \textit{where}$$
$$P_2(x,y) = y, \quad (x,y)\in X\times X, \quad \textit{and}$$
$$g(x,y) = \inf_{z\in E}\left\{2\langle G(z),y\rangle + \tfrac{1}{2}\|z-(x+y)\|^2\right\} + \tfrac{1}{2}\|x-y\|^2, \quad (x,y)\in X\times X.$$

## 2. Extensions of strongly biLipschitz homeomorphisms

In this section we consider *strongly biLipschitz homeomorphisms*, which appear naturally as derivatives of strongly convex $C^{1,1}$ functions, and we provide an extension result for this class of homeomorphisms. It is also worth mentioning that, in [5, Corollary 4.5], H. H. Bauschke, S. M. Moffat and X. Wang showed that strongly bilipchitz mappings are closely related to contractive mappings: a mapping $G: E\to X$ is strongly biLipschitz if and only if the *reflected resolvent* $N = 2(G+I)^{-1}-I : G(E)\to X$ of $G$ is a contractive operator. Strongly biLipschitz mappings may also be interesting in regard to some problems in computer vision or image processing where one needs to match points in $\mathbb{R}^n$: for instance, given two sets of points in $\mathbb{R}^n$ with equal cardinality, find a homeomorphism from $\mathbb{R}^n$ onto itself which does not distort distances too much and takes one set onto the other. Supposing that

the data satisfy the strongly biLipschitz condition, our explicit formula for such an extension can be useful.
It should also be noted that it is not generally true that a biLipschitz function whose domain and range is a subset of the same Hilbert space $X$ extends to a total homeomorphism.

**Example 4.** Let $C = \{x \in \mathbb{R}^n : |x| = 1\} \cup \{p\}$, where $p$ is a point in $\mathbb{R}^n$ with $|p| > 1$, and $g : C \to \mathbb{R}^n$ be defined by $g(x) = x$ for $|x| = 1$ and $g(p) = 0$. Then no continuous extension of $g$ to $\mathbb{R}^n$ can be one-to-one.

However, this is true for the class of *strongly biLipschitz functions*, and moreover, the extension can be performed without increasing what seems natural to call the *strong biLipschitz constant*, as we will show by using the extension formula for Lipschitz mappings given by Theorem 2.

**Definition 5.** Let $E$ be a subset of a Hilbert space $X$. We say that a mapping $G : E \to X$ is *strongly biLipschitz* provided that

$$\inf_{x \neq y} \frac{2\langle x - y, G(x) - G(y)\rangle}{\|x - y\|^2 + \|G(x) - G(y)\|^2} > 0.$$

**Proposition 6.** *If $G : E \to X$ is strongly biLipschitz then $G$ is biLipschitz.*

*Proof.* For every $x, y \in E$ we have

$$2\|x - y\|\|G(x) - G(y)\| \geq 2\langle x - y, G(x) - G(y)\rangle \geq \alpha \left(\|x - y\|^2 + \|G(x) - G(y)\|^2\right), \tag{7}$$

where $\alpha > 0$ is as in the previous definition. It follows that $G$ is one-to-one. Note that we can write (7) in the equivalent form

$$\left\|G(x) - G(y) - \tfrac{1}{\alpha}(x - y)\right\|^2 \leq \tfrac{1-\alpha^2}{\alpha^2}\|x - y\|^2. \tag{8}$$

Setting $\lambda := \dfrac{\|G(x) - G(y)\|}{\|x - y\|}$ for $x \neq y$, (7) holds if and only if

$$\lambda^2 - \tfrac{2}{\alpha}\lambda + 1 \leq 0, \tag{9}$$

which is equivalent to $\frac{1}{K} \leq \lambda \leq K$; where $K = \frac{1}{\alpha} + (\frac{1}{\alpha^2} - 1)^{1/2}$. This means that for any $x \neq y \in E$ we have

$$\frac{1}{K} \leq \frac{\|G(x) - G(y)\|}{\|x - y\|} \leq K.$$

Therefore $G$ is a biLipschitz mapping. □

**Remark 7.** (1) If $X = \mathbb{R}$, then the strongly biLipschitz functions are exactly the strictly increasing biLipschitz function.
(2) If $G$ is such that $\mathrm{SBilip}(G) = 1$, then $G$ is the restriction of a translation.
(3) If $G$ is an isometry such that

$$\alpha := \inf_{x \neq y} \frac{\langle x - y, G(x) - G(y)\rangle}{\|x - y\|^2} > 0,$$

then $G$ is a strongly biLipschitz function with $\mathrm{SBilip}(G) = \alpha$. However the composition of strongly biLipschitz isometries need not be strongly biLipschitz (for instance, if $r : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $r(z) = e^{\pi i/4}z$ then $r$ is strongly biLipschitz, but $r^2$ is not (and in fact $r^4 = -\mathrm{id}$ is not strongly biLipschitz locally on any disk).

