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Starlike Bodies and Deleting Diffeomorphisms in Banach Spaces

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1. Introduction

A subset K of X is said to be topologically negligible provided there exists a homeomorphism $h : X \to X \setminus K$. The homeomorphism h is usually required to be the identity outside a given neighborhood U of K. Here X can be a Banach space, a manifold, or just a topological space, but we will only consider the case when X is an infinite-dimensional Banach space and h is a diffeomorphism (recall that points are *not* topologically negligible in finite-dimensional spaces). Such h will be called a deleting diffeomorphism, and we will say that h has its support on U.

Deleting diffeomorphisms are very powerful tools in infinite-dimensional global analysis and nonlinear analysis. We do not intend to make a history of the development of topological negligibility and its applications, and we refer the reader to the introductions of the papers [5, 10, 22] and the references therein for a better insight and a glimpse of the many important applications of smooth negligibility. We should only add to those papers a recent result of the present authors [11]: in every Banach space X with a C^p smooth bump function and a Schauder basis, and for every compact subset K of X and every open set $U \supset K$, there is a C^p diffeomorphism $h : X \to X \setminus K$ so that h is the identity off U.

In this paper we will try to advance towards a solution of the most intriguing problem which remains open in this field despite all recent and old efforts: to provide a characterization of those infinite-dimensional Banach spaces X which admit diffeomorphisms deleting points with a bounded support. That is, if X is an infinite-dimensional Banach space, when does there exist a C^p diffeomorphism $h : X \to X \setminus \{0\}$ so that h is the identity outside a given neighborhood of the origin? As is easily seen, a necessary condition for the existence of such diffeomorphisms h is that X possesses a C^p smooth bump function. Then one could jump to the natural conjecture that this condition is sufficient as well. But nobody has been able to prove this. There are only partial results which could be summed up by saying that if X has a (not necessarily equivalent) smooth norm, or if X has smooth partitions of unity, then the conjecture is true, see [1, 10]. The general case of this problem seems to be quite difficult, because the assumption that X has a smooth bump is quite a weak one. The funny side of all this is that nobody knows of an example of a Banach space X with a C^p smooth bump function which does not fall into the category to which the combined efforts of [1, 10] apply. This is due to the fact that the only known examples of Banach spaces X which have smooth bump functions but do not possess equivalent smooth norms are those of Haydon's [25, 26], and Haydon's spaces do have smooth partitions of unity. In fact, the problem whether every Banach space with a C^p smooth bump has C^p smooth partitions of unity is wide open. If a positive solution to the latter is ever reached then the above conjecture would be true by virtue of the main result of [10]. Unfortunately, an eventual solution to the problem about the characterization of the existence of smooth partitions of unity seems to be even more difficult and far away than that of the existence of deleting diffeomorphisms.

Here we will provide several sufficient conditions for a Banach space X in order to have diffeomorphisms deleting points. These conditions are of a geometric flavor and involve the existence of certain families of smooth starlike bodies in the space X. All of these conditions are very general, and it is very likely that they are necessary as well. We do not know of any Banach space with a C^p smooth bump which does not meet these conditions.

At this point we need to introduce some terminology and notation concerning starlike bodies, which, apart from the statements of our results, play a key role in our proofs. We believe that the class of starlike bodies in Banach spaces is worth a much deeper study than the one it has been given up to now. The number of things that can be proved if one only knows that there are *enough* smooth starlike bodies in a space is somewhat amazing. We hope that this paper will justify these daring statements and will contribute to encourage the analysts and geometers to use and study starlike bodies, both as quite powerful tools and as objects of independent interest.

The next section is devoted to a study of the definitions and basic properties of starlike bodies, including several notions of inclusions between them, and a number of geometrical results (such as the Four Bodies Lemma and the Three Bodies Lemma) which will be used in the proofs of the main theorems of the paper. In section 3 we will state and prove our main theorems concerning the existence of diffeomorphisms deleting points in many Banach spaces.

2. A brief study of starlike bodies

Many of the proofs of the statements and results of this section are relatively easy and will be omitted, but the interested reader will be able to find them in A. Montesinos' PhD thesis [32], available on request by e-mail.

Definitions

A closed subset A of a Banach space X is said to be a starlike body if there exists a point a_0 in the interior of A such that every ray emanating from a_0 meets ∂A, the boundary of A, at most once. We will say that a_0 is a center of A. There can obviously exist many centers for a given starlike body. Up to a suitable translation, we can always assume that $a_0 = 0$ is the origin of X, and we will often do so, unless otherwise stated. For a starlike body A with center a_0, we define the characteristic cone of A as

$$ccA = \{x \in X \colon a_0 + r(x - a_0) \in A \text{ for all } r > 0\},$$

and the Minkowski functional of A with respect to the center a_0 as

$$\mu_{A,a_0}(x) = \mu_A(x) = \inf\{t > 0 \colon x - a_0 \in t(-a_0 + A)\} \quad \text{for all } x \in X.$$

Note that $\mu_A(x) = \mu_{-a_0+A}(x - a_0)$ for all $x \in X$. It is easily seen that μ_A is a continuous function which satisfies $\mu_A(a_0 + rx) = r\mu_A(a_0 + x)$ for every $r \geq 0$ and $x \in X$, and $\mu_A^{-1}(0) = ccA$. Moreover, $A = \{x \in X \colon \mu_A(x) \leq 1\}$, and $\partial A = \{x \in X \colon \mu_A(x) = 1\}$. Conversely, if $\psi : X \to [0, \infty)$ is continuous and satisfies $\psi(a_0 + \lambda x) = \lambda\psi(a_0 + x)$ for all $\lambda \geq 0$, then $A_\psi = \{x \in X \colon \psi(x) \leq 1\}$ is a starlike body with center a_0. More generally, for a continuous function $\psi : X \to [0, \infty)$ such that $\psi_x(\lambda) = \psi(a_0 + \lambda x)$, $\lambda > 0$, is increasing and $\sup\{\psi_x(\lambda) : \lambda > 0\} > \varepsilon$ for every $x \in X \setminus \psi^{-1}(0)$, the set $\psi^{-1}([0, \varepsilon])$ is a starlike body whose characteristic cone is $\psi^{-1}(0) \ni a_0$.

A familiar important class of starlike bodies are *convex bodies*, that is, starlike bodies that are convex. For a convex body U, ccU is always a convex set, but in general the characteristic cone of a starlike body is not convex. Starlike bodies can also be related to n-homogeneous polynomials, since the level sets of such polynomials are always boundaries of starlike bodies.

We will say that A is a C^p smooth starlike body provided its Minkowski functional μ_A is C^p smooth on the set $X \setminus ccA = X \setminus \mu_A^{-1}(0)$. This amounts to saying that ∂A is a C^p smooth one-codimensional submanifold of X such that no affine hyperplane tangent to ∂A contains a ray emanating from the center a_0. Throughout this paper, $p = 0, 1, 2, ..., \infty$, and C^0 smooth means just continuous.

We will also say that A is Lipschitz if μ_A is a Lipschitz function on X. It is easy to see that every *convex* body is Lipschitz with respect to any point in its interior, but this is no longer true if we drop convexity: even in the plane $\mathbb{R}^2$ there are starlike bodies which are not Lipschitz). For instance, it is easy to see that the body

$$A := \{(x, y) \in \mathbb{R}^2 : |x| \leq 1,\ |y| \leq 1,\ (|x| - 1)^2 + (|y| - 1)^2 \geq 1\}$$

is a bounded starlike body which is not Lipschitz.

Most of the starlike bodies that we will deal with in this paper are *radially bounded*. A starlike body A is said to be radially bounded provided that, for every ray emanating from the center a_0 of A, the intersection of this ray with A is a bounded set. This amounts to saying that $ccA = \{a_0\}$.

In finite dimensions every radially bounded starlike body is in fact bounded (because the Minkowski functional of the body attains an absolute minimum on the unit sphere, which is compact), but this is no longer true in infinite-dimensional Banach spaces. For instance, $A = \{x \in \ell_2 : \sum_{n=1}^{\infty} x_n^2/2^n \leq 1\}$ is a radially bounded convex body which is not bounded in the Hilbert space ℓ_2; the body A is the unit ball of the nonequivalent C^∞ smooth norm $\omega(x) = \big(\sum_{n=1}^{\infty} x_n^2/2^n\big)^{1/2}$ in ℓ_2.

For every *bounded* starlike body A in a Banach space $(X, \|\cdot\|)$ there are constants $M, m > 0$ such that

$$m\|x\| \leq \mu_A(x) \leq M\|x\| \text{ for all } x \in X.$$

If A is just *radially bounded* then we can only ensure that

$$\mu_A(x) \leq M\|x\| \quad \text{for all } x \in X,$$

for some $M > 0$.

Starlike bodies and bump functions

The following theorem is implicitly shown in [18, Proposition II.5.1].

THEOREM 2.1. *Let X be a Banach space. The following statements are equivalent:*

1. *X admits a bump function (resp., a Lipschitz bump function) of class C^p.*
2. *There are positive real numbers a, b and a continuous function (resp. Lipschitz) $\psi : X \to [0, \infty)$ which is positively homogeneous and symmetric, of class C^p on $X \setminus \{0\}$, and such that $a\|\cdot\| \le \psi \le b\|\cdot\|$.*
3. *X has a bounded symmetric (resp. and Lipschitz) starlike body of class C^p.*

As a corollary we get a result that links the existence of bumps on a quotient space X/Y to the fact that the distance function to the subspace Y of X can be approximated to some extent by a smooth function whose behavior is equivalent to that of the distance.

COROLLARY 2.2. *Let X be a Banach space, Y a closed subspace of X. Consider the following statements:*

1. *X/Y admits a bump function (resp., Lipschitz bump) of class C^p.*
2. *There are numbers $a, b > 0$ and a continuous (resp. Lipschitz) function $\psi : X \to [0, \infty)$ of class C^p on $X \setminus Y$, which is positively homogeneous and symmetric, and such that $a \cdot \operatorname{dist}(x, Y) \le \psi(x) \le b \cdot \operatorname{dist}(x, Y)$ for all $x \in X$.*
3. *X has a (Lipschitz) symmetric starlike body E of class C^p such that $cc(E) = Y$.*

Then, (1) $\implies$ (2) $\implies$ (3).

Proof. Let $\pi : X \to X/Y$ the natural projection. Assuming (1) is true, by the preceding Theorem 2.1 applied to the quotient X/Y, we get numbers $a, b > 0$ and a function $\Psi : X/Y \to [0, \infty)$ with the right properties so that the function $\psi := \Psi \circ \pi$ satisfies (2). Finally, in order to obtain (3) it is enough to define $E := \{x \in X : \psi(x) \le 1\}$. ∎

Three notions of inclusion

In the proofs of the results of the next section we will often require that the boundaries of certain starlike bodies are well separated. We next define and study three different notions of inclusion between starlike bodies, all of which are quite natural when different separation properties are required.

Definition 2.3. Let A, B be subsets of a Banach space X. We will say that B contains A in distance provided that $\operatorname{dist}(A, X \setminus B) > 0$, and we will write $A \subset_d B$.

Remark 2.4. The statement $A \subset_d B$ is equivalent to the existence of a number $\varepsilon > 0$ such that $A + \varepsilon B_X \subset B$. Assume that A is bounded. Then, for every $x \in X$, the set $-x + A$ is also bounded, hence there exists $\delta > 0$ so that $\delta(-x + A) \subset \varepsilon B_X$. Therefore

$$x + (1 + \delta)(-x + A) \subset A + \delta(-x + A) \subset A + \varepsilon B_X \subset B.$$

This observation leads us to the definition of the other two notions of inclusion.

Definition 2.5. Let A, B be subsets of a Banach space X.

(i) Assume that B is a starlike body with center x_0. We will say that B contains A in the Minkowski sense provided there exists $\delta > 0$ so that $x_0 + (1 + \delta)(-x_0 + A) \subseteq B$, and we will denote $A \subset_\mu B$.

(ii) Assume now that A is a starlike body with center x_0. We will say that A is Δ-contained in B if there is a $\delta > 0$ so that $x_0 + (1 + \delta)(-x_0 + A) \subseteq B$, and in this case we will write $A \subset_\Delta B$.

Remark 2.6. Note that the inclusions $\subset_\mu$ and $\subset_\Delta$ only differ on which of the two subsets is a starlike body with respect to x_0.

The next lemma links the inclusions $\subset_\mu$ and $\subset_\Delta$ to the Minkowski functional of the bodies.

Lemma 2.7. *Let A, B be subsets of a Banach space X.*

(i) *Suppose B is a starlike body with center x_0. Then*

$$A \subset_\mu B \Leftrightarrow \sup\{\mu_B(x) : x \in A\} < 1,$$

where μ_B is the Minkowski functional of B with respect to the center x_0.

(ii) *Suppose that A is a starlike body with center x_0. Then*

$$A \subset_\Delta B \Leftrightarrow \inf\{\mu_A(x) : x \in X \setminus B\} > 1.$$

Proof. The fact that $\sup\{\mu_B(x) \colon x \in A\} < 1$ is equivalent to the existence of $\delta \in (0,1)$ such that $\mu_B \leq 1-\delta$ on A, which is, in turn, obviously equivalente to the existence of $\delta \in (0,1)$ with $\frac{1}{1-\delta}A \subset B$ and, therefore, to $A \subset_\mu B$.

The proof of (ii) is analogous and does not offer any difficulty. ∎

Remark 2.8. If B is a starlike body, the functional μ_B measures the position with respect to B of the points of the space X. Since the interior of B is the set $\{x \in X : \mu_B(x) < 1\}$ and the boundary is given by $\{x \in X : \mu_B(x) = 1\}$, it seems quite natural to single out those subsets A of B which are separated from ∂B, where separation and closeness are given in terms of the values of μ_B.

We next present a special property of Lipschitz starlike bodies.

