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Viscosity solutions to second order partial differential equations on Riemannian manifolds ☆

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Abstract

We prove comparison, uniqueness and existence results for viscosity solutions to a wide class of fully nonlinear second order partial differential equations $F(x, u, du, d^2u) = 0$ defined on a finite-dimensional Riemannian manifold M. Finest results (with hypothesis that require the function F to be degenerate elliptic, that is nonincreasing in the second order derivative variable, and uniformly continuous with respect to the variable x) are obtained under the assumption that M has nonnegative sectional curvature, while, if one additionally requires F to depend on d^2u in a uniformly continuous manner, then comparison results are established with no restrictive assumptions on curvature.

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1. Introduction

The theory of viscosity solutions to nonlinear PDEs on $\mathbb{R}^n$ (and on infinite-dimensional Banach spaces) was introduced by M.G. Crandall and P.L. Lions in the 1980s. This theory quickly gained popularity and was enriched and expanded with numerous and important contributions from many mathematicians. We cannot mention all of the significant papers in the vast literature

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concerning viscosity solutions and Hamilton–Jacobi equations, so we will content ourselves with referring the reader to [4] and the references given therein.

More recently there have been various approaches to extend the theory of viscosity solutions of first order Hamilton–Jacobi equations, and the corresponding nonsmooth calculus, to the setting of Riemannian manifolds. This is a natural thing to do, because many functions arising from geometrical problems, such as the distance function to a given set of a Riemannian manifold, are not differentiable. Also, many important nonlinear equations full of geometrical meaning, such as the eikonal equations, have no classical solutions, and their *natural* solutions, which in this case we think are the viscosity solutions, are not differentiable (if some readers disagree with our saying that viscosity solutions are the natural notion of solution for eikonal equations, they might change their mind if they have a look at the recent paper [5], where the authors construct a 1-Lipschitz function u defined on the closed unit ball $\overline{B}$ of $\mathbb{R}^n$, $n \geqslant 2$, which is differentiable on the open ball B, and such that $\|\nabla u(x)\| = 1$ almost everywhere, but $\nabla u(0) = 0$; that is, the eikonal equation $\|\nabla u(x)\| = 1$ in B, $u = 0$ on ∂B, admits some exotic almost everywhere solutions which are everywhere differentiable and are very different from its unique viscosity solution, namely the distance function to the boundary ∂B, which is not everywhere differentiable but is much more natural from a geometric point of view).

Mantegazza and Mennucci [9] studied viscosity solutions to eikonal equations on Riemannian manifolds, in connection with regularity properties of the distance function to a compact subset of the manifold. In [2] a theory of (first order) nonsmooth calculus for Riemannian manifolds (possibly of infinite dimension) was introduced and applied to show existence and uniqueness of viscosity solutions to Hamilton–Jacobi equations on such manifolds. Simultaneously, Ledyaev and Zhu [8] developed a (first order) nonsmooth calculus on finite-dimensional Riemannian manifolds and applied it to the study of Hamilton–Jacobi equations from a somewhat different approach, related to control theory and differential inclusions.

The usefulness of nonsmooth analysis on Riemannian manifolds has been shown in [6], where viscosity solutions are employed as a technical tool to prove important results in conformal geometry.

However, to the best of our knowledge, no one has yet carried out a systematic study of second order viscosity subdifferentials and viscosity solutions to second order partial differential equations on Riemannian manifolds.

In this paper we will initiate such a study by establishing comparison, uniqueness and existence of viscosity solutions to second order PDEs of the form

$$F\bigl(x, u, du, d^2u\bigr) = 0,$$

where $u : M \to \mathbb{R}$ and M is a finite-dimensional complete Riemannian manifold. We will study the Dirichlet problem with a simple boundary condition of the type $u = f$ on $\partial\Omega$, where Ω is an open subset of M; and also the same equation, with no boundary conditions, on all of M.

Let us briefly describe the results of this paper. We begin with the natural definition of second order subjet of a function $u : M \to \mathbb{R}$, that is $J^{2,-}u(x) = \{(d\varphi(x), d^2\varphi(x))\colon\ \varphi \in C^2(M, \mathbb{R}),\ f - \varphi$ attains a local minimum at $x\}$. This is a nice definition from a geometric point of view, but it would be complicated and uneconomic to develop a nonsmooth calculus exclusively based on this definition. It is more profitable to try to localize the definition through charts and then use the second order nonsmooth calculus on $\mathbb{R}^n$ to establish the corresponding results on M. However, second derivatives of composite functions are complicated, so not every chart serves this purpose, and we have to work only with the exponential chart. It is not difficult to see that $(\zeta, A) \in J^{2,-}u(x)$ if and only if $(\zeta, A) \in J^{2,-}(u \circ \exp_x)(0)$.

When one turns to the limiting subjet $\overline{J}^{2,-}u(x)$ (defined as the set of limits of sequences (ζ_n, A_n), where $(\zeta_n, A_n) \in J^{2,-}u(x_n)$ and x_n converges to x), things become less obvious but, with the help of a lemma which relates the second derivatives of a function $\varphi : M \to \mathbb{R}$ to those of the function $\psi = \varphi \circ \exp_x$ (at points near the origin in TM_x), one can still show that $(\zeta, A) \in \overline{J}^{2,-}u(x)$ if and only if $(\zeta, A) \in \overline{J}^{2,-}(u \circ \exp_x)(0)$.

By using this characterization we can extend Theorem 3.2 of [4] to the Riemannian setting. This kind of result can be regarded as a sophisticated nonsmooth fuzzy rule for the superdifferential of the sum of two functions, and is the key to the proof of all the comparison results in [4] and in this paper. The result essentially says that if u_1, u_2 are two upper semicontinuous functions on M, φ is a C^2 smooth function on $M \times M$, and we assume that $\omega(x_1, x_2) = u_1(x_1) + u_2(x_2) - \varphi(x_1, x_2)$ attains a local maximum at $(\hat{x}_1, \hat{x}_2)$, then, for each $\varepsilon > 0$ there exist bilinear forms $B_i \in \mathcal{L}^2_s((TM)_{\hat{x}_i}, \mathbb{R})$, $i = 1, 2$, such that

$$\left(\frac{\partial}{\partial x_i}\varphi(\hat{x}_1, \hat{x}_2), B_i\right) \in \overline{J}^{2,+}u_i(\hat{x}_i)$$

for $i = 1, \ldots, k$, and the block diagonal matrix with entries B_i satisfies

$$-\left(\frac{1}{\varepsilon} + \|A\|\right)I \leqslant \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \leqslant A + \varepsilon A^2, \tag{$*$}$$

where $A = d^2\varphi(\hat{x}_1, \hat{x}_2) \in \mathcal{L}^2_s(TM_{\hat{x}_1} \times TM_{\hat{x}_2}, \mathbb{R})$. This is all done in Section 2 of the paper.

In the case $M = \mathbb{R}^n$ this result is usually applied with $\varphi(x, y) = \frac{\alpha}{2}\|x - y\|^2$, whose second order derivative is given by the matrix

$$\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

When applied to vectors of the form (v, v) in $\mathbb{R}^n \times \mathbb{R}^n$ this derivative vanishes, which allows one to derive from $(*)$ that $B_1 \leqslant B_2$ (as quadratic forms). This in turn provides a very general form of comparison result for viscosity solutions of the equation $F(x, u, du, d^2u) = 0$ in which the continuous function F is assumed to be degenerate elliptic (that is nonincreasing in the variable d^2u), strongly increasing in the variable u, and uniformly continuous with respect to x.

The natural approach in the Riemannian setting is then to consider $\varphi(x, y) = \frac{\alpha}{2}d(x, y)^2$, where d is the Riemannian distance in M. Two problems immediately arise. First, the function φ is not differentiable in general if the points x, y are not suitably close to each other. This is unimportant because, in the proof of the main comparison result, we only need φ to be C^2 smooth on a ball of small radius around a point x_0 which is the limit of two different sequences x_α and y_α, and we have to evaluate $d^2\varphi$ at the points (x_α, y_α).

The second problem, however, is substantial. The second derivative of the function φ is a quadratic form defined on $TM_x \times TM_y$, and what we would like is that, when applied to a vector of the form $(v, L_{xy}v)$, where L_{xy} is the parallel transport from TM_x to TM_y along the unique minimizing geodesic connecting x to y, this derivative is less than or equal to zero. This way condition $(*)$ would imply that $L_{\hat{x}_2\hat{x}_1}(B_2) \leqslant B_1$, where $L_{\hat{x}_2\hat{x}_1}(B_2)$ is the parallel transport of the quadratic form B_2 from $TM_{\hat{x}_2}$ to $TM_{\hat{x}_1}$ along the unique minimizing geodesic connecting $\hat{x}_2$ to $\hat{x}_1$, defined by

$$\langle L_{\hat{x}_2\hat{x}_1}(B_2)v, v\rangle := \langle B_2(L_{\hat{x}_1\hat{x}_2}v), L_{\hat{x}_1\hat{x}_2}v\rangle.$$

And therefore we should be able to conclude that, if F is continuous, strongly increasing in the variable u, and degenerate elliptic (that is $F(x,r,\zeta,B) \leqslant F(x,r,\zeta,A)$ whenever $A \leqslant B$), then a natural extension to $\mathcal{X} := \{(x,r,\zeta,A)\colon\ x \in M,\ r \in \mathbb{R},\ \zeta \in TM_x,\ A \in \mathcal{L}_s^2(TM_x)\}$ of the notion of uniform continuity of $F(x,r,\zeta,A)$ with respect to the variable x (namely, that

$$\left|F(y,r,L_{xy}\zeta,L_{xy}P) - F(x,r,\zeta,P)\right| \to 0 \quad \text{uniformly as } y \to x,$$

which we abbreviate by saying that F is *intrinsically uniformly continuous with respect to* x) would be enough to show that comparison holds.

However, as we will show in Section 3, one has that

$$d^2\varphi(x,y)(v,L_{xy}v)^2 \leqslant 0$$

for all $v \in TM_x$ if and only if M has nonnegative sectional curvature. Therefore, with this choice of φ, one can get results as sharp as those in $\mathbb{R}^n$ only when one deals with manifolds of nonnegative curvature. Nevertheless, if the sectional curvature K of M is bounded below, say $K \geqslant -K_0$, then one can show that

$$d^2\varphi(x,y)(v,L_{xy}v)^2 \leqslant 2K_0 d(x,y)^2\|v\|^2$$

for all $v \in TM_x$, and by using this estimation it is possible to deduce that, if one additionally assumes that F satisfies a certain uniform continuity assumption with respect to the variables x *and* D^2u of the kind "for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) \leqslant \delta$ and $P - L_{yx}Q \leqslant \delta I$ imply $F(y,r,L_{xy}\zeta,Q) - F(x,r,\zeta,P) \leqslant \varepsilon$ for all $\zeta \in TM_x^*$, $P \in \mathcal{L}_s^2(TM_x)$, $Q \in \mathcal{L}_s^2(TM_y)$, $r \in \mathbb{R}$," then the comparison principle holds for the equation $F = 0$ (either with the boundary condition $u = 0$ on $\partial\Omega$, or with the assumption that M has no boundary and the functions u, v for which one seeks comparison are bounded). This is all shown in Sections 4 and 5.

In Section 6 we see that Perron's method works perfectly well in the Riemannian setting. For instance one can show existence of viscosity solutions to the equation $u + G(x,du,d^2u) = 0$ on compact manifolds under the same continuity assumptions on G as those that we require for comparison.

In particular, we get the following: if M is a compact manifold and G is degenerate elliptic and uniformly continuous in the above sense, then there exists a unique viscosity solution of $u + G(x,du,d^2u) = 0$ on M. If one additionally assumes that M has nonnegative sectional curvature then the above uniform continuity assumption can be relaxed: it is enough to require that G is *intrinsically uniformly continuous with respect to* x, meaning that "for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) \leqslant \delta$ implies $G(y,L_{xy}\zeta,L_{xy}P) - G(x,\zeta,P) \leqslant \varepsilon$ for all $\zeta \in TM_x^*$, $P \in \mathcal{L}_s^2(TM_x)$."

We end the paper by discussing the applicability of the above theory to some particular examples of equations.

The notation we use is standard. $M = (M,g)$ will always be a finite-dimensional Riemannian manifold. The letters X, Y, Z, V, W will stand for smooth vector fields on the Riemannian manifold M, and $\nabla_Y X$ will always denote the covariant derivative of X along Y. The Riemannian curvature of M will be denoted by R. Geodesics in M will be denoted by γ, σ, and their velocity fields by γ', σ'. If X is a vector field along γ we will often denote $X'(t) = \frac{D}{dt}X(t) = \nabla_{\gamma'(t)}X(t)$. Recall that X is said to be parallel along γ if $X'(t) = 0$ for all t. The Riemannian distance in M will always be denoted by $d(x, y)$ (defined as the infimum of the lengths of all curves joining x to y in M).