**Theorem 8.** *Let $G : E \to X$ be a strongly biLipschitz one-one function. Then $G$ extends to a strongly biLipschitz mapping on $X$ preserving the strongly bilipchitz constant $SBilip(G)$. Moreover, if $\alpha = SBilip(G)$, the formula*

$$\widetilde{G}(x) := P_2\left(\nabla(\mathrm{conv}(g))(x,0)\right) + \tfrac{1}{\alpha}x, \quad x \in X; \quad \textit{where}$$

$$P_2(x,y) = y, \quad (x,y) \in X \times X, \quad \textit{and}$$

$$g(x,y) = \inf_{z\in E}\left\{\langle G(z), y\rangle - \tfrac{1}{\alpha}\langle z, y\rangle + \tfrac{1}{2\alpha}\sqrt{1-\alpha^2}\|x-z\|^2\right\} + \tfrac{\sqrt{1-\alpha^2}}{\alpha}\left(\tfrac{1}{2}\|x\|^2 + \|y\|^2\right), \quad (x,y) \in X \times X,$$

*defines such an extension.*

*Proof.* We know from the characterization (8) that $G - \frac{1}{\alpha}I$ is Lipschitz on $E$ with $\mathrm{Lip}\left(G - \frac{1}{\alpha}I\right) \leq \sqrt{\frac{1-\alpha^2}{\alpha^2}}$. By Theorem 2, the mapping $T : X \to X$ defined as

$$T(x) := P_2\left(\nabla(\mathrm{conv}(g))(x,0)\right) \quad x \in X; \quad \text{where}$$

$$P_2(x,y) = y \quad \text{for every} \quad (x,y) \in X \times X, \quad \text{and}$$

$$g(x,y) = \inf_{z\in E}\left\{\langle G(z), y\rangle - \tfrac{1}{\alpha}\langle z, y\rangle + \tfrac{1}{2\alpha}\sqrt{1-\alpha^2}\|x-z\|^2\right\} + \tfrac{\sqrt{1-\alpha^2}}{\alpha}\left(\tfrac{1}{2}\|x\|^2 + \|y\|^2\right), \quad (x,y) \in X \times X,$$

is an extension of $G - \frac{1}{\alpha}I$ to all of $X$ such that $\mathrm{Lip}(T) = \mathrm{Lip}(G - \frac{1}{\alpha}I) \leq \sqrt{\frac{1-\alpha^2}{\alpha^2}}$. Therefore, if we define the function $\widetilde{G} = T + \frac{1}{\alpha}I$, we have that

$$\|\widetilde{G}(x) - \widetilde{G}(y) - \tfrac{1}{\alpha}(x-y)\|^2 = \|T(x) - T(y)\|^2 \leq \frac{1-\alpha^2}{\alpha^2}\|x-y\|^2 \quad \text{for all} \quad x, y \in X.$$

We obtain from (8) that $\widetilde{G}$ is strongly biLipschitz on $X$ with $\mathrm{SBilip}(\widetilde{G}) = \alpha$. Also, since $T$ is an extension of $G - \frac{1}{\alpha}I$, it is obvious that $\widetilde{G}$ is an extension of $G$. □

## 3. $C^{1,1}$ STRONGLY CONVEX FUNCTIONS

Throughout this section $X$ denotes a Hilbert space with norm and inner product denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively.

**Definition 9.** Let $E \subseteq X$ be arbitrary, $(f, G) : E \to \mathbb{R} \times X$ be a 1-jet and $c \in \mathbb{R}$, $M > 0$ constants such that $M > c$. We say that $(f, G)$ satisfies condition $(SCW^{1,1})$ with constants $(c, M)$ provided that

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + \tfrac{c}{M-c}\langle G(x) - G(y), y-x\rangle + \tfrac{cM}{2(M-c)}\|x-y\|^2 + \tfrac{1}{2(M-c)}\|G(x) - G(y)\|^2$$

for every $x, y \in E$.