Lemma 2.9. (i) *Let A be a Lipschitz starlike body (with respect to the origin). Then, for every $\varepsilon > 0$ there exists $\delta > 0$ so that $A + \delta B_X \subset (1+\varepsilon)A$.*

(ii) *Let C be a convex body of a Banach space X with $0 \in \operatorname{int} C$. Then, for every $0 < \delta < 1$, we have $(1-\delta)\, C \subset_d C$.*

Proof. (i) Let M be the Lipschiz constant of μ_A. For a given $\varepsilon > 0$, choose $\delta > 0$ with $\delta M < \varepsilon$. Consider $x = y + z$, with $y \in A$, $z \in \delta B_X$. We have

$$\mu_A(x) = \mu_A(y+z) - \mu_A(y) + \mu_A(y) \leq M\|z\| + \mu_A(y) \leq M\delta + 1 < 1 + \varepsilon.$$

This shows that $A + \delta B_X \subset (1+\varepsilon)A$.

(ii) is immediate from (i) taking into account that every convex body is a Lipschitz starlike body. ∎

The next proposition relates the three notions of inclusion that we have established.

Proposition 2.10. *Let A and B be subsets of a Banach space X.*

(i) *If A is bounded, $A \subset_d B$ and B (resp. A) is a starlike body with center $\xi_0 \in X$, then $A \subset_\mu B$ (resp. $A \subset_\Delta B$).*
(ii) *If A and B are starlike bodies with center $\xi_0 \in X$, A is Lipschitz and $A \subset_\mu B$, then $A \subset_d B$.*

(iii) *If A is a Lipschitz starlike body and $A \subset_\Delta B$, then $A \subset_d B$.*

(iv) *If A and B are starlike bodies with center $\xi_0 \in X$, then $A \subset_\mu B$ if and only if $A \subset_\Delta B$.*

(v) *Assume that X is finite-dimensional, A and B are starlike bodies with the same center $\xi_0 \in X$, and the body A is bounded. Then, $A \subset_\Delta B$ if and only if $A \subset_d B$.*

Proof. (iv) is obvious, and (i) follows from Remark 2.4. Let us prove (ii). By definition, there is $\delta > 0$ so that $\xi_0 + (1+\delta)(-\xi_0 + A) \subset B$. Since A is Lipschitz, by Lemma 2.9, there is $\delta' > 0$ so that $(-\xi_0 + A) + \delta' B_X \subset (1+\delta)(-\xi_0 + A)$. Hence

$$A + \delta' B_X \subset \xi_0 + (1+\delta)(-\xi_0 + A) \subset B$$

y $A \subset_d B$. The proof of (iii) is analogous.

Let us show (v). Since $A \subset_d B$ and A is bounded, (i) implies that $A \subset_\Delta B$. Let us see the converse. Assume that A were not contained in distance in B. Then $\operatorname{dist}(A, X \setminus B) = 0 = \operatorname{dist}(\partial A, \partial B)$. Since ∂A is compact in the finite-dimensional space X, and $\operatorname{dist}(\partial A, \partial B) = 0$, we deduce that $\partial A \cap \partial B \neq \emptyset$. Take $x \in \partial A \cap \partial B$. As $A \subset_\Delta B$ and $x \in \partial B$, by Lemma 2.7, we get that $\mu_A(x) > 1$. This is a contradiction, because $x \in \partial A = \mu_A^{-1}(1)$. ∎

We next see a few examples showing that the relations established among the different notions of inclusions in the above proposition are optimal, in the sense that the results are false if one drops the boundedness assumption of parts (i) and (v), or the Lipschitz hypothesis in parts (ii) and (iii).

Example 2.11. The boundedness of A is essential in part (i) of Proposition 2.10. In the Hilbert space $X = \ell_2(\mathbb{N})$, with the usual norm $\|\cdot\|$, the associated inner product $(\,|\,)$, and the canonical basis $(e_k)_{k=1}^\infty$, let us consider the nonequivalent norm $\omega : X \to [0, \infty)$ defined as

$$\omega(x) = \Big(\sum_{n=1}^{\infty} \frac{x_n^2}{2^n}\Big)^{\frac{1}{2}},$$

for every $x = (x_n)_n \in X$. Its unit ball $A = \{x \in \ell_2(\mathbb{N}) \colon \omega(x) \leq 1\} \subset X$ is a C^∞ smooth symmetric convex body which is radially bounded but not bounded. In a similar way, the body $B = \{x \in \ell_2(\mathbb{N}) \colon \operatorname{dist}(x, A) \leq 1\}$ is a convex body which is not bounded. It is obvious, from the definition, that

$A \subset_d B$. And yet A is not Δ-contained in B (neither is A contained in the Minkowski sense in B). Indeed assume that there were a number $\delta > 0$ with $(1+\delta)A \subset B$. For each $k \in \mathbb{N}$, set $v_k = 2^{k/2} e_k$, and the affine hyperplane $H_k = \{x \in X \colon \big(x|v_k\big) = \big(v_k|v_k\big) = 2^k\}$. Clearly, $(v_k)_k \subset \partial A$. For each $k \in \mathbb{N}$ we have that $A \subset \{x \in X \colon \big(x|v_k\big) \le 2^k\}$, and therefore H_k is a supporting hyperplane of A at the point $v_k \in \partial A$ (note that $v_k \in H_k \cap \partial A$). Since $\big((1+\delta)v_k|v_k\big) > \big(v_k|v_k\big)$, it follows that H_k separates A from the point $(1+\delta)v_k$. Hence,

$$\operatorname{dist}\big((1+\delta)v_k, A\big) \ge \operatorname{dist}\big((1+\delta)v_k, H_k\big) = \operatorname{dist}\big((1+\delta)v_k, v_k\big) = \delta\|v_k\| = \delta 2^{k/2}.$$

By choosing $k_0 \in \mathbb{N}$ big enough, we get $\delta 2^{k_0/2} > 1$ and $\operatorname{dist}\big((1+\delta)v_{k_0}, A\big) > 1$. This means, according to the definition of B, that $(1+\delta)v_{k_0} \notin B$, a contradiction with the fact that $v_{k_0} \in A$ and the assumption that $(1+\delta)A \subset B$.

Example 2.12. In part (v) of Proposition 2.10 the boundedness assumption on A is essential. In $X = \mathbb{R}^2$ let us define the starlike bodies

$$A = \{(x,y) \in \mathbb{R}^2 \colon |xy| \le 1\} \quad \text{and} \quad B = \{(x,y) \in \mathbb{R}^2 \colon |xy| \le 2\}.$$

It is easy to see that $A \subset_\Delta B$ and $A \subset_\mu B$, but $\operatorname{dist}(A, X \backslash B) = \operatorname{dist}(\partial A, \partial B) = 0$, hence it is not true that $A \subset_d B$. In particular, neither A nor B are Lipschitz.

Example 2.13. (A porcupine body) The above example also shows that in parts (ii) and (iii) of Proposition 2.10 the Lipschitz assumption is essential. The bodies A and B cannot be bounded here because the ambient space is finite-dimensional. Next we exhibit a bounded counterexample in infinite dimensions. In the Hilbert space $X = \ell_2$, let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis. For each vector e_n we are going to construct a spike of length 3, directed by the vector e_n, in such a way that the bigger n, the narrower the spike. By gluing all those spikes together with the unit ball of X we will get a C^∞ smooth bounded starlike body A. For some $\delta > 0$, the body $B = (1+\delta)A$ satisfies that $A \subset_\Delta B$ and yet, because of the spiked structure of A, and with a wise choice of $\delta > 0$, one can have A not contained in distance in B. Let us give a formal definition of A. Fix a number

$$\varepsilon \in \Big(0, \frac{2}{3}\sqrt{1 - \Big(\frac{31}{32}\Big)^2}\ \Big),$$

and for each $n \in \mathbb{N}$, define $\varepsilon_n := \varepsilon^n$, and pick a function $g_n : \mathbb{R} \to (0,\infty)$ of class C^∞ satifying the following conditions:

1. $g_n(s) = g_n(-s)$ for each $s \in \mathbb{R}$.
2. $g_n(s) = 1/100$ if $|s| \geq \varepsilon$.
3. $g_n(s) = (1-s^2)^{1/2}$, if $\frac{1}{2}\varepsilon_n \leq s \leq \frac{3}{4}\varepsilon_n$.
4. $g_n(\frac{1}{4}\varepsilon_n) = 2$.
5. $g_n = 3$ on $[-\frac{1}{8}\varepsilon_n, \frac{1}{8}\varepsilon_n]$.
6. $g_n' \leq 0$ on $[0, \infty)$.

For every $m \in \mathbb{N}$, consider the hyperplane $H_m := \{x \in X : x_m = 0\}$ and the decomposition $X = H_m \oplus [e_m] = H_m \times \mathbb{R}$. Define the function $F_m : H_m \to (0, \infty)$ by $H_m \ni h \to g_m(\|h\|)$, and the sets

$$\begin{gathered} E_m = \{h + te_n \in X : h \in H_m,\ t \in \mathbb{R},\ F_m(h) \geq t\}, \\ \{h + te_n \in X : h \in H_m,\ t \in \mathbb{R},\ \|h\| \leq \varepsilon\}, \quad \text{and} \\ \{h + te_n \in X : h \in H_m,\ t \in \mathbb{R},\ t \geq -1/100\}. \end{gathered}$$

It is not difficult to see that these are starlike bodies with respect to the origin. Now let us define

$$U_m := \Big(\big\{h + te_n \in X : h \in H_m,\ t \in \mathbb{R},\ \|h\| \leq \varepsilon,\ t \geq -1/100\big\} \cap E_m\Big) \cup B_X.$$

One can show that U_m is a C^∞ smooth starlike body. Geometrically speaking, U_m is but a ball with a smooth spike of length 3 directed by the vector e_m, with the property that the diameter of the spike of U_m goes to zero as $n \to \infty$.

Then we consider the union of all of these balls with spikes, $A = \bigcup_{m \in \mathbb{N}} U_m$. Because of the construction of U_m, the set A is a C^∞ smooth bounded starlike body. Define $B = \frac{3}{2}A$. It is clear that B is a C^∞ smooth bounded starlike body as well. Finally, it is not difficult to see that $\mathrm{dist}(A, X \setminus B) = 0$, so A is not contained in distance in B. See [32] for the details.

Note that this example also shows the existence of C^∞ smooth bounded starlike bodies which are not Lipschitz (if A and B were Lipschitz then, by (iii) of Proposition 2.10, we would have $A \subset_d B$). This is a distinctive feature of infinite-dimensional spaces: in $\mathbb{R}^n$, every C^1 smooth bounded starlike body is Lipschitz.

Stability of starlike bodies

Now we will see that starlike bodies behave well (are stable) with respect to certain operations, such as radial diffeomorphisms, and inverse images of continuous linear mappings. We say that a mapping $h : X \to X$ is radial of center $x_0 \in X$ provided there exists a function $\eta : X \to [0, \infty)$ such that $h(x) = x_0 + \eta(x - x_0)(x - x_0)$ for every $x \in X$.

LEMMA 2.14. *Let X be a Banach space. Let $h : X \to X$ be a radial homeomorphism with center $x_0 \in X$. Then, for every starlike body E with center x_0, $h(E)$ is also a starlike body with center x_0. Moreover, if h is a C^p diffeomorphism, then h maps C^p starlike bodies with center x_0 onto C^p starlike bodies with center x_0.*

The proof is easy.

LEMMA 2.15. *Let $T : X \to Y$ be a continuous linear mapping between two Banach spaces. If $B' \subset Y$ is a C^p smooth starlike body with center $b' \in T(X)$ and with characteristic cone F', then $B := T^{-1}(B') \subset X$ is a C^p smooth starlike body with center $b \in T^{-1}(b')$ and with characteristic cone $T^{-1}(F')$.*

Proof. Set $b \in T^{-1}(b')$. Consider the function $\psi : X \to [0, \infty)$ defined by $\psi = \mu_{B'} \circ T$. This function is continuous, positively homogeneous (with respect to the point $b \in X$), of class C^p on $X \setminus \psi^{-1}(0)$, and such that $\psi^{-1}(0) = T^{-1}(F')$. Then

$$B := T^{-1}(B') = \{x \in X : \psi(x) \leq 1\}$$

is a C^p smooth starlike body with center b, and $cc(B) = T^{-1}\big(F'\big)$. ∎

LEMMA 2.16. *Let $T : X \to Y$ be a continuous linear mapping between two Banach spaces. Let A' and B' be starlike bodies with respect to a point $y_0 = T(x_0) \in T(X)$ such that $A' \subset_\mu B'$. Then $A := T^{-1}(A') \subset_\mu T^{-1}(B') := B$.*

Proof. According to the proof of the preceding lemma, $\mu_A = \mu_{A'} \circ T$, and $\mu_B = \mu_{B'} \circ T$. Then, for each $x \in A$,

$$\mu_B(x) = \mu_{B'}(T(x)) \leq \sup_{y \in A'} \mu_{B'}(y) < 1$$

because $A' \subset_\mu B'$, hence

$$\sup_{x \in A} \mu_B(x) \leq \sup_{y \in A'} \mu_{B'}(y) < 1.$$

that is $A \subset_\mu B$. ∎

LEMMA 2.17. *Let $T : X \to Y$ be a continuous linear mapping between two Banach spaces. Assume that A' is a starlike body with center $y_0 = T(x_0) \in T(X)$ such that $A' \subset_\Delta B'$. Then $A := T^{-1}(A') \subset_\Delta T^{-1}(B') := B$.*

Proof. By Lemma 2.15, $A := T^{-1}(A') \subset X$ is a starlike body with center $x_0 \in X$, and $A \subset B$. Besides, we have $\mu_A = \mu_{A'} \circ T$. Since $T(X \setminus B) \subset Y \setminus B'$, bearing in mind that $A' \subset_\Delta B'$, and applying Lemma 2.7, we get

$$\begin{aligned} \inf\{\mu_A(x) : x \in X \setminus B\} &= \inf\{\mu_{A'}\big(T(x)\big) : x \in X \setminus B\} \\ &\geq \inf\{\mu_{A'}(\xi) : \xi \in Y \setminus B'\} > 1. \end{aligned}$$

Now, again by Lemma 2.7, we conclude that $A \subset_\Delta B$. ∎

Smooth approximation of starlike bodies

Next we prove that in all Banach spaces with smooth partitions of unity every starlike body can be well approximated by smooth starlike bodies.