We will often identify (via the Riemannian metric) the tangent space of M at a point x, denoted by TM_x, with the cotangent space at x, denoted by TM_x^*. The space of bilinear forms on TM_x (respectively symmetric bilinear forms) will be denoted by $\mathcal{L}^2(TM_x)$ or $\mathcal{L}^2(TM_x, \mathbb{R})$ (respectively $\mathcal{L}_s^2(TM_x)$ or $\mathcal{L}_s^2(TM_x, \mathbb{R})$). Elements of $\mathcal{L}^2(TM_x)$ will be denoted by the letters A, B, P, Q, and those of TM_x^* by ζ, η, etc. Also, we will denote by $T_{2,s}(M)$ the tensor bundle of symmetric bilinear forms, that is

$$T_{2,s}(M) = \bigcup_{x \in M} \mathcal{L}_s^2(TM_x, \mathbb{R})$$

and $T_{2,s}(M)_x = \mathcal{L}_s^2(TM_x, \mathbb{R})$.

We will make extensive use of the exponential mapping $\exp_x$ and the parallel translation along a geodesic γ throughout the paper, and of Jacobi fields along γ only in Section 3. Recall that for every $x \in M$ there exists a mapping $\exp_x$, defined on a neighborhood of 0 in the tangent space TM_x, and taking values in M, which is a local diffeomorphism and maps straight line segments passing through 0 onto geodesic segments in M passing through x. The exponential mapping also induces a local diffeomorphism on the cotangent space TM_x^*, via the identification given by the metric, that will be denoted by $\exp_x$ as well.

On the other hand, for a minimizing geodesic $\gamma : [0, \ell] \to M$ connecting x to y in M, and for a vector $v \in TM_x$ there is a unique parallel vector field P along γ such that $P(0) = v$, this is called the parallel translation of v along γ. The mapping $TM_x \ni v \mapsto P(\ell) \in TM_y$ is a linear isometry from TM_x onto TM_y which we will denote by L_{xy}. Its inverse is of course L_{yx}. This isometry naturally induces an isometry (which we will still denote by L_{xy}), $TM_x^* \ni \zeta \mapsto L_{xy}\zeta \in TM_y^*$, defined by

$$\langle L_{xy}\zeta, v\rangle_y := \langle \zeta, L_{yx}v\rangle_x.$$

Similarly, L_{xy} induces an isometry $\mathcal{L}^2(TM_x, \mathbb{R}) \ni A \to L_{xy}(A) \in \mathcal{L}^2(TM_y, \mathbb{R})$ defined by

$$\big\langle L_{xy}(A)v, v\big\rangle_y := \big\langle A(L_{yx}v), L_{yx}v\big\rangle_x.$$

By $i_M(x)$ we will denote the injectivity radius of M at x, that is the supremum of the radius r of all balls $B(0_x, r)$ in TM_x for which $\exp_x$ is a diffeomorphism from $B(0_x, r)$ onto $B(x, r)$. Similarly, $i(M)$ will denote the global injectivity radius of M, that is $i(M) = \inf\{i_M(x)\colon x \in M\}$. Recall that the function $x \mapsto i_M(x)$ is continuous. In particular, if M is compact, we always have $i(M) > 0$.

For Jacobi fields and any other unexplained terms of Riemannian geometry used in Section 3, we refer the reader to [3,10].

2. Second order viscosity subdifferentials on Riemannian manifolds

Recall that the Hessian $D^2\varphi$ of a C^2 smooth function φ on a Riemannian manifold M is defined by

$$D^2\varphi(X,Y) = \langle \nabla_X \nabla\varphi, Y\rangle,$$

where $\nabla\varphi$ is the gradient of φ and X, Y are vector fields on M (see [10, p. 31]). The Hessian is a symmetric tensor field of type $(0,2)$ and, for a point $p \in M$, the value $D^2\varphi(X,Y)(p)$ only depends of f and the vectors $X(p), Y(p) \in TM_p$. So we can define the second derivative of φ at p as the symmetric bilinear form $d^2\varphi(p) : TM_p \times TM_p \to \mathbb{R}$

$$(v,w) \mapsto d^2\varphi(p)(v,w) := D^2\varphi(X,Y)(p),$$

where X, Y are any vector fields such that $X(p) = v, Y(p) = w$. A useful way to compute $d^2\varphi(p)(v,v)$ is to take a geodesic γ with $\gamma'(0) = v$ and calculate

$$\frac{d^2}{dt^2}\varphi\big(\gamma(t)\big)\bigg|_{t=0},$$

which equals $d^2\varphi(p)(v,v)$. We will often write $d^2\varphi(p)(v)^2$ instead of $d^2\varphi(p)(v,v)$.

Definition 2.1. Let M be a finite-dimensional Riemannian manifold, and $f : M \to (-\infty, +\infty]$ a lower semicontinuous function. We define the second order subjet of f at a point $x \in M$ by

$$J^{2,-}f(x) = \big\{\big(d\varphi(x), d^2\varphi(x)\big)\colon\ \varphi \in C^2(M,\mathbb{R}),\ f - \varphi \text{ attains a local minimum at } x\big\}.$$

If $(\zeta, A) \in J^{2,-}f(x)$, we will say that ζ is a first order subdifferential of f, and A is a second order subdifferential of f at x.

Similarly, for an upper semicontinuous function $g : M \to [-\infty, +\infty)$, we define the second order superjet of f at x by

$$J^{2,+}f(x) = \big\{\big(d\varphi(x), d^2\varphi(x)\big)\colon\ \varphi \in C^2(M,\mathbb{R}),\ f - \varphi \text{ attains a local maximum at } x\big\}.$$

Observe that $J^{2,-}f(x)$ and $J^{2,+}f(x)$ are subsets of $TM_x^* \times \mathcal{L}_s^2(TM_x,\mathbb{R})$, where $\mathcal{L}_s^2(TM_x,\mathbb{R}) \equiv \mathcal{L}_s^2(TM_x)$ stands for the symmetric bilinear forms on TM_x. It is also clear that $J^{2,-}f(x) = -J^{2,+}(-f)(x)$, and that we obtain the same definitions if we replace the condition "$\varphi \in C^2(M,\mathbb{R})$" with "$\varphi$ is C^2 smooth on a neighborhood of x."

By using the fact that a lower semicontinuous function f is bounded below on a neighborhood B of any point x with $f(x) < \infty$, one can easily find a function $\varphi \in C^2(M,\mathbb{R})$ such that $\inf_{y \in \partial B}(f - \varphi)(y) > f(x)$, hence $f - \varphi$ attains a local minimum at some point $z \in B$, and $(d\varphi(z), d^2\varphi(z)) \in J^{2,-}f(z)$. This shows that the set

$$\big\{z \in M\colon\ J^{2,-}f(z) \neq \emptyset\big\}$$

is dense in the set $\{x \in M\colon\ f(x) < \infty\}$. A similar statement is true of upper semicontinuous functions. Therefore, when dealing with semicontinuous functions, one has lots of points where these subjets or superjets are nonempty, that is lots of points of second order sub- or super-differentiability.

In the sequel M will always denote an n-dimensional Riemannian manifold. We next state and prove several results for subjets which also hold, with obvious modifications, for superjets.

Proposition 2.2. *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function. Let $\zeta \in TM_x^*$, $A \in \mathcal{L}_s^2(TM_x, \mathbb{R})$, $x \in M$. The following statements are equivalent*:

(1) $(\zeta, A) \in J^{2,-} f(x)$.
(2) $f(\exp_x(v)) \geqslant f(x) + \langle \zeta, v\rangle_x + \frac{1}{2}\langle Av, v\rangle_x + o(\|v\|^2)$.

Proof. (1) $\Rightarrow$ (2) If $(\zeta, A) \in J^{2,-} f(x)$, by definition there exists $\varphi \in C^2(M, \mathbb{R})$ such that $f - \varphi$ attains a local minimum at x and $\zeta = d\varphi(x)$, $A = d^2\varphi(x)$. We may obviously assume that $\varphi(x) = f(x)$, so we have

$$f(y) - \varphi(y) \geqslant 0$$

on a neighborhood of x. Let us consider the function $h(v) = \varphi(\exp_x(v))$ defined on a neighborhood of 0_x in TM_x. We have that

$$h(v) = h(0) + \big\langle dh(0), v\big\rangle_x + \frac{1}{2}\big\langle d^2h(0)v, v\big\rangle_x + o\big(\|v\|^2\big).$$

By taking $y = \exp_x(v)$ and combining this with the above inequality we get

$$f\big(\exp_x(v)\big) \geqslant f(x) + \big\langle dh(0), v\big\rangle_x + \frac{1}{2}\big\langle d^2h(0)v, v\big\rangle_x + o\big(\|v\|^2\big),$$

so we only need to show that $\zeta = dh(0)$ and $A = d^2h(0)$. To see this, let us fix $v \in TM_x$ and consider the geodesic $\gamma(t) = \exp_x(tv)$ and the function $t \mapsto \varphi(\gamma(t)) = h(tv)$. We have that

$$\frac{d}{dt}h(tv) = \big\langle d\varphi\big(\gamma(t)\big), \gamma'(t)\big\rangle$$

and

$$\frac{d^2}{dt^2}h(tv) = \big\langle d^2\varphi\big(\gamma(t)\big)\gamma'(t), \gamma'(t)\big\rangle.$$

In particular, for $t = 0$, we get

$$dh(0)(v) = \frac{d}{dt}h(tv)\bigg|_{t=0} = \big\langle d\varphi(x), v\big\rangle = \langle \zeta, v\rangle,$$

that is $dh(0) = \zeta$; and also

$$\big\langle d^2h(0)v, v\big\rangle = \frac{d^2}{dt^2}h(tv)\bigg|_{t=0} = \big\langle d^2\varphi(x)v, v\big\rangle = \langle Av, v\rangle,$$

that is $A = d^2h(0)$.

(2) ⇒ (1) Define $F(v) = f(\exp_x(v))$ for v in a neighborhood of $0_x \in TM_x$. We have that

$$F(v) \geqslant F(0) + \langle \zeta, v\rangle_x + \frac{1}{2}\langle Av, v\rangle_x + o\big(\|v\|^2\big).$$

The result we want to prove is known to be true in the case when $M = \mathbb{R}^n$, so there exists $\psi : TM_x \to \mathbb{R}$ such that $F - \psi$ attains a minimum at 0 and $d\psi(0) = \zeta$, $d^2\psi(0) = A$. Since minima are preserved by composition with diffemorphisms, the function $\varphi := \psi \circ \exp_x^{-1}$, defined on an open neighborhood of $x \in M$, has the property that $f - \varphi = (F - \psi) \circ \exp_x^{-1}$ attains a local minimum at $x = \exp_x^{-1}(0)$. Moreover, according to (1) ⇒ (2) above, we have that

$$d\varphi(x) = d\psi(0) \quad \text{and} \quad d^2\varphi(x) = d^2\psi(0),$$

so we get $d\varphi(x) = \zeta$ and $d^2\varphi(x) = A$. Finally, by using smooth partitions of unity we can extend φ from an open neighborhood of x to all of M. □

Corollary 2.3. *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function, and consider $\zeta \in TM_x^*$, $A \in \mathcal{L}_s^2(TM_x, \mathbb{R})$, $x \in M$. Then*

$$(\zeta, A) \in J^{2,-} f(x) \quad \Longleftrightarrow \quad (\zeta, A) \in J^{2,-}(f \circ \exp_x)(0_x).$$

Making use of the above characterization, one can easily extend many known properties of the sets $J^{2,-}f(x)$ and $J^{2,+}f(x)$ from the Euclidean to the Riemannian setting. For instance, one can immediately see that $J^{2,-}f(x)$ and $J^{2,+}f(x)$ are convex subsets of $TM_x^* \times \mathcal{L}_s^2(TM_x)$. They are not necessarily closed, but if one fixes a $\zeta \in TM_x^*$ then the set $\{A\colon (\zeta, A) \in J^{2,-}f(x)\}$ is closed. A useful property that also extends from Euclidean to Riemannian is the following: if ψ is C^2 smooth on a neighborhood of x then

$$J^{2,-}(f - \psi)(x) = \big\{\big(\zeta - d\psi(x),\ A - d^2\psi(x)\big)\colon (\zeta, A) \in J^{2,-}f(x)\big\}.$$

One can also see that f is twice differentiable at a point $x \in M$ (in the sense that for some unique $\zeta \in TM_x^*$, $A \in \mathcal{L}_s^2(TM_x, \mathbb{R})$ we have that $f(\exp_x(v)) = f(x) + \langle \zeta, v\rangle + \frac{1}{2}\langle Av, v\rangle + o(\|v\|^2)$ as $v \to 0$) if and only if $J^{2,-}f(x) \cap J^{2,+}f(x)$ is nonempty (in which case $J^{2,-}f(x) \cap J^{2,+}f(x) = \{(\zeta, A)\}$).

