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Calculus of Variations

Generalized motion of level sets by functions of their curvatures on Riemannian manifolds

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Abstract We consider the generalized evolution of compact level sets by functions of their normal vectors and second fundamental forms on a Riemannian manifold M. The level sets of a function $u : M \to \mathbb{R}$ evolve in such a way whenever u solves an equation $u_t + F(Du, D^2u) = 0$, for some real function F satisfying a geometric condition. We show existence and uniqueness of viscosity solutions to this equation under the assumptions that M has nonnegative curvature, F is continuous off $\{Du = 0\}$, (degenerate) elliptic, and locally invariant by parallel translation. We then prove that this approach is geometrically consistent, hence it allows to define a generalized evolution of level sets by very general, singular functions of their curvatures. For instance, these assumptions on F are satisfied when F is given by the evolutions of level sets by their mean curvature (even in arbitrary codimension) or by their positive Gaussian curvature. We also prove that the generalized evolution is consistent with the classical motion by the corresponding function of the curvature, whenever the latter exists. When M is not of nonnegative curvature, the same results hold if one additionally requires that F is uniformly continuous with respect to D^2u. Finally we give some counterexamples showing that several well known properties of the evolutions in $\mathbb{R}^n$ are no longer true when M has negative sectional curvature.

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1 Introduction

In the last 30 years there has been a lot of interest in the evolution of hypersurfaces of $\mathbb{R}^n$ by functions of their curvatures. In this kind of problem one is asked to find a one parameter family of orientable, compact hypersurfaces Γ_t which are boundaries of open sets U_t and satisfy

$$\begin{aligned} V &= -G(\nu, D\nu) \text{ for } t > 0,\ x \in \Gamma_t, \text{ and} \\ \Gamma_t|_{t=0} &= \Gamma_0 \end{aligned} \tag{1.1}$$

for some initial set $\Gamma_0 = \partial U_0$, where V is the normal velocity of Γ_t, $\nu = \nu(t, \cdot)$ is a normal field to Γ_t at each x, and G is a given (nonlinear) function.

Two of the most studied examples are the evolutions by mean curvature and by (positive) Gaussian curvature. In both cases, short time existence of classical solutions has been established. For strictly convex initial data U_0, it has been shown that U_t shrinks to a point in finite time, and moreover, Γ_t becomes spherical at the end of the contraction. See [3,14,15,20,21,23,24,34] and the references therein.

For dimension $n \geq 3$ it has been shown [19] that a hypersurface evolution Γ_t may develop singularities before it disappears. Hence it is natural to try to develop weak notions of solutions to (1.1) which allow to deal with singularities of the evolutions, and even with nonsmooth initial data Γ_0.

There are two mainstream approaches concerning weak solutions of (1.1): the first one uses geometric measure theory to construct (generally nonunique) varifold solutions, see [6,26], while the second one adapts the theory of second order viscosity solutions developed in the 1980s (see [8] and the references therein) to show existence and uniqueness of level-set weak solutions to (1.1).

In this paper we will focus on this second approach. The first works to develop a notion of *viscosity* level set solution to (1.1) were those of Evans and Spruck [10] and, independently developed, Chen et al. [7], [17]. This was followed by many important developments, which we find impossible to properly quote here; we refer the reader to the very comprehensive monograph [16] and the bibliography therein. This level set approach consists in observing that a smooth function $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with $Du := D_x u \neq 0$ has the property that all its level sets evolve by (1.1) if and only if u is a solution of

$$u_t + F(Du, D^2 u) = 0, \tag{1.2}$$

where F is related to G in (1.1) through of the following formula:

$$F(p, A) = |p|\, G\left(\frac{p}{|p|}, \frac{1}{|p|}\left(I - \frac{p \otimes p}{|p|^2}\right) A\right). \tag{1.3}$$

The function F is assumed to be continuous off $\{p = 0\}$ and (degenerate) elliptic, that is

$$F(p, B) \leq F(p, A) \quad \text{whenever } A \leq B. \tag{1.4}$$

Because of (1.3), F also has the following geometric property:

$$F(\lambda p, \lambda A + \mu p \otimes p) = \lambda F(p, A) \quad \text{for all } \lambda > 0, \mu \in \mathbb{R}. \tag{1.5}$$

The function F does not generally admit any continuous extension to $\mathbb{R}^n \times \mathbb{R}^{n^2}$ but, if it is bounded near $\{p = 0\}$ (this is the case of the mean curvature evolution equation), one can show that there is a unique viscosity solution to (1.2) with initial datum $u(0, x) = g(x)$ (for any continuous g such that $\Gamma_0 = \{x : g(x) = 0\}$). Next one can also see that if $\theta : \mathbb{R} \to \mathbb{R}$ is continuous and u is a solution of (1.2) then $\theta \circ u$ is a solution too, and this, together with a comparison principle, allows to show that the generalized geometric evolution

$$\Gamma_0 \to \Gamma_t := \{x : u(t, x) = 0\}$$

is well defined (that is, the zero level set of a solution to (1.2) only depends on the zero level set of its initial datum). It is also possible to show that this level set evolution agrees with any classical solution of (1.1).

When $F(p, A)$ is not bounded as $p \to 0$ (this is the case of more singular equations such as the Gaussian evolution), then the standard notion of viscosity solution to (1.2) (as is used, for instance, in [7]) at points $z = (t, x)$ where the test function φ satisfies $D\varphi(z) = 0$ is not suitable to tackle the problem. In this case two different modifications of the notion of solution have been proposed in the literature.

One possibility is simply not to specify any condition for the derivatives of a test function φ such that $u - \varphi$ attains a maximum or a minimum at a point (t_0, x_0) with $D\varphi(t_0, x_0) = 0$. This is Goto's approach in [18]. When one uses this definition of solution, the corresponding comparison theorem becomes harder to prove, and it is indeed a stronger statement since the class of solutions becomes bigger in this case, while the existence result is comparatively weaker.

The other possibility is to make the class of test functions φ smaller, in a clever way so that, if $z_k \to z_0$ and $D\varphi_k(z_k) \to 0$, one can show that $F(D\varphi_k(z_k), D^2\varphi_k(z_k))$ goes to 0, and then to demand that a subsolution u should satisfy that if $u - \varphi$ has a maximum at z_0 then $\varphi_t(z_0) \leq 0$. This is what Ishii and Souganidis did in [28]. The corresponding (sub)solutions are called $\mathcal{F}$-(sub)solutions in Giga's book [16]. In this approach the maximum principle is relatively easier to prove, while existence becomes harder (and is really a stronger result, because the class of solutions is smaller in this case).

The aim of this paper is to investigate to what extent one can develop a general theory of (viscosity) level-set solutions to the problem of the evolution of hypersurfaces by functions of their curvatures in a Riemannian manifold. To the best of our knowledge, the only work in this direction is Ilmanen's paper [25] (in fact this is the only paper we know of in which second order viscosity solutions are employed to deal with a second order evolution equation within the context of Riemannian manifolds). In [25] Ilmanen shows existence and uniqueness of a (standard) viscosity solution to the mean curvature evolution equation, that is (1.1) in the case when F is given by

$$F(p, A) = -\text{trace}\left(\left(I - \frac{p \otimes p}{|p|^2}\right) A\right),$$

with initial condition $u(0, x) = g(x)$, thus obtaining a corresponding generalized evolution by mean curvature, some of whose geometric properties he next studies. For instance, he proves that if noncompact initial data Γ_0 are allowed then one loses uniqueness of the generalized geometric evolution.

In recent years, an interest has grown in the use of viscosity solutions of (first order) Hamilton–Jacobi equations defined on Riemannian manifolds (in relation to dynamical systems, to geometric problems, or from a theoretical point of view), see [4,9,12,13,22,29,30], but no second order theory, apart from Ilmanen's paper, has apparently been developed for parabolic equations (in the case of stationary, degenerate elliptic equations, such a study was recently started in [5]).

We believe that a level set method for generalized evolution of hypersurfaces by functions of their curvatures can be useful in the setting of Riemannian manifolds. On the one hand we think that it is very natural, from a geometric point of view, to try to study the evolutions of level sets in a general Riemannian manifold M by their Gaussian (or by other functions of their) curvatures, in a way that is supple enough so that nonsmooth initial data and singularities of the evolutions are allowed. On the other hand, as one sees, for instance, by restricting to the case $M = \mathbb{R}^n$ endowed with a non Euclidean metric, the tools developed here allow to treat level set evolutions in inhomogeneous media, in which the function F depends (in a very special manner) on the position variable x.

Let us briefly describe the main results of this paper. In Sect. 2 we consider equations of the form (1.1), (1.2) for level sets of functions u defined on a Riemannian manifold M, and we show how the F's corresponding to the evolutions by mean curvature (even in arbitrary codimension, in the line of [2]) and by (positive) Gaussian curvature are extended to $J^2_0(M)$ in such a way that F is (degenerate) *elliptic, translation invariant, geometric*, and continuous off $\{Du = 0\}$ (see properties (**A - D**) in Sect. 2). Following [16,28], for each F we next define an appropriate class of test functions $\mathcal{A}(F)$ which allows us to deal with equation (1.2) on M, and we define the corresponding class of $\mathcal{F}$-solutions, see Definitions 2.4, 2.7. We also show that for all F which are continuous off $\{Du = 0\}$, elliptic, translation invariant and geometric, one has that $\mathcal{A}(F) \neq \emptyset$ provided that M is compact. Moreover, in the cases when F is given by the mean curvature or the Gaussian curvature evolution equations, we have $\mathcal{A}(F) \neq \emptyset$ no matter whether M is compact or not.

In Sect. 3 we present some technical results that will be used later on in the proofs of the main results.

Section 4 is devoted to proving a comparison result for viscosity solutions of (1.2) on M: under the above assumptions on F (namely, continuity, ellipticity, geometricity and translation invariance) we show that if M has nonnegative curvature u is a subsolution, v is a supersolution, $u \leq v$ on $\{0\} \times M$, and $\limsup_{(t,x)\to\infty}(u-v) \leq 0$ (this condition is understood to be requiring nothing when M is compact), then $u \leq v$ on $[0, T] \times M$. When M is not of nonnegative curvature, we have to additionally require that F be uniformly continuous with respect to D^2u.

In Sect. 5 we show that Perron's method (first used in [27]) works to produce $\mathcal{A}(F)$-solutions of (1.2) on a Riemannian manifold M, provided that comparison holds and $\mathcal{A}(F) \neq \emptyset$.

Therefore, for all such M and F, for every compact subset Γ_0 of M, and for every continuous function g on M such that $\Gamma_0 = \{x \in M : g(x) = 0\}$, there exists a unique solution of (1.2) on M with initial condition $u(0, \cdot) = g$. One can then define, for each compact Γ_0, an evolution $\Gamma_t = \{x \in M : u(t, x) = 0\}$, $t \geq 0$. In Sect. 6 we see that Γ_t does not depend on the function g chosen to represent Γ_0, and consequently the generalized geometric evolution $\Gamma_0 \mapsto \Gamma_t$ is well defined.

Next, in Sect. 7 we prove that this generalized evolution is consistent with the classical motion, whenever the latter exists. Namely, if $(\Gamma_t)_{t\in[0,T]}$ is a family of smooth, compact, orientable hypersurfaces in a Riemannian manifold M evolving according to a classical geometric motion, locally depending only on its normal vector fields and second fundamental forms according to an equation of the form (1.1), and Γ_0 can be represented as the zero level set of a smooth function g on M, then Γ_t coincides with the generalized level set evolution (with initial datum Γ_0) defined above.

Finally, in Sect. 8, we give some counterexamples showing that several well known properties of generalized solutions to the mean curvature flow cannot be extended from Euclidean spaces to Riemannian manifolds of negative sectional curvature. For instance, Ambrosio and Soner [1,2,32,33] showed that the distance function from $\Gamma_t \subset \mathbb{R}^n$ given by $|d|(t,x) = \text{dist}(x, \Gamma_t)$ is a supersolution of (1.2) when F corresponds to the mean curvature evolution equation. We show that this result fails when $\mathbb{R}^n$ is replaced with a manifold of negative sectional curvature. On the other hand, if M has negative curvature, then Eq. (1.2) does not preserve Lipschitz properties of the initial data, in contrast with [16, Chap. 3]. And, again in the case of the mean curvature flow, if Γ_0, $\hat{\Gamma}_0$ are smooth 1-codimensional submanifolds of a manifold M of negative curvature, then the function $t \mapsto \text{dist}(\Gamma_t, \hat{\Gamma}_t)$ can be decreasing, in contrast with [10, Theorem 7.3].

An the end of this article, the reader will find an appendix describing a comparison and an existence result for (standard) viscosity solutions to general evolution equations of the form

$$u_t + F(x, t, u, Du, D^2u) = 0$$

where F has no singularities. We omit the proofs because they resemble (and are easier than) those of the main comparison and existence result for $\mathcal{F}$-solutions of (1.2) given in Sects. 4 and 5.

Notation M will always be a finite-dimensional Riemannian manifold. We will write $\langle \cdot, \cdot \rangle$ for the Riemannian metric and $|\cdot|$ for the Riemannian norm on M. The tangent and cotangent space of M at a point x will be respectively denoted by TM_x and TM_x^*. We will often identify them via the isomorphism induced by the Riemannian metric. The space of bilinear forms on TM_x (respectively symmetric bilinear forms) will be denoted by $\mathcal{L}^2(TM_x)$ (resp. $\mathcal{L}_s^2(TM_x)$). Elements of $\mathcal{L}^2(TM_x)$ will be denoted by the letters A, B, P, Q, and those of TM_x^* by ζ, η, etc. Also, we will respectively denote the cotangent bundle and the tensor bundle of symmetric bilinear forms in M by

$$TM^* := \bigcup_{x \in M} TM_x^*, \quad T_{2,s}(M) := \bigcup_{x \in M} \mathcal{L}_s^2(TM_x).$$

We will also consider the two-jet bundles:

$$J^2M := \bigcup_{x \in M} TM_x^* \times \mathcal{L}_s^2(TM_x), \quad J_0^2(M) := \bigcup_{x \in M} \left(TM_x^* \setminus \{0_x\}\right) \times \mathcal{L}_s^2(TM_x).$$

The letters X, Y, Z will stand for smooth vector fields on M, and $\nabla_Y X$ will always denote the covariant derivative of X along Y. Curves and geodesics in M will be denoted by γ, σ, and their velocity fields by γ', σ'. If X is a vector field along γ we will often denote $X'(t) = \frac{d}{dt}X(t) = \nabla_{\gamma'(t)}X(t)$. Recall that X is said to be parallel along γ if $X'(t) = 0$ for all t. The Riemannian distance in M will always be denoted by $d(x, y)$ (defined as the infimum of the lengths of all curves joining x to y in M).

Given a smooth function $u : M \to \mathbb{R}$, we will denote its differential by $D_x u \in TM^*$; its gradient vector field will be written as ∇u, and its Hessian as $D_x^2 u$. Recall that, for any two vector fields X, Y satisfying $X(p) = v$, $Y(p) = w$ at some $p \in M$ we have:

$$D_x^2 u\,(X, Y) := \langle \nabla_Y \nabla u, X \rangle, \quad D_x^2 u\,(v, w) := D_x^2 u\,(X, Y)\,(p).$$

Given a function $v : M \to \mathbb{R}$ we will use the notation:

$$v^*(t, x) = \lim_{r \downarrow 0} \sup\{v(s, y) : y \in M,\ s > 0,\ |t - s| \le r,\ d(y, x) \le r\},$$

$$v_*(t, x) = \lim_{r \downarrow 0} \inf\{v(s, y) : y \in M,\ s > 0,\ |t - s| \le r,\ d(y, x) \le r\};$$

that is v^* denotes the upper semicontinuous envelope of v (the smallest upper semicontinuous function, with values in $[-\infty, \infty]$, satisfying $v \le v^*$), and similarly v_* stands for the lower semicontinuous envelope of v.