Proposition 2.18. *Let X be a Banach space with C^p smooth partitions of unity. Then, for every starlike body C with $cc(C) = \{0\}$ and for every $\delta > 0$, there exists a C^p smooth starlike body A with $cc(A) = \{0\}$ and such that $(1-\delta)C \subset A \subset (1+\delta)C$.*

Proof. Since X has C^p smooth partitions of unity, X has a C^p smooth bump as well, so by Theorem 2.1 there is a C^p smooth bounded symmetric starlike body B in X. Choose $\varepsilon_0 \in (0,1)$ so that

$$\frac{1}{1-\varepsilon_0} < 1+\delta \quad \text{and} \quad 1+\varepsilon_0 < \frac{1}{1-\delta}$$

and define $\varepsilon : X \setminus \{0\} \to (0,\infty)$ as

$$\varepsilon(x) = \varepsilon_0 \mu_C(x) \quad \text{for all } x \neq 0,$$

which is a strictly positive continuous function. Since X has C^p smooth partitions of unity, so does its open subset $X \setminus \{0\}$.

Hence, given the continuous function $\mu_C : X \setminus \{0\} \to (0,\infty)$, there exists a C^p smooth function $g : X \setminus \{0\} \to \mathbb{R}$ such that $|\mu_C(x) - g(x)| \leq \varepsilon(x)$ for all $x \neq 0$. Now define $\psi : X \longrightarrow \mathbb{R}$ by

$$\psi(x) = \mu_B(x) g\big(\frac{x}{\mu_B(x)}\big) \quad \text{if } x \neq 0,$$

and $\psi(0)=0$. The function ψ is clearly continuous on X, ψ is of class C^p on $X\setminus\{0\}$, and ψ is positively homogeneous. Moreover,

$$\begin{aligned}|\psi(x)-\mu_C(x)| &= \big|\mu_B(x)g\big(x/\mu_B(x)\big)-\mu_C(x)\big|\\ &= \big|\mu_B(x)g\big(x/\mu_B(x)\big)-\mu_B(x)\mu_C\big(x/\mu_B(x)\big)\big|\\ &\le \mu_B(x)\varepsilon\big(x/\mu_B(x)\big)\\ &= \varepsilon_0\mu_C(x)\end{aligned}$$

for all $x\neq 0$. In particular, $\psi(x)\ge(1-\varepsilon_0)\mu_C(x)>0$ if $x\neq 0$. Therefore,

$$A:=\{x\in X:\psi(x)\le 1\}$$

is a C^p smooth starlike body with respect to 0. Let us check that A approximates C as required. We have: if $x\in A$ then $(1-\varepsilon_0)\mu_C(x)\le 1$ which implies that $x\in\frac{1}{1-\varepsilon_0}C\subset(1+\delta)C$, so $A\subset(1+\delta)C$. On the other hand, if $x\in(1-\delta)C$, that is, $\mu_C(x)\le 1-\delta$, then we have

$$\psi(x)\le(1+\varepsilon_0)\mu_C(x)\le(1+\varepsilon_0)(1-\delta)<1,$$

hence $x\in A$. ∎

Corollary 2.19. *Let X be a Banach space with C^p smooth partitions of unity, C_1 a bounded starlike body with respect to a point c, and C_2 a mere subset of X such that $C_1\subset_d C_2$. Then there exist D_1 and D_2, C^p smooth starlike bodies with respect to c, which satisfy $C_1\subset_\mu D_1\subset_\mu D_2\subset_d C_2$.*

Proof. We may assume that $c=0$ and $C_2\subset B_X$. Let us pick $\varepsilon>0$ such that $\operatorname{dist}(C_1,X\setminus C_2)\ge\varepsilon$. According to Lemma 2.18, there exists a C^p smooth starlike body with respect to 0, A, satisfying

$$(1-\theta)(1+\varepsilon/2)C_1\subset A\subset(1+\theta)(1+\varepsilon/2)C_1,$$

where θ is any positive number such that $\varepsilon/2-\theta(1+\varepsilon/2)\ge\varepsilon/4$. Define $D_1:=A$. Since $(1-\theta)(1+\varepsilon/2)\ge 1+\varepsilon/4$ we have that $C_1\subset(1+\varepsilon/4)C_1\subset(1-\theta)(1+\varepsilon/2)C_1\subset A=D_1$, and in particular $C_1\subset_\mu D_1$. On the other hand,

$$A=D_1\subset(1+\theta)(1+\varepsilon/2)C_1\subset C_1+(\theta+\varepsilon/2+\theta\varepsilon/2)C_1\subset C_1+(\theta+\varepsilon/2+\theta\varepsilon/2)B_X,$$

which implies that

$$\begin{aligned}\operatorname{dist}(D_1,X\setminus C_2) &\ge \operatorname{dist}\big(C_1+(\theta+\varepsilon/2+\theta\varepsilon/2)B_X,X\setminus C_2\big)\\ &\ge \varepsilon-(\theta+\varepsilon/2+\theta\varepsilon/2)\ge\varepsilon/4.\end{aligned}$$

Define now $D_2 := (1+\varepsilon/8)D_1$. It is obvious that D_2 is a C^p smooth starlike body with respect to 0 satisfying $D_1 \subset_\mu D_2$. Finally, we have that $D_2 = (1+\varepsilon/8)D_1 \subset D_1 + \varepsilon/8 B_X$, and therefore

$$\begin{aligned}\operatorname{dist}\big(D_2, X\setminus C_2\big) &\geq \operatorname{dist}\big(D_1 + \frac{\varepsilon}{8}B_X, X\setminus C_2\big)\\ &\geq \operatorname{dist}\big(D_1, X\setminus C_2\big) - \frac{\varepsilon}{8}\\ &\geq \frac{\varepsilon}{4} - \frac{\varepsilon}{8} = \frac{\varepsilon}{8},\end{aligned}$$

which means that $D_2 \subset_d C_2$. ∎

The Four Bodies Lemma

It is well known that for every two C^p smooth convex bodies with the same characteristic cone there always exists a radial diffeomorphism carrying one body into the other. Next we are going to give a more sophisticated result for starlike bodies. Namely, given an ascending tower of four starlike bodies with the same characteristic cone and such that every body is contained in the interior of the following one, there is a diffeomorphism carrying the second body onto the third one and being the identity inside the first one and outside the fourth one. This will be an invaluable tool for us in order to prove the results of Section 3.

We begin with two technical lemmas.

Lemma 2.20. *Consider the region $\mathcal{G} = \{(a,b,x) \in \mathbb{R}^3 : a < b\}$ of $\mathbb{R}^3$. There exists a C^∞ smooth function $H : \mathcal{G} \longrightarrow [0,1]$ such that:*

1. *$H(a,b,x) = 0$ if $x \leq a$;*
2. *$H(a,b,x) = 1$ if $x \geq b$;*
3. *$\partial H/\partial x \geq 0$.*

Proof. Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ the C^∞ smooth function defined as

$$\alpha(x) = \begin{cases} 0 & \text{if } x \leq 0\\ \exp(-\frac{1}{x}) & \text{if } x > 0.\end{cases}$$

For every couple of numbers $a < b$, let us define

$$\beta_{a,b}(x) = \alpha(x-a)\alpha(b-x) = \begin{cases} 0 & \text{if } x \leq a\\ \exp(-\frac{1}{x-a})\exp(-\frac{1}{b-x}) & \text{if } a \leq x \leq b\\ 0 & \text{if } x \geq b.\end{cases}$$

It is clear that the function $\mathcal{G} \ni (a,b,x) \to \beta_{a,b}(x)$ is of class C^∞. Consider the region $\mathcal{V} := \{(x,y,z,w) \in \mathbb{R}^4 : z < w\}$, and define $F : \mathcal{V} \longrightarrow \mathbb{R}$,

$$F(x,y,z,w) = \int_y^x \beta_{z,w}(t)dt,$$

which is a C^∞ smooth function: this is very easily proved by taking into account that the function $(z,w,t) \to \beta_{z,w}(t)$ is C^∞ smooth, and by applying the Fundamental Theorem of Calculus, and differentiating under the integral. Consider the mappings

$$T_1(a,b,x) = (x,a,a,b), \qquad T_2(a,b,x) = (b,a,a,b);$$
$$H_1 := (F \circ T_1)|_{\mathcal{G}}, \qquad H_2 := (F \circ T_2)|_{\mathcal{G}},$$

which are C^∞ as well. Since H_2 is strictly positive on $\mathcal{G}$, the function $H : \mathcal{G} \longrightarrow \mathbb{R}$ defined as

$$H(a,b,x) = \frac{H_1(a,b,x)}{H_2(a,b,x)} = \frac{\int_a^x \beta_{a,b}(t)dt}{\int_a^b \beta_{a,b}(t)dt}$$

is also of class C^∞. It is immediate to verify that H satisfies conditions (1) and (2). Moreover,

$$\frac{\partial H}{\partial x} = \frac{\beta_{a,b}(x)}{\int_a^b \beta_{a,b}(t)dt} \geq 0,$$

so condition (3) is satisfied as well. ∎

NOTATION 2.21. In the remainder of this paragraph we will be using the following notation. Consider the region $\mathcal{U} := \{(t,s) \in \mathbb{R}^2 : 0 < t < s\}$ of $\mathbb{R}^2$. Let

r_1 be the line passing through the point $(1,1)$ with slope $tg(-\pi/6) = -\frac{1}{\sqrt{3}}$;

r_2 be the line passing through $(1,1)$ with slope $tg(-\pi/3) = -\sqrt{3}$.

For each $(t,s) \in \mathcal{U}$, let $\big(a_1(t,s), a_2(t,s)\big)$ be the point where r_1 intersects with the line passing through the origin and the point (t,s); and let $(b_1(t,s), b_2(t,s))$ be the point of intersection of r_2 with the same line. That is,

$$a_1(t,s) = \frac{t(1+\sqrt{3})}{t+s\sqrt{3}}, \qquad a_2(t,s) = \frac{s(1+\sqrt{3})}{t+s\sqrt{3}}$$
$$b_1(t,s) = \frac{t(1+\sqrt{3})}{s+t\sqrt{3}}, \qquad b_2(t,s) = \frac{s(1+\sqrt{3})}{s+t\sqrt{3}}.$$

It is clear that all of these functions are homogeneous of degree zero, and of class C^∞ on $\mathcal{U}$. Besides, for every $(t,s) \in \mathcal{U}$ we have $0 < (a_1^2 + a_2^2)(t,s) < (b_1^2 + b_2^2)(t,s)$.

Let us also define the following regions of the plane:

$$\begin{aligned}
\mathcal{U}_1 &= \{(t,s) \in \mathcal{U} : s \le 1 - (t-1)/\sqrt{3}\} \\
\mathcal{U}_2 &= \{(t,s) \in \mathcal{U} : 1 - (t-1)/\sqrt{3} \le s \le 1 - \sqrt{3}(t-1)\} \\
\mathcal{U}_3 &= \{(t,s) \in \mathcal{U} : 1 - \sqrt{3}(t-1) \le s\},
\end{aligned}$$

that is, the three subregions of $\mathcal{U}$ separated by the lines r_1 and r_2.

Lemma 2.22. *There is a C^∞ smooth function $\phi : \mathcal{U} \longrightarrow \mathbb{R}$ such that:*

1. *$0 \le \phi \le 1$;*
2. *$\phi(t,s) = 0$ if $0 < t^2 + s^2 \le (a_1^2 + a_2^2)(t,s)$, that is, if $(t,s) \in \mathcal{U}_1$;*
3. *$\phi(t,s) = 1$ if $(b_1^2 + b_2^2)(t,s) \le t^2 + s^2$, that is, if $(t,s) \in \mathcal{U}_3$;*
4. *the function $(0,\infty) \ni \lambda \to \phi(\lambda t, \lambda s)$ is nondecreasing for each $(t,s) \in \mathcal{U}$.*

Moreover, the function $\psi := 1 - \phi$ satisfies that $(0,\infty) \ni \lambda \to \psi(\lambda t, \lambda s)$ is nonincreasing, for every $(t,s) \in \mathcal{U}$.

Proof. Consider the region $\mathcal{G} = \{(a,b,x) \in \mathbb{R}^3 : a < b\}$. By Lemma 2.20, there is a C^∞ smooth function $H : \mathcal{G} \longrightarrow [0,1]$ such that $H(a,b,x) = 0$ if $x \le a$; $H(a,b,x) = 1$ if $x \ge b$; and $\partial H/\partial x \ge 0$. Define $\phi : \mathcal{U} \to \mathbb{R}$ as

$$\phi(t,s,\lambda) = H\big((a_1^2 + a_2^2)(t,s), (b_1^2 + b_2^2)(t,s), t^2 + s^2\big),$$

and let $\psi = 1 - \phi$. It is trivial to verify that ϕ and ψ have the required properties. ∎

Now we can state and prove the Four Bodies Lemma.