Next we have to define the closures of these set-valued mappings. Let us first recall that a sequence (A_n) with $A_n \in \mathcal{L}_s^2(TM_{x_n})$ is said to converge to $A \in \mathcal{L}_s^2(TM_x)$ provided x_n converges to x in M and for every vector field V defined on an open neighborhood of x we have that $\langle A_n V(x_n), V(x_n)\rangle$ converges to $\langle AV(x), V(x)\rangle$. Since we have $\langle AV, W\rangle = \frac{1}{2}(\langle A(V + W), V + W\rangle - \langle AV, V\rangle - \langle AW, W\rangle)$, it is clear that this is equivalent to saying that $\langle A_n V(x_n), W(x_n)\rangle$ converges to $\langle AV(x), W(x)\rangle$ for all vector fields V, W on a neighborhood of x in M.

Similarly, a sequence (ζ_n) with $\zeta_n \in TM_{x_n}^*$ converges to ζ provided that $x_n \to x$ and $\langle \zeta_n, V(x_n)\rangle \to \langle \zeta, V(x)\rangle$ for every vector field V defined on an open neighborhood of x.

Remark 2.4. It is not difficult to see that, if $M = \mathbb{R}^n$, then A_n (respectively ζ_n) converges to A (respectively ζ) in the above sense if and only if $\|A_n - A\| \to 0$ (respectively $\|\zeta_n - \zeta\| \to 0$) in $\mathcal{L}_s^2(\mathbb{R}^n, \mathbb{R})$ (respectively in $\mathbb{R}^n$).

It is also worth noting that $\|A_n - A\| \to 0$ (respectively $\|\zeta_n - \zeta\| \to 0$) in $\mathcal{L}_s^2(\mathbb{R}^n, \mathbb{R})$ (respectively in $\mathbb{R}^n$) if and only if $\langle A_n v, v\rangle \to \langle Av, v\rangle$ (respectively $\langle \zeta_n, v\rangle \to \langle \zeta, v\rangle$) for every $v \in \mathbb{R}^n$, that is pointwise convergence is equivalent to uniform convergence on bounded sets, as far as linear or bilinear maps on $\mathbb{R}^n$ are concerned.

Definition 2.5. Let f be a lower semicontinuous function defined on a Riemannian manifold M, and $x \in M$. We define

$$\overline{J}^{2,-} f(x) = \bigl\{(\zeta, A) \in TM_x^* \times \mathcal{L}_s(TM_x)\colon\ \exists x_n \in M,\ \exists (\zeta_n, A_n) \in J^{2,-} f(x_n) \text{ s.t. } \bigl(x_n, f(x_n), \zeta_n, A_n\bigr) \to \bigl(x, f(x), \zeta, A\bigr)\bigr\},$$

and for an upper semicontinuous function g on M we define $\overline{J}^{2,+}g(x)$ in an obvious similar way.

Remark 2.6. According to Remark 2.4, we have that, in the case $M = \mathbb{R}^n$, the sets $\overline{J}^{2,-}g(x)$ and $\overline{J}^{2,+}g(x)$ coincide with the subjets and superjets defined in [4].

In order to establish the analogue of Corollary 2.3 for the closure $\overline{J}^{2,-}g(x)$, we will use the following fact.

Lemma 2.7. *Let $\varphi : M \to \mathbb{R}$ be a C^2 smooth function, and define $\psi = \varphi \circ \exp_x$ on a neighborhood of $0 \in TM_x$. Let $\widetilde{V}$ be a vector field defined on a neighborhood of 0 in TM_x, and consider the vector field defined by $V(y) = d\exp_x(w_y)(\widetilde{V}(w_y))$ on a neighborhood of x in M, where $w_y := \exp_x^{-1}(y)$, and let*

$$\sigma_y(t) = \exp_x\bigl(w_y + t\widetilde{V}(w_y)\bigr).$$

Then we have that

$$D^2\psi(\widetilde{V}, \widetilde{V})(w_y) = D^2\varphi(V, V)(y) + \bigl\langle \nabla\varphi(y), \sigma_y''(0)\bigr\rangle.$$

Observe that $\sigma_x''(0) = 0$ so, when $y = x$, we obtain

$$d^2\psi(0)(v, v) = d^2\varphi(x)(v, v)$$

for every $v \in TM_x$.

Proof of Lemma 2.7. Fix y near x. We have that

$$\frac{d}{dt}\psi\bigl(w_y + t\widetilde{V}(w_y)\bigr) = \frac{d}{dt}\varphi\bigl(\sigma_y(t)\bigr) = \bigl\langle \nabla\varphi\bigl(\sigma_y(t)\bigr), \sigma_y'(t)\bigr\rangle$$

and

$$\frac{d^2}{dt^2}\psi\bigl(w_y + t\widetilde{V}(w_y)\bigr) = \frac{d^2}{dt^2}\varphi\bigl(\sigma_y(t)\bigr) = \bigl\langle \nabla_{\sigma_y'(t)}\nabla\varphi\bigl(\sigma_y(t)\bigr), \sigma_y'(t)\bigr\rangle + \bigl\langle \nabla\varphi\bigl(\sigma_y(t)\bigr), \sigma_y''(t)\bigr\rangle.$$

Note that $\sigma_y'(0)=V(y)$, hence by taking $t=0$ we get the equality in the statement. Observe that when $y=x$ the curve σ_x is a geodesic, so $\sigma_x''(0)=0$. □

Proposition 2.8. *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function, and consider $\zeta \in TM_x^*$, $A \in \mathcal{L}_s^2(TM_x, \mathbb{R})$, $x \in M$. Then*

$$(\zeta, A) \in \overline{J}^{2,-} f(x) \quad \Longleftrightarrow \quad (\zeta, A) \in \overline{J}^{2,-}(f \circ \exp_x)(0_x).$$

Proof. ($\Rightarrow$) If $(\zeta, A) \in \overline{J}^{2,-} f(x)$ there exist $x_n \to x$ and $(\zeta_n, A_n) \in J^{2,-} f(x_n)$ so that $\zeta_n \to \zeta$, $A_n \to A$, $f(x_n) \to f(x)$. Take $\varphi_n \in C^2(M)$ such that $f - \varphi_n$ attains a minimum at x_n and $\zeta_n = d\varphi_n(x_n)$, $A_n = d^2\varphi_n(x_n)$. Define $\psi_n = \varphi_n \circ \exp_x$ on a neighborhood of 0 in TM_x, and $v_n = \exp_x^{-1}(x_n)$. It is clear that $f \circ \exp_x - \psi_n$ attains a minimum at v_n. We then have that $(d\psi_n(v_n), d^2\psi_n(v_n)) \in J^{2,-}(f \circ \exp_x)(v_n)$, and since $v_n \to 0$ and $f \circ \exp_x(v_n) \to f(x)$, we only have to show that $d\psi_n(v_n) \to \zeta$ and $d^2\psi_n(v_n) \to A$.

Take a vector field $\widetilde{V}$ on TM_x, and define a corresponding vector field V on a neighborhood of x in M by

$$V(y) = d\exp_x(w_y)\big(\widetilde{V}(w_y)\big),$$

where $w_y = \exp_x^{-1}(y)$. We have that

$$\big\langle d\psi_n(v_n), \widetilde{V}(v_n)\big\rangle = \big\langle d\varphi_n(x_n) \circ d\exp_x(v_n), \widetilde{V}(v_n)\big\rangle = \big\langle d\varphi_n(x_n), V(x_n)\big\rangle,$$

so we get

$$\big\langle d\psi_n(v_n), \widetilde{V}(v_n)\big\rangle = \big\langle \zeta_n, V(x_n)\big\rangle \to \big\langle \zeta, V(x)\big\rangle = \big\langle \zeta, \widetilde{V}(0)\big\rangle,$$

which shows $d\psi_n(v_n) \to \zeta$. On the other hand, according to the preceding lemma, we also have that

$$d^2\psi_n(v_n)\big(\widetilde{V}(v_n), \widetilde{V}(v_n)\big) = A_n\big(V(x_n), V(x_n)\big) + \big\langle \zeta_n, \sigma_{x_n}''(0)\big\rangle,$$

where $\sigma_y(t) = \exp_x(w_y + t\widetilde{V}(w_y))$.

Notice that the mapping $y \mapsto \sigma_y''(0)$ defines a smooth vector field on a neighborhood of x in M (and in particular $\sigma_{x_n}''(0) \to \sigma_x''(0) = 0$ as $n \to \infty$). Since $A_n \to A$, $\zeta_n \to \zeta$, we get, by taking limits as $n \to \infty$ in the above equality, that

$$d^2\psi_n(v_n)\big(\widetilde{V}(v_n), \widetilde{V}(v_n)\big) \to A\big(V(x), V(x)\big) + 0 = A\big(\widetilde{V}(0), \widetilde{V}(0)\big),$$

which proves that $d^2\psi_n(v_n) \to A$.

($\Leftarrow$) If $(\zeta, A) \in \overline{J}^{2,-}(f \circ \exp_x)(0)$ there exist $v_n \to 0$ and $(\tilde{\zeta}_n, \widetilde{A}_n) \in J^{2,-}(f \circ \exp_x)(v_n)$ so that $\tilde{\zeta}_n \to \zeta$, $\widetilde{A}_n \to A$, $f(x_n) \to f(x)$, where $x_n = \exp_x(v_n)$. Take $\psi_n \in C^2(TM_x)$ such that $f \circ \exp_x - \psi_n$ attains a minimum at v_n and $\tilde{\zeta}_n = d\psi_n(v_n)$, $\widetilde{A}_n = d^2\psi(v_n)$. Define $\varphi_n = \psi_n \circ \exp_x^{-1}$ on a neighborhood of x in M. Then $f - \varphi_n$ attains a minimum at x_n, so $(d\varphi_n(x_n), d^2\varphi_n(x_n)) \in J^{2,-} f(x_n)$, and we only have to show that $d\varphi_n(x_n) \to \zeta$ and $d^2\varphi_n(x_n) \to A$. Take a vector field

V on a neighborhood of x in M, and define a corresponding vector field $\widetilde{V}$ on a neighborhood of 0 in TM_x by

$$\widetilde{V}(w_y) = d\exp_x^{-1}(y)\big(V(y)\big),$$

where $w_y = \exp_x^{-1}(y)$. Now we have that

$$\big\langle d\psi_n(v_n), \widetilde{V}(v_n)\big\rangle = \big\langle d\varphi_n(x_n), V(x_n)\big\rangle,$$

from which we deduce that $d\varphi_n(x_n) \to \zeta$; and also, by using this fact and the preceding lemma,

$$\begin{aligned} d^2\varphi_n\big(V(x_n), V(x_n)\big) &= \widetilde{A}_n\big(\widetilde{V}(v_n), \widetilde{V}(v_n)\big) - \big\langle d\varphi_n(x_n), \sigma''_{x_n}(0)\big\rangle \\ &\to \big\langle AV(x), V(x)\big\rangle - \big\langle \zeta, \sigma''_x(0)\big\rangle = \big\langle AV(x), V(x)\big\rangle - 0, \end{aligned}$$

concluding the proof. □

Remark 2.9. One can see, as in the case of $J^{2,-}f(x)$, that if ψ is C^2 smooth on a neighborhood of x then

$$\overline{J}^{2,-}(f-\psi)(x) = \big\{\big(\zeta - d\psi(x),\ A - d^2\psi(x)\big)\colon (\zeta, A) \in \overline{J}^{2,-}f(x)\big\}.$$

The following result is the Riemannian version of Theorem 3.2 in [4] and, as in that paper, will be the key to the proofs of comparison and uniqueness results for viscosity solutions of second order PDEs on Riemannian manifolds.

Theorem 2.10. *Let $M_1, \ldots, M_k$ be Riemannian manifolds, and $\Omega_i \subset M_i$ open subsets. Define $\Omega = \Omega_1 \times \cdots \times \Omega_k \subset M_1 \times \cdots \times M_k = M$. Let u_i be upper semicontinuous functions on Ω_i, $i = 1, \ldots, k$; let φ be a C^2 smooth function on Ω and set*

$$\omega(x) = u_1(x_1) + \cdots + u_k(x_k)$$

for $x = (x_1, \ldots, x_k) \in \Omega$. Assume that $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k)$ is a local maximum of $\omega - \varphi$. Then, for each $\varepsilon > 0$ there exist bilinear forms $B_i \in \mathcal{L}^2_s((TM_i)_{\hat{x}_i}, \mathbb{R})$, $i = 1, \ldots, k$, such that

$$\left(\frac{\partial}{\partial x_i}\varphi(\hat{x}), B_i\right) \in \overline{J}^{2,+}u_i(\hat{x}_i)$$

for $i = 1, \ldots, k$, and the block diagonal matrix with entries B_i satisfies

$$-\left(\frac{1}{\varepsilon} + \|A\|\right)I \leqslant \begin{pmatrix} B_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & B_k \end{pmatrix} \leqslant A + \varepsilon A^2,$$

where $A = d^2\varphi(\hat{x}) \in \mathcal{L}^2_s(TM_{\hat{x}}, \mathbb{R})$.