We will make frequent use of the exponential mapping $\exp_x$ and of the parallel translation along a geodesic γ. Recall that for every $x \in M$ there exists a mapping $\exp_x$, defined on a neighborhood of 0 in the tangent space TM_x, and taking values in M, which is a local diffeomorphism and maps straight line segments passing through 0 onto geodesic segments in M passing through x. The exponential mapping induces a local diffeomorphism on the cotangent space TM_x^*, via the identification given by the metric, that will be also denoted by $\exp_x$. On the other hand, for a minimizing geodesic $\gamma : [0, \ell] \to M$ connecting x to y in M, and for a vector $v \in TM_x$ there is a unique parallel vector field P along γ such that $P(0) = v$, this is called the parallel translation of v along γ. The mapping $TM_x \ni v \mapsto P(\ell) \in TM_y$ is a linear isometry from TM_x onto TM_y which we will denote by L_{xy}. This isometry naturally induces an isometry between the space of bilinear forms on TM_x and the space of bilinear forms on TM_y. Whenever we use the notation L_{xy} we assume implicitly that x and y are close enough to each other so that this makes sense.

By $i_M(x)$ we will denote the injectivity radius of M at x, that is the supremum of the radius r of all balls $B(0_x, r)$ in TM_x for which $\exp_x$ is a diffeomorphism from $B(0_x, r)$ onto $B(x, r)$. Similarly, $i(M)$ will denote the global injectivity radius of M, that is $i(M) = \inf\{i_M(x) : x \in M\}$. Recall that the function $x \mapsto i_M(x)$ is continuous. In particular, if M is compact, we always have $i(M) > 0$.

2 General curvature evolution equations on Riemannian manifolds

Consider the following evolution equation on a Riemannian manifold M, given by

$$\begin{aligned} u_t - F(Du, D^2u) &= 0 \quad \text{on } (0, T) \times M, \\ u(0, x) = g(x), &\quad \text{on } x \in M, \end{aligned} \tag{CEE}$$

where u is a function of $(t, x) \in [0, T) \times M$.

In what follows, u_t, Du and D^2u will stand for $D_t u$, $D_x u(t, x) \in TM_x^*$ and $D_x^2 u(t, x) \in \mathcal{L}_s^2(TM_x)$, respectively. The function F is assumed to be continuous on the normal vector to the level set $\Gamma_t = \{x \in M : u(t, x) = 0\}$ and on the curvature tensor, and having the form

$$F(\zeta, A) = |\zeta|\, G\left(\frac{\zeta}{|\zeta|}, \frac{1}{|\zeta|}\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right) A\right), \tag{2.1}$$

for all $\zeta \in TM_x^* \setminus \{0_x\}$ and $A \in \mathcal{L}_s^2(TM_x)$, where G is any (nonlinear) function such that:

(A) $F : J_0^2(M) \to \mathbb{R}$ is continuous;

(B) F is (*degenerate*) *elliptic*, that is,

$$A \le B \implies F(\zeta, B) \le F(\zeta, A),$$

for all $x \in M,\ \ \zeta \in TM_x^* \setminus \{0\},\ A, B \in \mathcal{L}_s^2(TM_x)$;

(C) F is *translation invariant*, meaning that there exists $\tau > 0$ such that

$$F(L_{xy}\zeta, A) = F(\zeta, L_{yx}(A)),$$

for every $x, y \in M,\ \ d(x, y) < \tau,\ \ \zeta \in TM_x^* \setminus \{0\},\ \ A \in \mathcal{L}_s^2(TM_y)$.

Notice that, because

$$\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right)(\zeta \otimes \zeta) = 0,$$

any function F of the form (2.1) also satisfies

(D) F is *geometric*, that is,

$$F(\lambda\zeta, \lambda A + \mu\zeta \otimes \zeta) = \lambda F(\zeta, A)$$

for every $\lambda > 0$, $\mu \in \mathbb{R}$.

Two very important problems where such functions F arise are the evolutions of level sets by mean curvature and by Gaussian curvature.

Example 2.1 **Motion of level sets by their mean curvature.**

If u is a function on $[0, T] \times M$ such that $Du(t, x) \neq 0$ for all t, x with $u(t, x) = c$, then each level set $\Gamma_t = \{u(t, \cdot) = c\}$ evolves according to its mean curvature if and only if u satisfies

$$\frac{u_t}{|Du|} = \text{div}\left(\frac{Du}{|Du|}\right)$$

(that is, the normal velocity of Γ_t at a point x equals $(n-1)$ times the mean curvature of Γ_t at x), which in turn is equivalent to

$$u_t - \text{trace}\left(\left(I - \frac{Du \otimes Du}{|Du|^2}\right) D^2u\right) = 0 \quad \text{on } (0, T) \times M. \tag{MCE}$$

That is, $u_t + F(Du, D^2u) = 0$, where

$$F(\zeta, A) = -\text{trace}\left(\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right) A\right). \tag{2.2}$$

It is not difficult to see that the function $F : J^2_0(M) \longrightarrow \mathbb{R}$ is continuous (though the function F remains undefined at $\zeta = 0$ and, in fact, there is no continuous extension of F to $J^2(M)$. Nevertheless, $F(\zeta, A)$ remains bounded as $\zeta \to 0$).

Let us now check that the function F is degenerate elliptic. If $P \leq Q$, since $R := I - \frac{\zeta \otimes \zeta}{|\zeta|^2} \geq 0$ and $S := Q - P \geq 0$, we obtain from the properties of the trace that $\text{trace}(RS) \geq 0$ and therefore

$$F(\zeta, P) - F(\zeta, Q) = \text{trace}\left(\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right)(Q - P)\right) \geq 0. \tag{2.3}$$

Finally, let us see that the function F in (2.2) is translation invariant. Notice that $\text{trace}(A) = \text{trace}(L_{xy}^{-1} \circ A \circ L_{xy}) = \text{trace}(L_{yx}(A))$, and

$$\begin{aligned}\text{trace}\left(\frac{\zeta \otimes \zeta}{|\zeta|^2} \circ L_{yx}(A)\right) &= \text{trace}\left(L_{xy} \circ \frac{\zeta \otimes \zeta}{|\zeta|^2} \circ L_{yx}(A) \circ L_{xy}^{-1}\right) \\ &= \text{trace}\left(L_{xy} \circ \frac{\zeta \otimes \zeta}{|\zeta|^2} \circ L_{xy}^{-1} \circ A\right).\end{aligned}$$

On the other hand,

$$L_{xy} \circ \frac{\zeta \otimes \zeta}{|\zeta|^2} \circ L_{xy}^{-1} = \frac{L_{xy}\zeta \otimes L_{xy}\zeta}{|L_{xy}\zeta|^2}, \tag{2.4}$$

hence we immediately deduce that $F(L_{xy}\zeta, A) = F(\zeta, L_{yx}(A))$ whenever $d(x, y) < i(M)$, $\zeta \in TM_x$, $A \in \mathcal{L}_s^2(TM_y)$.

Example 2.2 **Motion of level sets by their Gaussian curvature.**

Now, if u is a function on $[0, T] \times M$ such that $Du(t, x) \neq 0$ for all t, x with $u(t, x) = c$, then all level sets $\Gamma_t = \{u(t, \cdot) = c\}$ evolve according to their Gaussian curvature if and only if u satisfies

$$\frac{u_t}{|Du|} = \det\left(\nabla^T\left(\frac{\nabla u}{|\nabla u|}\right)\right),$$

where ∇^T stands for the orthogonal projection onto $T\Gamma_t$ of the covariant derivative in M. This equation is equivalent to

$$u_t - |Du|\det\left(\frac{1}{|Du|}\left(I - \frac{Du \otimes Du}{|Du|^2}\right)D^2u + \frac{Du \otimes Du}{|Du|^2}\right) = 0. \tag{GCE}$$

That is, $u_t + H(Du, D^2u) = 0$, where

$$H(\zeta, A) = -|\zeta|\det\left(\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right)A + \frac{\zeta \otimes \zeta}{|\zeta|^2}\right).$$

However, the function H is not elliptic, so this problem cannot be treated, in its most general form, with the theory of viscosity solutions. Nevertheless, if our initial data $u(0, x) = g(x)$ satisfies that $D^2g(x) \geq 0$ (that is, if the initial hypersurface $\Gamma_0 = \{x \in M : g(x) = c\}$ has nonnegative Gaussian curvature) then it is reasonable, and consistent with the classical motion of convex surfaces by their Gaussian curvature, to assume that $D^2u(t, x) \geq 0$ for all (t, x) with $u(t, x) = c$ (that is, Γ_t will have nonnegative Gaussian curvature as long as it exists). In this case our equation becomes

$$u_t - |Du|\det_+\left(\frac{1}{|Du|}\left(I - \frac{Du \otimes Du}{|Du|^2}\right)D^2u + \frac{Du \otimes Du}{|Du|^2}\right) = 0, \tag{+GCE}$$

where $\det_+$ is defined by

$$\det_+(A) = \prod_{j=1}^{n} \max\{\lambda_j, 0\}$$

if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. That is,

$$u_t + F(Du, D^2u) = 0,$$

where

$$F(\zeta, A) = -|\zeta|\det_+\left(\frac{1}{|\zeta|}\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right)A + \frac{\zeta \otimes \zeta}{|\zeta|^2}\right). \tag{2.5}$$

As in the case of the mean curvature, it is not difficult to see that F is elliptic and translation invariant, and that F is continuous off $\{\zeta = 0\}$ (this time the singularities at $\zeta = 0$ are of higher order, as $F(\zeta, A)$ generally tends to $\pm\infty$ as ζ goes to 0).

Example 2.3 **Motion by mean curvature in arbitrary codimension.**

If Γ_0 is a k-codimensional surface of an n-dimensional Riemannian manifold M, we choose a continuous function v_0 with $\Gamma_0 = v_0^{-1}(0)$, and consider

$$u_t + F(Du, D^2u) = 0, \quad u(0, x) = v_0(x),$$

where

$$F(\zeta, A) = \sum_{i=1}^{d-k} \lambda_i(Q)$$

and

$$\lambda_1(Q) \leq \lambda_2(Q) \leq \cdots \leq \lambda_{d-1}(Q)$$

are the eigenvalues of $Q := S_\zeta A S_\zeta$, with

$$S_\zeta := \left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right),$$

corresponding to eigenvectors orthogonal to ζ (note that ζ is an eigenvector corresponding to the eigenvalue 0 of Q).

The same proof as in [2] shows that F is elliptic, the key observation is that

$$\lambda_i(Q) = \max\left\{\min_{\eta \in E} \frac{\langle Q\eta, \eta\rangle}{|\eta|^2} \;:\; E \subset TM_x,\ \operatorname{codim}(E) \leq i - 1\right\}.$$

On the other hand, it is easy to see, as in Example 2.1, that F is translation invariant.

Our aim is to establish comparison, existence and uniqueness of viscosity solutions to the general curvature evolution equation CEE, and then to prove that the resulting generalized motion is consistent with the corresponding classical motion (whenever the latter exists). However, because this equation is, in general, highly singular, one has to define very carefully what a viscosity solution to CEE is at points where $Du = 0$. Here we will adapt Ishii-Souganidis' definition [28] (see also [16]) from the Euclidean to the Riemannian setting. This requires a slight change in the definition of the set of test functions φ.

Definition 2.4 Let $F : J_0^2(M) \to \mathbb{R}$ be continuous, (degenerate) elliptic, translation invariant and geometric. Denote by $\mathcal{F} = \mathcal{F}(F)$ the set of functions $f \in C^2([0, \infty))$ such that $f(0) = f'(0) = f''(0) = 0$ and $f''(s) > 0$ for $s > 0$ which satisfy

$$\lim_{|\zeta| \to 0} \frac{f'(|\zeta|)}{|\zeta|} F(\zeta, 2I) = \lim_{|\zeta| \to 0} \frac{f'(|\zeta|)}{|\zeta|} F(\zeta, -2I) = 0. \tag{2.6}$$

It is clear that $\mathcal{F}$ is a cone (that is, $f + g \in \mathcal{F}$ and $\lambda f \in \mathcal{F}$ whenever $f, g \in \mathcal{F}, \lambda \in [0, \infty)$).

Proposition 2.5 *If M is compact and $F : J_0^2(M) \to \mathbb{R}$ is continuous, elliptic, translation invariant, and geometric, then $\mathcal{F}(F) \neq \emptyset$.*

Proof One can adapt the proof given in [28, p. 229] for the case $M = \mathbb{R}^n$. The only difference (apart from the replacement of I with $2I$) is that $|\zeta| = |\zeta|_x$ depends on the point x such that $\zeta \in TM_x$, and one has to be cautious about this dependence (as a matter of fact, that is why we require compactness of M). Let us give the essential details for the reader's convenience.

Since F is continuous on $J_0^2(M)$ and the sets $\{(\zeta_x, \pm 2I) : |\zeta_x|_x = 1, x \in M\}$ are compact in $J_0^2(M)$, there exists a continuous function $c : (0, \infty) \to (0, \infty)$ such that

$$-c(|\zeta|) \leq F(\zeta, 2I) \leq F(\zeta, -2I) \leq c(|\zeta|)$$

for all $\zeta \in TM^* \setminus \{0_x : x \in M\}$. Without loss of generality one can then assume that c is C^1 on $(0, \infty)$ and satisfies $(1/c)' > 0$ in $(0, 1]$, $\lim_{r \to 0^+} c(r) = \infty$, and $\lim_{r \to 0^+} (1/c)'(r) = 0$. Then it is not difficult to show that an appropriate extension to $[0, \infty)$ of the function f defined on $[0, 1]$ by

$$f(r) = \begin{cases} \int_0^r \frac{s^2}{c(s)} ds, & \text{if } 0 < r \leq 1; \\ 0, & \text{if } r = 0, \end{cases}$$

belongs to $\mathcal{F}(F)$. □

For many interesting choices of the function F it is easy to show that $\mathcal{F}(F) \neq \emptyset$ without requiring M to be compact:

Example 2.6 If F is given by (2.2) (corresponding to the mean curvature evolution equation), then we may take $f \in \mathcal{F}(F)$ of the form

$$f(t) = t^4.$$

On the other hand, when F is associated to the Gaussian curvature evolution equation [that is, F is given by (2.5)] then

$$f(t) = t^{2n}$$

belongs to $\mathcal{F}(F)$ (here n is the dimension of M).

Definition 2.7 We define the set $\mathcal{A}(F)$ of *admissible test functions* for the equation (CEE) as the set of all functions $\varphi \in C^2((0, T) \times M)$ such that, for every $z_0 = (t_0, x_0) \in (0, T) \times M$ with $D\varphi(z_0) = 0$ there exist some $\delta > 0$, $f \in \mathcal{F}$, $w \in C([0, \infty))$ satisfying $\lim_{r \to 0^+} w(r)/r = 0$ and

$$|\varphi(z) - \varphi(z_0) - \varphi_t(z_0)(t - t_0)| \leq f(d(x, x_0)) + w(|t - t_0|)$$

for all $z = (t, x) \in B(z_0, \delta)$.