**Remark 10.** Assume that $(f, G) : E \to \mathbb{R} \times X$ satisfies condition $(SCW^{1,1})$ with constants $(c, M)$.
(1) For every $x, y \in E$ we have

$$(c+M)\langle G(x) - G(y), x-y\rangle \geq cM\|x-y\|^2 + \|G(x) - G(y)\|^2.$$

(2) $G$ is Lipschitz with $c \leq \mathrm{Lip}(G) \leq M$.
(3) If $c > 0$, then $G$ is strongly biLipschitz with $\mathrm{SBilip}(G) \geq \frac{2}{c+M}\min\{1, cM\}$.
(4) For $c = -M$ we recover Wells's condition $W^{1,1}$ considered in [24, 18, 3]. For $c = 0$, $(SCW^{1,1})$ is just condition $(CW^{1,1})$ of [2, 3]. For $c \in (0, M]$ we have what can be called a $C^{1,1}$ strongly convex 1-jet, which in the extreme case $c = M$ becomes the restriction of a parabola to $E$.

*Proof.* $(1)$ : Let $x, y \in E$. By summing the inequalities

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + \tfrac{c}{M-c}\langle G(x)-G(y), y-x\rangle + \tfrac{cM}{2(M-c)}\|x-y\|^2 + \tfrac{1}{2(M-c)}\|G(x)-G(y)\|^2$$
$$f(y) \geq f(x) + \langle G(x), y-x\rangle + \tfrac{c}{M-c}\langle G(y)-G(x), x-y\rangle + \tfrac{cM}{2(M-c)}\|x-y\|^2 + \tfrac{1}{2(M-c)}\|G(x)-G(y)\|^2$$

we obtain

$$0 \geq \left(1 + \tfrac{2c}{M-c}\right)\langle G(x)-G(y), y-x\rangle + \tfrac{cM}{M-c}\|x-y\|^2 + \tfrac{1}{M-c}\|G(x)-G(y)\|^2,$$

which is equivalent to the desired estimation.

$(2)$ : Let $x, y \in E$ be such that $x \neq y$. Writing $\lambda = \|G(x)-G(y)\|/\|x-y\|$, the inequality in (1) yields $\lambda^2 - (c+M)\lambda + cM \leq 0$, which in turn implies $c \leq \lambda \leq M$.

$(3)$ : It follows immediately from (1) and Definition 5. $\square$

We say that a 1-jet $(f, G) : E \to \mathbb{R} \times X$ satisfies condition $(CW^{1,1})$ with constant $M > 0$ on $E$ provided that

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + \tfrac{1}{2M}\|G(x)-G(y)\|^2 \quad \text{for every} \quad x, y \in E.$$

In [3, Theorem 2.4] it was shown that $(CW^{1,1})$ is a necessary and sufficient condition on $(f, G)$ for the existence of a $C^{1,1}(X)$ convex extension $F$ of $f$ with $\nabla F = G$ on $E$.

**Lemma 11.** The 1-jet $(f, G)$ satisfies $(SCW^{1,1})$ with constants $(c, M)$ on $E$ if and only if the 1-jet $(\widetilde{f}, \widetilde{G}) = (f - \frac{c}{2}\|\cdot\|^2, G - cI)$ satisfies condition $(CW^{1,1})$ with constant $M - c$ on $E$.

*Proof.* Assume first that $(f, G)$ satisfies $(SCW^{1,1})$ with constants $(c, M)$ on $E$. We have

$$\widetilde{f}(x) - \widetilde{f}(y) - \langle \widetilde{G}(y), x-y\rangle - \tfrac{1}{2(M-c)}\|\widetilde{G}(x)-\widetilde{G}(y)\|^2 = f(x) - f(y) - \langle G(y), x-y\rangle - \tfrac{c}{2}\|x-y\|^2$$
$$- \tfrac{1}{2(M-c)}\left(\|G(x)-G(y)\|^2 + c^2\|x-y\|^2 + 2c\langle G(x)-G(y), y-x\rangle\right) = f(x) - f(y) - \langle G(y), x-y\rangle$$
$$- \tfrac{c}{M-c}\langle G(x)-G(y), y-x\rangle - \tfrac{cM}{2(M-c)}\|x-y\|^2 - \tfrac{1}{2(M-c)}\|G(x)-G(y)\|^2 \geq 0.$$