Recall that, for $\zeta \in TM^*$, $A \in \mathcal{L}(TM_x \times TM_x, \mathbb{R})$, the norms $\|\zeta\|_x$ and $\|A\|_x$ are defined by

$$\|\zeta\|_x = \sup\big\{\langle \zeta, v\rangle_x \colon\ v \in TM_x,\ \|v\|_x \leqslant 1\big\}$$

and

$$\|A\|_x = \sup\big\{\big|\langle Av, v\rangle_x\big| \colon\ v \in TM_x,\ \|v\|_x \leqslant 1\big\} = \sup\big\{|\lambda| \colon\ \lambda \text{ is an eigenvalue of } A\big\}.$$

Proof of Theorem 2.10. The result is proved in [4] in the case when all the manifolds M_i are Euclidean spaces, and we are going to reduce the problem to this situation. By taking smaller neighborhoods of the x_i if necessary, we can assume that the Ω_i are diffeomorphic images of balls by the exponential mappings $\exp_{\hat{x}_i} : B(0, r_i) \to \Omega_i = B(\hat{x}_i, r_i)$, and that $\exp_{\hat{x}}$ maps diffeomorphically a ball in $TM_{\hat{x}}$ onto a ball containing Ω. The exponential map $\exp_{\hat{x}}$ from this ball in $TM_{\hat{x}} = (TM_1)_{\hat{x}_1} \times \cdots \times (TM_k)_{\hat{x}_k}$ into M is given by

$$\exp_{\hat{x}}(v_1, \ldots, v_k) = \big(\exp_{\hat{x}_1}(v_1), \ldots, \exp_{\hat{x}_k}(v_k)\big).$$

Now define functions on open subsets of Euclidean spaces by $\tilde{\omega}(v) = \omega(\exp_{\hat{x}}(v))$ and $\tilde{u}_i(v_i) = u_i(\exp_{\hat{x}_i}(v_i))$. We have that $\tilde{\omega}(v_1, \ldots, v_k) = \tilde{u}_1(v_1) + \cdots + \tilde{u}_k(v_k)$, and $0_{\hat{x}} = (0_{\hat{x}_1}, \ldots, 0_{\hat{x}_k})$ is a local maximum of $\tilde{\omega} - \psi$, where $\psi = \varphi \circ \exp_{\hat{x}}$.

Then, by the known result for Euclidean spaces, for each $\varepsilon > 0$ there exist bilinear forms $B_i \in \mathcal{L}_s^2((TM_i)_{\hat{x}_i}, \mathbb{R})$, $i = 1, \ldots, k$, such that

$$\left(\frac{\partial}{\partial v_i}\psi(0_{\hat{x}}), B_i\right) \in \overline{J}^{\,2,+}\tilde{u}_i(0_{\hat{x}_i})$$

for $i = 1, \ldots, k$, and the block diagonal matrix with entries B_i satisfies

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) I \leqslant \begin{pmatrix} B_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & B_k \end{pmatrix} \leqslant A + \varepsilon A^2,$$

where $A = d^2\psi(0_{\hat{x}}) \in \mathcal{L}_s^2(TM_{\hat{x}}, \mathbb{R})$. According to Proposition 2.8 we have that

$$\left(\frac{\partial}{\partial v_i}\psi(0_{\hat{x}}), B_i\right) \in \overline{J}^{\,2,+}\tilde{u}_i(0_{\hat{x}_i}) \quad\iff\quad \left(\frac{\partial}{\partial v_i}\psi(0_{\hat{x}}), B_i\right) \in \overline{J}^{\,2,+}u_i(\hat{x}_i),$$

so we are done if we only see that

$$\frac{\partial}{\partial v_i}\psi(0_{\hat{x}}) = \frac{\partial}{\partial x_i}\varphi(\hat{x}) \quad \text{and} \quad d^2\psi(0_{\hat{x}}) = d^2\varphi(\hat{x}).$$

But this is a consequence of Lemma 2.7. □

Now we extend the notion of viscosity solution to a Hamilton–Jacobi equation on a Riemannian manifold. In the sequel we will denote

$$\mathcal{X} := \big\{(x, r, \zeta, A) \colon\ x \in M,\ r \in \mathbb{R},\ \zeta \in TM_x,\ A \in \mathcal{L}_s^2(TM_x)\big\}.$$

Definition 2.11 *(Viscosity solution).* Let M be a Riemannian manifold, and $F : \mathcal{X} \to \mathbb{R}$. We say that an upper semicontinuous function $u : M \to \mathbb{R}$ is a viscosity subsolution of the equation $F = 0$ provided that

$$F\big(x, u(x), \zeta, A\big) \leqslant 0$$

for all $x \in M$ and $(\zeta, A) \in J^{2,+}u(x)$. Similarly, a viscosity supersolution of $F = 0$ on M is a lower semicontinuous function $u : M \to \mathbb{R}$ such that

$$F\big(x, u(x), \zeta, A\big) \geqslant 0$$

for every $x \in M$ and $(\zeta, A) \in J^{2,-}u(x)$. If u is both a viscosity subsolution and a viscosity supersolution of $F = 0$, we say that u is a viscosity solution of $F = 0$ on M.

Remark 2.12. If u is a solution of $F \leqslant 0$ and F is continuous on $\mathcal{X}$ then $F(x, u(x), \zeta, A) \leqslant 0$ for every $(\zeta, A) \in \overline{J}^{2,+}u(x)$. A similar observation applies to solutions of $F \geqslant 0$ and solutions of $F = 0$.

Definition 2.13 *(Degenerate ellipticity).* We will say that a function $F : \mathcal{X} \to \mathbb{R}$ is *degenerate elliptic* provided that

$$A \leqslant B \quad \Longrightarrow \quad F(x, r, \zeta, B) \leqslant F(x, r, \zeta, A)$$

for all $x \in M$, $r \in \mathbb{R}$, $\zeta \in TM_x$, $A, B \in \mathcal{L}^2_s(TM_x)$.

Example 2.14. If we canonically identify the space of symmetric bilinear forms on TM_x with the space of self-adjoint linear mappings from TM_x into TM_x, we have that

$$L_{yx}(Q) = L_{xy}^{-1} Q L_{xy}.$$

Hence

$$\operatorname{trace}(L_{yx} Q) = \operatorname{trace}(Q) \quad \text{and} \quad \det_+\big(L_{yx}(Q)\big) = \det_+(Q)$$

(where $\det_+ A$ is defined as the product of the nonnegative eigenvalues of A), and it is immediately seen that the functions $G(r, \zeta, A) = -\det_+(A)$ and $H(r, \zeta, A) = -\operatorname{trace}(A)$ are degenerate elliptic and, moreover, are invariant by parallel translation, in the sense that

$$G(r, \zeta, A) = G(r, L_{xy}\zeta, L_{xy}A)$$

for all $r \in \mathbb{R}$, $\zeta \in TM_x$, $A \in \mathcal{L}^2_s(TM_x)$. The same can be said of all nonincreasing, symmetric functions of the eigenvalues of A. Thus one may combine such functions to construct many interesting examples of equations to which our results apply, as we will see later on.

Remark 2.15. *If the function F is degenerate elliptic, then every classical solution of $F = 0$ is a viscosity solution of $F = 0$*, as is immediately seen. However this may be not true if F is not degenerate elliptic; for instance when $M = \mathbb{R}$ the function $u(x) = x^2 - 2$ is a classical solution of $u''(x) + u(x) - x^2 = 0$ but is not a viscosity solution.

In order that the theory of viscosity solutions applies to an equation $F = 0$, the following condition is usually required.

Definition 2.16 *(Properness).* We will say that a function $F : \mathcal{X} \to \mathbb{R}$, $(x, r, \zeta, A) \mapsto F(x, r, \zeta, A)$, is *proper* provided

(i) F is degenerate elliptic, and
(ii) F is nondecreasing in the variable r.

3. A key property of the Hessian of the function $(x, y) \mapsto d(x, y)^2$

When trying to establish comparison results for viscosity solutions of second order PDEs on a Riemannian manifold M we will need to know how the Hessian of the function $\varphi : M \times M \to \mathbb{R}$,

$$\varphi(x, y) = d(x, y)^2$$

behaves. More precisely we will need to know on which manifolds M one has that

$$d^2\varphi(x, y)(v, L_{xy}v)^2 \leqslant 0 \tag{$\sharp$}$$

for all $v \in TM_x$, with $x, y \in M$ close enough to each other so that $d(x, y) < \min\{i_M(x), i_M(y)\}$.

Let us calculate this derivative. We have that

$$\frac{\partial\varphi}{\partial x}(x, y) = 2d(x, y)\frac{\partial d}{\partial x}(x, y) = -2\exp_x^{-1}(y). \tag{1}$$

The second equality can be checked, for instance, by using the first variation formula of the arc-length (see [10, p. 90]). Indeed, if $\alpha(t, s)$ is a variation through geodesics of a minimizing geodesic $\gamma(t)$ with $y = \gamma(0)$ and $x = \gamma(\ell)$, where $\ell = d(x, y)$, and if $L(s)$ denotes the length of the geodesic $t \mapsto \alpha(t, s)$, then

$$\frac{d}{ds}L(s)\bigg|_{s=0} = \left[\langle V, T\rangle\big|_0^\ell - \int_0^\ell \langle V, \nabla_T T\rangle\, dt\right] = \big(\langle V(\ell), T(\ell)\rangle - \langle V(0), T(0)\rangle\big),$$

where $T = \partial\alpha/\partial t$ (so $\nabla_T T = 0$) and $V = \partial\alpha/\partial s$. Taking an α such that V is the Jacobi field along γ satisfying $V(0) = 0$, $V(\ell) = v$, we get

$$\frac{\partial d}{\partial x}(x, y)(v) = \frac{d}{ds}L(s)\bigg|_{s=0} = \frac{1}{\ell}\big\langle v, -\exp_x^{-1}(y)\big\rangle.$$

Similarly, we have

$$\frac{\partial\varphi}{\partial y}(x, y) = 2d(x, y)\frac{\partial d}{\partial y}(x, y) = -2\exp_y^{-1}(x). \tag{2}$$

Observe that

$$\frac{\partial\varphi}{\partial y}(x, y) + L_{xy}\left(\frac{\partial\varphi}{\partial x}(x, y)\right) = 0 = \frac{\partial d}{\partial y}(x, y) + L_{xy}\left(\frac{\partial d}{\partial x}(x, y)\right). \tag{3}$$

By differentiating again in (1) and (2), we get

$$\frac{\partial^2\varphi}{\partial x^2}(x,y)(v)^2 = 2\left(\frac{\partial d}{\partial x}(x,y)(v)\right)^2 + 2d(x,y)\frac{\partial^2 d}{\partial x^2}(x,y)(v)^2,$$

$$\frac{\partial^2\varphi}{\partial x\partial y}(x,y)(v,w) = 2\frac{\partial d}{\partial x}(x,y)(v)\frac{\partial d}{\partial y}(x,y)(w) + 2d(x,y)\frac{\partial^2 d}{\partial x\partial y}(x,y)(v,w),$$

$$\frac{\partial^2\varphi}{\partial y^2}(x,y)(w)^2 = 2\left(\frac{\partial d}{\partial y}(x,y)(w)\right)^2 + 2d(x,y)\frac{\partial^2 d}{\partial y^2}(x,y)(w)^2,$$

so, if we take $w = L_{xy}v$ and we sum the two first equations, and then we use (3), we get that

$$\frac{\partial^2\varphi}{\partial x^2}(x,y)(v)^2 + \frac{\partial^2\varphi}{\partial x\partial y}(x,y)(v,L_{xy}v) = 2d(x,y)\left[\frac{\partial^2 d}{\partial x^2}(x,y)(v)^2 + \frac{\partial^2 d}{\partial x\partial y}(x,y)(v,L_{xy}v)\right]$$

and we get a similar equation by changing x for y. By summing these two equations we get

$$d^2\varphi(x,y)(v,L_{xy}v)^2 = 2d(x,y)d^2(d)(x,y)(v,L_{xy}v)^2,$$

so it is clear that condition (♯) holds if and only if

$$d^2(d)(x,y)(v,L_{xy}v)^2 \leqslant 0 \qquad (\flat)$$

for all $v \in TM_x$.