Lemma 2.23. (The Four Bodies Lemma) *Let X be a Banach space, and let A, B, C and D be four C^p smooth radially bounded starlike bodies with respect to a point x_0, Assume that*

$$A \subset \mathrm{int}(B) \subset B \subset \mathrm{int}(C) \subset C \subset \mathrm{int}(D).$$

Then there exists a C^p diffeomorphism $h : X \to X$ such that:

1. *$h(B) = C$;*
2. *h is the identity on $A \cup (X \setminus D)$;*

3. *h is a radial diffeomorphism; in particular h maps C^p smooth starlike bodies with respect to x_0 onto starlike bodies with the same center.*

Proof. We may assume $x_0 = 0$. Let ϕ and ψ be the functions defined on $\mathcal{U}$ as in Lemma 2.22. Define $f : X \to X$ by

$$f(x) = \Big(\phi\big(\mu_B(x), \mu_A(x)\big)\frac{\mu_B(x)}{\mu_C(x)} + 1 - \phi\big(\mu_B(x), \mu_A(x)\big)\Big)x, \quad \text{if } x \neq 0,$$
$$f(0) = 0.$$

Note that for every $x \neq 0$ we have $\mu_B(x) < \mu_A(x)$ because $A \subset \mathrm{int}(B)$. Let us first see that f is a C^p diffeomorphism of X. Consider the function $G : (X \setminus \{0\}) \times (0, \infty) \longrightarrow \mathbb{R}$ defined by

$$G(y, \lambda) = \Big(\phi\big(\lambda\mu_B(y), \lambda\mu_A(y)\big)\frac{\mu_B(y)}{\mu_C(y)} + 1 - \phi\big(\lambda\mu_B(y), \lambda\mu_A(y)\big)\Big)\lambda$$

for every $y \in X \setminus \{0\}$, and $\lambda \in (0, \infty)$. For a given vector $y \neq 0$, define the partial function $G_y(\lambda) = G(y, \lambda)$ for all $\lambda > 0$. A simple calculation shows that

$$\begin{aligned}\frac{\partial G_y}{\partial \lambda} =& \Big(\phi\big(\lambda\mu_B(y), \lambda\mu_A(y)\big)\frac{\mu_B(y)}{\mu_C(y)} + 1 - \phi\big(\lambda\mu_B(y), \lambda\mu_A(y)\big)\Big) \\ &+ \frac{\partial}{\partial \lambda}\Big(\phi\big(\lambda\mu_B(y), \lambda\mu_A(y)\big)\Big)\Big(\frac{\mu_B(y)}{\mu_C(y)} - 1\Big).\end{aligned}$$

Fix $y \in X \setminus \{0\}$. From the above expression for $\frac{\partial G_y}{\partial \lambda}$, and taking into account (4) of Lemma 2.22, we deduce that

$$\begin{aligned}\frac{\partial G_y}{\partial \lambda} &\geq \Big(\phi\big(\lambda\mu_B(y), \lambda\mu_A(y)\big)\frac{\mu_B(y)}{\mu_C(y)} + 1 - \phi\big(\lambda\mu_B(y), \lambda\mu_A(y)\big)\Big) \\ &\geq \min\Big\{1, \frac{\mu_B(y)}{\mu_C(y)}\Big\} = 1,\end{aligned}$$

and therefore the function $\lambda \in (0, \infty) \to G_y(\lambda)$ is strictly increasing. This function also satisfies $\lim_{\lambda \to 0^+} G_y(\lambda) = 0$, and $\lim_{\lambda \to \infty} G_y(\lambda) = \infty$. It follows that for every $y \in X \setminus \{0\}$, there exists a unique $\lambda(y) > 0$ for which $G_y(\lambda(y)) = 1$, that is, $f(\lambda(y)y) = y$. Hence $f : X \setminus \{0\} \longrightarrow X \setminus \{0\}$ is a bijection, with $f^{-1}(y) = \lambda(y)y$ for all $y \neq 0$. Since $f(0) = 0$, the mapping $f : X \longrightarrow X$ is a bijection.

Now, for every $y_0 \neq 0$ and $\lambda_0 > 0$, we have

$$\frac{\partial G}{\partial \lambda}(y_0, \lambda_0) = \frac{\partial G_y}{\partial \lambda}(y_0, \lambda_0) \geq 1,$$

hence, by the Implicit Function Theorem, $y \to \lambda(y)$ is of class C^p on $X \setminus \{0\}$, and so is f^{-1}.

In view of the properties of ϕ, if $0 < \mu_A(x) \leq 1$, then

$$\big(\mu_B(x), \mu_A(x)\big) \in \mathcal{U}_1 \subset \phi^{-1}(0),$$

and $f(x) = x$. Therefore, f is the identity on A, which is a neighborhood of 0.

From all of this we get that $f : X \longrightarrow X$ is as C^p diffeomorphism. Moreover, when $x \notin B$ we have

$$\big(\mu_B(x), \mu_A(x)\big) \in \mathcal{U}_3 \subset \phi^{-1}(1),$$

so that, $f(x) = (\mu_B(x)/\mu_C(x))x$, and $f(X \setminus B) = X \setminus C$ and $f(B) = C$.

Now let us consider the mapping $g : X \setminus \{0\} \to X \setminus \{0\}$ defined by

$$g(x) = \Big(\psi\big(\mu_D(x), \mu_C(x)\big)\frac{\mu_C(x)}{\mu_B(x)} + 1 - \psi\big(\mu_D(x), \mu_C(x)\big)\Big)x, \text{ if } x \neq 0.$$

An analogous argument to the one used for f shows that g is a C^p smooth self-diffeomorphism of $X \setminus \{0\}$ and g is the identity on $X \setminus D$. Let us collect the properties of the mappings f and g:

$A \xrightarrow{f} A : x \to x$
$X \setminus B \xrightarrow{f} X \setminus C : x \to \frac{\mu_B(x)}{\mu_C(x)}x$
$f(B) = C$
$X \setminus D \xrightarrow{g} X \setminus D : x \to x$
$C \setminus \{0\} \xrightarrow{g} B \setminus \{0\} : x \to \frac{\mu_C(x)}{\mu_B(x)}x$
$g(X \setminus C) = X \setminus B.$

Next we study the behavior of f and g on ∂B. For a given point $x \in \partial B$, we have $1 = \mu_B(x) < \mu_A(x)$, so $(\mu_B(x), \mu_A(x)) \in \mathrm{int}(\mathcal{U}_3) \subset \mathrm{int}(\phi^{-1}(1))$. Bearing in mind that μ_B and μ_A are continuous, we get a $\delta_x > 0$ so that

$$\big(\mu_B(\xi), \mu_A(\xi\big) \in \mathrm{int}(\mathcal{U}_3) \subset \mathrm{int}(\phi^{-1}(1))$$

for every $\xi \in X$ with $\|\xi - x\| \leq \delta_x$. Then for every $x \in \partial B$ there exists $\delta_x > 0$ such that

$$\|\xi - x\| \leq \delta_x \Rightarrow f(\xi) = \frac{\mu_B(\xi)}{\mu_C(\xi)}\xi \ .$$

Take again $x \in \partial B$. Since g is a C^p diffeomorphism $X \setminus \{0\}$ and $g(\partial C) = \partial B$, there exists $y \in \partial C$ such that $g(y) = x$. By a similar reasoning to the above, there exists an open neighborhood V^y of y such that $g(\xi) = \frac{\mu_C(\xi)}{\mu_B(\xi)}\xi$ whenever $\xi \in V^y$. It follows that

$$g^{-1}(\xi) = \frac{\mu_B(\xi)}{\mu_C(\xi)}\xi \ \text{ if } \xi \in W^x,$$

where $W^x := g(V^y)$, which is an open neighborhood of x. Summing up,

($*$) For all $x \in \partial B$ there exists U^x, open neighborhood of $x \in \partial B$, such that $f = g^{-1}$ on U^x.

Now we can define the diffeomorphism h of the statement. Let $h : X \to X$ be defined as

$$h(x) = \begin{cases} f(x) & \text{if } \mu_B(x) \leq 1 \\ g^{-1}(x) & \text{if } \mu_B(x) \geq 1. \end{cases}$$

Since $f : X \to X$ and $g : X\backslash\{0\} \to X\backslash\{0\}$ are C^p, and taking ($*$) into account, it is clear that h is well defined and is, at least locally, a C^p diffeomorphism. Let us see that h is a bijection. By the definition,

$$h(\text{int}(B)) = f(\text{int}(B)) = \text{int}(C), \qquad h(X \setminus B) = g^{-1}(X \setminus B) = X \setminus C,$$

so that $h(\text{int}(B)) \cap h(X \setminus B) = \emptyset$, and h is an injection. That h is onto follows from the facts that $h(B) = f(B) = C$ and $h(X \setminus B) = g^{-1}(X \setminus B) = X \setminus C$. Since h is locally a diffeomorphism and is bijective, h is a diffeomorphism.

It is clear that h satisfies conditions (1) and (2). Furthermore, since f and g are radial mappings, so is h, and, by Lemma 2.14, (3) is true. ∎

Remark 2.24. It is clear from the above proof that in the statement of the Four Bodies Lemma one can drop the assumption that the bodies are radially bounded. It is enough to demand that they all have the same characteristic cone.

Remark 2.25. From the above lemma it is easily deduced that every two C^p smooth starlike body with the same characteristic cone are C^p diffeomorphic.

Smooth squares

A smooth square is an ordinary square V in $\mathbb{R}^2$, but with rounded corners, so that it becomes a C^∞ smooth symmetric convex body that is very useful to combine two different convex (or starlike) bodies A and B of different subspaces Y and Z of a Banach space $X = Y \oplus Z$ into a new body D (defined by $\mu_D(y,z) = \mu_V(\mu_A(y), \mu_B(z)))$, which is equivalent to the sum (or sup) combination of the old bodies, with no loss of smoothness.

Definition 2.26. A subset $\mathcal{V}$ of the plane $\mathbb{R}^2$ is a smooth square provided that:

(i) $\mathcal{V} \subset \mathbb{R}^2$ is a C^∞ smooth bounded convex body, with the origin as an interior point.
(ii) $(y,z) \in \partial\mathcal{V} \Leftrightarrow (\epsilon_1 y, \epsilon_2 z) \in \partial\mathcal{V}$ for each couple $(\epsilon_1, \epsilon_2) \in \{-1,1\}^2$.
(iii) $[-\frac{1}{2}, \frac{1}{2}] \times \{-1,1\} \cup \{-1,1\} \times [-\frac{1}{2}, \frac{1}{2}] \subset \partial\mathcal{V}$.
(iv) $\mathcal{V} \subset [-1,1] \times [-1,1]$.

The following lemma enumerates the essential properties of a smooth square. Property (4), which may seem rather technical, means that the unit sphere of V is locally flat and orthogonal to the axes on a neighborhood of the intersection of $\partial\mathcal{V}$ with the lines $\{x = 0\} \cup \{y = 0\}$. This property is fundamental in order not to lose differentiability properties when combining starlike bodies by means of the smooth square, as we will soon see.

Lemma 2.27. *Let $\mathcal{V} \subset \mathbb{R}^2$ be a smooth square. Then its Minkowski functional $\mu_\mathcal{V} : \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ smooth norm on $\mathbb{R}^2$ such that*

1. $\mu_\mathcal{V} \leq \|\cdot\|_1 \leq 2\mu_\mathcal{V}$.
2. $\|\cdot\|_\infty \leq \mu_\mathcal{V} \leq 2\|\cdot\|_\infty$.
3. $\mu_\mathcal{V}(0,z) = |z|, \quad \mu_V(y,0) = |y|$.
4. *For every $(y,z) \in (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$, there exists $\sigma > 0$ so that $\mu_\mathcal{V}(y',z') = |y'|$ if $\|(y'-y, z')\|_\infty \leq \sigma$ and $\mu_\mathcal{V}(y',z') = |z'|$ if $\|(y', z'-z)\|_\infty \leq \sigma$.*
5. *The functions $t \in (0,\infty) \to \mu_\mathcal{V}(y,tz)$ and $t \in (0,\infty) \to \mu_\mathcal{V}(ty,z)$ are both nondecreasing.*

Proof. It is clear that μ_V is a C^∞ smooth norm. Since

$$[-\tfrac{1}{2},\tfrac{1}{2}]\times[-\tfrac{1}{2},\tfrac{1}{2}]\subset\mathcal{V}\subset[-1,1]\times[-1,1],$$

we have that $\|\cdot\|_\infty\le\mu_{\mathcal{V}}\le 2\|\cdot\|_\infty$, that is (2). In a similar way one can show (1). Property (3) is immediately deduced from the fact that the points $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$ belong to $\partial\mathcal{V}$.

Let us show (4). Let $\sigma := 1/4\min\{|y|,|z|\}$, which is a strictly positive number. If $\|(y'-y,z')\|_\infty\le\sigma$, then

$$\frac{|z'|}{|y'|}\le\frac{\sigma}{|y|-\sigma}\le\frac{\frac{1}{4}|y|}{|y|-\frac{1}{4}|y|}=\frac{1}{3}<\frac{1}{2},$$

so that

$$\Big(\frac{y'}{|y'|},\frac{z'}{|y'|}\Big)\in\{-1,1\}\times\Big[-\frac{1}{2},\frac{1}{2}\Big]\subset\partial\mathcal{V},$$

and therefore $\mu_V(y',z')=|y'|$. By symmetry it is clear that $\mu_{\mathcal{V}}(y',z')=|z'|$ if $\|(y',z'-z)\|_\infty\le\sigma$.

Finally, let us check (5). Pick $(y,z)\in\mathbb{R}^2$. The function $g:[0,\infty)\to\mathbb{R}$ defined by $g(t)=\mu_{\mathcal{V}}(y,tz)$ is obviously convex. Therefore, in order to prove that g is nondecreasing, it is enough to see that g has a minimum at the point $t=0$. Thanks to the symmetry properties of the smooth square this is obvious: for each $v\in\mathbb{R}$, we have that

$$\mu_{\mathcal{V}}(y,0)=\mu_{\mathcal{V}}\Big(\tfrac{1}{2}(y,v)+\tfrac{1}{2}(y,-v)\Big)\le\tfrac{1}{2}\mu_{\mathcal{V}}(y,v)+\tfrac{1}{2}\mu_{\mathcal{V}}(y,-v)=\mu_{\mathcal{V}}(y,v),$$

that is g attain its minimum at 0. ∎

Now we can state and prove the above mentioned result about combining Minkowski functionals into equivalent functionals without losing smoothness properties.

LEMMA 2.28. *Let $X=Y\oplus Z$ be a Banach space, $\rho_Y:Y\to\mathbb{R}$ and $\rho_Z:Z\to\mathbb{R}$ continuous functionals which are of class C^p on $Y\setminus\{0\}$ and $Z\setminus\{0\}$, respectively. Then the functional $\rho:X=Y\oplus Z\to\mathbb{R}$ defined by*

$$\rho(x)=\rho(y,z)=\mu_{\mathcal{V}}(\rho_Y(y),\rho_Z(z))$$

is of class C^p on $X\setminus\{0\}$, where $\mathcal{V}$ is any smooth square of $\mathbb{R}^2$.