Notice that in particular, for all $\varphi \in \mathcal{A}(F)$ we have that

$$D\varphi(z) = 0 \implies D^2\varphi(z) = 0.$$

Proposition 2.8 *If M is a compact Riemannian manifold then the class $\mathcal{A}(F)$ of admissible test functions is dense in the space $\mathcal{C}(M)$ of continuous functions on M.*

Proof It is not difficult to check that the class $\mathcal{A}(F)$ satisfies the hypotheses of the Stone–Weierstrass theorem. □

Definition 2.9 We will say that an upper semicontinuous function $u : [0, T) \times M \to \mathbb{R}$ is a viscosity subsolution of (CEE) provided that, for every $\varphi \in \mathcal{A}(F)$ and every maximum point $z = (t, x)$ of $u - \varphi$, we have

$$\begin{cases} \varphi_t + F(D\varphi(z), D^2\varphi(z)) \leq 0 & \text{if } D\varphi(z) \neq 0, \\ \varphi_t(z) \leq 0 & \text{otherwise.} \end{cases}$$

Similarly, we will say that a lower semicontinuous function $u : [0, T) \times M \to \mathbb{R}$ is a viscosity supersolution of (CEE) if, for every $\varphi \in \mathcal{A}(F)$ and every minimum point $z = (t, x)$ of $u - \varphi$, we have

$$\begin{cases} \varphi_t + F(D\varphi(z), D^2\varphi(z)) \geq 0 & \text{if } D\varphi(z) \neq 0, \\ \varphi_t(z) \geq 0 & \text{otherwise.} \end{cases}$$

A viscosity solution of (CEE) is a continuous function $u : [0, T) \times M \to \mathbb{R}$ which is both a viscosity subsolution and a viscosity supersolution of (CEE).

In [16] this kind of solution is called an $\mathcal{F}$-solution, but here we will simply call it a solution. It is clear that one can always assume that the minimum or maximum in these definitions are strict.

Notice that the set of test functions φ we are using is smaller than the standard one in the general theory of viscosity solutions, and that we here require that φ is C^2 with respect to the variables t and x (while in the usual definition of the parabolic semijets one demands C^1 differentiability with respect to t and C^2 differentiability with respect to x).

It is easy to check that this definition is consistent with u being a classical solution. Indeed, if u is a classical solution then we have $Du(z) \neq 0$ and $u_t(z) + F(Du(z), D^2u(z)) = 0$ for all z. Then, if $\varphi \in \mathcal{A}(F)$ is such that $u - \varphi$ attains a minimum at z, we have $\varphi_t(z) = u_t(z)$, $D\varphi(z) = Du(z) \neq 0$, and $D^2u(z) \geq D^2\varphi(z)$. Since F is elliptic we get

$$\varphi_t(z) + F(D\varphi(z), D^2\varphi(z)) \geq u_t(z) + F(Du(z), D^2u(z)) = 0,$$

that is, u is a supersolution at z. A similar argument shows that u is a subsolution.

It can be proved, as in the Euclidean case [16], that if the lower and upper semicontinuous envelopes of F (denoted by $\underline{F}$ and $\overline{F}$ respectively) are finite and $\underline{F}(0, 0) = \overline{F}(0, 0) = 0$, then every standard viscosity solution is an $\mathcal{F}$-solution, and conversely. This is the case of the F associated to the mean curvature evolution.

3 Some technical tools

In this section we collect some rather technical results that will be needed in the proof of the main comparison theorem.

First, we will need to use the following variant of the maximum principle for semicontinuous functions already used in [5], which we restate here for the reader's convenience.

Theorem 3.1 *Let $M_1, \ldots, M_k$ be Riemannian manifolds, and $\Omega_i \subset M_i$ open subsets. Define $\Omega = \Omega_1 \times \cdots \times \Omega_n \subset M_1 \times \cdots \times M_k = M$. Let u_i be upper semicontinuous functions on Ω_i, $i = 1, \ldots, k$; let φ be a C^2 smooth function on Ω and set*

$$\omega(x) = u_1(x_1) + \cdots + u_n(x_k)$$

for $x = (x_1, \ldots, x_k) \in \Omega$. Assume that $(\hat{x}_1, \ldots, \hat{x}_k)$ is a local maximum of $\omega - \varphi$. Then, for each $\varepsilon > 0$ there exist bilinear forms $B_i \in \mathcal{L}^2_s((TM_i)_{\hat{x}_i})$, $i = 1, \ldots, k$, such that

$$\left(D_{x_i}\varphi(\hat{x}), B_i\right) \in \overline{J}^{2,+}u_i(\hat{x}_i)$$

for $i = 1, \dots, k$, *and the block diagonal matrix with entries* B_i *satisfies*

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) I \leq \begin{pmatrix} B_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_k \end{pmatrix} \leq A + \varepsilon A^2,$$

where $A = D^2\varphi(\hat{x}) \in \mathcal{L}^2_s(TM_{\hat{x}})$.

We recall that

$$J^{2,+}f(x) = \{(D\varphi(x), D^2\varphi(x)) \,:\, \varphi \in C^2(M, \mathbb{R}),\; f - \varphi \text{ attains a local maximum at } x\},$$

and

$$\overline{J}^{2,+}f(x) = \{(\zeta, A) \in TM^*_x \times \mathcal{L}_s(TM_x) \,:\, \exists (x_k, \zeta_k, A_k) \in M \times TM^*_{x_k} \times \mathcal{L}_s(TM_{x_k})$$

$$\text{s.t. } (\zeta_k, A_k) \in J^{2,+}f(x_k),\; (x_k, f(x_k), \zeta_k, A_k) \to (x, f(x), \zeta, A)\},$$

see [5].

Another important ingredient of the proof of our main comparison result is the following Proposition, established in [5, Proposition 3.3].

Proposition 3.2 *Consider the function* $\Psi(x, y) = d(x, y)^2$*, defined on* $M \times M$*. Assume that the sectional curvature* K *of* M *is bounded below, say* $K \geq -K_0$*. Then*

$$D^2_{x,y}\Psi(x, y)(v, L_{xy}v)^2 \leq 2K_0 d(x, y)^2\|v\|^2$$

for all $v \in TM_x$ *and* $x, y \in M$ *with* $d(x, y) < \min\{i_M(x), i_M(y)\}$.

In particular, if $-K_0 \geq 0$ (*that is* M *has nonnegative sectional curvature*) *one has that the restriction of* $D^2_{x,y}\Psi(x, y)$ *to the subspace* $\mathcal{D} = \{(v, L_{xy}v) : v \in TM_x\}$ *of* $TM_x \times TM_y$ *is negative semidefinite.*

We will also need the following auxiliary result.

Lemma 3.3 *Let* $\phi \in USC(M)$*,* $\psi \in LSC(M)$*,* $f \in \mathcal{F}(F)$*, and*

$$m_\alpha := \sup_{M \times M} \left(\phi(x) - \psi(y) - \alpha\, f\left(d(x, y)^2\right)\right)$$

for $\alpha > 0$. *Suppose* $m_\alpha < \infty$ *for large* α *and let* (x_α, y_α) *be such that*

$$\lim_{\alpha\to\infty} \left(m_\alpha - (\phi(x_\alpha) - \psi(y_\alpha) - \alpha f(d(x_\alpha, y_\alpha)^2))\right) = 0.$$

Then we have:

(1) $\lim_{\alpha\to\infty} \alpha f(d(x_\alpha, y_\alpha)^2) = 0$, *and*
(2) *if* $\widehat{x} \in M$ *is a limit point of* x_α *as* $\alpha \to \infty$ *then*

$$\lim_{\alpha\to\infty} m_\alpha = \phi(\widehat{x}) - \psi(\widehat{x}) = \sup_{x\in M} (\phi(x) - \psi(x)).$$

Proof A more general form of this result is proved in [8, Theorem 3.7] in the case when M is an Euclidean space, and the same proof clearly works in a general metric space. □

Let us now define $\mathcal{P}^{2,+}$, $\mathcal{P}^{2,-}$, $\overline{\mathcal{P}}^{2,+}$, and $\overline{\mathcal{P}}^{2,-}$, the "parabolic" variants of the semijets $J^{2,+}$, $J^{2,-}$, $\overline{J}^{2,+}$, $\overline{J}^{2,-}$ introduced in [5] for functions defined on a Riemannian manifold.

Definition 3.4 Let $f : (0, T) \times M \to (-\infty, +\infty]$ be a lower semicontinuous (LSC) function. We define the parabolic second order subjet of f at a point $(t_0, x_0) \in (0, T) \times M$ by

$$\begin{aligned}\mathcal{P}^{2,-} f(t_0, x_0) :=& \{(D_t\varphi(t_0, x_0), D_x\varphi(t_0, x_0)), D_x^2\varphi(t_0, x_0)) : \varphi \text{ is once continuously}\\ & \text{differentiable in } t \in (0, T),\ \text{twice continuously differentiable in } x \in M\\ & \text{and } f - \varphi \text{ attains a local minimum at } (t_0, x_0)\}.\end{aligned}$$

Similarly, for an upper semicontinuous (USC) function $f : (0, T) \times M \to [-\infty, +\infty)$, we define the parabolic second order superjet of f at (t_0, x_0) by

$$\begin{aligned}\mathcal{P}^{2,+} f(t_0, x_0) :=& \{(D_t\varphi(t_0, x_0), D_x\varphi(t_0, x_0)), D_x^2\varphi(t_0, x_0)) : \varphi \text{ is once continuously}\\ & \text{differentiable in } t \in (0, T),\ \text{twice continuously differentiable in } x \in M\\ & \text{and } f - \varphi \text{ attains a local maximum at } (t_0, x_0)\}.\end{aligned}$$

Observe that $\mathcal{P}^{2,-} f(t, x)$ and $\mathcal{P}^{2,+} f(t, x)$ are subsets of $\mathbb{R} \times TM_x^* \times \mathcal{L}_s^2(TM_x)$. Notice that we can assume that the auxiliary functions φ are defined on a neighborhood of (t_0, x_0). We may as well assume (just by adding a function of the form $\pm\varepsilon d(x, x_0)^4$) that the minima or maxima in these definitions are strict. It is also easily seeing that the min or max can always be supposed to be global.

Definition 3.5 Let $f : (0, T) \times M \longrightarrow (-\infty, +\infty]$ be a LSC function and $(t, x) \in (0, T) \times M$. We define $\overline{\mathcal{P}}^{2,-} f(t, x)$ as the set of the $(a, \zeta, A) \in \mathbb{R} \times TM_x^* \times \mathcal{L}_s^2(TM_x)$ such that there exist a sequence (x_k, a_k, ζ_k, A_k) in $M \times \mathbb{R} \times TM_{x_k}^* \times \mathcal{L}_s^2(TM_{x_k})$ satisfying:

(i) $(a_k, \zeta_k, A_k) \in \mathcal{P}^{2,-} f(t_k, x_k)$,

(ii) $\lim_k (t_k, x_k, f(t_k, x_k), a_k, \zeta_k, A_k) = (t, x, f(t, x), a, \zeta, A)$.

The corresponding definition of $\overline{\mathcal{P}}^{2,+} f(t, x)$ when f is an upper semicontinuous function is then clear.

The next two lemmas are needed to establish the parabolic version of the maximum principle we state as follows.

Lemma 3.6 ([5]) *Let $U \subset M$ be an open subset, $(t, z) \in (0, T) \times U$ and a function $\varphi : (0, T) \times U \to \mathbb{R}$. Assume that φ is once continuously differentiable in $(0, T)$ and twice continuously differentiable in U. Define $\psi(s, v) = \varphi(s, \exp_z v)$ on a neighborhood of $0 \in TM_z$. Let $\widetilde{V}$ be a vector field defined on a neighbourhood of 0 in TM_z, and consider the vector field defined by $V(y) = D\exp_z(w_y)(\widetilde{V}(w_y))$ on a neighbourhood of z, where $w_y := \exp_z^{-1}(y)$, and let*

$$\sigma_y(r) = \exp_z(w_y + r\widetilde{V}(w_y)).$$

Then we have that

$$D_v^2\psi(\widetilde{V}, \widetilde{V})(t, w_y) = D_x^2\varphi(V, V)(t, y) + \langle \nabla_x\varphi(t, y), \sigma_y''(0)\rangle.$$

Observe that $\sigma_z''(0) = 0$ so, when $y = z$, we obtain

$$D_v^2\psi(t, 0) = D_x^2\varphi(t, z).$$

Proof Analogous to [5, Lemma 2.7]. □

Lemma 3.7 *Let $U \subset M$ be an open subset, $(t, z) \in (0, T) \times U$ and $u : (0, T) \times M \to [-\infty, \infty)$ be an upper semicontinuous function and consider a neighbourhood V of $0 \in TM_z$ and $\widetilde{u} : (0, T) \times V \to [-\infty, \infty)$ defined as $\widetilde{u}(s, v) = u(s, \exp_z v)$. Then, if $(b, \zeta, A) \in \mathbb{R} \times TM_z^* \times \mathcal{L}_s^2(TM_z)$,*

$$(b, \zeta, A) \in \overline{\mathcal{P}}^{2,+} u(t, z) \iff (b, \zeta, A) \in \overline{\mathcal{P}}^{2,+} \widetilde{u}(t, 0).$$

Proof Use the above Lemma as in the proof of [5, Proposition 2.8]. □

As in [25] in the case of the mean curvature evolution equation and [5] in the case of general (nonsingular) stationary equations, the following result is one of the keys to the proof of the comparison result for general (nonsingular) evolution equations which we give in the Appendix.

Theorem 3.8 *Let $M_1, \ldots, M_k$ be Riemannian manifolds, and $\Omega_i \subset M_i$ open subsets. Define $\Omega = (0, T) \times \Omega_1 \times \cdots \times \Omega_k$. Let u_i be upper semicontinuous functions on $(0, T) \times \Omega_i$, $i = 1, \ldots, k$; let φ be a function defined on Ω such that it is once continuously differentiable in $t \in (0, T)$ and twice continuously differentiable in $x := (x_1, \ldots, x_k) \in \Omega_1 \times \cdots \times \Omega_k$ and set*

$$\omega(t, x_1, \ldots, x_k) = u_1(t, x_1) + \cdots + u_k(t, x_k)$$

for $(t, x_1, \ldots, x_k) \in \Omega$. Assume that $(\widehat{t}, \widehat{x}_1, \ldots, \widehat{x}_k)$ is a maximum of $\omega - \varphi$ in Ω. Assume, moreover, that there is an $\tau > 0$ such that for every $M > 0$ there is $C > 0$ such that for $i = 1, \ldots, k$,

$$\begin{cases} a_i \le C \;\text{ whenever }\; (a_i, \zeta_i, A_i) \in \overline{\mathcal{P}}_{M_i}^{2,+} u_i(t, x_i) \\ d(x_i, \widehat{x}_i) + |t - \widehat{t}| \le \tau \;\text{ and }\; |u_i(t, x_i)| + |\zeta_i| + ||A_i|| \le M. \end{cases} \tag{3.1}$$

Then, for each $\varepsilon > 0$ there exist real numbers b_i and bilinear forms $B_i \in \mathcal{L}_s^2((TM_i)_{\widehat{x}_i})$, $i = 1, \ldots, k$, such that

$$\left(b_i, D_{x_i}\varphi(\widehat{t}, \widehat{x}_1, \ldots, \widehat{x}_k), B_i\right) \in \overline{\mathcal{P}}_{M_i}^{2,+} u_i(\widehat{t}, \widehat{x}_i)$$

for $i = 1, \ldots, k$, and the block diagonal matrix with entries B_i satisfies

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) I \le \begin{pmatrix} B_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & B_k \end{pmatrix} \le A + \varepsilon A^2,$$

where $A = D_x^2\varphi(\widehat{t}, \widehat{x}_1, \ldots, \widehat{x}_k)$ and $b_1 + \cdots + b_k = \frac{\partial \varphi}{\partial t}(\widehat{t}, \widehat{x}_1, \ldots, \widehat{x}_k)$.