Another way to write conditions (♯) or (♭) is

$$\frac{d^2}{dt^2}\big(d\big(\sigma_x(t),\sigma_y(t)\big)\big)\bigg|_{t=0} \leqslant 0, \qquad (\natural)$$

where σ_x and σ_y are geodesics with $\sigma_x(0) = x$, $\sigma_y(0) = y$, $\sigma_x'(0) = v$ and $\sigma_y'(0) = L_{xy}v$. The function $t \mapsto h(t) := d(\sigma_x(t), \sigma_y(t))$ measures the distance between the geodesics σ_x and σ_y (which have the same velocity and are parallel at $t = 0$) evaluated at a point moving along any of these geodesics.

We are going to show that the second derivative $h''(0)$ is negative (that is, condition (♮) holds) if and only if M has positive sectional curvature.

In particular, by combining this fact with Eq. (3) (which tells us that $h'(0) = 0$), we see that the function $h(t)$ attains a local maximum at $t = 0$ if and only if M has positive sectional curvature. This corresponds to the intuitive notion that two geodesics that are parallel at their starting points will get closer if the sectional curvature is positive, while they will spread apart if the sectional curvature is negative.

Proposition 3.1. *Condition* (♯) (*equivalently* (♭)*, or* (♮)) *holds for a Riemannian manifold M if and only if M has nonnegative sectional curvature. In fact one has, for the function $\varphi(x,y) = d(x,y)^2$ on $M \times M$, that*:

(1) *If M has nonnegative sectional curvature then*

$$d^2\varphi(x, y)(v, L_{xy}v)^2 \leqslant 0$$

for all $v \in TM_x$, with $x, y \in M$ close enough to each other so that $d(x, y) < \min\{i_M(x), i_M(y)\}$.

(2) *If M has nonpositive sectional curvature then*

$$d^2\varphi(x, y)(v, L_{xy}v)^2 \geqslant 0$$

for all $v \in TM_x$, $x, y \in M$ such that $d(x, y) < \min\{i_M(x), i_M(y)\}$.

This fact must be known to the specialists in Riemannian geometry, but we have been unable to find a reference for part (1), so we provide a proof. Let us begin by reviewing some standard facts about the second variation of the arc-length and the energy functionals.

Take two points $x_0, y_0 \in M$ with $d(x_0, y_0) < \min\{i_M(x_0), i_M(y_0)\}$, and let γ be the unique minimizing geodesic, parameterized by arc-length, connecting x_0 to y_0. Denote $\ell = d(x_0, y_0)$, the length of γ. Consider $\alpha(t, s)$, a smooth variation of γ, that is a smooth mapping $\alpha : [0, \ell] \times [-\varepsilon, \varepsilon] \to M$ such that $\alpha(t, 0) = \gamma(t)$ for all $t \in [0, \ell]$. Consider the length and the energy functionals, defined by

$$L(s) = L(\alpha_s) = \int_0^\ell \left\| \alpha_s'(t) \right\| dt$$

and

$$E(s) = E(\alpha_s) = \int_0^\ell \left\| \alpha_s'(t) \right\|^2 dt,$$

where α_s is the variation curve defined by $\alpha_s(t) = \alpha(t, s)$ for every $t \in [0, \ell]$. According to the Cauchy–Schwarz inequality (applied to the functions $f \equiv 1$ and $g(t) = \|\alpha_s'(t)\|$ on the interval $[0, \ell]$) we have that

$$L(s)^2 \leqslant \ell E(s),$$

with equality if and only if $\|\alpha_s'(t)\|$ is constant. Therefore, in the case when α_s is a geodesic for each s (that is α is a variation of γ through geodesics) we have that

$$L(s)^2 = \ell E(s)$$

for every $s \in [-\varepsilon, \varepsilon]$.

Now take a vector $v \in TM_{x_0}$, set $w = L_{x_0 y_0} v$, and consider the geodesics $\sigma_{x_0}, \sigma_{y_0}$ defined by

$$\sigma_{x_0}(s) = \exp_{x_0}(sv), \qquad \sigma_{y_0}(s) = \exp_{y_0}(sw).$$

We want to calculate

$$d^2\varphi(x_0, y_0)(v, w)^2 = \frac{d^2}{ds^2}\varphi\big(\sigma_{x_0}(s), \sigma_{y_0}(s)\big)\Big|_{s=0},$$

where $\varphi(x, y) = d(x, y)^2$. To this end let us denote by $\alpha_s : [0, \ell] \to M$ the unique minimizing geodesic joining the point $\sigma_{x_0}(s)$ to the point $\sigma_{y_0}(s)$ (now, for $s \neq 0$, α_s is not necessarily parameterized by arc-length), and let us define $\alpha : [0, \ell] \times [-\varepsilon, \varepsilon] \to M$ by $\alpha(t, s) = \alpha_s(t)$. Then α is a smooth variation through geodesics of $\gamma(t) = \alpha(t, 0)$ and, according to the above discussion, we have

$$\varphi\big(\sigma_{x_0}(s), \sigma_{y_0}(s)\big) = L(s)^2 = \ell E(s),$$

and therefore

$$d^2\varphi(x_0, y_0)(v, w)^2 = \ell E''(0). \tag{4}$$

If we denote $X(t) = \partial\alpha(t, 0)/\partial s$, the variational field of α, then the formula for the second variation of energy (see [3, p. 197]) tells us that

$$\frac{1}{2}E''(0) = -\int_0^\ell \big\langle X, X'' + R(\gamma', X)\gamma'\big\rangle\, dt + \big\langle X(t), X'(t)\big\rangle\Big|_{t=0}^{t=\ell} + \left\langle \frac{D}{ds}\frac{\partial\alpha}{\partial s}(t, 0), \gamma'(t)\right\rangle\bigg|_{t=0}^{t=\ell}, \tag{5}$$

or equivalently

$$\frac{1}{2}E''(0) = \int_0^\ell \big(\langle X', X'\rangle - \big\langle R(\gamma', X)\gamma', X\big\rangle\big)\, dt + \left\langle \frac{D}{ds}\frac{\partial\alpha}{\partial s}(t, 0), \gamma'(t)\right\rangle\bigg|_{t=0}^{t=\ell}, \tag{6}$$

where we denote $X' = \nabla_{\gamma'(t)}X$, and $X'' = \nabla_{\gamma'(t)}X'$.

Note that, since the variation field of a variation through geodesics is always a Jacobi field, and since the points x_0 and y_0 are not conjugate, the field X is in fact the unique Jacobi field along γ satisfying that $X(0) = v$, $X(\ell) = w$, that is X is the unique vector field along γ satisfying

$$X''(t) + R\big(\gamma'(t), X(t)\big)\gamma'(t) = 0 \quad \text{and} \quad X(0) = v, \quad X(\ell) = w,$$

where R is the curvature of M. On the other hand, since the curves $s \to \alpha(0, s) = \sigma_{x_0}(s)$ and $s \to \alpha(\ell, s) = \sigma_{y_0}(s)$ are geodesics, we have that

$$\left\langle \frac{D}{ds}\frac{\partial\alpha}{\partial s}(t, 0), \gamma'(t)\right\rangle\bigg|_{t=0}^{t=\ell} = 0.$$

These observations allow us to simplify (5) and (6) by dropping the terms that vanish, thus obtaining that

$$\frac{1}{2}E''(0) = \big\langle X(\ell), X'(\ell)\big\rangle - \big\langle X(0), X'(0)\big\rangle \tag{7}$$

and also

$$\frac{1}{2}E''(0) = \int_0^\ell \big(\langle X', X'\rangle - \langle R(\gamma', X)\gamma', X\rangle\big)\,dt. \tag{8}$$

Recall that the right-hand side of (8) is called the index form and is denoted by $I(X, X)$.

By combining (4), (7) and (8) we get

$$\begin{aligned} d^2\varphi(x_0, y_0)(v, w)^2 &= 2\ell\big(\langle X(\ell), X'(\ell)\rangle - \langle X(0), X'(0)\rangle\big) \\ &= 2\ell \int_0^\ell \big(\langle X', X'\rangle - \langle R(\gamma', X)\gamma', X\rangle\big)\,dt. \end{aligned}$$

Therefore condition ($\sharp$) holds if and only if, for every Jacobi field X along γ with $X(0) = v$, $X(\ell) = w = L_{x_0 y_0} v$, one has that

$$\langle X(\ell), X'(\ell)\rangle - \langle X(0), X'(0)\rangle \leqslant 0 \tag{$\diamondsuit$}$$

or, equivalently,

$$\int_0^\ell \big(\langle X', X'\rangle - \langle R(\gamma', X)\gamma', X\rangle\big)\,dt \leqslant 0$$

for the same Jacobi fields.

Proof of Proposition 3.1. The proof of (2) is immediate and is well referenced (see for instance [7, Theorem IX.4.3]): if M has nonpositive sectional curvature then we have $\langle R(\gamma', X)\gamma', X\rangle \leqslant 0$, hence, according to the above formulas,

$$d^2\varphi(x_0, y_0)(v, w)^2 = 2\ell \int_0^\ell \big(\langle X', X'\rangle - \langle R(\gamma', X)\gamma', X\rangle\big)\,dt \geqslant 2\ell \int_0^\ell \langle X', X'\rangle\,dt \geqslant 0,$$

which proves (2). Note that in this case we do not use that $w = L_{x_0 y_0} v$, so this holds for all v, w.

Our proof of (1) uses the following lemma, which is a restatement of Corollary 10 in Chapter 8 of [11].

Lemma 3.2. *Let $\gamma : [0, \ell] \to M$ be a geodesic without conjugate points, X a Jacobi field along γ, and Z a piecewise smooth vector field along γ such that $X(0) = Z(0)$ and $X(\ell) = Z(\ell)$. Then*

$$I(X, X) \leqslant I(Z, Z),$$

and equality holds only when $Z = X$.

That is, among all vector fields along γ with the same boundary conditions, the unique Jacobi field along γ determined by those conditions minimizes the index form. Recall that

$$I(Z,Z)=\int_0^\ell \big(\langle Z',Z'\rangle-\big\langle R(\gamma',Z)\gamma',Z\big\rangle\big)\,dt,$$

but this number is not equal to $\langle Z(\ell),Z'(\ell)\rangle-\langle Z(0),Z'(0)\rangle$ unless Z is a Jacobi field.

Let X be the unique Jacobi field with $X(0)=v$, $X(\ell)=w=L_{x_0y_0}(v)$. Define $Z=P(t)$, where $P(t)$ is the parallel translation along γ with $P(0)=v$ (hence $P(\ell)=w$). The field Z is not necessarily a Jacobi field, but it has the considerable advantage that $Z'(t)=0$ for all t, so we have that

$$I(Z,Z)=\int_0^\ell \big(\langle Z',Z'\rangle-\big\langle R(\gamma',Z)\gamma',Z\big\rangle\big)\,dt=-\int_0^\ell \big\langle R(\gamma',Z)\gamma',Z\big\rangle\,dt\leqslant 0$$

because M has nonnegative sectional curvature. We then deduce from the above lemma that

$$\big\langle X(\ell),X'(\ell)\big\rangle-\big\langle X(0),X'(0)\big\rangle=I(X,X)\leqslant I(Z,Z)\leqslant 0,$$

which, according to the above remarks (see ($\diamondsuit$)), concludes the proof. $\square$

Even though we will not have $d^2\varphi(x,y)(v,L_{xy}v)^2\leqslant 0$ when M has negative curvature, we can estimate this quantity and show that it is bounded by a term of the order of $d(x,y)^2$, provided that the curvature is bounded below. This will also be used in the next section to deduce a comparison result which holds for all Riemannian manifolds (assuming that F is uniformly continuous).

Proposition 3.3. *Let M be a Riemannian manifold. Consider the function $\varphi(x,y)=d(x,y)^2$, defined on $M\times M$. Assume that the sectional curvature K of M is bounded below, say $K\geqslant -K_0$. Then*

$$d^2\varphi(x,y)(v,L_{xy}v)^2\leqslant 2K_0 d(x,y)^2\|v\|^2$$

for all $v\in TM_x$ and $x,y\in M$ with $d(x,y)<\min\{i_M(x),i_M(y)\}$.

Note that for $K_0=0$ we recover part (1) of Proposition 3.1.

Proof. Let X, Z be as in the proof of (1) of the preceding proposition. With the same notations, we have that

$$\begin{aligned}d^2\varphi(x,y)(v,L_{xy}v)^2&=2\ell I(X,X)\leqslant 2\ell I(Z,Z)\\&=-2\ell\int_0^\ell \big\langle R(\gamma',Z)\gamma',Z\big\rangle\,dt\leqslant 2\ell\int_0^\ell K_0\big|\gamma'(t)\wedge Z(t)\big|^2\,dt\end{aligned}$$

$$\leqslant 2\ell \int_0^\ell K_0 \|\gamma'(t)\|^2 \|Z(t)\|^2 = 2\ell \int_0^\ell K_0 \|v\|^2\, dt = 2\ell^2 K_0 \|v\|^2,$$

which proves the result. □

4. Comparison results for the Dirichlet problem

In this section and throughout the rest of the paper we will often abbreviate saying that u is an upper semicontinuous function on a set Ω by writing $u \in USC(\Omega)$. Similarly, $LSC(\Omega)$ will stand for the set of lower semicontinuous functions on Ω.