Proof. It is clear that ρ is of class C^p on the set $\{(y,z) \in X : y \neq 0 \neq z\}$. Now take a point (y_0, z_0) in $(\{y = 0\} \cup \{z = 0\}) \setminus \{0\}$. Suppose that $y_0 \neq 0 = z_0$. From the continuity of ρ_Y y ρ_Z, and from property (4) of Lemma 2.27, there exist a neighborhood of the point $(y_0, z_0) = (y_0, 0)$, contained in $\{(y,z) : y \neq 0\}$, and a number $\kappa > 0$ so that

$$\rho(y,z) = \kappa \rho_Y(y)$$

in such neighborhood. This shows that ρ is C^p smooth on $\{(y,0) : y \neq 0\}$. By symmetry it follows that ρ is also of class C^p on $\{(0,z) : z \neq 0\}$. Hence ρ is of class C^p on $X \setminus \{0\}$. ∎

Translating diffeomorphisms with bounded supports

Here we will see that if a Banach space X admits a C^p smooth bump function then, for every two points $a, b \in X$, and for every $R > 0$ there exists a C^p diffeomorphism $\phi : X \to X$ which is the identity outside the set $\{x \in X : \text{dist}\big(x, [a,b]\big) \leq R\}$, and which behaves like a translation directed by the vector $b - a$ when restricted to balls of center a and radius r (where $r < R$ in a ratio depending only on the space X but not on the points a, b). These translating diffeomorphisms with bounded supports are essential tools for us to prove the Three Bodies Lemma of the following paragraph, which in turn plays a key role in the proofs of our main theorems of Section 3.

We first show the existence of such diffeomorphisms when $X = \mathbb{R}^2$. In this case the diffeomorphism is obtained by means of a global flow associated to a vector field.

Lemma 2.29. *Let $a \neq b$ be two points of $\mathbb{R}^2$, $\varepsilon > 0$. Define $C = \{x \in \mathbb{R}^2 : \text{dist}(x, [a,b]) \leq 2\varepsilon\}$, where $[a,b]$ is the segment joining a to b. Then there exists a C^∞ diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

1. *$h(x) = x + (b - a)$ for each $x \in B(a, \varepsilon)$, and in particular $h(B(a,\varepsilon)) = B(b, \varepsilon)$;*
2. *h is the identity on $\mathbb{R}^2 \setminus C$;*
3. *h is of the form $h(x) = x + g(x)(b - a)$, where $g : \mathbb{R}^2 \to \mathbb{R}$; therefore, h takes each line parallel to the segment $[a,b]$ onto itself.*

Proof. With no loss of generality we may assume that $a = 0$ and $b = (b_1, 0)$, with $b_1 > 0$. Define

$$C_1 = \{x \in \mathbb{R}^2 : \text{dist}(x, [a,b]) \leq \varepsilon\}.$$

By the smooth version of Urysohn's Lemma, there exists a C^∞ smooth function $f : \mathbb{R}^2 \to [0,1]$ such that $f = 0$ on $\mathbb{R}^2 \setminus C$ and $f = 1$ on C_1. Now, for each $y = (y_1, y_2) \in \mathbb{R}^2$, consider the ordinary differential equation

$$\begin{cases} x_1'(t) = f(x_1(t), x_2(t)) \\ x_2'(t) = 0 \\ x_1(0) = y_1, \quad x_2(0) = y_2 \end{cases}$$

It is easily seen that this equation has a unique solution $x^{(y)}$ defined on the whole of $\mathbb{R}$. A closer look at this equation reveals that

(i) if $y \notin C$, then $x^{(y)}(t) = y$ for each $t \in \mathbb{R}$;
(ii) if $y \in B(a, \varepsilon)$, then $x^{(y)}(t) = (t + y_1, y_2)$ whenever $0 \le t \le b_1$;
(iii) $x^{(y)}$ is of the form $x^{(y)}(t) = (x_1^{(y)}(t), y_2)$.

The family of solutions $\{x^{(y)} : y \in \mathbb{R}^2\}$ are the integral curves of the vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x_1, x_2) = (f(x_1, x_2), 0)$. Since F is bounded, of class C^∞, Lipschitz, and vanishes outside the bounded set C, it is clear that F generates a unique uniparametric group of diffeomorphisms of $\mathbb{R}^2$. That is, there exists a unique global flow $\Psi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of class C^∞ associated to the vector field F (see [31, Lemma 2.4] o [29, Proposition IV.2.5]). This means that for every $t \in \mathbb{R}$, the mapping $\Psi_t : \mathbb{R}^2 \to \mathbb{R}^2, x \to \Psi(t, x)$ is a C^∞ diffeomorphism with inverse $(\Psi_t)^{-1} = \Psi_{-t}$, and $\Psi_{s+t} = \Psi_s \circ \Psi_t$ whenever $s, t \in \mathbb{R}^2$. Moreover, $\Psi(t, y) = x^{(y)}(t)$ for each $(y, t) \in \mathbb{R}^2 \times \mathbb{R}$.

Let us define $h := \Psi_{b_1}$. Bearing in mind the properties (i)-(ii)-(iii) above, it is easily seen that $h : X \to X$ is a C^∞ diffeomorphism satisfying properties (1) and (2) of the statement. To check (3) it is enough to write

$$h(y) = \Psi_{b_1}(y) = x^y(b_1) = (x_1^{(y)}(b_1), y_2) = (y_1, y_2) + \tfrac{1}{b_1} x_1^{(y)}(b_1)(b - a),$$

and define $g(y) = x_1^{(y)}(b_1)/b_1$. ∎

Now we can pass to the case of a general Banach space X.

Proposition 2.30. *Let X be a Banach space having a C^p smooth bump function. There exists a constant $K > 1$, depending only on the space X, such that for every pair of points $a, b \in X$ and every $R > 0$, there exists a C^p diffeomorphism $\phi : X \to X$ satisfying:*

1. *$\phi(x) = x + (b - a)$ for every $x \in B(a, r)$, if $0 < r \le R/K$;*

2. *ϕ is the identity on $X \setminus C$, where $C := \{x \in X : \mathrm{dist}(x, [a,b]) \leq R\}$;*
3. *ϕ is of the form $\phi(x) = x + g(x)(b-a)$, and in particular ϕ maps each line parallel to the segment $[a,b]$ onto itself.*

Proof. We may assume that $a = 0$. Since X has a C^p smooth bump function, there exists a C^p smooth symmetric bounded starlike body A (see Theorem 2.1). Because A is bounded and has the origin as an interior point, there are numbers m_1 y M_1 such that $m_1\|x\| \leq \mu_A(x) \leq M_1\|x\|$ for all $x \in X$.

Set $v = b/\|b\|$. By the Hahn-Banach theorem, we may choose $v^* \in X^*$ so that $\|v^*\| = \|v\| = v^*(v) = 1$, and we may decompose the space as $X = [v] \oplus H$, with $H = \mathrm{Ker}(v^*)$. In the remainder of the proof we identify X to $\mathbb{R} \times H$, with the norm

$$\rho(t,x) = \left[t^2 + \|x\|^2\right]^{1/2},$$

which is equivalent to the original norm $\|\cdot\|$ of X. Thus each $\xi \in X$ is written as $(t,x) \in \mathbb{R} \times H$, with $\xi = tv + x = t(b/\|b\|) + x$.

Next we introduce a new starlike body D. Let, for every $(t,x) \in \mathbb{R} \times H = X$

$$\eta(t,x) = \left[t^2 + \mu_A(x)^2\right]^{1/2}.$$

Then η is the Minkowski functional of a starlike body D, which is bounded, symmetric and of class C^1, defined as $D := \{(t,x) \in X : \eta(t,x) \leq 1\}$. The body D is equivalent to $\|\cdot\|$, in the sense that there are positive constants m_2 and M_2 so that $m_2\|x\| \leq \mu_D(x) \leq M_2\|x\|$ for every $x \in X$.

From the equivalence of the norms $\|\cdot\|$ and ρ and the Minkowski functionals of A and D, it follows that there is a constant $M \geq 1$ such that $1/M\|x\| \leq \mu_A(x) \leq M\|x\|$, $1/M\|x\| \leq \mu_D(x) \leq M\|x\|$ and $1/M\|x\| \leq \rho(x) \leq M\|x\|$, for all $x \in X$. Define $\alpha = R/2M^2$, and take a C^∞ smooth function $\theta : \mathbb{R}^2 \to [0, \infty)$ so that

$$\begin{aligned} &\theta(t) = 0 && \text{if } t \leq \alpha/2 \\ &\theta(t) = t - \alpha && \text{if } t \geq 2\alpha \\ &t - \alpha \leq \theta(t) \leq t && \text{if } t \geq 0. \end{aligned}$$

Now fix $\varepsilon = R/4M^2$. According to Lemma 2.29, there exists a C^∞ diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying:

(i) h translates the ball $B((0,0),\varepsilon)$ onto the ball $B((\|b\|,0),\varepsilon)$.
(ii) h is the identity outside the set $\{(t,s) \in \mathbb{R}^2 : \mathrm{dist}((t,s),[0,\|b\|] \times \{0\} \leq 2\varepsilon\}$.

Now we can define a diffeomorphism $\phi : X = \mathbb{R} \times H \longrightarrow X$ by

$$\phi(t, x) = \big(h_1(t, (\theta \circ \mu_A)(x)), x\big),$$

where h_1 is the first coordinate function of the diffeomorphism $h = (h_1, h_2)$. It is immediately checked that $\phi : X \longrightarrow X$ is a bijection with inverse

$$\phi^{-1}(t, x) = \Big((h^{-1})_1\big(t, (\theta \circ \mu_A)(x)\big), x\Big),$$

where $(h^{-1})_1$ is the first coordinate function of the inverse h^{-1} of h. Clearly, both ϕ and ϕ^{-1} are C^p smooth, and therefore ϕ is a C^p smooth diffeomorphism. It only remains to check that ϕ satisfies the required properties.

Let us begin with (2). Assume that $\mathrm{dist}\big((t, x), [a, b]\big) > R$. Then, for each $s \in [0, \|b\|]$, the point of coordinates $(0, s)$ in $X = H \times \mathbb{R}$ belongs to the segment $[a, b] \subset X$, and therefore

$$\begin{aligned} R^2 &< \|(t, x) - (s, 0)\|^2 \leq M\rho\big((t, x) - (s, 0)\big)^2 \\ &= M[(t - s)^2 + \|(0, x)\|^2] \leq M[(t - s)^2 + M^2 \mu_A(x)^2] \\ &\leq M^4[(t - s)^2 + \mu_A(x)^2]. \end{aligned}$$

Hence

$$[(t - s)^2 + \mu_A(x)^2]^{\frac{1}{2}} > \frac{R}{M^2} \quad \text{for every } s \in [0, \|b\|].$$

This means that

$$\mathrm{dist}_{\mathbb{R}^2}\Big(\big(t, \mu_A(x)\big), [0, \|b\|] \times \{0\}\Big) > \frac{R}{M^2}.$$

Since the function $\mathrm{dist}_{\mathbb{R}^2}(\cdot, [0, \|b\|] \times \{0\})$ is 1-Lipschitz 1, and $|\theta(s) - s| \leq \alpha$ whenever $s \geq 0$, it follows that

$$\begin{aligned} &\mathrm{dist}_{\mathbb{R}^2}\Big(\big(t, \theta(\mu_A(x))\big), [0, \|b\|] \times \{0\}\Big) \\ &\qquad \geq \mathrm{dist}_{\mathbb{R}^2}\Big(\big(t, \mu_A(x)\big), [0, \|b\|] \times \{0\}\Big) - \alpha \\ &\qquad \geq \frac{R}{M^2} - \alpha \geq 2\varepsilon, \end{aligned}$$

and this implies $h_1\Big(t, \theta(\mu_A(0, x))\Big) = t$, whence $\phi(t, x) = (t, x)$. This shows that ϕ is the identity off C.

Now define $K := 4M^3$ and let us see that, for every choice of r with $0 < r \le R/K$, ϕ satisfies (1). Take $(t,x) \in B\big((0,0),r\big)$. Since $0 \le \theta(\mu_A(x)) \le \mu_A(x)$, we can estimate

$$
\begin{aligned}
r \ge \|(t,x)\| &\ge \frac{1}{M}\mu_D(t,x) = \frac{1}{M}(t^2+\mu_A(x)^2)^{\frac{1}{2}} \\
&\ge \frac{1}{M}\Big(t^2+\theta\big(\mu_A(x)\big)^2\Big)^{\frac{1}{2}} = \frac{1}{M}\Big\|\Big(t,\theta(\mu_A(x))\Big)\Big\|_{\mathbb{R}^2},
\end{aligned}
$$

that is

$$
\Big\|\Big(t,\theta(\mu_A(x))\Big)\Big\|_{\mathbb{R}^2} \le rM \le \frac{RM}{K} = \frac{R}{4M^2} = \varepsilon,
$$

which, bearing in mind property (i) of h, implies that $h\big(t,\theta(\mu_A(x))\big) = (\|b\|,0) + (t,\theta(\mu_A(x)))$, and in particular, $h_1\big(t,\theta(\mu_A(x))\big) = \|b\| + t$, whence

$$
\phi(t,x) = (\|b\|+t,x) = (\|b\|,0) + (t,x) = b + (t,x).
$$

This shows that ϕ translates the ball $B(a,r)$ onto $B(b,r)$. Finally, property (3) is immediate. ∎

The Three Bodies Lemmas

The Four Bodies Lemma 2.23 allows to work with towers of starlike bodies with the same center. However, it tells us nothing when the centers are different. In this paragraph we will see how we can combine the Four Bodies Lemma and the preceding proposition 2.30 to obtain the Three Bodies Lemma, a result that allows to transform every radially bounded starlike body onto another one by means of a diffeomorphism which restricts to the identity outside a given starlike body containing the other two bodies.

We begin with a first (seemingly weaker) version of the result.

Lemma 2.31. (The Three Bodies Lemma, first version) *Let X be a Banach space with a C^p smooth bump function. Let P, Q, L be three subsets of X, and a, b two points of X satisfying the following conditions:*

(i) *P and L are C^p smooth radially bounded starlike bodies with center a.*
(ii) *Q is a radially bounded C^p starlike body with center b.*
(iii) *$P \subset \operatorname{int}(L)$.*
(iv) *$Q \subset_\Delta L$.*

Then there exists a C^p diffeomorphism $\Psi : X \longrightarrow X$ such that:

1. $\Psi(P) = Q$;
2. Ψ *is the identity on* $X \setminus L$;
3. *if* $S \subset P$ *is a starlike body with center* $a \in X$, *then* $\Psi(S) \subset Q$ *is a starlike body with center* $b \in X$.