Proof The result is proved in [8] for $M_i = \mathbb{R}^{n_i}$, $i = 1, \ldots, k$. As in the stationary case [5], we can reduce the problem to this situation by an adecuate composition with the exponential mappings. Let us give some details for completeness. We may assume (by taking smaller neighborhoods of x_i, if necessary), that the sets Ω_i are diffeomorphic images of balls by the exponential mappings $\exp_{\widehat{x}_i} : B(0_{\widehat{x}_i}, r_i) \to \Omega_i := B(\widehat{x}_i, r_i)$. Consider the Riemannian manifold $M := M_1 \times \cdots \times M_k$ and $\widehat{x} := (\widehat{x}_1, \ldots, \widehat{x}_k) \in \Omega_1 \times \cdots \times \Omega_k$. Recall that if $v := (v_1, \ldots, v_k) \in B(0_{\widehat{x}_1}, r_1) \times \cdots \times B(0_{\widehat{x}_k}, r_k)$ the exponential map $\exp_{\widehat{x}}$ is defined as $\exp_{\widehat{x}}(v) = \left(\exp_{\hat{x}_1}(v_1), \ldots, \exp_{\hat{x}_k}(v_k)\right)$. Then $\exp_{\widehat{x}}$ maps diffeomorphically the open set $B(0_{\widehat{x}_1}, r_1) \times \cdots \times B(0_{\widehat{x}_k}, r_k) \subset TM_{\widehat{x}} = (TM_1)_{\widehat{x}_1} \times \cdots \times (TM_k)_{\widehat{x}_k}$ onto $\Omega_1 \times \cdots \times \Omega_k$.

We consider the functions, defined on suitable open subsets of euclidean spaces,

$$\widetilde{\omega}(t,v) := \omega(t, \exp_{\widehat{x}}(v)), \quad \psi(t,v) := \varphi(t, \exp_{\widehat{x}}(v)), \quad \widetilde{u}_i(t,v_i) := u_i(t, \exp_{\widehat{x}_i}(v_i)).$$

We have that

$$\widetilde{\omega}(t, v_1, \dots, v_k) = \widetilde{u}_1(t, v_1) + \cdots + \widetilde{u}_k(t, v_k),$$

and $(\widehat{t}, 0_{\widehat{x}}) = (\widehat{t}, 0_{\widehat{x}_1}, \dots, 0_{\widehat{x}_k})$ is the maximum of $\widetilde{\omega} - \psi$. Therefore, we apply [8, Theorem 8.3] to ensure, for every $\varepsilon > 0$, the existence of real numbers b_i and bilinear forms $B_i \in \mathcal{L}^2_s(\mathbb{R}^{n_i})$, $i = 1, \dots, k$, such that

$$\big(b_i, D_{v_i}\psi(\widehat{t}, 0_{\widehat{x}}), B_i\big) \in \overline{\mathcal{P}}^{2,+}\widetilde{u}_i(\widehat{t}, 0_{\widehat{x}_i})$$

for $i = 1, \dots, k$, and the block diagonal matrix with entries B_i satisfies

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) I \le \begin{pmatrix} B_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_k \end{pmatrix} \le A + \varepsilon A^2,$$

where $A = D_v^2\psi(\widehat{t}, 0_{\widehat{x}})$ and $b_1 + \cdots + b_k = \frac{\partial\psi}{\partial t}(\widehat{t}, 0_{\widehat{x}})$. Clearly

$$\frac{\partial\psi}{\partial t}(\widehat{t}, 0_{\widehat{x}}) = \frac{\partial\varphi}{\partial t}(\widehat{t}, \widehat{x}), \qquad D_{v_i}\psi(\widehat{t}, 0_{\widehat{x}}) = D_{x_i}\varphi(\widehat{t}, \widehat{x}),$$

and Lemma 3.6 provides the equality $D_v^2\psi(\widehat{t}, 0_{\widehat{x}}) = D_x^2\varphi(\widehat{t}, \widehat{x})$. To conclude this proof it remains to apply Lemma 3.7 to obtain the equivalence

$$\big(b_i, D_{v_i}\psi(\widehat{t}, 0_{\widehat{x}}), B_i\big) \in \overline{\mathcal{P}}^{2,+}\widetilde{u}_i(\widehat{t}, 0_{\widehat{x}_i}) \iff \big(b_i, D_{x_i}\varphi(\widehat{t}, \widehat{x}), B_i\big) \in \overline{\mathcal{P}}^{2,+}u_i(\widehat{t}, \widehat{x}_i).$$

□

4 Comparison

Let us state and prove our main comparison result for viscosity solutions of (CEE).

Theorem 4.1 *Let M be a compact Riemannian manifold of nonnegative sectional curvature, and let $F : J^2_0(M) \to \mathbb{R}$ be continuous, elliptic, translation invariant and geometric. Let $u \in USC([0,T) \times M)$ be a subsolution and $v \in LSC([0,T) \times M)$ be a supersolution of (CEE) on M. Then $u \le v$ on $[0,T) \times M$ whenever $u \le v$ on $\{0\} \times M$.*

Proof Since M is compact we know that M has injectivity radius $i_M > 0$.

Let us start noting that we may assume u and $-v$ bounded above on $[0,T) \times M$. Otherwise, for every $0 < S < T$, consider u and $-v$ defined on the compact set $[0,S] \times M$, where they are also u.s.c. and thus bounded above. Then, we apply the arguments of the proof to u and $-v$ in $[0,S) \times M$.

Next, let us observe that for $\varepsilon > 0$, the function $\widetilde{u} = u - \frac{\varepsilon}{T-t}$ is also a subsolution of $u_t + F(Du, D^2u) = 0$ on $[0,T) \times M$. Moreover,

$$\widetilde{u}_t + F(D\widetilde{u}, D^2\widetilde{u}) \le -\frac{\varepsilon}{T^2} \quad \text{for} \quad D\widetilde{u} \ne 0, \tag{4.1}$$

$$\widetilde{u}_t \le -\frac{\varepsilon}{T^2} \quad \text{for} \quad D\widetilde{u} = 0, \text{ and} \tag{4.2}$$

$$\lim_{t \to T^-} \widetilde{u}(t,x) = -\infty \text{ uniformly on } M. \tag{4.3}$$

Since the assertion $\widetilde{u} \leq v$ for every $\varepsilon > 0$ implies $u \leq v$, it will suffice to prove the comparison result under the assumptions given in (4.1–4.3).

Assume to the contrary that $\sup_{[0,T)\times M}(u - v) > 0$. Take $f \in \mathcal{F}$. Since M is compact, u and $-v$ are u.s.c. and (4.3) holds, we can consider for every $\alpha \in \mathbb{N}$,

$$m_\alpha := \sup_{\substack{0\leq s,t<T \\ x,y\in M}} \{u(s,x) - v(t,y) - \alpha f\left(d(x,y)^2\right) - \alpha(t-s)^2\},$$

which is attained at some $(s_\alpha, t_\alpha, x_\alpha, y_\alpha) \in [0,T) \times [0,T) \times M \times M$. Clearly,

$$m_\alpha \geq \sup_{[0,T)\times M}(u - v) > 0.$$

If $t_\alpha = 0$ for infinitely many α's, which we may assume are all α, then we have

$$0 < \sup_{[0,T)\times M}(u - v) \leq m_\alpha = \sup_{s,x,y}\left(u(s,x) - v(0,y) - \alpha f\left(d(x,y)^2\right) - \alpha s^2\right)$$

We deduce from Lemma 3.3 that $\lim_{\alpha\to\infty} \alpha f\left(d(x_\alpha, y_\alpha)^2\right) = 0$ and $\lim_{\alpha\to\infty} \alpha(t_\alpha - s_\alpha)^2 = 0$. By compactness, we can assume that a subsequence of $(t_\alpha, s_\alpha, x_\alpha, y_\alpha)$, which we still denote $(t_\alpha, s_\alpha, x_\alpha, y_\alpha)$, converges to a point (s_0, t_0, x_0, y_0). By Lemma 3.3 we have that $x_0 = y_0$ and $s_0 = t_0 = 0$, and $\lim_{\alpha\to\infty} m_\alpha = u(0,x_0) - v(0,x_0) = \sup_{x\in M}(u(0,x) - v(0,x)) \leq 0$, which is a contradiction.

A completely analogous argument leads us to a contradiction if $s_\alpha = 0$ for infinitely many α's.

Thus we may assume that there exist $\alpha_0 > 0$ such that $s_\alpha > 0$ and $t_\alpha > 0$ for $\alpha > \alpha_0$. By compactness and Lemma 3.3 we may also assume that x_α and y_α converge to the same point $x_0 = y_0$, and in particular that $x_\alpha, y_\alpha \in B(x_0, r/2)$ for all $\alpha > \alpha_0$, where $r > 0$ is small enough such that $0 < r < i_M$ and conditions (**B**, **C**) of Sect. 2 hold whenever $d(x,y) < r$. Therefore the function $d(x,y)^2$ and hence the functions

$$\Phi_\alpha(x,y) := \alpha f(d(x,y)^2), \qquad \varphi_\alpha(s,t,x,y) := \Phi_\alpha(x,y) + \alpha(t-s)^2$$

are C^2 smooth on $(0,T) \times (0,T) \times B(x_0, r/2) \times B(x_0, r/2)$.

Recall that $\overline{\mathcal{P}}^{2,-}v(t_\alpha, y_\alpha) = -\overline{\mathcal{P}}^{2,+}(-v)(t_\alpha, y_\alpha)$, and if we consider the function

$$\Psi(x,y) := d(x,y)^2$$

we obtain from [5, Sect. 3] that

$$D_x\Psi(x_\alpha, y_\alpha) = -2\exp_{x_\alpha}^{-1}(y_\alpha), \text{ and } D_y\Psi(x_\alpha, y_\alpha) = -2\exp_{y_\alpha}^{-1}(x_\alpha).$$

Now we cannot directly apply Theorem 3.8, because condition (3.1) is not generally satisfied due to the singularity of F (one has a serious difficulty when $\overline{\mathcal{P}}^{2,+}u(s_\alpha, x_\alpha)$ contains triplets of the form $(a, 0, A)$: in this case one cannot use the fact that u is a subsolution to guarantee that $a \leq C$, since $F(\zeta, A) \to \infty$ as $\zeta \to 0$). Instead we will use Theorem 3.1, treating the variables s, t as if they were spatial variables in the stationary case, and then ignoring the information that this result gives about the second derivatives with respect to the variables t, s, which we do not need here. Bearing in mind that $(s_\alpha, t_\alpha, x_\alpha, y_\alpha)$ is the maximum of the function $(s,t,x,y) \to u(s,x) - v(t,y) - \varphi_\alpha(s,t,x,y)$, and setting

$$A_\alpha := D^2_{x,y}\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha), \qquad \varepsilon := \varepsilon_\alpha = \frac{1}{1 + ||A_\alpha||},$$

we obtain this way two bilinear forms $P_\alpha \in \mathcal{L}^2_s(TM_{x_\alpha})$, and $Q_\alpha \in \mathcal{L}^2_s(TM_{y_\alpha})$ such that

$$\left(\frac{\partial}{\partial s}\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha), D_x\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha), P_\alpha\right) \in \overline{\mathcal{P}}^{2,+}u(s_\alpha, x_\alpha), \tag{4.4}$$

$$\left(-\frac{\partial}{\partial t}\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha), -D_y\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha), Q_\alpha\right) \in \overline{\mathcal{P}}^{2,-}v(t_\alpha, y_\alpha), \tag{4.5}$$

and

$$-\left(\frac{1}{\varepsilon_\alpha} + \|A_\alpha\|\right) I \leq \begin{pmatrix} P_\alpha & 0 \\ 0 & -Q_\alpha \end{pmatrix} \leq A_\alpha + \varepsilon_\alpha A_\alpha^2. \tag{4.6}$$

These inequalities can be deduced from the corresponding ones in Theorem 3.1 (just bear in mind the special form of our function φ_α, and apply the inequalities given by Theorem 3.1 to vectors of the form $(0, 0, v, w)$, where the zeros correspond to the variables s and t).

In our case we have

$$a_\alpha := \frac{\partial}{\partial s}\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha) = -2\alpha(t_\alpha - s_\alpha), \tag{4.7}$$

$$-b_\alpha := -\frac{\partial}{\partial t}\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha) = -2\alpha(t_\alpha - s_\alpha),$$

$$D_x\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha) = -2\alpha f'(d(x_\alpha, y_\alpha)^2)\exp^{-1}_{x_\alpha}(y_\alpha), \tag{4.8}$$

$$-D_y\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha) = 2\alpha f'(d(x_\alpha, y_\alpha)^2)\exp^{-1}_{y_\alpha}(x_\alpha),$$

and in particular we see that

$$a_\alpha + b_\alpha = 0. \tag{4.9}$$

Let us now distinguish two cases.

Case 1. Assume that $x_\alpha \neq y_\alpha$. Let us consider the non-zero vectors

$$\zeta_\alpha := -2\alpha f'(d(x_\alpha, y_\alpha)^2)\exp^{-1}_{x_\alpha}(y_\alpha),$$

and notice that

$$L_{x_\alpha y_\alpha}\zeta_\alpha = 2\alpha f'(d(x_\alpha, y_\alpha)^2)\exp^{-1}_{y_\alpha}(x_\alpha).$$

Since u is a strict subsolution and v is a supersolution of $u_t + F(Du, D^2u) = 0$, we have that

$$a_\alpha + F(\zeta_\alpha, P_\alpha) \leq \frac{-\varepsilon}{T^2} < 0 \leq -b_\alpha + F(L_{x_\alpha y_\alpha}\zeta_\alpha, Q_\alpha)$$

(notice that here we used continuity of F off $\{\zeta = 0\}$, and the important observation that if $(\zeta, A) \in \overline{\mathcal{P}}^{2,+}u(z)$ and $\zeta \neq 0$ then (ζ, A) is a limit of a sequence (ζ_k, A_k) with $(\zeta_k, A_k) = (D\varphi_k(z_k), D^2\varphi_k(z_k))$ for some $\varphi_k \in \mathcal{A}(F)$ and $z_k \to z$).