Conversely, if $(\widetilde{f}, \widetilde{G}) = (f - \frac{c}{2}\|\cdot\|^2, G - cI)$ satisfies condition $(CW^{1,1})$ with constant $M - c$, we can write

$$\begin{aligned} f(x) - f(y) - \langle G(y), x-y\rangle - \tfrac{c}{M-c}\langle G(x)-G(y), y-x\rangle - \tfrac{cM}{2(M-c)}\|x-y\|^2 - \tfrac{1}{2(M-c)}\|G(x)-G(y)\|^2 \\ = \widetilde{f}(x) - \widetilde{f}(y) - \langle \widetilde{G}(y), x-y\rangle + \tfrac{c}{2}\|x-y\|^2 - \tfrac{c}{M-c}\langle \widetilde{G}(x)-\widetilde{G}(y), y-x\rangle + \tfrac{c^2}{M-c}\|x-y\|^2 \\ - \tfrac{cM}{2(M-c)}\|x-y\|^2 - \tfrac{1}{2(M-c)}\left(\|\widetilde{G}(x)-\widetilde{G}(y)\|^2 + c^2\|x-y\|^2 + 2c\langle \widetilde{G}(x)-\widetilde{G}(y), x-y\rangle\right) \\ = \widetilde{f}(x) - \widetilde{f}(y) - \langle \widetilde{G}(y), x-y\rangle - \tfrac{1}{2(M-c)}\|\widetilde{G}(x)-\widetilde{G}(y)\|^2 \geq 0. \end{aligned}$$

$\square$

**Proposition 12.** *Let $F \in C^{1,1}(X)$ be such that* $\mathrm{Lip}(\nabla F) \leq M$ *and $g := F - \frac{c}{2}\|\cdot\|^2$ is a convex function, where $c \in \mathbb{R}$, $M > 0$ are constants (we call such a function a $C^{1,1}$ (globally) strictly convex function). Then:*
(1) *We have $M \geq c$ and if $M = c$, then $F$ is a parabola.*
(2) *$g$ is of class $C^{1,1}(X)$ with* $\mathrm{Lip}(\nabla g) \leq M - c$.
*Assume from now on that $M > c$.*
(3) *For every $x, y \in X$ we have*

$$F(x) \geq F(y) + \langle \nabla F(y), x-y\rangle + \tfrac{c}{M-c}\langle \nabla F(x)-\nabla F(y), y-x\rangle + \tfrac{cM}{2(M-c)}\|x-y\|^2 + \tfrac{1}{2(M-c)}\|\nabla F(x)-\nabla F(y)\|^2.$$

*Therefore, according to Definition 9, $(F, \nabla F)$ satisfies condition $(SCW^{1,1})$ on $X$ with constants $(c, M)$.*

*Proof.*
(1) and (2) : Given $x, h \in X$, the fact that $\mathrm{Lip}(\nabla F) \leq M$ and the convexity of $g$ yield

$$\begin{aligned} 0 &\leq g(x+h) + g(x-h) - 2g(x) \\ &= F(x+h) + F(x-h) - 2F(x) - \tfrac{c}{2}\left(\|x+h\|^2 + \|x-h\|^2 - 2\|x\|^2\right) \leq (M-c)\|h\|^2. \end{aligned}$$

This shows that $c \leq M$ and, again by convexity of $g$, that $\mathrm{Lip}(\nabla g) \leq M - c$. Finally observe that if $M = c$, then $g$ is affine and therefore $F$ is a parabola.

(3) : By virtue of (2), the convexity of $g$ implies that $(g, \nabla g)$ satisfies condition $(CW^{1,1})$ with constant $M - c$ on $X$. Thus Lemma 11 gives the desired inequality. $\square$

**Theorem 13.** *Let $E$ be an arbitrary subset of a Hilbert space $X$, $f : E \to \mathbb{R}$, $G : E \to X$ be two functions, and $c \in \mathbb{R}$, $M > 0$ be two constants. There exists a function $F \in C^{1,1}(X)$ such that $F = f$, $\nabla F = G$ on $E$, $\mathrm{Lip}(\nabla F) \leq M$, and $F - \frac{c}{2}\|\cdot\|^2$ is a convex function if and only if the jet $(f, G)$ satisfies condition $(SCW^{1,1})$ with constants $(c, M)$ on $E$. In fact, $F$ can be defined by means of the formula*

$$\begin{aligned} &F = \mathrm{conv}(g) + \tfrac{c}{2}\|\cdot\|^2, \\ &g(x) = \inf_{y \in E}\{f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}\|x-y\|^2\} - \tfrac{c}{2}\|x\|^2, \quad x \in X. \end{aligned}$$