Proof. Since Q is Δ-contained in L, there exists $\delta > 0$ such that $b + (1 + \delta)(-b + Q) \subset L$. As L is starlike with respect to $a \in X$ and $b \in \mathrm{int}(Q) \subset \mathrm{int}(L)$, the segment $[a, b]$ is a compact subset of $\mathrm{int}(L)$, and therefore there exists $R > 0$ such that

$$\mathcal{C} := \{x \in X : \mathrm{dist}(x, [a, b]) \leq R\} \subset L$$
$$B(a, R) \subset P, \quad B(b, R) \subset Q.$$

By Lemma 2.30, there exist $0 < r < R$ and a C^p diffeomorphism $\phi : X \longrightarrow X$ such that ϕ is the identity on $X \setminus \mathcal{C}$ and ϕ is a translation directed by $b - a$ when restricted to the ball $B(a, r)$.

Take a C^p smooth bounded starlike body $E \subset X$ with center a and such that $E \subset B(a, r)$ (this body exists because the space X has a C^p smooth bump function). Since $E \subset B(a, r) \subset \mathrm{int}(B(a, R)) \subset \mathrm{int}(P) \subset P \subset \mathrm{int}(L)$, we can consider the tower of starlike bodies

$$a + \tfrac{1}{2}(-a + E) \subset \mathrm{int}(E) \subset E \subset \mathrm{int}(P) \subset P \subset \mathrm{int}(L),$$

and apply the Four Bodies Lemma to get a radial diffeomorphism $h_1 : X \to X$ of class C^p so that $h_1(P) = E$ and h_1 is the identity on $X \setminus L$.

It is clear that $\phi(E) = E + (b - a)$ is a C^p smooth bounded starlike body with center b. Besides, we have

$$\begin{aligned} &\phi(E) \subset (b - a) + B(a, r) = B(b, r) \subset \mathrm{int}(B(b, R)) \subset \mathrm{int}(Q) \\ &\subset Q \subset \mathrm{int}\left[b + (1 + \delta)(-b + Q)\right] \subset \left[b + (1 + \delta)(-b + Q)\right] \subset L. \end{aligned}$$

By applying the Four Bodies Lemma again 2.23, this time to the tower

$$b+\tfrac{1}{2}\big(-b+\phi(E)\big) \subset \mathrm{int}(\phi(E)) \subset \phi(E) \subset \mathrm{int}(Q) \subset Q \subset \mathrm{int}\Big(b+(1+\delta)(-b+Q)\Big),$$

we deduce the existence of a radial diffeomorphism $h_2 : X \to X$ of class C^p so that $h_2(\phi(E)) = Q$ and h_2 is the identity on $X \setminus [b + (1 + \delta)(-b + Q)] \supset X \setminus L$.

Define $\Psi := h_2 \circ \phi \circ h_1$. It is obvious that Ψ is a C^p diffeomorphism so that $\Psi(P) = Q$ and Ψ is the identity on $X \setminus L$ (bear in mind that h_1 and h_2 are the identity on $X \setminus L$, $\mathcal{C} \subset L$ and ϕ is the identity on $X \setminus \mathcal{C}$). The rest of the properties are also trivial. ∎

Remark 2.32. As Q and L do not have the same center, one cannot apply the Four Bodies Lemma directly. To do so, there must be some room between Q and L so that we can put a little enlargement of Q inside L. This roominess is provided by the Δ-inclusion hypothesis.

Now we can deduce the second form of the Three Bodies Lemma.

Corollary 2.33. (The Three Bodies Lemma, second version) *Let X be a Banach space with a C^p smooth bump function. Let L_1, L_2, L_3 be three subsets of X satisfying:*

(i) *L_1, L_2 y L_3 are radially bounded C^p smooth starlike bodies with centers $x_1 \in X$, $x_2 \in X$ and $x_3 \in X$ respectively.*
(ii) *Both L_1 and L_2 are Δ-contained in L_3 (see Definition* 2.5).

Then there exists a C^p diffeomorphism $\Psi : X \longrightarrow X$ such that

1. $\Psi(L_1) = L_2$;
2. *Ψ is the identity on $X \setminus L_3$;*
3. *if $S \subset L_1$ is a starlike body with center $x_1 \in X$, then $\Psi(S) \subset L_2$ is a starlike body with center $x_2 \in X$.*

Proof. By applying a suitable contraction of center $x_3 \in X$ and ratio less than one to the body L_3 we obtain a radially bounded C^p smooth starlike body C with center $x_3 \in X$ and such that $C \subset \mathrm{int}(L_3)$. By the preceding Lemma 2.31, taking $P = C$, $Q = L_1$ and $L = L_3$, we deduce the existence of a C^p diffeomorphism $\Psi_1 : X \to X$ such that $\Psi_1(C) = L_1$ and Ψ_1 is the identity outside L_3. A similar argument proves the existence of a C^p diffeomorphism Ψ_2 such that $\Psi_2(C) = L_2$ and Ψ_2 is the identity outside L_3. Then the diffeomorphism $\Psi := \Psi_2 \circ \Psi_1^{-1}$ satisfies what we want. ∎

The reader might want to consult the references [2, 4, 7, 9] for other properties of starlike bodies.

3. Diffeomorphisms deleting points in Banach spaces

The following result shows that in a Banach space X there are always C^p diffeomorphisms deleting points, provided that the space X is rich enough in C^p smooth starlike bodies.

Theorem 3.1. *Let X be a Banach space. Assume that there exist a bounded C^p smooth starlike body $B \subset X$, a family $(A_n)_{n\geq 1}$ of subsets of X, and a sequence $(a_n)_{n\geq 1}$ of points of X such that, for every n :*

(i) *A_n is a radially bounded C^p smooth starlike body with center a_n;*
(ii) *$A_n \subset_\Delta A_{n-1}$;*
(iii) *$\bigcap_{k=1}^\infty A_k = \emptyset$.*

Then, for each $\varepsilon > 0$, there exists a C^p diffeomorphism $\Phi_\varepsilon : X \longrightarrow X \setminus \{0\}$ which is the identity on $X \setminus \varepsilon B_X$.

Proof. In the proofs of the results of this section we will use the following notation.

Notation 3.2. For a family of mappings $(g_i)_{i=1}^n$ from the space X into itself, the symbol $\bigcirc_{i=1}^n g_i$ will stand for the composite mapping $g_n \circ g_{n-1} \circ \ldots \circ g_2 \circ g_1$.

We may assume that $B \subset \varepsilon B_X \subset \mathrm{int}(A_1)$ and the origin is a center of both A_1 and B. In particular $a_1 = 0$. For each $n \geq 1$, let us define $B_n = \frac{1}{n}B$, which is a C^p smooth bounded starlike body. It is clear that

$$\bigcap_{n\geq 1} B_n = \{0\}.$$

Assume that there is a family $(\Psi_n)_{n\geq 1}$ of C^p diffeomorphisms so that Ψ_1 is the identity and:

1. $(\bigcirc_{i=1}^n \Psi_i)(B_m)$ is a C^p smooth radially bounded starlike body with center a_n, for each $m \geq n \geq 1$;
2. $\Psi_n(A'_n) = A_n$, where $A'_n := (\bigcirc_{i=1}^{n-1} \Psi_i)(B_n)$, for each $n \geq 2$;
3. if $S \subset A'_n$ is a C^p smooth starlike body with center a_{n-1}, then $\Psi_n(S) \subset A_n$ is a C^p smooth starlike body with center, for each $n \geq 2$;
4. Ψ_n is the identity on $X \setminus A_{n-1}$, for each $n \geq 2$;
5. $A'_n \subset \mathrm{int}(A_{n-1})$, for each $n \geq 2$.

With such a family we can define our diffeomorphism Φ_ε. Indeed, for each $n \geq 2$, define the diffeomorphism $F_n := \bigcirc_{i=1}^{n} \Psi_i$. Let us check that F_n and F_{n+1} coincide on the region $X \setminus B_n$. Take $x \in X \setminus B_n$. Then $(\bigcirc_{i=1}^{n-1} \Psi_i)(x) \in X \setminus A'_n$, so by condition (2) above it follows that

$$F_n(x) = (\Psi_n \circ \bigcirc_{i=1}^{n-1} \Psi_i)(x) \in X \setminus A_n,$$

which, thanks to condition (4), implies that $F_{n+1}(x) = \Psi_{n+1}(F_n(x)) = F_n(x)$.

Taking into account that $\bigcap_{n=1}^{\infty} B_n = \emptyset$, that

$$X \setminus B_n \subset X \setminus B_{n+1} \subset X \setminus \{0\}, \quad \forall n \in \mathbb{N},$$

that $X \setminus \{0\} = \bigcup_{n=2}^{\infty} X \setminus B_n$, and that for each $n \geq 2$, the functions F_n and F_{n+1} coincide on $X \setminus B_n$, we may define a mapping $F : X \setminus \{0\} \to X$ by the relation

$$F|_{X \setminus B_n} = F_n|_{X \setminus B_n}.$$

It is clear that F is an injection and a local diffeomorphism of class C^p. Let us check that F is surjective, hence a C^p diffeomorphism. For each $n \geq 2$, we have that

$$F(X \setminus B_n) = F_n(X \setminus B_n) = (\Psi_n \circ \bigcirc_{i=1}^{n-1} \Psi_i)(X \setminus B_n) = \Psi_n(X \setminus A'_n) = X \setminus A_n,$$

and therefore

$$F(X \setminus \{0\}) = F(\bigcup_{n \geq 2} X \setminus B_n) = \bigcup_{n \geq 2} F(X \setminus B_n) = \bigcup_{n \geq 2} X \setminus A_n = X,$$

because $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Take $x \in X \setminus A_1 \subset X \setminus B_2$. From condition (4) above we get that

$$F(x) = (\Psi_2 \circ \Psi_1)(x) = (\Psi_2)(x) = x.$$

Hence $F^{-1} : X \to X \setminus \{0\}$ is a C^p diffeomorphism which restrict to the identity on $X \setminus A_1$.

Now consider

$$\tfrac{1}{2}B \subset \operatorname{int}(B) \subset B \subset \operatorname{int}(A_1).$$

By the Four Bodies Lemma 2.23, there exists a C^p diffeomorphism $h : X \to X$ such that $h(B) = A_1$ and h is the identity on $\frac{1}{2}B$. It is clear that $h^{-1} \circ F^{-1} \circ h$ is a diffeomorphism from X onto $X \setminus \{0\}$. Besides,

$$x \notin \varepsilon B_X \Rightarrow x \notin B \Rightarrow h(x) \notin A_1$$
$$\Rightarrow F^{-1}(h(x)) = h(x) \Rightarrow (h^{-1} \circ F^{-1} \circ h)(x) = x.$$

Then we can define

$$\Phi_\varepsilon := h^{-1} \circ F^{-1} \circ h,$$

which is the mapping we want.

CONSTRUCTION OF THE FAMILY $(\Psi_n)_{n\geq 1}$. By induction on n.

• *First step.* We define Ψ_1 as the identity mapping.

• *Second step*: *definition of* Ψ_2. Since $A_2' = \Psi_1(B_2) = B_2$, it is clear that A_2' is a C^p smooth radially bounded starlike body with $A_2' \subset \operatorname{int}(A_1)$. By applying the Three Bodies Lemma 2.31 with $P = A_2'$, $Q = A_2$, $L = A_1$, $a = a_1$, $b = a_2$, there exists a C^p diffemorphism $\Psi_2 : X \to X$ satisfying conditions (2), (3) and (4). From (3) one can immediately get that $\Psi_2 \circ \Psi_1$ satisfies (1) as well.

• $(n+1)$*-th step*: *definition of* Ψ_{n+1}. Assume the diffeomorphisms $(\Psi_j)_{j=1}^n$ are already constructed in such a way that they satisfy the four conditions above. By (1) we get that $(\bigcirc_{i=1}^{n-1}\Psi_i)(B_{n+1})$ is a radially bounded C^p smooth starlike body with center $a_{n-1} \in X$. Besides,

$$(\bigcirc_{i=1}^{n-1}\Psi_i)(B_{n+1}) \subset \operatorname{int}[A_n'] = \operatorname{int}[(\bigcirc_{i=1}^{n-1}\Psi_i)(B_n)],$$

from which we deduce, by using (3), that $A_{n+1}' = (\Psi_n \circ \bigcirc_{i=1}^{n-1}\Psi_i)(B_{n+1})$ is a C^p smooth starlike body with center $a_n \in X$ and such that $A_{n+1}' \subset \operatorname{int}(A_n)$. Then we can apply Lemma 2.31 with $P = A_{n+1}'$, $Q = A_{n+1}$, $L = A_n$, $a = a_n$, $b = a_{n+1}$, to find a C^p diffeomorphism Ψ_{n+1} satisfying the required conditions. ∎

The rest of this section is devoted to exhibit three natural classes of spaces in which one can construct decreasing towers of radially bounded smooth starlike bodies with empty intersection, so that the preceding theorem applies and we can thus deduce the existence of diffeomorphisms deleting points. These classes contain all the spaces to which the previously known results on deleting diffeomorphisms apply, namely, those of all Banach spaces having smooth norms or smooth partitions of unity.

THEOREM 3.3. *Let X be an infinite-dimensional Banach space. Assume that X satisfies at least one of the following conditions:*

1. *X has a (not necessarily equivalent) C^p smooth norm and a C^p smooth bump function; or*
2. *there exists Y a nonreflexive separable subspace of X, such that both X and X/Y admit a Lipschitz bump function of class C^p; or*
3. *X has C^p smooth partitions of unity.*

Then there exists a family $(A_n)_{n\in\mathbb{N}}$ of C^p smooth radially bounded starlike bodies such that

(i) $A_n \subset_\Delta A_{n-1}$, *for every* $n \geq 2$;
(ii) $\bigcap_{n=1}^\infty A_n = \emptyset$.