Thus, there is $c := \frac{\varepsilon}{T^2}$ such that

$$0 < c \leq F(L_{x_\alpha y_\alpha}\zeta_\alpha, Q_\alpha) - F(\zeta_\alpha, P_\alpha). \tag{4.10}$$

On the other hand, since F is translation invariant, we deduce

$$0 < c \leq F(L_{x_\alpha y_\alpha}\zeta_\alpha, Q_\alpha) - F(\zeta_\alpha, P_\alpha) = F(\zeta_\alpha, L_{y_\alpha x_\alpha}(Q_\alpha)) - F(\zeta_\alpha, P_\alpha) \tag{4.11}$$

Recall that $A_\alpha = D^2\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha)$ and $\Phi_\alpha = \alpha f \circ \Psi$. Then

$$D_{x,y}\Phi_\alpha(x, y) = \alpha f'(\Psi(x, y)) \left(D_x\Psi(x, y), D_y\Psi(x, y)\right)$$
$$= -2\alpha f'(\Psi(x, y)) \left(\exp_x^{-1} y, \exp_y^{-1} x\right). \tag{4.12}$$

Now $D^2_{x,y}\varphi_\alpha$, the Hessian of φ_α, satisfies for every vector fields X, Y on $M \times M$

$$\begin{aligned} D^2_{x,y}\varphi_\alpha(s, t, X, Y) &= \langle \nabla_X(D\varphi_\alpha), Y\rangle = \langle \nabla_X(\alpha f'(\Psi)D\Psi),\ Y\rangle \\ &= \alpha\langle f'(\Psi)\nabla_X(D\Psi) + X(f'(\Psi))D\Psi,\ Y\rangle \\ &= \alpha f'(\Psi)\langle\nabla_X(D\Psi), Y\rangle + \alpha X(f'(\Psi))\langle D\Psi, Y\rangle \\ &= \alpha f'(\Psi)D^2_{x,y}\Psi(X, Y) + \alpha f''(\Psi)X(\Psi)\langle D\Psi, Y\rangle \\ &= \alpha f'(\Psi)D^2_{x,y}\Psi(X, Y) + \alpha f''(\Psi)(D\Psi \otimes D\Psi)(X, Y). \end{aligned} \tag{4.13}$$

In particular for every two points $x, y \in M$ such that $d(x, y) < \min\{i_M(x), i_M(y)\}$ and every $v \in TM_x$, we consider $X = Y$ with $X(x, y) = (v, L_{xy}v) \in TM_x \times TM_y$ and we obtain

$$X(\Psi)(x, y) = D^2_{x,y}\Psi(x, y)(v, L_{xy}v) = D_x\Psi(x, y)(v) + D_y\Psi(x, y)(L_{xy}v) = 0.$$

The last equality in the above expression is proved in [5, Sect. 3]. Therefore, if M has sectional curvature bounded below by some constant $-K_0 \leq 0$, we obtain from equation (5.5) and Proposition 3.2 that

$$\begin{aligned} A_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha)(v, L_{x_\alpha y_\alpha}v)^2 &= D^2\varphi_\alpha(s_\alpha, t_\alpha, x_\alpha, y_\alpha)(v, L_{x_\alpha y_\alpha}v)^2 \\ &= \alpha f'(\Psi(x_\alpha, y_\alpha))D^2\Psi(x_\alpha, y_\alpha)(v, L_{x_\alpha y_\alpha}v)^2 \\ &\leq \alpha f'(\Psi(x_\alpha, y_\alpha))2K_0\Psi(x_\alpha, y_\alpha)||v||^2, \end{aligned} \tag{4.14}$$

for every $v \in TM_{x_\alpha}$. Let us denote by $\lambda_1 \leq \cdots \leq \lambda_n$ the eigenvalues of the restriction of A_α to the subspace $\mathcal{D} = \{(v, L_{x_\alpha y_\alpha}v) : v \in TM_{x_\alpha}\}$ of $TM_{x_\alpha} \times TM_{y_\alpha}$. The above inequality implies that $\lambda_1, \ldots, \lambda_n \leq 2\alpha K_0\Psi(x_\alpha, y_\alpha)f'(\Psi(x_\alpha, y_\alpha))$. With our choice of ε_α, we have that

$$\lambda_i + \varepsilon_\alpha\lambda_i^2 \leq \lambda_i + \frac{1}{1 + \sup_{1\leq j\leq n}|\lambda_j|}\lambda_i^2 \leq \lambda_i + |\lambda_i| \leq 2\max\{0, \lambda_n\}, \quad i = 1, \ldots, n.$$

Since $\lambda_i + \varepsilon_\alpha\,\lambda_i^2, i = 1, \ldots, n$, are the eigenvalues of $\left(A_\alpha + \varepsilon_\alpha A_\alpha^2\right)|_{\mathcal{D}}$, this means that when M has nonnegative sectional curvature, that is $K_0 = 0$, or equivalently $\lambda_n \leq 0$, we have

$$\left(A_\alpha + \varepsilon_\alpha A_\alpha^2\right)(v, L_{x_\alpha y_\alpha}v)^2 \leq 0.$$

Therefore, the second inequality in (4.6) implies $P_\alpha - L_{y_\alpha x_\alpha}(Q_\alpha) \leq (A_\alpha + \varepsilon_\alpha A_\alpha^2)|_{\mathcal{D}} \leq 0$. Thus Eq. (4.11), and the fact that F is elliptic imply that

$$0 < c \leq F(\zeta_\alpha, L_{y_\alpha x_\alpha}(Q_\alpha)) - F(\zeta_\alpha, P_\alpha) \leq 0,$$

a contradiction.

Case 2. If we are not in Case 1 then we may assume $x_\alpha = y_\alpha$ for every $\alpha > \alpha_0$. We know that

$$u(s, x) - v(t, y) - \alpha f(d(x, y)^2) - \alpha(t - s)^2 \leq u(s_\alpha, x_\alpha) - v(t_\alpha, y_\alpha) - \alpha(t_\alpha - s_\alpha)^2$$

for all (s, t, x, y). By taking $y = y_\alpha$, $t = t_\alpha$ we get that the function $(s, x) \mapsto u(s, x) - \alpha f(d(x, y_\alpha)^2) - \alpha(t_\alpha - s)^2$ has a maximum at (s_α, x_α), which (bearing in mind that $f'(0) = 0 = f''(0)$) yields

$$(-2\alpha(t_\alpha - s_\alpha),\ 0, 0) \in \mathcal{P}^{2,+}u(s_\alpha, x_\alpha).$$

Similarly, we also deduce that

$$(-2\alpha(t_\alpha - s_\alpha),\ 0,\ 0) \in \mathcal{P}^{2,-}v(t_\alpha, y_\alpha).$$

Since u is a strict subsolution and v is a supersolution, we get

$$-2\alpha(t_\alpha - s_\alpha) \leq \frac{-\varepsilon}{T^2} < 0 \leq -2\alpha(t_\alpha - s_\alpha),$$

a contradiction. □

The preceding proof can be easily modified to yield the following more general results.

Remark 4.2 One can replace the compactness of M in the statement of Theorem 4.1 by the following condition on the behavior of u and v at ∞:

$$\limsup_{(t,x)\to\infty} u(t, x) - v(t, x) \leq 0 \tag{4.15}$$

(this condition is meant to be empty when M is compact).

In the case when M does not have positive curvature, one can prove the following.

Theorem 4.3 *Let M be a complete Riemannian manifold with sectional curvature bounded below and positive injectivity radius. Let F satisfy conditions (**A - D**) of Sect. 2. Assume furthermore that there exist $f \in \mathcal{F}(F)$ and $C > 0$ such that*

$$tf'(t) \leq Cf(t) \quad \textit{for all} \quad t > 0, \tag{4.16}$$

and that F satisfies the following uniform continuity assumption with respect to the variable D^2u:

$$F(\zeta, P - \delta I) - F(\zeta, P) \xrightarrow{\delta\to 0} 0 \quad \textit{uniformly on} \quad \zeta, P. \tag{4.17}$$

Let $u \in USC([0, T) \times M)$ be a subsolution and $v \in LSC([0, T) \times M)$ be a supersolution of (CEE) *on M. Suppose that $u \leq v$ on $\{0\} \times M$ and*

$$\limsup_{(t,x)\to\infty} u(t, x) - v(t, x) \leq 0. \tag{4.18}$$

Then $u \leq v$ on $[0, T) \times M$.

Proof Assume that the sectional curvature of M is bounded below by $-K_0$, with $K_0 > 0$. We have, with the notation used in the proof of Theorem 4.1, case 1, following equation (4.14), that $\lambda_n > 0$, and

$$\lambda_i + \varepsilon_\alpha \lambda_i^2 \leq 2\lambda_n \leq 4\alpha K_0 \Psi(x_\alpha, y_\alpha) f'(\Psi(x_\alpha, y_\alpha)),$$

hence

$$\left(A_\alpha + \varepsilon_\alpha A_\alpha^2\right)(v, L_{x_\alpha y_\alpha} v)^2 \leq 4\alpha K_0 \Psi(x_\alpha, y_\alpha) f'(\Psi(x_\alpha, y_\alpha))||v||^2.$$

Thus, inequality (4.6) and condition (4.16) imply that

$$\begin{aligned} P_\alpha(v)^2 - L_{y_\alpha x_\alpha}(Q_\alpha)(v)^2 &= P_\alpha(v)^2 - Q_\alpha(L_{x_\alpha y_\alpha} v)^2 \\ &\leq \left(A_\alpha + \varepsilon_\alpha A_\alpha^2\right)(v, L_{x_\alpha y_\alpha} v)^2 \\ &\leq 4\alpha K_0 \Psi(x_\alpha, y_\alpha) f'(\Psi(x_\alpha, y_\alpha))||v||^2 \\ &\leq 4\alpha K_0 C f(\Psi(x_\alpha, y_\alpha))||v||^2. \end{aligned} \tag{4.19}$$

Let us denote

$$\delta_\alpha := 4\alpha K_0 C f(\Psi(x_\alpha, y_\alpha)).$$

We have that $\lim_{\alpha\to\infty} \delta_\alpha = 0$. From (4.19) we obtain $P_\alpha - \delta_\alpha I \leq L_{y_\alpha x_\alpha}(Q_\alpha)$. Then, Eq. (4.11), the fact that F is elliptic, and condition (4.17) imply that

$$\begin{aligned} 0 < c &\leq F(\zeta_\alpha, L_{y_\alpha x_\alpha}(Q_\alpha)) - F(\zeta_\alpha, P_\alpha) \\ &\leq F(\zeta_\alpha, P_\alpha - \delta_\alpha I) - F(\zeta_\alpha, P_\alpha) \xrightarrow{\alpha\to\infty} 0, \end{aligned}$$

which again leaves us with a contradiction. The proof of case 2 parallels that in Theorem 4.1. □

Remark 4.4 Condition (4.16) is always met when one is able to take an f of the form

$$f(t) = t^k,$$

with $k \geq 2$. Therefore, in the cases when F is given by the evolutions by mean curvature or by Gaussian curvature (4.16) is automatically satisfied.

On the other hand, condition (4.17) is also clearly met by the function F associated to the mean curvature evolution problem. Indeed, in this case the function $A \mapsto F(\zeta, A)$ is linear, so we have

$$F(\zeta, P - \delta I) - F(\zeta, P) = -\delta F(\zeta, I) = \delta \operatorname{trace}\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right) \leq \delta(n-1),$$

where n is the dimension of M. We thus recover Ilmanen's Theorem from [25]:

Corollary 4.5 (Ilmanen) *Let M be complete, with sectional curvature bounded below and positive injectivity radius. Let F be given by (2.2). Let $u \in USC([0, T) \times M)$ be a subsolution and $v \in LSC([0, T) \times M)$ be a supersolution of (MCE) on M. Suppose that $u \leq v$ on $\{0\} \times M$ and* $\limsup_{(t,x)\to\infty} u(t, x) - v(t, x) \leq 0$. *Then $u \leq v$ on $[0, T) \times M$.*

Unfortunately, condition (4.17) in Theorem 4.3 is not satisfied by the function F given by (2.5) corresponding to the evolution of level sets by Gaussian curvature. In this case, we can only apply Theorem 4.1 in order to deduce a comparison result for manifolds of nonnegative curvature:

Corollary 4.6 *Let M be a complete Riemannian manifold of nonnegative sectional curvature and positive injectivity radius. Let F be given by (2.5). Let $u \in USC([0, T) \times M)$ be a subsolution and $v \in LSC([0, T) \times M)$ be a supersolution of (+GCE) on M. Suppose that $u \leq v$ on $\{0\} \times M$ and* $\limsup_{(t,x)\to\infty} u(t, x) - v(t, x) \leq 0$. *Then $u \leq v$ on $[0, T) \times M$.*

Given the form of the equation (CEE), it immediately follows that, in all cases where comparison holds, one has continuous dependence of solutions with respect to initial data.

Remark 4.7 If u, v are solutions with initial conditions g and h respectively, and $\|g - h\|_{L^\infty(M)} \leq \varepsilon$, then $\|u - v\|_{L^\infty(M\times[0,T))} \leq \varepsilon$.

5 Existence by Perron's method

We will have to use the following estimation for the second derivative of the distance to a fixed point.

Lemma 5.1 [31, p. 153] *Let M be a complete Riemannian manifold whose sectional curvature K satisfies $\delta \le K \le \Delta$. Suppose $0 < r < \min\{i_M(x_0), \pi/2\sqrt{\Delta}\}$. Then, for all $x \in B(x_0, r)$ and $v \perp \nabla d(\cdot, x_0)(x)$, one has*

$$\frac{c_\Delta(d(x,x_0))}{s_\Delta(d(x,x_0))}\langle v, v\rangle \le D^2 d(\cdot, x_0)(x)(v,v) \le \frac{c_\delta(d(x,x_0))}{s_\delta(d(x,x_0))}\langle v, v\rangle,$$

and the gradient $\nabla d(\cdot, x_0)(x)$ belongs to the null space of $D^2 d(\cdot, x_0)(x)$.

Here s_δ and c_δ are defined by

$$s_\delta(t) := \begin{cases} (\sin(\sqrt{\delta}\, t))/\sqrt{\delta}, & \delta > 0; \\ t, & \delta = 0; \\ (\sinh(\sqrt{|\delta|}\, t))/\sqrt{\delta}, & \delta < 0, \end{cases}$$

and

$$c_\delta(t) := \begin{cases} \cos(\sqrt{\delta}\, t), & \delta > 0; \\ 1, & \delta = 0; \\ \cosh(\sqrt{|\delta|}\, t), & \delta < 0. \end{cases}$$

Notice that

$$\lim_{t \to 0} \frac{t\, c_\Delta(t)}{s_\Delta(t)} = 1. \tag{5.1}$$

Proposition 5.2 *Let F satisfy conditions (**A - D**) of Sect. 2, and assume $\mathcal{F}(F) \neq \emptyset$. Let $\mathcal{S}$ be a nonempty family of subsolutions of*

$$u_t + F(Du, D^2 u) = 0, \tag{5.2}$$

and define

$$W(z) := \sup\{v(z) : v \in \mathcal{S}\}.$$

Suppose that $W^*(z) < +\infty$ for all $z \in [0,T) \times M$. Then W^* is a subsolution of (5.2) on $[0,T) \times M$.

Proof Let $\varphi \in \mathcal{A}(F)$ be such that $W^* - \varphi$ has a strict maximum at $z_0 = (t_0, x_0)$. We may assume that $W^*(z_0) - \varphi(z_0) = 0$.
Case 1. Suppose first that $D\varphi(z_0) \neq 0$, and let us see that

$$\varphi_t(z_0) + F(D\varphi(z_0), D^2\varphi(z_0)) \le 0.$$

Define $\psi(t,x) := \varphi(t,x) + f(d(x,x_0)) + (t - t_0)^4$, where $f \in \mathcal{F}(F)$, and observe that

$$W^*(z) - \psi(z) \le -f(d(x,x_0)) - (t-t_0)^4. \tag{5.3}$$

Also notice that $\psi \in \mathcal{A}(F)$, $\psi_t(z_0) = \varphi_t(z_0)$, $D\psi(z_0) = D\varphi(z_0)$, and $D^2\psi(z_0) = D^2\varphi(z_0)$.
By definition of W^* there exist z_k' such that $\lim_{k\to\infty} z_k' = z_0$ and

$$\alpha_k := W^*(z_k') - \psi(z_k') \to W^*(z_0) - \psi(z_0) = 0.$$

Now, by definition of W, there exists a sequence $(v_k) \subset \mathcal{S}$ such that $v_k(z'_k) > W(z'_k) - \frac{1}{k}$, which implies

$$(v_k - \psi)(z'_k) > a_k - \frac{1}{k}. \tag{5.4}$$

Since $v_k \leq W$ (5.3) implies

$$(v_k - \psi)(z) \leq -f(d(x, x_0)) - (t - t_0)^4 \text{ for all } z. \tag{5.5}$$

Let B be a closed ball of center z_0. Since $v_k - \psi$ is upper semicontinuous it attains its maximum on B at some point $z_k \in B$. From (5.4) and (5.5) we get

$$\alpha_k - \frac{1}{k} < (v_k - \psi)(z'_k) \leq (v_k - \psi)(z_k) \leq -f(d(x_k, x_0)) - (t_k - t_0)^4 \leq 0,$$

and since $\alpha_k \to 0$ we deduce that $z_k \to z_0$ and $t_k \to t_0$. Moreover, $v_k - \psi$ has a local maximum at z_k.