The following lemma will be used in the proof of the main comparison result for the Dirichlet problem

$$F\big(x, u(x), du(x), d^2u(x)\big) = 0 \quad \text{on } \Omega, \qquad u = f \quad \text{on } \partial\Omega. \tag{DP}$$

Lemma 4.1. *Let Ω be a subset of a Riemannian manifold M, $u \in USC(\overline{\Omega})$, $v \in LSC(\overline{\Omega})$ and*

$$m_\alpha := \sup_{\Omega\times\Omega} \left(u(x) - v(y) - \frac{\alpha}{2} d(x, y)^2 \right)$$

for $\alpha > 0$. Let $m_\alpha < \infty$ for large α and (x_α, y_α) be such that

$$\lim_{\alpha\to\infty} \left(m_\alpha - \left(u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2} d(x_\alpha, y_\alpha)^2 \right) \right) = 0.$$

Then we have:

(1) $\lim_{\alpha\to\infty} \alpha d(x_\alpha, y_\alpha)^2 = 0$, *and*
(2) $\lim_{\alpha\to\infty} m_\alpha = u(\hat{x}) - v(\hat{x}) = \sup_{x\in\Omega}(u(x) - v(x))$ *whenever $\hat{x} \in \Omega$ is a limit point of x_α as $\alpha \to \infty$.*

Proof. The result is proved in [4, Lemma 3.1] in the case $M = \mathbb{R}^n$, and the same proof clearly works in the generality of the statement (in fact this holds in any metric space). □

Now we can prove the main comparison result for the Dirichlet problem.

Theorem 4.2. *Let Ω be a bounded open subset of a complete finite-dimensional Riemannian manifold M, and $F : \mathcal{X} \to \mathbb{R}$ be proper, continuous, and satisfy*:

(1) *there exists $\gamma > 0$ such that*

$$\gamma(r - s) \leqslant F(x, r, \zeta, Q) - F(x, s, \zeta, Q)$$

for $r \geqslant s$; and

(2) *there exists a function* $\omega : [0, \infty] \to [0, \infty]$ *with* $\lim_{t\to 0^+} \omega(t) = 0$ *and such that*

$$F\big(y, r, \alpha \exp_y^{-1}(x), Q\big) - F\big(x, r, -\alpha \exp_x^{-1}(y), P\big) \leqslant \omega\big(\alpha d(x, y)^2 + d(x, y)\big)$$

for all $x, y \in \Omega$, $r \in \mathbb{R}$, $P \in T_{2,s}(M)_x$, $Q \in T_{2,s}(M)_y$ *with*

$$-\left(\frac{1}{\varepsilon_\alpha} + \|A_\alpha\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} P & 0 \\ 0 & -Q \end{pmatrix} \leqslant A_\alpha + \varepsilon_\alpha A_\alpha^2, \tag{$*$}$$

where A_α *is the second derivative of the function* $\varphi_\alpha(x, y) = \frac{\alpha}{2} d(x, y)^2$ *at the point* $(x, y) \in M \times M$,

$$\varepsilon_\alpha = \frac{1}{2(1 + \|A_\alpha\|)},$$

and the points x, y *are assumed to be close enough to each other so that* $d(x, y) < \min\{i_M(x), i_M(y)\}$.

Let $u \in USC(\overline{\Omega})$ *be a subsolution and* $v \in LSC(\overline{\Omega})$ *a supersolution of* $F = 0$ *on* Ω, *and* $u \leqslant v$ *on* $\partial\Omega$.

Then $u \leqslant v$ *holds on all of* $\overline{\Omega}$.

In particular the Dirichlet problem (*DP*) *has at most one viscosity solution.*

Proof. Assume to the contrary that there exists $z \in \Omega$ with $u(z) > v(z)$. By compactness of $\overline{\Omega}$ and upper semicontinuity of $u - v$, and according to Lemma 4.1, there exist x_α, y_α so that, with the notation of Lemma 4.1,

$$u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2} d(x_\alpha, y_\alpha)^2 = m_\alpha \geqslant \delta := u(z) - v(z) > 0 \tag{3}$$

and

$$\alpha d(x_\alpha, y_\alpha)^2 \to 0 \quad \text{as } \alpha \to \infty. \tag{4}$$

Again by compactness of $\overline{\Omega}$ we can assume that a subsequence of (x_α, y_α), which we will still denote (x_α, y_α) (and suppose $\alpha \in \mathbb{N}$), converges to a point $(x_0, y_0) \in \overline{\Omega} \times \overline{\Omega}$. By Lemma 4.1 we have that $x_0 = y_0$ and

$$\delta \leqslant \lim_{\alpha\to\infty} m_\alpha = u(x_0) - v(x_0) = \sup_{\overline{\Omega}}\big(u(x) - v(x)\big),$$

and in view of the condition $u \leqslant v$ on $\partial\Omega$ we have that $x_0 \in \Omega$, and $x_\alpha, y_\alpha \in \Omega$ for large α.

Fix $r_0 > 0$ and $R_0 > 0$ such that, for every $x \in B(x_0, r_0)$, $\exp_x$ is a diffeomorphism from $B(0, R_0) \subset TM_x$ onto $B(x, R_0) \supset B(x_0, r_0)$ (see [3, Theorem 3.7 of Chapter 3]). Then, for every $x, y \in B(x_0, r_0)$ we have that $d(x, y) < \min\{i_M(x), i_M(y)\}$, the vectors $\exp_x^{-1}(y) \in TM_x \equiv TM_x^*$ and $\exp_y^{-1}(x) \in TM_y \equiv TM_y^*$ are well defined, and the function $\varphi(x, y) = d(x, y)^2$ is C^2 smooth on $B(x_0, r_0) \times B(x_0, r_0) \in M \times M$. Taking a subsequence if necessary, we can assume that $x_\alpha, y_\alpha \in B(x_0, r_0)$ for all α.

Now, for each α, we can apply Theorem 2.10 with $\Omega_1 = \Omega_2 = B(x_0, r_0)$, $u_1 = u$, $u_2 = -v$, $\varphi(x, y) = \varphi_\alpha(x, y) := \frac{\alpha}{2} d(x, y)^2$, and for

$$\varepsilon = \varepsilon_\alpha := \frac{1}{2(1 + \|d^2\varphi_\alpha(x_\alpha, y_\alpha)\|)}.$$

Since (x_α, y_α) is a local maximum of the function $(x, y) \mapsto u(x) - v(y) - \varphi(x, y)$, we obtain bilinear forms $P \in \mathcal{L}^2_s(TM_{x_\alpha}, \mathbb{R})$, and $Q \in \mathcal{L}^2_s(TM_{y_\alpha}, \mathbb{R})$ such that

$$\left(\frac{\partial}{\partial x}\varphi(x_\alpha, y_\alpha), P\right) \in \overline{J}^{2,+}u(x_\alpha),$$
$$\left(-\frac{\partial}{\partial y}\varphi(x_\alpha, y_\alpha), Q\right) \in \overline{J}^{2,-}v(y_\alpha)$$

(recall that $\overline{J}^{2,-}v(y_\alpha) = -\overline{J}^{2,+}(-v)(y_\alpha)$), and

$$-\left(\frac{1}{\varepsilon_\alpha} + \|A_\alpha\|\right) I \leqslant \begin{pmatrix} P & 0 \\ 0 & -Q \end{pmatrix} \leqslant A_\alpha + \varepsilon_\alpha A_\alpha^2,$$

where $A_\alpha = d^2\varphi(x_\alpha, y_\alpha) \in \mathcal{L}^2_s(TM_{(x,y)}, \mathbb{R})$, so we get that condition $(*)$ holds for $x = x_\alpha$, $y = y_\alpha$. Therefore, according to condition (2), we have that

$$F\big(y_\alpha, r, \alpha \exp^{-1}_{y_\alpha}(x_\alpha), Q\big) - F\big(x_\alpha, r, -\alpha \exp^{-1}_{x_\alpha}(y_\alpha), P\big) \leqslant \omega\big(\alpha d(x_\alpha, y_\alpha)^2 + d(x_\alpha, y_\alpha)\big). \tag{5}$$

On the other hand, from Eq. (1) in the preceding section we have that

$$\frac{\partial}{\partial x}\varphi(x_\alpha, y_\alpha) = -\alpha \exp^{-1}_{x_\alpha}(y_\alpha) \quad \text{and} \quad -\frac{\partial}{\partial y}\varphi(x_\alpha, y_\alpha) = \alpha \exp^{-1}_{y_\alpha}(x_\alpha),$$

hence $(-\alpha \exp^{-1}_{x_\alpha}(y_\alpha), P) \in \overline{J}^{2,+}u(x_\alpha)$, $(\alpha \exp^{-1}_{y_\alpha}(x_\alpha), Q) \in \overline{J}^{2,-}v(y_\alpha)$. Since u is subsolution and v is supersolution, and F is continuous we then have, according to Remark 2.12, that

$$F\big(x_\alpha, u(x_\alpha), -\alpha \exp^{-1}_{x_\alpha}(y_\alpha), P\big) \leqslant 0 \leqslant F\big(y_\alpha, v(y_\alpha), \alpha \exp^{-1}_{y_\alpha}(x_\alpha), Q\big). \tag{6}$$

By combining Eqs. (3)–(6) above, and using condition (1) too, we finally get

$$\begin{aligned}
0 < \gamma\delta &\leqslant \gamma\big(u(x_\alpha) - v(y_\alpha)\big) \\
&\leqslant F\big(x_\alpha, u(x_\alpha), -\alpha \exp^{-1}_{x_\alpha}(y_\alpha), P\big) - F\big(x_\alpha, v(y_\alpha), -\alpha \exp^{-1}_{x_\alpha}(y_\alpha), P\big) \\
&\leqslant F\big(x_\alpha, u(x_\alpha), -\alpha \exp^{-1}_{x_\alpha}(y_\alpha), P\big) - F\big(y_\alpha, v(y_\alpha), \alpha \exp^{-1}_{y_\alpha}(x_\alpha), Q\big) \\
&\quad + F\big(y_\alpha, v(y_\alpha), \alpha \exp^{-1}_{y_\alpha}(x_\alpha), Q\big) - F\big(x_\alpha, v(y_\alpha), -\alpha \exp^{-1}_{x_\alpha}(y_\alpha), P\big) \\
&\leqslant \omega\big(\alpha d(x_\alpha, y_\alpha)^2 + d(x_\alpha, y_\alpha)\big),
\end{aligned}$$

and the contradiction follows by letting $\alpha \to \infty$. □

Remark 4.3. Observe that, since $\alpha \exp_y^{-1}(x) = L_{xy}(-\alpha \exp_x^{-1}(y))$, condition (2) of Theorem 4.2 can be replaced with a stronger but simpler assumption, namely that

$$F(y, r, L_{xy}\zeta, Q) - F(x, r, \zeta, P) \leqslant \omega\big(\alpha d(x, y)^2 + d(x, y)\big)$$

for all $x, y \in \Omega$, $r \in \mathbb{R}$, $P \in T_{2,s}(M)_x$, $Q \in T_{2,s}(M)_y$, $\zeta \in TM_x^*$ satisfying $(*)$.

Remark 4.4. If we want to compare two solutions u and v of $F = 0$ and we know that these functions are bounded by some $R > 0$ (e.g. when M is compact) then it is obvious from the above proof that it suffices to require that conditions (1) and (2) of Theorem 4.2 be satisfied for all r, s in the interval $[-R, R]$.

