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Fixed Points and Zeros for Set Valued Mappings on Riemannian Manifolds: A Subdifferential Approach

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Abstract In this paper we establish several results which allow to find fixed points and zeros of set-valued mappings on Riemannian manifolds. In order to prove these results we make use of subdifferential calculus. We also give some useful applications.

Keywords Fixed point · Set valued mapping · Subdifferential · Graphical derivative

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1 Introduction

It is well known that graphical derivatives of set-valued mappings can be very useful in order to obtain fixed-point and inverse-function-like results. We could cite [1] for instance, where a calculus of contingent derivatives is introduced and applied to prove an adaptation of the inverse function theorem due to Ekeland (which we generalize here to the setting of Riemannian manifolds, see Theorem 10 below). Along the same line we should cite Mordukovich and Outrata's paper [10], where contingent derivatives of the subdifferential of a given function are studied and applied to the contact problem with non-monotonous friction.

In the present paper we introduce a notion of graphical derivative of set-valued mappings defined on Riemannian manifolds, and we obtain a very general result

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(see Theorem 5 below) which ensures the existence of solutions to inclusions where the set-valued mapping, in the case when the manifold has a Lie group structure, is the sum of a mapping satisfying a Lipschitz condition and another mapping whose graphical derivative satisfies a certain condition. From this main theorem we deduce many applications. For instance we give some conditions which guarantee Lipschitz dependence of such solutions with respect to parameters (see Corollary 8). We also establish a result which allows us to find fixed points of set-valued mappings under some assumptions on the graphical derivatives of such mappings (see Theorem 13).

We should also observe that, in the case when the function is single-valued, our results improve the main theorems of [2].

Let us now explain the notation and some of the tools that we will be using. Throughout this paper we will assume that M is a complete Riemannian manifold and that the set-valued mappings $F\colon M \rightrightarrows M$ are proper and satisfy the following conditions:

1. $d(z, F(x))$ is attained for every $x, z \in M$.
2. For every $z_0 \in M$, the function $\varphi(x) = d(z_0, F(x))$ is lower semicontinuous.

Condition (1) requires $F(x)$ closed, and it is met if $F(x)$ is compact or finite dimensional for instance.

We require condition (2) in order to ensure the existence of the subdifferential of the function φ in a dense set of points. This is not a strong restriction; it holds under very natural assumptions, such as upper semicontinuity of F.

Let us introduce some notation that we will be using throughout this paper. We will let i_M stand for the injectivity radius of M, and i_p denote the injectivity radius of M at $p \in M$. As usual, $\exp_x : TM_x \to M$ will denote the exponential mapping at a point $x \in M$. We refer the reader to [6, 8] for the definitions of injectivity radii, exponential mapping, parallel transport and other standard terms of differential geometry. Recall that, for a given curve $\gamma\colon I \to M$, numbers $t_0, t_1 \in I$ and a vector $V_0 \in TM_{\gamma(t_0)}$ there exists a unique parallel vector field $V(t)$ along γ such that $V(t_0) = V_0$, and the mapping defined by $V_0 \mapsto V(t_1)$ is a linear isometry between the tangent spaces $TM_{\gamma(t_0)}$ and $TM_{\gamma(t_1)}$. In the case when γ is a minimizing geodesic and $x = \gamma(t_0)$, $y = \gamma(t_1)$ we denote this mapping by $L_{xy} : TM_x \to TM_y$, the parallel transport from TM_x to TM_y along γ. The parallel transport allows us to measure the length of the "difference" between vectors (or forms) which are in different tangent spaces (or in duals of tangent spaces, that is, different fibers of the cotangent bundle), and to do so in a natural way. Indeed, let γ be a minimizing geodesic connecting two points $x, y \in M$, and let L_{xy} the parallel transport along γ. For any two vectors $v \in TM_x$, $w \in TM_y$ we can define a natural distance between v and w as the number

$$\|v - L_{yx}(w)\|_x = \|w - L_{xy}(v)\|_y$$

(this equality holds because L_{xy} is a linear isometry between the two tangent spaces, with inverse L_{yx}). Since the spaces T^*M_x and T^*M_y are isometrically identified by the formula $v \equiv \langle v, \cdot\rangle$, we can obviously use the same method to measure distances between forms $\zeta \in T^*M_x$ and $\eta \in T^*M_y$ lying on different fibers of T^*M.

We will write the subindices for the norm and the scalar product whenever it is necessary in order to avoid confusion about the tangent space we work on.

We will also use the definition of the proximal subdifferential $\partial_P f(x_0)$ of a function f defined on a Riemannian manifold, which was introduced in [2] and in [3], as well as the following characterization: $\zeta \in \partial_P f(x_0)$ if and only if there exists a positive number σ and a neighborhood of x_0 on which

$$f(x) \geqslant f(x_0) + \left\langle \zeta, exp_{x_0}^{-1}(x) \right\rangle - \sigma d(x, x_0)^2$$

holds (this is called the *proximal inequality*).