Since $D\psi(z_0) = D\varphi(z_0) \neq 0$, we have $D\psi(z) \neq 0$ for all z in a neighborhood (which we may assume to be B) of z_0. Because v_k is a subsolution and $D\varphi(z_k) \neq 0$, we get

$$\psi_t(z_k) + F(D\psi(z_k), D^2\psi(z_k)) \leq 0.$$

Therefore, by taking limits and using the continuity of F off $\{\zeta = 0\}$ and the continuity of $\psi_t, D\psi, D^2\psi$, we obtain

$$\varphi_t(z_0) + F(D\varphi(z_0), D^2\varphi(z_0)) = \psi_t(z_0) + F(D\psi(z_0), D^2\psi(z_0)) \leq 0,$$

and we conclude that W^* is a subsolution of (5.2) at z_0.

Case 2. Assume now that $D\varphi(z_0) = 0$, and let us check that $\varphi_t(z_0) \leq 0$. Since $\varphi \in \mathcal{A}(F)$, there exist $\delta_0 > 0$, $\omega \in C(\mathbb{R})$ with $\omega(r) = o(r)$, and $f \in \mathcal{F}(F)$ such that

$$|\varphi(x, t) - \varphi(z_0) - \varphi_t(z_0)(t - t_0)| \leq f(d(x, x_0)) + \omega(t - t_0) \tag{5.6}$$

for all $z = (t, x) \in B := B(z_0, \delta_0)$. We may assume that $\omega \in C^1(\mathbb{R})$, $\omega(0) = 0 = \omega'(0)$, and $\omega(r) > 0$ for $r > 0$. Let us define

$$\psi(t, x) := \varphi_t(z_0)(t - t_0) + 2f(d(x, x_0)) + 2\omega(t - t_0), \text{ and}$$

$$\psi_k(t, x) := \varphi_t(z_0)(t - t_0) + 2f(d(x, x_0)) + 2\omega_k(t - t_0),$$

where (ω_k) is a sequence of C^2 functions on $\mathbb{R}$ such that $\omega_k \to \omega$ and $\omega'_k \to \omega'$ uniformly on $\mathbb{R}$.

From (5.6) we deduce that $W^* - \psi$ has a local strict maximum at z_0. On the other hand it is clear that $(\psi_k) \subset \mathcal{A}(F)$, and $\psi_k \to \psi$ uniformly. Arguing as in Case 1, we may find a sequence of subsolutions $(v_k) \subset \mathcal{S}$ and a sequence of points z_k such that $z_k \to z_0$ and $v_k - \psi_k$ attains a maximum at z_k. Since v_k is a subsolution we have

$$(\psi_k)_t(z_k) + F(D\psi_k(z_k), D^2\psi_k(z_k)) \leq 0 \text{ for all } k, \text{ when } x_k \neq x_0, \text{ and} \tag{5.7}$$

$$(\psi_k)_t(z_k) \leq 0, \text{ when } x_k = x_0. \tag{5.8}$$

Notice that

$$\lim_{k\to\infty} (\psi_k)_t(z_k) = \varphi_t(z_0). \tag{5.9}$$

If $x_k = x_0$ for infinitely many k's, we immediately deduce from (5.8) and (5.9) that $\varphi_t(z_0) \leq 0$.

Therefore we may assume that $x_k \neq x_0$ for all k. If we set

$$\zeta_k := -\exp_{x_k}^{-1}(x_0),$$
$$A_k := D^2 d(\cdot, x_0)(x_k)$$

then we have that $|\zeta_k| = d(x_k, x_0)$, and

$$(\psi_k)_t(z_k) = \varphi_t(z_0) + 2\omega_k'(t_k - t_0)$$
$$D\psi_k(z_k) = \frac{2}{|\zeta_k|} f'(|\zeta_k|)\, \zeta_k,$$
$$D^2\psi_k(z_k) = 2f''(|\zeta_k|)\, \frac{\zeta_k \otimes \zeta_k}{|\zeta_k|^2} + 2f'(|\zeta_k|) A_k.$$

Since F is geometric we have

$$F(D\psi_k(z_k), D^2\psi_k(z_k)) = 2f'(|\zeta_k|) F\left(\frac{\zeta_k}{|\zeta_k|}, A_k\right). \tag{5.10}$$

Next, because $B = B(z_0, \delta_0)$ is compact, we may find numbers $\Delta, \delta > 0$ such that the sectional curvature K of M satisfies $\delta \leq K \leq \Delta$ on B. We may of course assume $\delta_0 < \min\{i_M(x_0), \pi/2\sqrt{\Delta}\}$, so that we can apply Lemma 5.1: we obtain that $A_k(\zeta_k, \zeta_k) = 0$, and for all $v \in TM_{x_k}$ such that $v \perp \zeta_k$ we have

$$\frac{c_\Delta(|\zeta_k|)}{s_\Delta(|\zeta_k|)} \langle v, v\rangle \leq A_k(v, v) \leq \frac{c_\delta(|\zeta_k|)}{s_\delta(|\zeta_k|)} \langle v, v\rangle.$$

This implies

$$\frac{c_\Delta(|\zeta_k|)}{s_\Delta(|\zeta_k|)} \left(I - \frac{\zeta_k \otimes \zeta_k}{|\zeta_k|^2}\right) \leq A_k \leq \frac{c_\delta(|\zeta_k|)}{s_\delta(|\zeta_k|)} \left(I - \frac{\zeta_k \otimes \zeta_k}{|\zeta_k|^2}\right). \tag{5.11}$$

On the other hand, Eq. (5.1) tells us that

$$\frac{c_\Delta(t)}{s_\Delta(t)} \geq \frac{1}{2t} \quad \text{and} \quad \frac{c_\delta(t)}{s_\delta(t)} \leq \frac{2}{t}$$

if $t > 0$ is small enough. Hence we have

$$\frac{c_\Delta(|\zeta_k|)}{s_\Delta(|\zeta_k|)} \geq \frac{1}{2|\zeta_k|} \quad \text{and} \quad \frac{c_\delta(|\zeta_k|)}{s_\delta(|\zeta_k|)} \leq \frac{2}{|\zeta_k|}$$

for k large enough, which we may assume are all k. By plugging these inequalities into (5.11) we obtain

$$\frac{1}{2|\zeta_k|} \left(I - \frac{\zeta_k \otimes \zeta_k}{|\zeta_k|^2}\right) \leq A_k \leq \frac{2}{|\zeta_k|} \left(I - \frac{\zeta_k \otimes \zeta_k}{|\zeta_k|^2}\right) \tag{5.12}$$

Bearing in mind that F is elliptic and geometric, we get

$$\begin{aligned}
\frac{1}{|\zeta_k|} F(\zeta_k, 2I) &= F\left(\frac{\zeta_k}{|\zeta_k|}, \frac{2}{|\zeta_k|} I\right) = F\left(\frac{\zeta_k}{|\zeta_k|}, \frac{2}{|\zeta_k|}\left(I - \frac{\zeta_k \otimes \zeta_k}{|\zeta_k|^2}\right)\right) \\
&\leq F\left(\frac{\zeta_k}{|\zeta_k|}, A_k\right) \leq F\left(\frac{\zeta_k}{|\zeta_k|}, \frac{1}{2|\zeta_k|}\left(I - \frac{\zeta_k \otimes \zeta_k}{|\zeta_k|^2}\right)\right) \\
&= F\left(\frac{\zeta_k}{|\zeta_k|}, \frac{1}{2|\zeta_k|} I\right) \leq \frac{1}{|\zeta_k|} F(\zeta_k, -2I),
\end{aligned}$$

which combined with (5.10) yields

$$2\frac{f'(|\zeta_k|)}{|\zeta_k|}F(\zeta_k, 2I) \leq F(D\psi_k(z_k), D^2\psi_k(z_k)) \leq 2\frac{f'(|\zeta_k|)}{|\zeta_k|}F(\zeta_k, -2I), \tag{5.13}$$

which, thanks to condition (2.6), allows to conclude that

$$\lim_{k\to\infty} F(D\psi_k(z_k), D^2\psi_k(z_k)) = 0. \tag{5.14}$$

Finally, from (5.7), (5.9) and (5.14), it follows that

$$\varphi_t(z_0) \leq 0.$$

In either case we see that W^* is a subsolution of (5.2). □

Theorem 5.3 *Let F satisfy conditions* (**A - D**) *of Sect.* 2*, and assume that $\mathcal{F}(F) \neq \emptyset$ and comparison holds for the equation*

$$\begin{cases} u_t + F(Du, D^2u) = 0 \\ u(0, x) = g(x). \end{cases} \tag{5.15}$$

Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of (5.15)*, respectively, satisfying $\underline{u}_*(0, x) = \overline{u}^*(0, x) = g(x)$. Then $w = \sup\{v : \underline{u} \leq v \leq \overline{u},\ \ v \text{ is a subsolution}\}$ is a solution of* (5.15).

Proof From $= \underline{u}_* \leq w_* \leq w \leq w^* \leq \overline{u}^*$, we deduce that $w_*(0, x) = w(0, x) = w^*(0, x) = g(x)$. On the other hand w^* is a subsolution by Proposition 5.2, and $w^* \leq \overline{u}$ by comparison, hence $w^* = w$ by definition of w. We claim that w_* is a supersolution. This implies $w^* \leq w_*$ by comparison, hence $w_* = w = w^*$ and consequently w is a solution.

Let us prove the claim. Suppose to the contrary that w_* is not a supersolution. Then there exist $z_0 = (t_0, x_0)$ and a C^2 function φ such that $(w_* - \varphi)(z) \geq 0 = (w_* - \varphi)(z_0)$ for all z, and either

$$\varphi_t(z_0) + F(D\varphi(z_0), D^2\varphi(z_0)) < 0, \quad \text{when } D\varphi(z_0) \neq 0, \text{ or} \tag{5.16}$$

$$\varphi_t(z_0) < 0, \quad \text{when } D\varphi(z_0) = 0. \tag{5.17}$$

By replacing $\varphi(t, x)$ with the function $\varphi(t, x) + d(x, x_0)^4 + (t - t_0)^4$ on a neighborhood of z_0 we can furthermore assume that

$$(w_* - \varphi)(t, x) \geq d(x, x_0)^4 + (t - t_0)^4. \tag{5.18}$$

Let us denote

$$U_\delta := \{(t, x) : d(x, x_0)^4 + (t - t_0)^4 \leq \delta^4\}.$$

Case 1. In the case when (5.16) holds, by continuity of φ_t, $D\varphi$, $D^2\varphi$ and F, we can find $r > 0$ such that

$$\varphi_t(z) + F(D\varphi(z), D^2\varphi(z)) < 0$$

for all $z \in U_{2r}$, that is φ is a subsolution on U_{2r}, and obviously the same is true of $\widetilde{\varphi} := \varphi + r^4/2$.

From (5.18) we have that

$$w(z) \geq w_*(z) - \frac{r^4}{2} \geq \varphi(z) + \frac{r^4}{2} \quad \text{for all } z \in U_{2r} \setminus U_r. \tag{5.19}$$

Now let us define

$$W(z) = \begin{cases} \max\{\widetilde{\varphi}(z), w(z)\}, & \text{if } z \in U_r; \\ w(z), & \text{otherwise.} \end{cases}$$

By using Proposition 5.2 and Eq. (5.19), it is immediately checked that W is a subsolution. We have $W = w$ outside $B(z_0, r)$, but

$$\sup(W - w) > 0 \tag{5.20}$$

because, by definition of w_*, there exits a sequence $\{(t_n, x_n)\}$ converging to (t_0, x_0) such that $\lim w(t_n, x_n) = w_*(t_0, x_0)$, and consequently we have

$$\lim(W(t_n, x_n) - w(t_n, x_n)) \geq \lim(\widetilde{\varphi}(t_n, x_n) - w(t_n, x_n)) = r^4/2 > 0.$$

On the other hand, $W(0, x) = w(0, x) = g(x)$, because we could of course have taken $r > 0$ small enough such that $(0, x) \notin B(z_0, r)$. We deduce (from comparison again) that $W \leq \overline{u}$ and consequently $W \leq w$, which contradicts (5.20).

Case 2. On the other hand, in the case when (5.17) holds, since $\varphi \in \mathcal{A}(F)$, there exist $\delta_0 > 0$, $\omega \in C(\mathbb{R})$ with $\omega(r) = o(r)$, and $f \in \mathcal{F}(F)$ such that

$$|\varphi(x, t) - \varphi(z_0, t_0) - \varphi_t(t_0, x_0)(t - t_0)| \leq f(d(x, x_0)) + \omega(t - t_0)$$

for all $z = (t, x) \in B := B(z_0, \delta_0)$. We may assume that $\omega \in C^1(\mathbb{R})$, $\omega(0) = 0 = \omega'(0)$, and $\omega(r) > 0$ for $r > 0$. Let us define

$$\psi(t, x) = \varphi(z_0) + \varphi_t(z_0)(t - t_0) - 2f(d(x, x_0)) - 2\omega(t - t_0).$$

Then $w_* - \psi$ attains a strict minimum at z_0. Also notice that $D\psi(z) \neq 0$ for $z \neq z_0$. Arguing as in Case 2 of the proof of Proposition 5.2, one can show that

$$\lim_{z \to z_0} F(D\psi(z), D^2\psi(z)) = 0. \tag{5.21}$$

By combining this with the continuity of ψ_t and the fact that $\psi_t(z_0) = \varphi_t(z_0) < 0$, we can find an $r > 0$ such that

$$\psi_t(z) + F(D\psi(z), D^2\psi(z)) < 0$$

for all $z \in U_{2r}$, $z \neq z_0$. The rest of the proof is identical to that of Case 1 (just replace φ with ψ). □

Let us now show how to apply the above Theorem in order to construct solutions of (5.15). We will need to use the following stability result.

Lemma 5.4 *Assume that u.s.c.* (*respectively l.s.c.*) *functions u_k are subsolutions* (*supersolutions, respectively*) *of* (CEE). *Assume also that $\{u_k\}$ converges locally uniformly to a function u. Then u is subsolution* (*supersolution, respectively*) *of* (CEE).

Proof Suppose that $\varphi \in \mathcal{A}(F)$ and $u - \varphi$ attains a strict local maximum at (t_0, x_0). The convergence of the subsolutions u_k allows us to find a sequence of local maxima (t_k, x_k) of $u_k - \varphi$ which converges to (t_0, x_0). Then, by a similar argument to that of the proof of Proposition 5.2, one can show that u is a subsolution of (CEE) at (t_0, x_0). □

Let M be a compact Riemannian manifold. Assume that comparison holds for the equation (5.15). Let us first produce solutions of (5.15) for initial data g in the class $\mathcal{A}(F)$.