Proposition 4.5. *If M has nonnegative sectional curvature, then condition $(*)$ implies that $P \leqslant L_{yx}(Q)$.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the restriction of A_α to the subspace $\mathcal{D} = \{(v, L_{xy}v)\colon\ v \in TM_x\}$ of $TM_x \times TM_y$. By Proposition 3.1 we have that $A_\alpha(v, L_{xy}v)^2 \leqslant 0$ for all $v \in TM_x$, that is $(A_\alpha)|_{\mathcal{D}} \leqslant 0$, or equivalently $\lambda_i \leqslant 0$ for $i = 1, \ldots, n$. With our choice of ε_α, this implies that

$$\lambda_i + \varepsilon_\alpha \lambda_i^2 \leqslant \lambda_i + \frac{1}{2(1 + \sup_{1 \leqslant j \leqslant n} |\lambda_j|)} \lambda_i^2 \leqslant \lambda_i + \frac{|\lambda_i|}{2} = \frac{\lambda_i}{2} \leqslant 0$$

and since $\lambda_i + \varepsilon_\alpha \lambda_i^2$, $i = 1, \ldots, n$, are the eigenvalues of $(A_\alpha + \varepsilon_\alpha A_\alpha^2)|_{\mathcal{D}}$, this means that

$$\big(A_\alpha + \varepsilon_\alpha A_\alpha^2\big)(v, L_{xy}v)^2 \leqslant 0.$$

Then condition $(*)$ implies that

$$P(v)^2 - Q(L_{xy}v)^2 \leqslant \big(A_\alpha + \varepsilon_\alpha A_\alpha^2\big)(v, L_{xy}v)^2 \leqslant 0$$

for all $v \in TM_x$, which means that $P \leqslant L_{yx}(Q)$. $\square$

Therefore, *if M has nonnegative curvature and F is degenerate elliptic* then $(*)$ automatically implies that

$$F(x, r, \zeta, L_{yx}Q) - F(x, r, \zeta, P) \leqslant 0,$$

hence

$$\begin{aligned} &F(y, r, L_{xy}\zeta, Q) - F(x, r, \zeta, P) \\ &\quad = F(y, r, L_{xy}\zeta, Q) - F(x, r, \zeta, L_{yx}Q) + F(x, r, \zeta, L_{yx}Q) - F(x, r, \zeta, P) \\ &\quad \leqslant F(y, r, L_{xy}\zeta, Q) - F(x, r, \zeta, L_{yx}Q), \end{aligned}$$

and we see that condition (2) of the theorem is satisfied if we additionally require, for instance, that

$$F(y, r, \eta, Q) - F(x, r, L_{yx}\eta, L_{yx}Q) \leqslant \omega\big(d(x, y)\big). \tag{2$\sharp$}$$

Note that, in the case $M = \mathbb{R}^n$ we have $L_{yx}\eta \equiv \eta$ and $L_{yx}Q \equiv Q$, and condition (2♯) simply means that $F(x, u, du, d^2u)$ is uniformly continuous with respect to the variable x. Therefore we can regard condition (2♯) as the natural extension to the Riemannian setting of the Euclidean notion of uniform continuity of F with respect to x. This justifies the following

Definition 4.6. We will say that $F : \mathcal{X} \to \mathbb{R}$ is *intrinsically uniformly continuous with respect to the variable* x if condition (2♯) above is satisfied.

Remark 4.7. As we saw in Example 2.14 above, many interesting examples of equations involving nonincreasing symmetric functions of the eigenvalues of d^2u (such as the trace and the positive determinant $\det_+$) automatically satisfy condition (2♯) as long as they do not depend on x. In fact, since the eigenvalues of A are the same as those of $L_{yx}AL_{xy}$, any function of the form

$$F(x, r, \zeta, A) = G\big(r, \|\zeta\|_x, \text{ eigenvalues of } A\big)$$

is intrinsically uniformly continuous with respect to x.

Therefore, for manifolds of nonnegative curvature, we do not need to impose that F depends on $d^2u(x)$ in a uniformly continuous manner: the assumptions that F is degenerate elliptic and intrinsically uniformly continuous with respect to x are sufficient. Let us sum up what we have just shown.

Corollary 4.8. *Let Ω be a bounded open subset of a complete finite-dimensional Riemannian manifold M with nonnegative sectional curvature, and $F : \mathcal{X} \to \mathbb{R}$ be continuous, degenerate elliptic, and satisfy*:

(1) *F is strongly increasing, that is there exists $\gamma > 0$ such that, if $r \geqslant s$ then*

$$\gamma(r - s) \leqslant F(x, r, \zeta, Q) - F(x, s, \zeta, Q);$$

(2) *F is intrinsically uniformly continuous with respect to x (that is there exists a function $\omega : [0, \infty] \to [0, \infty]$ with $\lim_{t\to 0^+} \omega(t) = 0$ and such that*

$$F(y, r, \eta, Q) - F(x, r, L_{yx}\zeta, L_{yx}Q) \leqslant \omega\big(d(x, y)\big) \tag{2$\sharp$}$$

for all x, y, r, ζ, Q with $d(x, y) < \min\{i_M(x), i_M(y)\}$).

Let $u \in USC(\overline{\Omega})$ be a subsolution and $v \in LSC(\overline{\Omega})$ a supersolution of $F = 0$ on Ω, and $u \leqslant v$ on $\partial\Omega$.

Then $u \leqslant v$ on all of $\overline{\Omega}$.

In particular, the Dirichlet problem (DP) has at most one viscosity solution.

When M has negative curvature, condition $(*)$ does not imply $P \leqslant L_{yx}Q$, and degenerate ellipticity together with fulfillment of $(2\sharp)$ is not enough to ensure that condition (2) of Theorem 4.2 is satisfied. In this case condition (2) of Theorem 4.2 involves kind of a uniform continuity assumption on the dependence of F with respect to $d^2u(x)$. Let us be more explicit.

Proposition 4.9. *Assume that M has sectional curvature bounded below by some constant $-K_0 \leqslant 0$. Then condition $(*)$ in Theorem 4.2 implies that*

$$P - L_{yx}(Q) \leqslant \frac{3}{2} K_0 \alpha d(x,y)^2 I,$$

where $I(v)^2 = \langle v, v\rangle = \|v\|^2$.

Proof. We have that $A_\alpha = (\alpha/2)d^2\varphi(x,y)$, where $\varphi(x,y) = d(x,y)^2$. According to Proposition 3.3 we have

$$d^2\varphi(x,y)(v, L_{xy}v)^2 \leqslant 2K_0 d(x,y)^2\|v\|^2$$

for all $v \in TM_x$ and $x, y \in M$ with $d(x,y) < \min\{i_M(x), i_M(y)\}$. Therefore

$$A_\alpha(v, L_{xy}v)^2 \leqslant \alpha K_0 d(x,y)^2\|v\|^2.$$

This means that the maximum eigenvalue of the restriction of A_α to $\mathcal{D} := \{(v, L_{xy}v)\colon v \in TM_x\}$, which we denote λ_n, satisfies

$$\lambda_n \leqslant \alpha K_0 d(x,y)^2.$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $(A_\alpha)|_{\mathcal{D}}$ then $\lambda_i + \varepsilon_\alpha \lambda_i^2$, $i = 1, \ldots, n$, are those of $(A_\alpha + \varepsilon_\alpha A_\alpha^2)|_{\mathcal{D}}$. For a given $i = 1, \ldots, n$, if $\lambda_i \leqslant 0$ then $\lambda_i + \varepsilon_\alpha\lambda_i^2 \leqslant 0$ as in the proof of Proposition 4.5. In particular, if $\lambda_n \leqslant 0$ then $\lambda_j \leqslant 0$ for all $j = 1, \ldots, n$, so we get that $\lambda_j + \varepsilon_\alpha\lambda_j^2 \leqslant 0$ for all $j = 1, \ldots, n$, which means that

$$\big(A_\alpha + \varepsilon_\alpha A_\alpha^2\big)(v, L_{xy}v)^2 \leqslant 0.$$

On the other hand, if $\lambda_n \geqslant 0$ then $\lambda_n + \varepsilon_\alpha\lambda_n^2 \geqslant 0$, and because the function $[0, +\infty) \ni s \mapsto s + \varepsilon_\alpha s^2 \in [0, +\infty)$ is increasing, the maximum eigenvalue of $(A_\alpha + \varepsilon_\alpha A_\alpha^2)|_{\mathcal{D}}$ is precisely $\lambda_n + \varepsilon_\alpha\lambda_n^2$. This means that

$$\big(A_\alpha + \varepsilon_\alpha A_\alpha^2\big)(v, L_{xy}v)^2 \leqslant \lambda_n + \varepsilon_\alpha\lambda_n^2$$

for all $v \in TM_x$. Besides we have, by the choice of ε_α, that

$$\lambda_n + \varepsilon_\alpha\lambda_n^2 \leqslant \lambda_n + \frac{1}{2(1 + \sup_{1\leqslant j\leqslant n}|\lambda_n|)}\lambda_n^2 \leqslant \lambda_n + \frac{\lambda_n}{2} = \frac{3}{2}\lambda_n,$$

hence

$$\big(A_\alpha + \varepsilon_\alpha A_\alpha^2\big)(v, L_{xy}v)^2 \leqslant \frac{3}{2}\lambda_n \leqslant \frac{3}{2}\alpha K_0 d(x,y)^2\|v\|^2.$$

In any case (no matter what the sign of λ_n is) we get that the above inequality holds. Therefore condition $(*)$ implies

$$P(v)^2 - Q\big(L_{xy}(v)\big)^2 \leqslant \big(A_\alpha + \varepsilon_\alpha A_\alpha^2\big)(v, L_{xy}v)^2 \leqslant \frac{3}{2}K_0\alpha d(x,y)^2\|v\|^2. \qquad \square$$

Corollary 4.10. *Let M be a complete Riemannian manifold (no assumption on curvature), and Ω be a bounded open subset of M. Suppose that $F : \mathcal{X} \to \mathbb{R}$ is proper, continuous, and satisfies the following uniform continuity assumption: for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d(x,y) \leqslant \delta, \quad P - L_{yx}(Q) \leqslant \delta I \quad \Longrightarrow \quad F(y,r,L_{xy}\zeta,Q) - F(x,r,\zeta,P) \leqslant \varepsilon \tag{2$\flat$}$$

for all $x, y \in M$ with $d(x,y) < i_\Omega$, $r \in \mathbb{R}$, $\zeta \in TM_x^*$, $P \in \mathcal{L}_s^2(TM_x,\mathbb{R})$, and $Q \in \mathcal{L}_s^2(TM_y,\mathbb{R})$. Assume also that there is $\gamma > 0$ such that

$$\gamma(r-s) \leqslant F(x,r,\zeta,Q) - F(x,s,\zeta,Q) \quad \textit{for all } r \geqslant s.$$

Then there is at most one viscosity solution of the Dirichlet problem (DP).

Proof. Since M is complete, we know from the Hopf–Rinow Theorem that $\overline{\Omega}$ is compact, hence we have that $i_\Omega = \inf_{x\in\overline{\Omega}} i_M(x) > 0$. Take a number r with $0 < 2r < \inf_{x\in\overline{\Omega}} i_M(x)$. Also by compactness of $\overline{\Omega}$, there exists $K_0 > 0$ such that the sectional curvature is bounded below by $-K_0$ on $\overline{\Omega}$. Therefore we have that $\varphi(x,y) = d(x,y)^2$ is C^∞ smooth on the set $\{(x,y) \in \overline{\Omega} \times \overline{\Omega}\colon d(x,y) < r\}$ and, according to the preceding remark, if P, Q satisfy condition $(*)$ of Theorem 4.2 we get $P - L_{yx}(Q) \leqslant \frac{3}{2}K_0\alpha\, d(x,y)^2 I$ whenever $d(x,y) < r$. Then the uniform continuity assumption on F yields the existence of a function $\omega : [0,\infty] \to [0,\infty]$ with $\lim_{t\to 0^+}\omega(t) = 0$ and such that

$$F\big(y,r,\alpha\exp_y^{-1}(x),Q\big) - F\big(x,r,-\alpha\exp_x^{-1}(y),P\big) \leqslant \omega\big(\alpha d(x,y)^2 + d(x,y)\big),$$

hence the result follows from Theorem 4.2. $\square$

Remark 4.11. As is usual with comparison principles, the proof of Theorem 4.2 can easily be adapted to show that the viscosity solutions u of the equations $F = 0$ depend continuously on F. That is, if u is solution of $F = 0$ and v is solution of $G = 0$, then

$$\sup_{\overline{\Omega}}\big|u(x) - v(x)\big| \leqslant \sup_{(x,r,\zeta,A)\in X}\big|F(x,r,\zeta,A) - G(x,r,\zeta,A)\big| + \sup_{\partial\Omega}|u - v|.$$

5. Comparison results without boundary conditions

The same argument as in the proof of Theorem 4.2, with some small changes, yields the following.

Theorem 5.1. *Let M be a connected, complete Riemannian manifold (without boundary) such that $i(M) > 0$, and $F : \mathcal{X} \to \mathbb{R}$ be proper, continuous, and satisfy assumptions (1) and (2) of Theorem 4.2. Let u be a subsolution, and v a supersolution, of $F = 0$. Assume that u and v are uniformly continuous and $\lim_{x\to\infty} u(x) - v(x) \leqslant 0$. Then $u \leqslant v$ on M. In particular, if M is compact, there is at most one viscosity solution of $F = 0$ on M.*

Uniform continuity and the inequality at infinity guarantee that the m_α are attained, so the only difference with the proof of Theorem 4.2 is that now we cannot assume that x_α and y_α converge to some point x_0, but we do have that $d(x_\alpha, y_\alpha) < i(M)$ for large α, hence all the computations and estimations in the proof of Theorem 4.2 are still valid.