*Moreover, if $H$ is another function of class $C^{1,1}(X)$ satisfying the above properties, then $H \leq F$ on $X$.*

*Proof.* The necessity of the condition $(SCW^{1,1})$ with constants $(c, M)$ on the jet $(f, G)$ follows immediately from Proposition 12 (3). Conversely, assume that $(f, G)$ satisfies condition $(SCW^{1,1})$ with constants $(c, M)$ on $E$. By Lemma 11, the jet $(\widetilde{f}, \widetilde{G}) = (f - \frac{c}{2}\|\cdot\|^2, G - cI)$ satisfies condition $(CW^{1,1})$ with constant $M - c$ and we can apply [3, Theorem 2.4] to obtain that

$$\widetilde{F} = \mathrm{conv}(g), \quad \text{where}$$

$$g(x) = \inf_{y \in E}\{\widetilde{f}(y) + \langle \widetilde{G}(y), x - y\rangle + \tfrac{M-c}{2}\|x-y\|^2\} = \inf_{y \in E}\{f(y) + \langle G(y), x-y\rangle + \tfrac{M}{2}\|x-y\|^2\} - \tfrac{c}{2}\|x\|^2$$

is convex and of class $C^{1,1}(X)$ with $\mathrm{Lip}(\nabla \widetilde{F}) \leq M - c$ and $(\widetilde{F}, \nabla \widetilde{F}) = (\widetilde{f}, \widetilde{G})$ on $E$. If we consider the function $F := \widetilde{F} + \frac{c}{2}\|\cdot\|^2$, Lemma 11 says that $(F, \nabla F)$ satisfies condition $(SCW^{1,1})$ with constants $(c, M)$ on $X$ (because $(\widetilde{F}, \nabla \widetilde{F})$ satisfies $(CW^{1,1})$ with constant $M - c$ on $X$) and $(F, \nabla F) = (f, G)$ on $E$. It is obvious that $F - \frac{c}{2}\|\cdot\|^2$ is convex on $X$ and, by Remark 10 (2), $\mathrm{Lip}(\nabla F) \leq M$.
Finally, if $H$ is a function of class $C^{1,1}(X)$ such that $(H, \nabla H) = (f, G)$ on $E$, $\mathrm{Lip}(\nabla H) \leq M$ and $\widetilde{H} := H - \frac{c}{2}\|\cdot\|^2$ is convex, then it is easy to see (using the same calculations as in the proof of Proposition 12 (2)) that $\mathrm{Lip}(\nabla \widetilde{H}) \leq M - c$, and obviously $(\widetilde{H}, \nabla \widetilde{H}) = (\widetilde{f}, \widetilde{G})$ on $E$. We thus have from [3, Theorem 2.4] that $\widetilde{H} \leq \widetilde{F}$ on $X$, and therefore $H \leq F$ on $X$. $\square$

## Acknowledgements

D. Azagra and C. Mudarra were partially supported by Grant MTM2015-65825-P and by ICMAT Severo Ochoa project SEV-2015-0554 (MINECO). C. Mudarra also acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the "Severo Ochoa for Centres of Excellence in R&D" (SEV-2015-0554).

## References


[1] M. Aschenbrenner and A. Fischer, *Definable versions of theorems by Kirszbraun and Helly.* Proc. London Math. Soc (3) 102 (2011) 468-502.

[2] D. Azagra and C. Mudarra, *Whitney Extension Theorems for convex functions of the classes $C^1$ and $C^{1,\omega}$*, Proc. London Math. Soc. 114 (2017), no.1, 133–158.

[3] D. Azagra, E. Le Gruyer and C. Mudarra, *Explicit formulas for $C^{1,1}$ and $C^{1,\omega}_{conv}$ extensions of 1-jets in Hilbert and superreflexive spaces.* J. Funct. Anal. 274 (2018), 3003-3032.

[4] H. H. Bauschke, *Fenchel duality, Fitzpatrick functions and the extension of firmly nonexpansive mappings.* Proc. Amer. Math. Soc. 135, pp. 135-139, 2007.