Moreover, when X satisfies (1) *above, the bodies A_n can be taken to be convex.*

In any case, by Theorem 3.1, *for every $\varepsilon > 0$ there exists a C^p diffeomorphism $h : X \to X \setminus \{0\}$ so that $h(x) = x$ if $\|x\| \geq \varepsilon$.*

Proof. **Case 1.** Assume that X satisfies (1). As shown in [1] and [5], there exists an *asymmetric noncomplete norm* $\omega : X \to [0, \infty)$ of class C^p (that is, there exists a C^p smooth radially bounded convex body, not necessarily symmetric, with Minkowski functional ω), and there exists a path $p : (0, \infty) \to X$ of class C^∞ such that $\omega(p(\alpha) - p(\beta)) \leq 1/2(\beta - \alpha)$, if $\beta \geq \alpha > 0$, and $\limsup_{t\to 0^+} \omega(x - p(t)) > 0$, for all $x \in X$. For each $n \in \mathbb{N}$, define $A_n := \{x \in X : \omega(x - p(1/n)) \leq 1/n\}$. Each A_n is clearly a radially bounded convex body of class C^p. Moreover, $\bigcap_{n=1}^\infty A_n = \emptyset$. Indeed, if there was a point $y \in \bigcap_{n\geq 1} A_n$, then we would have that $\omega(y - p(1/n)) \leq 1/n$ for all n, hence that $\lim_n \omega(y - p(1/n)) = 0$, which contradicts $\limsup_{t\to 0^+} \omega(x - p(t)) > 0$ for each $x \in X$. Therefore (ii) is proved.

Let us check (i), that is $A_n \subset_\Delta A_{n-1}$. For a given $n \geq 2$, define $\delta_n := 1/2(n-1)$. Take $x \in A_n$. By the definition of A_n, we have $\omega(x-p(1/n)) \leq 1/n$. Bearing in mind the properties of the functional ω and the path p, we conclude that

$$\begin{aligned}
&\omega\Big((1+\delta_n)\big(x - p(1/n)\big) + p\big(1/n\big) - p\big(1/(n-1)\big)\Big) \\
&\quad \leq (1+\delta_n)\omega\Big(\big(x - p(1/n)\big)\Big) + \omega\Big(p\big(1/n\big) - p\big(1/(n-1)\big)\Big) \\
&\quad \leq \frac{1+\delta_n}{n} + \frac{1}{2}\Big(\frac{1}{n-1} - \frac{1}{n}\Big) \leq \frac{1}{n-1},
\end{aligned}$$

which means $(1+\delta_n)(-p(1/n) + A_n) + p(1/n) \subset A_{n-1}$. Hence $A_n \subset_\Delta A_{n-1}$, and (i) is proved.

Case 2. Assume that X satisfies (2). This case contains the most complex proof of this paper and uses most of the machinery we have developed, so it will be worth giving the gist of the argument before proceeding with the details. In the nonreflexive separable subspace Y we consider a descending tower of bounded convex bodies $(C_n)_n$ with empty intersection (such a family always exists in a nonreflexive Banach space). Next we use the main result of [8], which provides a smooth Lipschitz approximation of every Lipschitz function defined on Y. By applying this result to the Minkowski functionals of the bodies C_n we regularize this family of convex bodies into a tower of C^p smooth Lipschitz starlike bodies $(D_n)_n$ of Y with empty intersection. Then, by combining (through a smooth square) the Minkowski functionals of the D_n with a smooth approximation of the distance function $\operatorname{dist}(\cdot, Y)$ (here is where we use the existence of a Lipschitz smooth bump on the quotient X/Y), we can extend the family (D_n) to a tower (A_n) of C^p smooth starlike bodies with empty intersection.

The following lemma ensures the existence of the family (C_n) in Y.

Lemma 3.4. *Let Y be a nonreflexive Banach space. There exists a family $(C_n)_{n\geq 1}$ of bounded convex bodies such that:*

1. $\bigcap_{n=1}^{\infty} C_n = \emptyset$,
2. $C_{n+1} \subset_d C_n \subset B_Y$ *for every* $n \in \mathbb{N}$.

Proof. Fix $0 < \delta_0 < 1/2$. We have that $\operatorname{dist}\big((1-\delta_0)B_Y, X \setminus B_Y\big) = \delta_0 > 0$. Since X is nonreflexive, by James' theorem we can take a continuous linear functional $T \in X^*$ which does not attain its sup on the ball $(1-2\delta_0)B_Y$. Let

$$\alpha := \sup\{T(x) \colon x \in (1-2\delta_0)C\},$$

and define

$$H_n := \{x \in (1-2\delta_0)C \colon T(x) \geq \alpha - 1/n\}$$

for each $n \in \mathbb{N}$. We have that $\bigcap_{n=1}^{\infty} H_n = \emptyset$, $H_{n+1} \subset H_n$ for each $n \in \mathbb{N}$, and $H_1 \subset (1-2\delta_0)B_Y \subset_d (1-\delta_0)B_Y$. Choose $\varepsilon > 0$ so that $H_1 + \varepsilon B_Y \subset (1-\delta_0)B_Y$ and $3\varepsilon < \delta_0$. Now, for each $n \in \mathbb{N}$, let

$$C_n := \big\{x \in X \colon \operatorname{dist}(x, H_n) \leq \frac{\varepsilon}{2^n}\big\}.$$

It is easy to check that $(C_n)_{n\geq 1}$ is a sequence of bounded convex bodies of satisfying $\bigcap_{n=1}^{\infty} C_n = \emptyset$, and $C_{n+1} \subset_d C_n$ for every $n \in \mathbb{N}$. ∎

Let us continue with the proof of the theorem. By Theorem 2.1 we can take a C^p smooth bounded symmetric Lipschitz starlike body $E \subset B_X$. By

Corollary 2.2 there exist $M > 1$ and a Lipschitz function $\psi : X \to [0, \infty)$, symmetric and positively homogeneous, of class C^p on $X \setminus Y$, such that

$$\operatorname{dist}(x, Y) \leq \psi(x) \leq M \cdot \operatorname{dist}(x, Y) \quad \text{for all } x \in X.$$

By the preceding Lemma 3.4 we can take a tower $(C_n)_{n\in\mathbb{N}}$ of bounded convex bodies of Y such that

$$\bigcap_{n\in\mathbb{N}} C_n = \emptyset; \quad C_{n+1} \subset_d C_n \subset Y \cap \frac{1}{10} B_X, \quad \forall n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let $c_n \in Y$ be a center of C_n, and let $\mu_n : Y \to [0, \infty)$ be the function defined by

$$\mu_n(x) = \mu_{C_n}(x + c_n) \quad \text{for every } x \in Y,$$

that is, μ_n is the Minkowski functional of the convex body $-c_n + C_n$ with respect to the origin (which is an interior point of this body because c_n is a center of C_n).

Now let us take a sequence $(\gamma_n)_{n\in\mathbb{N}} \subset (0, 1/4)$ converging to 0 and such that

$$\gamma_1 \leq \frac{\operatorname{dist}(C_2, \partial C_1)}{2 + \operatorname{dist}(C_2, \partial C_1)};$$
$$\gamma_n \leq \min\left\{\frac{1}{4}\ \operatorname{dist}(C_n, \partial C_{n-1}),\ \frac{\operatorname{dist}(C_{n+1}, \partial C_n)}{2 + \operatorname{dist}(C_{n+1}, \partial C_n)}\right\}, \quad \text{if } n \geq 2.$$

For each $n \in \mathbb{N}$, pick $r_n \in (0, 1/2)$ so that

$$\frac{1}{1 + r_n} \geq 1 - \gamma_n\ ; \quad \frac{1}{1 - r_n} \leq 1 + \gamma_n.$$

Define then $\varepsilon_n := r_n \inf\{\mu_n(x) : x \in Y \cap \partial E\}$. Since both $Y \cap E$ and C_n are bounded subsets of Y, the number ε_n is strictly positive. Besides, since $C_n \subset Y$ is convex, the functional μ_n is a Lipschitz function. Now, by the main result of [8], for the given number $\varepsilon_n > 0$ and the Lipschitz function μ_n, there exists a C^p smooth Lipschitz function $g_n : X \to \mathbb{R}$ so that

$$\left|g_n(\xi) - \mu_n(\xi)\right| \leq \varepsilon_n, \quad \forall\ \xi \in E \cap Y.$$

In particular, if $\xi \in \partial E \cap Y$, then

$$g_n(\xi) \geq \mu_n(\xi) - \varepsilon_n \geq \frac{\varepsilon_n}{r_n} - \varepsilon_n = \varepsilon_n\Big(\frac{1}{r_n} - 1\Big) \geq \varepsilon_n > 0.$$

Define now $h_n : X \to \mathbb{R}$ by

$$h_n(x) = \begin{cases} \mu_E(x) g_n\Big(\frac{x}{\mu_E(x)}\Big) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is plain that h_n is Lipschitz, positively homogeneous, symmetric, and of class C^p on $X \setminus \{0\}$. Let L_n stand for the Lipschitz constant of h_n.

Since $g_n|_{\partial E \cap Y} \geq \varepsilon_n > 0$, we have that $h_n|_Y \geq 0$. Let us define, for each $n \in \mathbb{N}$,

$$D_n := \{x \in Y : h_n(x - c_n) \leq 1\} = c_n + \{x \in Y : h_n(x) \leq 1\},$$

which is a starlike body in with center $c_n \in Y$.

Claim 3.5. *We have that:*

1. $(1-\gamma_n)(-c_n+C_n)+c_n \subset D_n \subset (1+\gamma_n)(-c_n+C_n)+c_n$ *for each* $n \in \mathbb{N}$*;*
2. $(1+\gamma_{n+1})(-c_{n+1}+C_{n+1})+c_{n+1} \subset_d (1-\gamma_n)(-c_n+C_n)+c_n$ *for each* $n \in \mathbb{N}$*;*
3. $D_{n+1} \subset_d D_n \subset_d \frac{1}{2}B_X \cap Y$ *for each* $n \in \mathbb{N}$*, and* $\bigcap_{n\in\mathbb{N}} D_n = \emptyset$*.*

Let us assume for a while that the claim is proved, and see how we can finish the proof of the theorem. Let $\mathcal{V} \subset \mathbb{R}^2$ be a smooth square. By induction we are going to construct a family $(A_n)_{n\in\mathbb{N}}$ such that

1. A_n is a bounded starlike body of class C^p with center $a_n := c_n$, for each $n \in \mathbb{N}$;
2. $A_n \cap Y = D_n$, for each $n \in \mathbb{N}$;
3. $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$;
4. $A_{n+1} \subset_\Delta A_n$, for each $n \in \mathbb{N}$.

This family satisfies the statement of the theorem.

Construction of the family $(A_n)_{n\in\mathbb{N}}$.

Step 1: *definition of* A_1. Let b_1 be a number such that $b_1 > 2$. Define

$$A_1 := c_1 + \{x \in X : \mu_{\mathcal{V}}\big(h_1(x), b_1\psi(x)\big) \leq 1\}.$$

Since $\mu_{\mathcal{V}}$ is a norm of the plane $\mathbb{R}^2$, and both h_1 and ψ are positively homogeneous Lipschitz functions, the mapping $x \in X \longrightarrow \mu_{\mathcal{V}}\big(h_1(x), b_1\psi(x)\big)$

is Lipschitz and positively homogeneous. Therefore A_1 is a Lipschitz starlike body with center c_1. Let us see that A_1 is radially bounded. Indeed, if $\mu_{\mathcal{V}}\big(h_1(x), b_1\psi(x)\big) = 0$, then $\operatorname{dist}(x, Y) = 0 = \psi(x)$ and $h_1(x) = 0$, that is,

$$x \in Y, \quad \mu_E(x) g_1\Big(\frac{x}{\mu_E(x)}\Big) = 0.$$

Since $\frac{x}{\mu_E(x)} \in \partial E \cap Y$, we have that

$$g_1\Big(\frac{x}{\mu_E(x)}\Big) > 0.$$

Hence $\mu_E(x) = 0$ and $x = 0$. Thus $cc(A_1) = \{c_1\}$, and A_1 is radially bounded. In fact the body A_1 is *bounded*: the proof is analogous to the one used below to show the boundedness of A_2.

Let us see that A_1 is of class C^p. This amounts to showing that the function $x \in X \longrightarrow \mu_{\mathcal{V}}\big(h_1(x), b_1\psi(x)\big)$ is of class C^p on $X \setminus \{0\}$. Fix $x \in X \setminus \{0\}$. Since $g_1 > 0$ on $\partial E \cap Y$, it follows that $h_1(x) > 0$ if $x \in Y$. Therefore $(h_1(x), b_1\psi(x)) \neq (0, 0)$, for each $x \in X \setminus \{0\}$. According to the properties of smooth squares, see Lemmas 2.27 and 2.28, and bearing in mind that h_1 is C^p smooth away from the origin and ψ is C^p smooth on $X \setminus Y$, we may conclude that A_1 is of class C^p.

Step 2: *definition of* A_2. Let us first see that $2\delta_2 := \inf\{\operatorname{dist}(x, D_2) \colon x \in \partial A_1\} > 0$. Assume, on the contrary, that $\delta_2 = 0$. Then there are two sequences $(y^j)_{j\in\mathbb{N}} \subset D_2$ and $(x^j)_{j\in\mathbb{N}} \subset \partial A_1$ such that $\|x^j - y^j\| < \frac{1}{jL_1}$ for every $j \in \mathbb{N}$. As $x^j \in \partial A_1$, we have that

$$\mu_{\mathcal{V}}\big(h_1(x^j - c_1), b_1\psi(x^j - c_1)\big) = 1.$$

From this, taking into account that

$$b_1\psi(x^j - c_1) \leq b_1 M \operatorname{dist}(x^j, Y) \leq b_1 M \|x^j - y^j\| \to 0$$

for $j \to \infty$, and recalling the properties of the smooth square, namely, $\partial V \cap \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \leq 1/2\} = \{1, -1\} \times [-1/2, 1/2]$, it follows that $|h_1(x^j - c_1)| = 1$ if $j \geq j_0$ for some j_0 big enough. By passing to a subsequence if necessary, we may assume $j_0 = 1$. Since h_1 is L_1-Lipschitz, so

$$h_1(x^j - c_1) \geq h_1(y^j - c_1) - L_1\|x^j - y^j\| > -1/j,$$

we get $h_1(x^j - c_1) = 1$ (note that $h_1(y^j - c_1) \geq 0$ because $y^j - c_1 \in Y$). Therefore

$$h_1(y^j - c_1) \geq h_1(x^j - c_1) - L_1\|x^j - y^j\| = 1 - L_1\|x^j - y^j\| \geq 1 - 1/j,$$

for each $j \geq 1$. Moreover, since $y^j \in D_2 \subset D_1$, we have that $h_1(y^j - c_1) \leq 1$. From this, and bearing in mind that $\|y^j - c_1\| \to 0$ because $\{y^j - c_1\} \subset D_1$ and the body D_1 is bounded, we deduce that

$$\begin{aligned}
\operatorname{dist}(D_2, X \setminus D_1) = \operatorname{dist}(D_2, \partial D_1) &\leq \left\| y^j - \left[c_1 + \frac{y^j - c_1}{h_1(y^j - c_1)} \right] \right\| \\
&= \|y^j - c_1\| \cdot \left| 1 - \frac{1}{h_1(y^j - c_1)} \right| \\
&\leq \|y^j - c_1\| \left| 1 - (1 - \tfrac{1}{j})^{-1} \right| \to 0
\end{aligned}$$

as $j \to \infty$, which is a contradiction: by the Claim, we know that $\operatorname{dist}(D_2, X \setminus D_1) > 0$. Thus $\delta_2 > 0$.