Corollary 5.2. *Let M be a compact Riemannian manifold of nonnegative sectional curvature, and let $F : \mathcal{X} \to \mathbb{R}$ be continuous, degenerate elliptic, strongly increasing in u, and intrinsically uniformly continuous with respect to x (that is, F satisfies conditions $(1-2)$ of Corollary 4.8). Let u be a subsolution, and v a supersolution of $F = 0$. Then $u \leqslant v$ on M.*

Proof. The same considerations as in Proposition 4.5 apply. □

Example 5.3. As we remarked above, condition (2) of Corollary 4.8 is easily satisfied when $F(x, r, \zeta, A)$ does not depend on ζ and A themselves, but on $\|\zeta\|$ and the eigenvalues of A. For instance, the function

$$F(x, r, \zeta, A) = r - \big(\det_+(A)\big)^3 \|\zeta\|^2 - f(x)\big(\mathrm{trace}(A)\big)^5$$

satisfies (1) and (2) of the above corollary provided that $f \geqslant 0$ and f is uniformly continuous. Therefore the equation

$$u - \big(\det_+\big(D^2 u\big)\big)^3 \|\nabla u\|^2 - (\Delta u)^5 f = 0$$

has at most one viscosity solution on any compact manifold of positive curvature if we only require that f is continuous and nonnegative.

Corollary 5.4. *Let M be a compact Riemannian manifold (no assumption on curvature). Suppose that $F : \mathcal{X} \to \mathbb{R}$ satisfies the uniform continuity assumption, and the growth assumption, of Corollary 4.10. Let u be a subsolution, and v a supersolution of $F = 0$. Then $u \leqslant v$ on M. In particular there is at most one viscosity solution of $F = 0$.*

Proof. Since M is compact the sectional curvature of M is bounded on all M, say $K \geqslant -K_0$. Take a number r with $0 < 2r < i(M)$. The function $\varphi(x, y) = d(x, y)^2$ is C^∞ on the set $\{(x, y) \in M \times M \colon d(x, y) \leqslant 2r\}$. Suppose that P and Q satisfy $(*)$ of Theorem 4.2. Then, from Proposition 4.9, we get that $P - L_{yx} Q \leqslant \frac{3}{2} K_0 \alpha\, d(x, y)^2 I$ provided that $d(x, y) < r$. Therefore the uniform continuity property of F gives us a function $\omega : [0, \infty] \to [0, \infty]$ with $\lim_{t \to 0^+} \omega(t) = 0$ and such that

$$F\big(y, r, \alpha \exp_y^{-1}(x), Q\big) - F\big(x, r, -\alpha \exp_x^{-1}(y), P\big) \leqslant \omega\big(\alpha d(x, y)^2 + d(x, y)\big).$$

Hence we can apply Theorem 5.1 and conclude the result. □

6. Existence results

Perron's method can easily be adapted to the Riemannian setting to establish existence of viscosity solutions to the Dirichlet problem. The proof goes exactly as in [4] with appropriate changes. The only step which is not completely obvious is the proof of the following

Proposition 6.1. *Let $(\zeta, A) \in J^{2,+} f(z)$ Suppose that f_n is a sequence of upper semicontinuous functions such that*

(i) *there exists x_n such that $(x_n, f_n(x_n)) \to (x, f(x))$, and*
(ii) *if $y_n \to y$, then $\limsup_{n\to\infty} f_n(y_n) \leqslant f(y)$.*

Then there exist $\hat{x}_n$ and $(\zeta_n, A_n) \in J^{2,+} f_n(\hat{x}_n)$ such that $(\hat{x}_n f_n(\hat{x}_n), \zeta_n, A_n) \to (x, f(x), \zeta, A)$.

Proof. Consider the functions $f \circ \exp_x$ and $f_n \circ \exp_x$ defined on a neighborhood of 0 in TM_x. These functions satisfy properties (i) and (ii) of the statement (when they take the roles of f and f_n and M is replaced with TM_x). By Corollary 2.3 we have that $(\zeta, A) \in J^{2,+}(f \circ \exp_x)(0)$. And of course the result is known in the case when $M = \mathbb{R}^n$, so we get a sequence $\hat{v}_n$ and $(\tilde{\zeta}_n, \widetilde{A}_n) \in J^{2,+}(f_n \circ \exp_x)(\hat{v}_n)$ such that

$$\big(\hat{v}_n, f_n \circ \exp_x(\hat{v}_n), \tilde{\zeta}_n, \widetilde{A}_n\big) \to \big(0, f \circ \exp_x(0), \zeta, A\big).$$

Set $\hat{x}_n = \exp_x(\hat{v}_n)$. We have that $\hat{x}_n \to x$ and $f_n(\hat{x}_n) \to f(x)$. Since $(\tilde{\zeta}_n, \widetilde{A}_n) \in J^{2,+}(f_n \circ \exp_x)(\hat{v}_n)$ there exist functions ψ_n such that $f_n \circ \exp_x - \psi_n$ attains a maximum at $\hat{v}_n$, $\tilde{\zeta}_n = d\psi_n(\hat{v}_n)$ and $\widetilde{A}_n = d^2\psi_n(\hat{v}_n)$. Let us define $\varphi = \psi_n \circ \exp_x^{-1}$ on a neighborhood of x. Then $f_n - \varphi_n$ attains a maximum at $\hat{x}_n$ so, if we set $\zeta_n = d\varphi(\hat{x}_n)$, $A_n = d^2\varphi(\hat{x}_n)$, we have that $(\zeta_n, A_n) \in J^{2,+} f_n(\hat{x}_n)$. It only remains to show that $\zeta_n \to \zeta$ and $A_n \to A$. But this is exactly what was shown in ($\Leftarrow$) of the proof of Proposition 2.8. □

By using this proposition one can prove, as in [4], existence of viscosity solutions to the Dirichlet problem

$$F\big(x, u, du, d^2u\big) = 0 \quad \text{in } \Omega, \qquad u = f \quad \text{on } \partial\Omega, \tag{DP}$$

where Ω is an open bounded subset of a complete Riemannian manifold M.

Theorem 6.2. *Let comparison hold for (DP), i.e., if w is a subsolution of (DP) and v is a supersolution of (DP), then $w \leqslant v$. Suppose also that there exists a subsolution $\underline{u}$ and a supersolution $\bar{u}$ of (DP) that satisfy the boundary condition $\underline{u}_*(x) = \bar{u}^*(x) = f(x)$ for $x \in \partial\Omega$. Then*

$$W(x) = \sup\big\{w(x)\colon\ \underline{u} \leqslant w \leqslant \bar{u} \text{ and } w \text{ is a subsolution of (DP)}\big\}$$

is a solution of (DP).

Here we used the following notation:

$$u^*(x) = \lim_{r\downarrow 0} \sup\big\{u(y)\colon\ y \in \Omega \text{ and } d(y, x) \leqslant r\big\},$$

$$u_*(x) = \lim_{r\downarrow 0} \inf\big\{u(y)\colon\ y \in \Omega \text{ and } d(y, x) \leqslant r\big\},$$

that is u^* denotes the upper semicontinuous envelope of u (the smallest upper semicontinuous function, with values in $[-\infty, \infty]$, satisfying $u \leqslant u^*$), and similarly u_* stands for the lower semicontinuous envelope of u.

One can also easily adapt the proof of [2, Theorem 6.17] to the second order situation, obtaining the following.

Corollary 6.3. *Let M be a compact Riemannian manifold, and $G(x, du, d^2u)$ be degenerate elliptic and uniformly continuous in the sense of Corollary* 4.10. *Then there exists a unique viscosity solution of the equation $u + G(x, du, d^2u) = 0$ on M.*

Again, if M has nonnegative curvature, the assumptions that F is elliptic and intrinsically uniformly continuous with respect to x are sufficient in order to get an analogous result.

7. Examples

Most of the examples of proper F's given in [4] remain valid in the Riemannian setting. In particular, as we have already seen, the functions $(x, r, \zeta, A) \mapsto -\det_+(A)$ and $(x, r, \zeta, A) \mapsto -\operatorname{trace}(A)$ are degenerate elliptic and intrinsically uniformly continuous with respect to x. The same is true of all many symmetric functions of the eigenvalues of A, such as minus the minimum (or the maximum) eigenvalue, and of course nondecreasing combinations and sums of these are degenerate elliptic too. One can find lots of examples of nonlinear equations for which the results of this paper yield existence and uniqueness of viscosity solutions. For instance, one can easily show that, for every compact manifold of positive curvature, the equation

$$\max\bigl\{u - \lambda_1\bigl(D^2u\bigr)\|\nabla u\|^p - (\Delta u)^{2q+1}\|\nabla u\|^r - \bigl(\det_+\bigl(D^2u\bigr)\bigr)^{2k+1} f^2,\ u - g\bigr\} = 0$$

(where λ_1 denotes the minimum eigenvalue function and $p, q, r, k \in \mathbb{N}$) has a unique viscosity solution if we only require that f and g are continuous. This gives an idea of the generality of the above results.

Of course this example is rather unnatural. Let us finish this paper by examining what our results yield in the case of a classic equation, that of Yamabe's, which has been extensively studied and completely solved by using variational methods. We do not claim that the following discussion gives any new insight into Yamabe's problem, we only want to study, from the point of view of the viscosity solutions theory, a well-known example of a nonlinear equation arising from an important geometrical problem.

Example 7.1 *(The Yamabe equation).* A fundamental problem in conformal geometry is to know whether or not there exists a conformal metric g' with constant scalar curvature S' on a given compact n-dimensional Riemannian manifold (M, g), with $n \geqslant 3$, see [1,12]. This is equivalent to solving the equation

$$-4\frac{n-1}{n-2}\Delta u + S(x)u = S' u^{\frac{n+2}{n-2}}, \tag{Y}$$

where S is the scalar curvature of g. One can write this equation in the form $F = 0$, where

$$F(x, r, \zeta, A) = S(x)r - S' r^{\frac{n+2}{n-2}} - 4\frac{n-1}{n-2}\operatorname{trace}(A) = 0.$$

It is clear that F is degenerate elliptic. Assume that S is everywhere positive and that $S' \leqslant 0$. Then, by compactness, there exists $\gamma > 0$ such that $S(x) \geqslant \gamma$ for all $x \in M$. According to Remark 4.4, in order to check conditions (1) and (2) of Theorem 5.1 we may assume that r, s lie on a bounded interval. We have that

$$F(y, r, \eta, Q) - F(x, r, L_{yx}\eta, L_{yx}Q) \leqslant r\left|S(y) - S(x)\right|,$$

hence, because S is uniformly continuous on M and r is bounded, we deduce that F satisfies (2) of Corollary 5.2. On the other hand, if $r \geqslant s$ then

$$F(x, r, \zeta, A) - F(x, s, \zeta, A) = S(x)(r - s) - S'\left(r^{\frac{n+2}{n-2}} - s^{\frac{n+2}{n-2}}\right) \geqslant \gamma(r - s),$$

so condition (1) is also satisfied. It follows that there is at most one viscosity solution of $F = 0$. Existence can be shown by using Perron's method. In all, we see that if S is everywhere positive and $S' \leqslant 0$ then there exists a unique viscosity solution u of (Y).

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References

[1] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer Monogr. Math., Springer-Verlag, Berlin, 1998.
[2] D. Azagra, J. Ferrera, F. López-Mesas, Nonsmooth analysis and Hamilton–Jacobi equations on Riemannian manifolds, J. Funct. Anal. 220 (2) (2005) 304–361.
[3] M.P. do Carmo, Riemannian Geometry, Math. Theory Appl., Birkhäuser, Boston, 1992.
[4] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[5] R. Deville, E. Matheron, Infinite games, Banach space geometry and the eikonal equation, Proc. London Math. Soc. (3) 95 (1) (2007) 49–68.
[6] M.J. Gursky, J.A. Viaclovsky, Prescribing symmetric functions of the eigenvalues of the Ricci tensor, Ann. of Math. (2) 166 (2) (2007) 475–531.
[7] S. Lang, Fundamentals of Differential Geometry, Grad. Texts in Math., vol. 191, Springer-Verlag, New York, 1999.
[8] Yu.S. Ledyaev, Q.J. Zhu, Nonsmooth analysis on smooth manifolds, Trans. Amer. Math. Soc. 359 (8) (2007) 3687–3732.
[9] C. Mantegazza, A.C. Mennucci, Hamilton–Jacobi equations and distance functions on Riemannian manifolds, Appl. Math. Optim. 47 (1) (2003) 1–25.
[10] T. Sakai, Riemannian Geometry, Transl. Math. Monogr., vol. 149, Amer. Math. Soc., 1992.
[11] M. Spivak, A Comprehensive Introduction to Differential Geometry, vol. IV, second ed., Publish or Perish, Wilmington, DE, 1979.
[12] J. Viaclovsky, Conformal geometry and fully nonlinear equations, World Sci. Memorial Volume for S.S. Chern, 2006.