Let us define

$$\underline{u}(t,x) = -Kt + g(x) \quad \text{and} \quad \overline{u}(t,x) = Kt + g(x),$$

where $K := \sup_{x\in M} |F(Dg(x), D^2g(x))|$ (which is finite because $g \in \mathcal{A}(F)$ and M is compact). It is immediately seen that $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution of (5.15), and obviously $\underline{u}_*(0,x) = \overline{u}^*(0,x) = g(x)$. According to Theorem 5.3 and comparison, there exists a unique solution u of (5.15).

Now take g a continuous function on M. According to Proposition 2.8, we can find a sequence g_k of functions in $\mathcal{A}(F)$ such that $g_k \to g$ uniformly on M. Let u_k be the unique solution of (5.15) with initial datum g_k. By Remark 4.7, (u_k) is a Cauchy sequence in $\mathcal{C}([0,\infty)\times M)$, hence it converges to some $u \in \mathcal{C}([0,\infty)\times M)$ uniformly on $[0,\infty)\times M$. Then by Lemma 6.1 it follows that u is a solution with initial datum $u(0,x) = g(x)$.

Therefore we can combine this argument with Theorems 4.1 and 4.3 to obtain the following corollaries.

Corollary 5.5 *Let M be a compact Riemannian manifold of nonnegative sectional curvature, $g : M \to \mathbb{R}$ a continuous function, and let $F : J_0^2(M) \to \mathbb{R}$ be continuous, elliptic, translation invariant and geometric. Then there exists a unique solution of* (CEE) *on* $[0,\infty)\times M$.

Corollary 5.6 *Let M be a compact Riemannian manifold, $g : M \to \mathbb{R}$ a continuous function, and let F satisfy conditions (**A - D**) of Sect. 2. Assume furthermore that* (4.16) *and* (4.17) *are satisfied. Then there exists a unique solution of* (CEE) *on* $[0,\infty)\times M$.

Corollary 5.7 (Ilmanen) *Let M be a compact Riemannian manifold, $g : M \to \mathbb{R}$ continuous. There exists a unique solution u of the mean curvature evolution equation* (MCE) *such that* $u(0,x) = g(x)$.

Corollary 5.8 *Let M be a compact Riemannian manifold, $g : M \to \mathbb{R}$ continuous. There exists a unique solution u of the positive Gaussian curvature evolution equation* (+GCE) *such that* $u(0,x) = g(x)$.

Corollary 5.9 *Let M be a compact Riemannian manifold, $g : M \to \mathbb{R}$ continuous. There exists a unique solution u of the mean curvature evolution equation in arbitrary codimension (given in Example 2.3) such that* $u(0,x) = g(x)$.

When M is not compact, analogous corollaries can be established if one additionally demands that the initial datum g be a (positive) constant outside some bounded set of M, and that $i(M) > 0$. The proof is similar (replacing uniform convergence on M with uniform convergence on compact subsets of M).

6 Geometric consistency and level set method

Theorem 6.1 *Let $\theta : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let u be a bounded continuous solution of* (CEE). *Then $v = \theta \circ u$ is also a solution. Moreover, if θ is nondecreasing and u is a subsolution (resp. supersolution) then $v = \theta \circ u$ is a subsolution (resp. supersolution) as well.*

Proof Assume first that θ is monotone. We may consider a sequence of smooth functions θ_k with nonvanishing derivatives, converging uniformly to θ over the bounded range of u.

Hence by Lemma 5.4, we may directly assume that $\theta' \neq 0$. Notice that $g = \theta^{-1}$ satisfies $g' \neq 0$ too.

Suppose first that $\theta' > 0$. Let $\varphi \in \mathcal{A}(F)$ and assume that $\theta \circ u - \varphi$ attains a local maximum at z_0. If we denote $\psi = g \circ \varphi$, it is not difficult to check that $\psi \in \mathcal{A}(F)$, and $u - \psi$ clearly attains a local maximum at (t_0, x_0). Consequently

$$\psi_t(z_0) + F(D\psi(z_0), D^2\psi(z_0)) \leq 0$$

if $D\psi(t_0, x_0) \neq 0$, and $\psi_t(t_0, x_0) \leq 0$ otherwise. But $D\psi(z_0) = 0$ if and only if $D\varphi(z_0) = 0$, and

$$\begin{aligned} \psi_t &= g'(\varphi)\varphi_t \\ D\psi &= g'(\varphi)D\varphi \\ D^2\psi &= g''(\varphi)D\psi \otimes D\psi + g'(\varphi)D^2\varphi. \end{aligned}$$

Since F is geometric and $g' > 0$, one immediately sees that

$$\varphi_t(z_0) + F(D\varphi(z_0), D^2\varphi(z_0)) = \frac{1}{g'(\varphi)(z_0)} \left(\psi_t(z_0) + F(D\psi(z_0), D^2\psi(z_0))\right) \leq 0$$

if $D\varphi(z_0) \neq 0$, and $\varphi_t(t_0, x_0) = \frac{1}{g'(\varphi(z_0))}\psi_t(z_0) \leq 0$ otherwise. This shows that $\theta \circ u$ is a subsolution.

If $\theta' < 0$, the same argument tells us that if u is subsolution (respectively supersolution), then v is supersolution (respectively subsolution). In order to establish the result for continuous functions, it is enough to observe that a continuous function can be uniformly approximated by a sequence of locally monotone functions. Then a local application of Lemma 5.4 yields the result. □

Now one can show that, if comparison and existence hold for (CEE) (e.g. when M is a compact Riemannian manifold of nonnegative curvature), then for every compact level set Γ_0 there is a unique, well-defined, level set evolution Γ_t of Γ_0 by the geometric curvature evolution equation corresponding to (CEE).

Let g be a continuous function on M with $\Gamma_0 = \{x \in M : g(x) = 0\}$, and assume that Γ_0 is compact. We may also assume that g is constant outside a bounded neighborhood of Γ_0, and in particular bounded. Let u be the unique solution of (CEE) with $u(0, \cdot) = g$. We define

$$\Gamma_t = \{x \in M : u(t, x) = 0\}.$$

Theorem 6.2 *Assume that comparison and existence hold for* (CEE). *Let* $\hat{g} : M \to \mathbb{R}$ *be a continuous function satisfying* $\Gamma_0 = \{x \in M : \hat{g}(x) = 0\}$ *and such that* $\hat{g}$ *is constant outside a bounded neighborhood of* Γ_0. *Let* $\hat{u}$ *be the unique continuous solution of* (CEE) *with initial condition* $\hat{g}$. *Then*

$$\Gamma_t = \{x \in M : \hat{u}(t, x) = 0\}.$$

Proof This is a consequence of Theorem 6.1 and the comparison principle. It follows exactly as in the case $M = \mathbb{R}^n$, see [10, Theorem 5.1], or [16, Chap. 4], for instance. □

Corollary 6.3 *The definition of* $\Gamma_t = \{x \in M : u(t, x) = 0\}$ *does not depend upon the particular choice of the function* g *satisfying* $\Gamma_0 = \{x \in M : g(x) = 0\}$.

It can also be checked that the evolution $\Gamma_0 \mapsto \mathcal{K}(t)\Gamma_0 := \Gamma_t$ thus defined has the semigroup property

$$\mathcal{K}(t+s) = \mathcal{K}(t)\mathcal{K}(s).$$

Some other properties of the evolutions can be established as in the case $M = \mathbb{R}^n$. For instance, in the case of the evolution by mean curvature, it is possible to show that if $\Gamma_0 = \partial U$ is a smooth connected hypersurface with positive mean curvature with respect to the inner unit normal field, then Γ_t continues to have positive mean curvature as long as it exists, in the sense that

$$\Gamma_t = \{x \in M : v(x) = t\},$$

where v is the solution of the stationary problem

$$\begin{cases} -\text{trace}\left(\left(I - \frac{Du \otimes Du}{|Du|^2}\right) D^2 u\right) = 1, & \text{on } U; \\ v = 0, & \text{on } \Gamma_0 = \partial U, \end{cases}$$

(which admits a unique viscosity solution, see [5]).

However, one has to be very cautious and not take it for granted that all the usual geometrical properties of the generalized evolutions by mean curvature or by Gaussian curvature could be immediately extended from the Euclidean to the Riemannian setting. As a matter of fact, many of these properties are very likely to fail in the case of manifolds of negative curvature. We will present several counterexamples and related conjectures in Sect. 8.

7 Consistency with the classical motion

In this section we suppose that equation (CEE) arises from a classical geometric evolution for hypersurfaces in M. We establish the consistency of the level set evolution equation with this classical geometric motion.

More precisely, suppose $(\Gamma_t)_{t\in[0,T]}$ is a family of smooth, compact, orientable hypersurfaces in M evolving according to a classical geometric motion, locally depending only on its normal vector fields and second fundamental forms. In particular, we shall assume that Γ_t is the boundary of a bounded open set $U_t \subset M$ and that there exists a family of diffeomorphisms of manifolds with boundary

$$\phi^t : \overline{U_0} \to \overline{U_t}, \qquad t \in [0, T],$$

such that:

(i) ϕ^0 =Id, and,
(ii) for every $x \in \Gamma_0$ the following holds:

$$\frac{d}{dt}\phi^t(x) = G\left(\nu\left(t, \phi^t(x)\right), \nabla^\Gamma \nu\left(t, \phi^t(x)\right)\right), \tag{7.1}$$

where $\nu(t, \cdot)$ is a unit normal vector field to Γ_t, and the linear map

$$\nabla^\Gamma \nu(t, x) : (T\Gamma_t)_x \ni \xi \mapsto \nabla^T_\xi \nu(t, x) \in (T\Gamma_t)_x$$

and ∇^T stands for the orthogonal projection onto $(T\Gamma_t)_x$ of covariant derivative in M.

Classical motion by mean curvature corresponds to taking $f\left(\nu, \nabla^{\Gamma}\nu\right) = \operatorname{tr}\left(-\nabla^{\Gamma}\nu\right)\nu$, whereas classical motion by Gaussian curvature is defined by $f\left(\nu, \nabla^{\Gamma}\nu\right) = \det\left(-\nabla^{\Gamma}\nu\right)\nu$. The level set evolution equation induced by (7.1) is of the form (CEE) where F is related to G through formula (2.1). As before, we assume that F is elliptic, translation invariant and geometric. In this case we already know that F is continuous, and in fact smooth off $\{\zeta = 0\}$, because F is of the form (2.1) with G smooth.

Define $d : [0, T] \times M \to \mathbb{R}$, the signed distance function from Γ_t, as:

$$d(t, x) := \begin{cases} \operatorname{dist}(x, \Gamma_t) & \text{if } x \in U_t \\ -\operatorname{dist}(x, \Gamma_t) & \text{if } x \in M \backslash U_t. \end{cases}$$

Lemma 7.1 *There exist constants $K, \delta > 0$ such that d is smooth in*

$$I_\delta := \{(t, x) \in [0, T] \times M \ : \ |d(t, x)| < \delta\}$$

and

$$\left|d_t + F\left(Dd, D^2 d\right)\right| \le K\,|d| \quad in \quad I_\delta. \tag{7.2}$$

Proof We consider geodesic normal coordinates from Γ_t,

$$\Phi(t, s, x) := \exp_x(s\nu(t, x)),$$

assuming that $\nu(t, x)$ points towards the interior U_t for every $x \in \Gamma_t$. Clearly, for s small enough,

$$d(t, \Phi(t, s, x)) = s.$$

Given $x_0 \in \Gamma_0$ there exists a neighborhood V of x_0 in Γ_0 and an interval $(-r, r)$ such that $\Phi(t, \cdot, \cdot)$ is a diffeomorphism from $(-r, r) \times \phi^t(V)$ onto its image $X_t := \Phi\big((-r, r) \times \phi^t(V)\big)$ for $t \in [0, T]$. Note that Φ is also smooth in t. Denote by $\Psi(t, \cdot)$ the inverse of $\Phi(t, \cdot, \cdot)$ and write

$$\Psi(t, y) := (\rho(t, y), X(t, y)).$$

Now for $x \in \phi^t(V)$ and $s \in (-r, r)$ we have $X(t, \Phi(t, s, x)) = x$ and $\rho(t, \Phi(t, s, x)) = s$. Both X and ρ are smooth in t, and clearly $\rho = d$ in $\bigcup_{t \in [0,T]} \{t\} \times X_t = I_\delta$.

In order to prove (7.2) it suffices to note that, since, for $x \in \Gamma_t$, $Dd(t, x) = \nu(t, x) \neq 0$,

$$r(t, x) := d_t + F\left(Dd, D^2 d\right)$$

is a smooth function vanishing for $x \in \Gamma_t$. This gives (7.2) locally; a global bound then follows by the compactness of Γ_t. □

Next we state and prove the main result of this section.

Theorem 7.2 *Let u be the unique viscosity solution to the level set equation (CEE) with initial datum $u|_{t=0} = d|_{t=0}$. Then, for every $t \in [0, T]$, the zero level set of $u(t, \cdot)$ coincides with Γ_t:*

$$\Gamma_t = \{x \in M \ : \ u(t, x) = 0\}.$$

Proof Define

$$v(t, x) := e^{tK}\left((d(t, x) \vee 0) \wedge \delta/2\right),$$

where K is the constant given by (7.2). We shall prove that v is a viscosity *supersolution* to equation (CEE). Clearly, $v|_{t=0} \geq u|_{t=0} \wedge \delta/2$, and, by Theorem 6.1, $u \wedge (\delta/2)$ is a viscosity solution to (CEE) as well. The comparison principle (Theorem 4.1) then will ensure that $v \geq u \wedge (\delta/2)$. In particular,

$$\{x \in M \ : \ u\,(t,x) > 0\} \subseteq U_t, \qquad t \in [0,T]\,. \tag{7.3}$$

On the other hand, we shall prove that

$$w\,(t,x) := e^{-tK}\,((d\,(t,x) \wedge 0) \vee (-\delta/2))$$

is a viscosity *subsolution* to (CEE). Now $w|_{t=0} \leq u|_{t=0} \vee (-\delta/2)$, and the comparison principle will imply that $w \leq u \vee (-\delta/2)$. This, together with (7.3) yields

$$\{x \in M \ : \ u\,(t,x) > 0\} = U_t, \qquad t \in [0,T]\,.$$

Now take $\varepsilon > 0$ and let $u_\varepsilon := u + \varepsilon$; this is again a viscosity solution to (CEE). It turns out that

$$v_\varepsilon\,(t,x) := e^{tK}\,(((d\,(t,x) + \varepsilon) \vee 0) \wedge \delta/2)\,,$$

$$w_\varepsilon\,(t,x) := e^{-tK}\,(((d\,(t,x) + \varepsilon) \wedge 0) \vee (-\delta/2))\,,$$

are respectively super and subsolutions to (CEE) provided ε is much smaller than δ – namely, small enough to ensure that v_ε and w_ε are smooth in the regions $0 < d\,(t,x) + \varepsilon < \delta/2$ and $-\delta/2 < d\,(t,x) + \varepsilon < 0$, respectively. Applying the comparison principle as we did before we ensure that

$$\{x \in M \ : \ u\,(t,x) > -\varepsilon\} = \{x \in M \ : \ d\,(t,x) > -\varepsilon\}\,.$$

Letting ε go to zero we conclude that the zero level set of $u\,(t,\cdot)$ is precisely Γ_t, as we wanted to prove.