Now let us choose a number $b_2 > 2^2$ with $D_2 + \frac{L_2}{b_2} B_X \subset B_X$ (note that $D_2 \subset_d B_X$), and such that

$$\frac{1}{b_2} \leq \frac{\delta_2}{2} \quad \text{and} \quad 0 < 1 - \frac{b_2}{b_2 + L_2} \leq \frac{\delta_2}{2}.$$

Then define

$$A_2 := c_2 + \{x \in X \colon \mu_{\mathcal{V}}\big(h_2(x), b_2\psi(x)\big) \leq 1\}.$$

As in Step 1, it is easy to prove that A_2 is a C^p smooth radially bounded starlike body with center $c_2 \in X$. Moreover, the body A_2 satisfies

$$A_2 \subset \{x \in X : \operatorname{dist}(x, D_2) \leq \delta_2\}.$$

Indeed, let x be an interior point of A_2. By the definition of A_2, we have that

$$\|\big(h_2(x - c_2), b_2\psi(x - c_2)\big)\|_\infty \leq \mu_{\mathcal{V}}\big(h_2(x - c_2), b_2\psi(x - c_2)\big) < 1,$$

hence $\operatorname{dist}(x, Y) \leq \psi(x - c_2) < b_2^{-1}$. Choose $y_x \in Y$ such that $\|x - y_x\| < b_2^{-1}$. Then

$$h_2(y_x - c_2) \leq h_2(x - c_2) + L_2\|x - y_x\| \leq 1 + L_2/b_2,$$

and so $y_x \in (1 + L_2/b_2)(-c_2 + D_2) + c_2$. Finally,

$$\begin{aligned}
\operatorname{dist}(x, D_2) &\leq \left\| x - \left[c_2 + \frac{y_x - c_2}{1 + L_2/b_2} \right] \right\| \\
&\leq \|x - y_x\| + \left| 1 - \frac{b_2}{b_2 + L_2} \right| \|y_x - c_2\| \\
&\leq 1/b_2 + \tfrac{1}{2}\delta_2\|y_x - c_2\| \leq \tfrac{1}{2}\delta_2 + \tfrac{1}{2}\delta_2 = \delta_2.
\end{aligned}$$

Thus we have $A_2 \subset \{x \in X : \operatorname{dist}(x, D_2) \leq \delta_2\}$. From this we may deduce two facts. First, A_2 is bounded. This is obvious because D_2 is bounded. Second, $A_2 \subset_d A_1$: it is enough to note that $A_2 \subset \{x \in X : \operatorname{dist}(x, D_2) \leq \delta_2\}$, and $2\delta_2 := \inf\{\operatorname{dist}(x, D_2) \colon x \in \partial A_1\} > 0$. From the boundedness of A_2, together with the fact that $A_2 \subset_d A_1$, it follows that $A_2 \subset_\Delta A_1$ (see Proposition 2.10).

Step (n+1) : definition of A_{n+1}. Suppose we have already defined two sequences $(b_j)_{j=1}^n$ or numbers and $(A_j)_{j=1}^n$ of C^p smooth bounded starlike bodies such that:

c_j is a center of A_j, for $j = 1, \ldots, n$;

$b_j > 2^j$, for $j = 1, \ldots, n$;

$A_j = c_j + \{x \in X : \mu_{\mathcal{V}}\big(h_j(x), b_j \psi(x)\big) \leq 1\}$, for $j = 1, \ldots, n$;

$A_{j+1} \subset_\Delta A_j$, for $j = 1, \ldots, n-1$.

As in Steps 1 and 2, one shows that

$$2\delta_{n+1} := \inf\{\operatorname{dist}(x, D_{n+1}) \colon x \in \partial A_n\} > 0,$$

hence we can choose $b_{n+1} > 2^{n+1}$ so that

$$A_{n+1} = c_{n+1} + \{x \in X : \mu_{\mathcal{V}}\big(h_{n+1}(x), b_{n+1}\psi(x)\big) \leq 1\}$$

is a C^p smooth bounded starlike body with center $c_{n+1} \in X$, such that $A_{n+1} \subset_\Delta A_n$.

Therefore, the induction process provides us with a family $(A_n)_{n \in \mathbb{N}}$ which obviously satisfies conditions (1) and (4). Let us see that it also satisfies (2) and (3). We have that $A_n \cap Y = D_n$ for each $n \in \mathbb{N}$. Indeed, according to the properties of smooth squares,

$$\begin{aligned} A_n \cap Y &= \{x \in Y : \mu_{\mathcal{V}}\big(h_n(x - c_n), b_n \psi(x - c_n)\big) \leq 1\} \\ &= \{x \in Y : \mu_{\mathcal{V}}\big(h_n(x - c_n), 0\big) \leq 1\} \\ &= \{x \in Y : h_n(x - c_n) \leq 1\} = D_n, \end{aligned}$$

hence (2) is true. To check (3), suppose there exists $\xi \in \cap_{n \in \mathbb{N}} A_n$. By the definition of A_n, we have that

$$b_n \psi(x - c_n) \leq \mu_{\mathcal{V}}\big(h_n(x - c_n), b_n \psi(x - c_n)\big) \leq 1,$$

so that

$$\operatorname{dist}(\xi, Y) = \operatorname{dist}(\xi - c_n, Y) \leq \psi(x - c_n) \leq b_n^{-1}, \quad \text{for each } n \in \mathbb{N}.$$

But $b_n^{-1} \to 0$, and therefore $\operatorname{dist}(\xi, Y) = 0$, which means $\xi \in Y$. Hence

$$\xi \in \bigcap_{n\in\mathbb{N}} (A_n \cap Y) = \bigcap_{n\in\mathbb{N}} D_n,$$

which is a contradiction since the Claim tells us that, $\bigcap_{n\in\mathbb{N}} D_n = \emptyset$.

It only remains to prove the Claim.

Proof of Claim 3.5. *Property* (1). We have to show that $(1-\gamma_n)(-c_n + C_n) + c_n \subset D_n \subset (1+\gamma_n)(-c_n + C_n) + c_n$ for each $n \in \mathbb{N}$. For any $x \in Y \setminus \{0\}$, we have that

$$\begin{aligned} |h_n(x) - \mu_n(x)| &= \big|\mu_E(x) g_n\big(\frac{x}{\mu_E(x)}\big) - \mu_n(x)\big| \\ &= \big|\mu_E(x) g_n\big(\frac{x}{\mu_E(x)}\big) - \mu_E(x)\mu_n\big(\frac{x}{\mu_E(x)}\big)\big| \\ &\le r_n \inf\{\mu_n(x) : x \in Y \cap \partial E\}\mu_E(x) \\ &\le r_n \mu_E(x)\mu_n\Big(\frac{x}{\mu_E(x)}\Big) = r_n\mu_n(x). \end{aligned}$$

Let us see how the starlike body $D_n = \{x \in Y : h_n(x - c_n) \le 1\}$ aproximates C_n as is required. Indeed, for each $x \in Y$, taking into account that $\frac{1}{1+r_n} \ge 1 - \gamma_n$ and $\frac{1}{1-r_n} \le 1 + \gamma_n$, we have that

$$\begin{aligned} &x \in D_n \iff h_n(x - c_n) \le 1 \implies \mu_n(x - c_n) \le 1 + r_n\mu_n(x - c_n) \\ &\implies (1 - r_n)\mu_n(x - c_n) \le 1 \implies x - c_n \in \frac{1}{1 - r_n}(-c_n + C_n) \\ &\implies x \in c_n + \frac{1}{1 - r_n}(-c_n + C_n) \subset c_n + (1 + \gamma_n)(-c_n + C_n), \end{aligned}$$

hence $D_n \subset c_n + (1 + \gamma_n)(-c_n + C_n)$. On the other hand, if

$$x \in (1 - \gamma_n)(-c_n + C_n) + c_n \subset \frac{1}{1 + r_n}(-c_n + C_n) + c_n,$$

then $\mu_n(x - c_n) \le (1 + r_n)^{-1}$, hence $h_n(x - c_n) \le (1 + r_n)\mu_n(x - c_n) \le 1$, that is, $x \in D_n$, and (1) is proved.

Property (2). Take $x \in C_{n+1}$ and $z \in \partial\Big[(1 - \gamma_n)(-c_n + C_n) + c_n\Big] \subset C_n$.

By the choice of z, we have that $c_n + \frac{1}{1-\gamma_n}(z - c_n) \in \partial C_n$. Therefore

$$\begin{aligned}\operatorname{dist}(C_{n+1}, \partial C_n) &\leq \left\| x - \left\{ c_n + \frac{1}{1-\gamma_n}(z - c_n) \right\} \right\| \\ &\leq \|x - z\| + \|z - c_n\| \left| \frac{\gamma_n}{1-\gamma_n} \right| \\ &\leq \|x - z\| + \left| \frac{\gamma_n}{1-\gamma_n} \right| \\ &\leq \|x - z\| + \tfrac{1}{2} \operatorname{dist}(C_{n+1}, \partial C_n).\end{aligned}$$

Notice that, by the choice of γ_n, we know that

$$\gamma_n \leq \frac{\operatorname{dist}(C_{n+1}, \partial C_n)}{2 + \operatorname{dist}(C_{n+1}, \partial C_n)},$$

which implies that $|\frac{\gamma_n}{1-\gamma_n}| \leq \frac{1}{2} \operatorname{dist}(C_{n+1}, \partial C_n)$. Then we have

$$\|x - z\| \geq \tfrac{1}{2} \operatorname{dist}(C_{n+1}, \partial C_n) > 0,$$

and consequently

$$\operatorname{dist} \Big\{ C_{n+1}, \partial \big[(1 - \gamma_n)(-c_n + C_n) + c_n \big] \Big\} \geq \tfrac{1}{2} \operatorname{dist}(C_{n+1}, \partial C_n) > 0.$$

Therefore

$$\begin{gathered}(1 + \gamma_{n+1})(-c_{n+1} + C_{n+1}) + c_{n+1} \subset C_{n+1} + \gamma_{n+1} B_X \\ \subset C_{n+1} + \tfrac{1}{4} \operatorname{dist}(C_{n+1}, \partial C_n) B_X \subset_d (1 - \gamma_n)(-c_n + C_n) + c_n.\end{gathered}$$

Property (3). By (1) and (2) we immediately deduce that, for each $n \in \mathbb{N}$,

$$D_{n+1} \subset (1 + \gamma_{n+1})(-c_{n+1} + C_{n+1}) + c_{n+1} \subset_d (1 - \gamma_n)(-c_n + C_n) + c_n \subset D_n,$$

hence that $D_{n+1} \subset_d D_n$. Moreover, bearing in mind that $\gamma_1 \leq 1/4$, we obtain that

$$D_1 \subset (1+\gamma_1)(-c_1+C_1)+c_1 \subset C_1+\gamma_1(-c_1+C_1) \subset C_1+\gamma_1 B_X \subset \tfrac{1}{10} B_X + \tfrac{1}{4} B_X,$$

so that $D_1 \subset_d \frac{1}{2} B_X$.

Finally let us show that $\bigcap_{n \in \mathbb{N}} D_n = \emptyset$. We know that

$$\bigcap_{n \in \mathbb{N}} D_n \subset \bigcap_{n \in \mathbb{N}} (1 + \gamma_n)(-c_n + C_n) + c_n \subset \bigcap_{n \in \mathbb{N}} (C_n + \gamma_n B_X).$$

Thus we only have to see that $\bigcap_{n\in\mathbb{N}}(C_n+\gamma_n B_X)=\emptyset$. Suppose there exists $x\in\bigcap_{n\in\mathbb{N}}(C_n+\gamma_n B_X)$. Then there is a sequence $(z_n)_n\subset B_X$ so that $(x-\gamma_n z_n)\in C_n$ for each $n\in\mathbb{N}$. For a given $j\in\mathbb{N}$, taking into account that $\gamma_n\to 0$, that $(z_n)_n\subset B_X$, that C_j is closed, and that $(x-\gamma_n z_n)_{n\geq j}\subset C_j$,

$$x=\|\cdot\|-\lim(x-\gamma_n z_n)\in C_j.$$

As this is true for every $j\in\mathbb{N}$, we conclude that $x\in\bigcap_j C_j=\emptyset$, a contradiction.

Case 3. Finally let us consider the case when X has C^p smooth partitions of unity. We consider two situations.

First, if X is reflexive then X can be linearly injected into some $c_0(\Gamma)$, that is there exists a continuous linear injection $T:X\to c_0(\Gamma)$. Since $c_0(\Gamma)$ has an equivalent C^∞ smooth norm $|\cdot|$, we can define a (not necessarily equivalent) C^∞ smooth norm ρ on X by the formula $\rho(x)=|T(x)|$. Thus we can use Case 1 to conclude the existence of the family (A_n).

Second, if X is nonreflexive, then, by Lemma 3.4 there exists a sequence (C_n) of bounded convex bodies with $C_{n+1}\subset_d C_n$ for every $n\in\mathbb{N}$ and $\bigcap_{k=1}^\infty C_k=\emptyset$. Then we can use Corollary 2.19 to find C^p smooth starlike bodies A_n such that $C_{n+1}\subset A_n\subset_d C_n$ for each $n\in\mathbb{N}$, and then Proposition 2.10 to conclude that $A_{n+1}\subset_\Delta A_n$ for all $n\in\mathbb{N}$. In particular, since $\bigcap_{n=1}^\infty C_n=\emptyset$ and $A_n\subset C_n$ for every n, we also have that $\bigcap_{n=1}^\infty A_n=\emptyset$. ∎

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