[5] H. H. Bauschke, S. M. Moffat, and X. Wang, *Firmly nonexpansive mappings and maximally monotone operators: correspondence and duality.* Set-Valued Anal. 20 (2012) 131–153.

[6] H. H. Bauschke and X. Wang, *The kernel average for two convex functions and its applications to the extension and representation of monotone operators.* Trans. Amer. Math. Soc. 361 (2009) 5947–5965.

[7] H. H. Bauschke and X. Wang, *Firmly nonexpansive and Kirszbraun-Valentine extensions: a constructive approach via monotone operator theory.* Nonlinear analysis and optimization, Contemporary Mathematics (eds. A. Leizarowitz et al.; American Mathematical Society, Providence, RI, 2010), 55-64.

[8] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis.* Amer. Math. Soc. Coll. Pubs. 48, Providence, RI, 2000.

[9] Y. Brudnyi and P. Shvartsman, *Whitney's extension problem for multivariate $C^{1,\omega}$-functions.* Trans. Am. Math. Soc. 353 (2001), 2487–2512.

[10] J. M. Borwein and J. D. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counterexamples.* Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2010.

[11] A. Daniilidis, M. Haddou, E. Le Gruyer and O. Ley, *Explicit formulas for $C^{1,1}$ Glaeser-Whitney extensions of 1-fields in Hilbert spaces.* Proc. Amer. Math. Soc. 146 (2018), 4487-4495.

[12] H. Federer, *Geometric measure theory.* Springer-Verlag, New York, 1969.

[13] C. Fefferman, *A sharp form of Whitney's extension theorem.* Ann. of Math. (2) 161 (2005), no. 1, 509–577.

[14] C. Fefferman, *Whitney's extension problem for $C^m$.* Ann. of Math. (2) 164 (2006), no. 1, 313–359.

[15] C. Fefferman, *Whitney's extension problems and interpolation of data.* Bull. Amer. Math. Soc. (N.S.) 46 (2009), no. 2, 207–220.

[16] F. Grünbaum and E. H. Zarantonello, *On the extension of uniformly continuous mappings.* Michigan Math. J. 15 (1968), 65–74.

[17] M. D. Kirszbraun, *Über die zusammenziehenden und Lipschitzschen Transformationen.* Fund. Math. 22 (1934), 77-108.

[18] E. Le Gruyer, *Minimal Lipschitz extensions to differentiable functions defined on a Hilbert space.* Geom. Funct. Anal 19(4) (2009), 1101-1118.

[19] E. Le Gruyer and Thanh-Viet Phan, *Sup-Inf explicit formulas for minimal Lipschitz extensions for* 1*-fields on* $\mathbb{R}^n$. J. Math. Anal. Appl. 424 (2015), 1161-1185.

[20] E. J. Mickle, *On the extension of a transformation.* Bull. Amer. Math. Soc. 55 (1949), 160-164.

[21] S. Reich and S. Simons, *Fenchel duality, Fitzpatrick functions and the Kirszbraun-Valentine extension theorem.* Proceedings of the American Mathematical Society, vol. 133 (2005), 2657-2660.

[22] T. Rockafellar, *Convex Analysis.* Princeton Univ. Press, Princeton, NJ, 1970.

[23] F.A. Valentine, *A Lipschitz condition preserving extension for a vector function.* Amer. J. Math., 67 No. 1 (1945), 83–93.

[24] J.C. Wells, *Differentiable functions on Banach spaces with Lipschitz derivatives.* J. Differential Geometry 8 (1973), 135–152.

ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático y Matemática Aplicada, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain. **DISCLAIMER:** The first-named author is affiliated to Universidad Complutense de Madrid, but this does not mean this institution has offered him all the support he expected; on the contrary, the Biblioteca Complutense has hampered his research by restricting his access to many books.
*E-mail address*: `azagra@mat.ucm.es`

INSA de Rennes & IRMAR, 20, Avenue des Buttes de Coësmes, CS 70839 F-35708, Rennes Cedex 7, France
*E-mail address*: `Erwan.Le-Gruyer@insa-rennes.fr`

ICMAT (CSIC-UAM-UC3-UCM), Calle Nicolás Cabrera 13-15. 28049 Madrid, Spain
*E-mail address*: `carlos.mudarra@icmat.es`