We now show our claim that v is a supersolution to (CEE). Start noticing that Lemma 7.1 ensures that, for $t \in [0,T]$ the following holds:

$$\begin{cases} v_t\,(t,x) + F\left(Dv\,(t,x)\,, D^2 v\,(t,x)\right) \geq 0, & \text{if } 0 < d\,(t,x) < \delta/2, \\ v_t\,(t,x) \geq 0, & \text{if } d\,(t,x) < 0 \text{ or } d\,(t,x) > \delta/2. \end{cases} \tag{7.4}$$

Let $(t_0, x_0) \in [0,T] \times M$ and $\varphi \in \mathcal{A}\,(F)$ be such that $v - \varphi$ has a local minimum at (t_0, x_0). Without loss of generality, we may assume that

$$\varphi\,(t_0, x_0) = v\,(t_0, x_0)\,, \tag{7.5}$$

and that

$$\varphi \leq v, \qquad \text{locally around } (t_0, x_0)\,. \tag{7.6}$$

Since the level sets $d\,(t,x) = c \in (-\delta, \delta)$ are smooth hypersurfaces, necessarily $d\,(t_0, x_0) \neq \delta/2$. If moreover $(t_0, x_0) \notin \Gamma_t$, then (7.5), (7.6) and the smoothness of v imply that, at (t_0, x_0) one has $\varphi_t = v_t$ and $D\varphi = Dv$. Therefore, using (7.4) we conclude that:

$$\begin{cases} \varphi_t\,(t_0, x_0) + F\left(D\varphi\,(t_0, x_0)\,, D^2\varphi\,(t_0, x_0)\right) \geq 0, & \text{if } D\varphi\,(t_0, x_0) \neq 0, \\ \varphi_t\,(t_0, x_0) \geq 0, & \text{otherwise.} \end{cases} \tag{7.7}$$

Now we shall prove that the above identity also holds when $(t_0, x_0) \in \Gamma_t$. Let $Q := \bigcup_{t \in [0,T]} \{t\} \times \Gamma_t$; this is precisely the set of zeroes of d. As $|Dd| = 1$ on I_δ, we infer that Q is a smooth hypersurface of $[0, T] \times M$; since (7.5) and (7.6) imply that $v - \varphi$ has a minimum at (t_0, x_0), the following holds for every tangent vector $(\tau, \xi) \in TQ_{(t_0,x_0)}$:

$$\varphi_t(t_0, x_0)\,\tau + D\varphi(t_0, x_0)(\xi) = 0; \tag{7.8}$$

moreover, for every curve $\gamma : (-1, 1) \to \Gamma_{t_0}$ with $\gamma(0) = x_0$, and $\gamma'(0) = \xi$ we have,

$$\frac{d^2}{ds^2}\varphi(t_0, \gamma(s))\,|_{s=0} \le 0. \tag{7.9}$$

From identity (7.8) we deduce that

$$(\varphi_t(t_0, x_0), D\varphi(t_0, x_0)) = \lambda\,(d_t(t_0, x_0), Dd(t_0, x_0)), \tag{7.10}$$

for some $\lambda \in \mathbb{R}$, whereas (7.9) merely states that:

$$\left\langle D^2\varphi(t_0, x_0)\,\xi, \xi\right\rangle \le -\left\langle \nabla\varphi(t_0, x_0), \gamma''(0)\right\rangle.$$

Taking into account that $\left\langle \nabla d(t_0, \gamma(0)), \gamma'(0)\right\rangle = 0$, we obtain:

$$\begin{aligned}\left\langle D^2\varphi(t_0, x_0)\,\xi, \xi\right\rangle &\le -\left\langle \nabla\varphi(t_0, x_0), \gamma''(0)\right\rangle\\ &= -\lambda\left\langle \nabla d(t_0, x_0), \gamma''(0)\right\rangle = \lambda\left\langle D^2 d(t_0, x_0)\,\xi, \xi\right\rangle.\end{aligned} \tag{7.11}$$

Given a smooth curve $\eta : (-1, 1) \to Q$ such that $\eta(0) = (t_0, x_0)$ and $\eta'(0) = -\big(d_t(t_0, x_0), \nabla d(t_0, x_0)\big)$ the following holds:

$$\frac{d}{dt}\varphi(\eta(t))\,|_{t=0} = -\lambda\left(d_t(t_0, x_0)^2 + |Dd(t_0, x_0)|^2\right).$$

Since necessarily $v(\eta(t)) = v(\eta(0)) = 0$ for $t \ge 0$ sufficiently small (7.5) and (7.6) imply that $\lambda \ge 0$.

Now (7.7) trivially holds if $\lambda = 0$. Suppose $\lambda > 0$, then using (7.10), (7.11) we deduce

$$\begin{aligned}-F\left(D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)\right) &\le -F\left(\lambda Dd(t_0, x_0), \lambda D^2 d(t_0, x_0)\right)\\ &= -\lambda F\left(Dd(t_0, x_0), D^2 d(t_0, x_0)\right),\end{aligned}$$

and

$$\varphi_t(t_0, x_0) = \lambda d_t(t_0, x_0) = -\lambda F\left(Dd(t_0, x_0), D^2 d(t_0, x_0)\right).$$

Therefore, we conclude that φ satisfies (7.7) at (t_0, x_0).

A completely analogous proof shows that v_ε is a supersolution, and w and w_ε are subsolutions to (CEE). □

Remark 7.3 In the very special case of the evolution by mean curvature in arbitrary codimension (see Example 2.3) the above proof breaks down because the sets Γ_t are no longer hypersurfaces of M, but k-codimensional submanifolds. The consistency of the generalized motion (given, e.g., by Corollary 5.9) with the classical evolution thus remains an open problem (whose solution would probably require a careful analysis of the properties of the distance functions to the submanifolds Γ_t, and of the eigenvalues of their Hessians, similar to the study carried out in [2] for the case $M = \mathbb{R}^n$).

8 Counterexamples and conjectures

In this final section we provide some counterexamples showing that many well known properties of the evolutions by mean curvature in $\mathbb{R}^n$ fail when M is a Riemannian manifold of negative curvature.

Example 8.1 When (M, g) is the Euclidean space equipped with the canonical metric, Ambrosio and Soner have proved in [2] that the distance function $|d|$ is always a supersolution to the mean curvature equation (MCE). This is no longer the case of a general Riemannian manifold, as the following example shows.

Let (M, g) be a surface of revolution embedded in $\mathbb{R}^3$, locally parameterized by:

$$I \times (-\pi, \pi) \ni (s, \theta) \mapsto (r(s)\cos\theta, r(s)\sin\theta, s) \in \mathbb{R}^3,$$

where $I \subseteq \mathbb{R}$ is an open interval and $r \in C^\infty(I)$ with $r \geq \rho > 0$. In these coordinates, the metric takes the form:

$$\begin{pmatrix} r'(s)^2 + 1 & 0 \\ 0 & r(s)^2 \end{pmatrix}.$$

Suppose $0 \in I$ and $r'(0) = 0$; then the "Equator" $s = 0$ is a geodesic of M, denote it by Γ_0. The classical evolution by mean curvature starting from Γ_0 is constant in time. Therefore, the corresponding signed distances satisfy $d(t, \cdot) = d(0, \cdot) \equiv d$ for every $t \in \mathbb{R}$ (we shall assume $d > 0$ for $s > 0$). Let us next explicitly compute d. The geodesics of M that are orthogonal to Γ_0 are of the form $(s(t), \theta)$ with $\theta \in (-\pi, \pi)$ constant. Take such a geodesic and assume that is parameterized by arc length and satisfies $s' > 0$. In particular, since its tangent vector $(s', 0)$ must be of norm one,

$$v(s(t))^2 \left(s'(t)\right)^2 = 1, \qquad v(s) := \sqrt{r'(s)^2 + 1}. \tag{8.1}$$

Clearly, $d(s(t), \theta) = t$ and $\partial_s d(s(t), \theta)\, s'(t) = 1$. Identity (8.1) and our assumption $s' > 0$ allow us to conclude that $\partial_s d(s, \theta) = v(s)$ and $Dd = (1/v, 0)$. Finally, a direct computation gives:

$$|Dd| \operatorname{div}\left(\frac{Dd}{|Dd|}\right) = \frac{r'}{rv}.$$

The function $|d|$ will be a supersolution to the mean curvature equation provided $r'(s)\operatorname{sign}(s) \leq 0$. This is always the case if the curvature of M remains nonnegative. On the other hand, $|d|$ will a subsolution whenever the curvature of M is nonpositive everywhere. Finally, taking for instance $r(s) = 1 + \cos^2 s$ we are able to produce a $|d|$ that is not a supersolution, neither a subsolution to the mean curvature equation.

Conjecture If M has nonnegative sectional curvature then $|d|$ is always a supersolution. If M has negative curvature then there always exists Γ_0 such that $|d|$ is not a supersolution.

Example 8.2 When $M = \mathbb{R}^n$, Evans and Spruck [10, Theorem 7.3] showed that if Γ_0, $\hat{\Gamma}_0$ are compact level sets and Γ_t, $\hat{\Gamma}_t$ are the corresponding generalized evolutions by mean curvature, then

$$\operatorname{dist}(\Gamma_0, \hat{\Gamma}_0) \leq \operatorname{dist}(\Gamma_t, \hat{\Gamma}_t)$$

for all $t > 0$. This result fails for manifolds of negative curvature. For instance, let $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 + z^2\}$ be a hyperboloid of revolution embedded in $\mathbb{R}^3$. Let

$$\Gamma_0 = \{(x, y, z) \in M : z = 0\},$$

and

$$\hat{\Gamma}_0 = \{(x, y, z) \in M : z = 1\}.$$

Then

$$\Gamma_t = \Gamma_0 \text{ for all } t > 0, \qquad \text{and} \qquad \mathrm{dist}(\Gamma_0, \hat{\Gamma}_0) > 0,$$

but

$$\mathrm{dist}(\Gamma_t, \hat{\Gamma}_t) = \mathrm{dist}(\Gamma_0, \hat{\Gamma}_t) \to 0 \quad \text{as } t \to \infty.$$

Conjecture Evans-Spruck's [10, Theorem 7.3] result holds true for all manifolds of nonnegative sectional curvature, but fails for all manifolds of negative curvature.

Example 8.3 In the case $M = \mathbb{R}^n$ it is well known that equation (CEE) preserves Lipschitz properties of the initial data. Namely, if g is L-Lipschitz and u is the unique solution of (CEE) then $u(t, \cdot)$ is L-Lipschitz too, for all $t > 0$; see [16, Chap. 3]. Since the proof of Theorem 7.3 in [10] remains valid for any manifold provided that one assumes the Lipschitz preserving property of (CEE), the preceding example also shows that (CEE) does not preserve Lipschitz constants when M is a hyperboloid of revolution.

Conjecture: The equation (CEE) has the Lipschitz preserving property if and only if M has nonnegative sectional curvature.

9 Appendix: Existence and uniqueness of viscosity solutions to a (nonsingular) general parabolic equation

In this appendix we present the standard definition of viscosity solution and state existence and comparison result for viscosity solutions to non-singular parabolic fully nonlinear equations.

Definition 9.1 Let M be a finite-dimensional Riemannian manifold, and a continuous function $F : (0, T) \times M \times \mathbb{R} \times J^2 M \to \mathbb{R}$. Consider the parabolic equation

$$u_t + F(t, x, u, Du, D^2 u) = 0, \tag{9.1}$$

where u is a function of (t, x). We say that an USC function $u : (0, T) \times M \to \mathbb{R}$ is a viscosity subsolution of the partial differential evolution equation provided that $a + F(t, x, u(t, x), \zeta, A) \leq 0$ for all $(t, x) \in (0, T) \times M$ and $(a, \zeta, A) \in \mathcal{P}^{2,+}u(t, x)$. Similarly, a viscosity supersolution of (9.1) is a LSC function $u : (0, T) \times M \to \mathbb{R}$ such that $a + F(t, x, u(t, x), \zeta, A) \geq 0$ for every $(t, x) \in (0, T) \times M$ and $(a, \zeta, A) \in \mathcal{P}^{2,-}u(t, x)$. If u is both a viscosity subsolution and a viscosity supersolution of $u_t + F(t, x, u, Du, D^2 u) = 0$, we say that u is a viscosity solution.

Remark 9.2 f u is a subsolution of $u_t + F(t, x, u, Du, D^2 u) = 0$ and F is lower semicontinuous, then $a + F(t, x, u(t, x), \zeta, A) \leq 0$ for every $(a, \zeta, A) \in \overline{\mathcal{P}}^{2,+}u(t, x)$ and every $(t, x) \in (0, T) \times M$. A similar observation applies to supersolutions when F is upper semicontinuous, and to solutions when F is continuous.

Theorem 9.3 *Let M be a compact and $F : (0, T) \times M \times \mathbb{R} \times J^2M \to \mathbb{R}$ be continuous, proper, and such that*

(1) *there exists $\gamma > 0$ with*

$$\gamma(r - s) \leq F(t, x, r, \zeta, Q) - F(t, x, s, \zeta, Q)$$

for all $r \geq s$, $(t, x, \zeta, Q) \in (0, T) \times M \times J^2M)$;

(2) *there exists a function $\omega : [0, \infty] \to [0, \infty]$ with $\lim_{t \to 0^+} \omega(t) = 0$ and such that, for every $\alpha > 0$,*

$$F(t, y, r, \alpha \exp_y^{-1}(x), Q) - F(t, x, r, -\alpha \exp_x^{-1}(y), P) \leq \omega\left(\alpha d(x, y)^2 + d(x, y)\right)$$

for all $t \in (0, T)$, $x, y \in M$, $r \in \mathbb{R}$, $P \in \mathcal{L}_s^2(TM_x)$, $Q \in \mathcal{L}_s^2(TM_y)$ with

$$-\left(\frac{1}{\varepsilon_\alpha} + \|A_\alpha\|\right) I \leq \begin{pmatrix} P & 0 \\ 0 & -Q \end{pmatrix} \leq A_\alpha + \varepsilon_\alpha A_\alpha^2$$

and $\varepsilon_\alpha = (2 + 2\|A_\alpha\|)^{-1}$, where A_α is the second derivative of the function $\varphi_\alpha(x, y) = \frac{\alpha}{2} d(x, y)^2$ at the point $(x, y) \in M \times M$ with $d(x, y) < \min\{i_M(x), i_M(y)\}$.

Assume that $u \in USC([0, T) \times M)$ is a subsolution and $v \in LSC([0, T) \times M)$ is a supersolution of (9.1) on M and $u \leq v$ on $\{0\} \times M$. Then $u \leq v$ on $[0, T) \times M$.

The proof of this result is a combination of the ideas in the proof of [8, Theorem 8.2] with the new techniques for second order nonsmooth analysis on manifolds developed when dealing with the stationary case in [5, Theorem 4.2]. See also Theorem 3.8, which should be used in this proof. We leave the details to the reader's care.

Analogous results to Corollaries 4.6 and 4.8 stated in [5] can also be obtained in a similar way for these general evolution equations. We say in such cases that "comparison holds".

As it is customary in these cases, whenever the Comparison Theorem holds there is no difficulty in applying Perron's method (see [8,27]) to show that the problem

$$\begin{cases} u_t + F(t, x, u(t, x), Du(t, x), D^2u(t, x)) = 0, \\ u(0, x) = g(x). \end{cases} \tag{GEE}$$

has a unique bounded viscosity solution u on M, provided that comparison holds and one is able to find a viscosity subsolution $\underline{u}$ and a viscosity supersolution $\overline{u}$ such that $\underline{u}_*(0, x) = \overline{u}^*(0, x) = g(x)$. In fact u is given by

$$W(t, x) = \sup\{w(t, x) : \underline{u} \leq w \leq \overline{u} \text{ and } w \text{ is a subsolution of (9.1)}\}.$$

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