Definition 1 Let M be a Riemannian manifold. Consider a set-valued mapping $G\colon M \rightrightarrows M$, and a point $x_0 \in M$. The graphical derivative (or contingent derivative) of G at x_0 for $y_0 \in G(x_0)$ is the set-valued mapping $DG(x_0 \mid y_0) : T_{x_0}M \to T_{y_0}M$ defined by

$$v \in DG(x_0 \mid y_0)(h) \text{ if and only if } \exists h_n \in T_{x_0}M, \exists v_n \in T_{y_0}M, \exists t_n \downarrow 0 \text{ such that}$$

$$(h, v) = lim_n(h_n, v_n) \text{ and } exp_{y_0}(t_n v_n) \in G(exp_{x_0}(t_n h_n))$$

In the case when $M = H$, where H is a Hilbert space, Definition 1 is equivalent to the definition of the graphical derivative via the tangent cone introduced by J.P. Aubin in [1]. The graphical derivative of a singled-valued function agrees with its differential whenever the function is differentiable.

2 Main Results

Lemma 2 *Let M be a Riemannian manifold. Let $G\colon M \rightrightarrows M$ be a set-valued mapping, and define $\varphi\colon M \to \mathbb{R}$ by $\varphi(x) = d(z_0, G(x))$. Assume that $\zeta \in D^-\varphi(x_0)$, with $\varphi(x_0) > 0$. If $y_0 \in G(x_0)$ satisfies that $\varphi(x_0) = d(y_0, z_0)$, $v \in DG(x_0|y_0)(h)$ and the function $x \to d^2(x, z_0)$ is C^2 at y_0, then we have*

$$\langle \zeta, h \rangle_{x_0} \leqslant \left\langle \frac{-exp_{y_0}^{-1}(z_0)}{d(y_0, z_0)}, v \right\rangle_{y_0},$$

and consequently, for $h \neq 0$, we have

$$||\zeta|| \geqslant \left\langle \frac{exp_{y_0}^{-1}(z_0)}{d(y_0, z_0)}, \frac{v}{||h||} \right\rangle$$

while

$$\left\langle \frac{-exp_{y_0}^{-1}(z_0)}{d(y_0, z_0)}, v \right\rangle \geqslant 0$$

when $h = 0$.

Proof We have that $(h, v) = \lim(h_n, v_n)$ with $exp_{y_0}(t_n v_n) \in G(exp_{x_0}(t_n h_n))$ for a sequence $\{t_n\} \downarrow 0$.

On the other hand, since $\zeta \in D^-\varphi(x_0)$, we have that for any chart $f\colon U \subset M \to H$ and $x_0 \in U$ (where U is an open set in M),

$$\liminf_{v\to 0} \frac{(\varphi \circ f^{-1})(f(x_0)+v) - \varphi(x_0) - \langle \eta, v\rangle}{||v||} \geqslant 0$$

where $\eta = \zeta \circ df^{-1}(f(x_0))$ (see [4]).

Consequently, we obtain

$$(\varphi \circ f^{-1})(f(x_0)+v) \geqslant \varphi(x_0) + \langle \eta, v\rangle + o(||v||)$$

for $||v||$ small enough. Now, if we take $f = exp_{x_0}^{-1}$, then we get $\eta = \zeta$, because $\eta = \zeta \circ dexp_{x_0}(0) = \zeta \circ id_{T_{x_0}M} = \zeta$. If $v = exp_{x_0}^{-1}(x)$ then

$$\varphi(x) \geqslant \varphi(x_0) + \left\langle \zeta, exp_{x_0}^{-1}(x)\right\rangle + o(d(x,x_0))$$

for x near x_0. We deduce

$$d(exp_{y_0}(t_n v_n), z_0) \geqslant \varphi(exp_{x_0}(t_n h_n)) \geqslant d(y_0,z_0) + \langle \zeta, t_n h_n\rangle + o(t_n||h_n||)$$

hence

$$\langle \zeta, h_n\rangle \leqslant \frac{d(exp_{y_0}(t_n v_n), z_0) - d(y_0,z_0)}{t_n} + \frac{1}{t_n} o(t_n||h_n||),$$

which is equivalent to

$$\langle \zeta, h_n\rangle \leqslant \frac{d^2(exp_{y_0}(t_n v_n), z_0) - d^2(y_0,z_0)}{t_n(d(exp_{y_0}(t_n v_n), z_0) + d(y_0,z_0))} + \frac{1}{t_n} o(t_n||h_n||).$$

Now we define $\psi(x) = d^2(x,z_0)$ so

$$\psi'(y_0) = 2d(y_0,z_0)D_1 d(y_0,z_0)$$

and we obtain

$$\begin{aligned} d^2(exp_{y_0}(t_n v_n), z_0) - d^2(y_0,z_0) &= 2t_n d(y_0,z_0) D_1 d(y_0,z_0)(v_n) \\ &\quad + \psi''(y_0)(t_n v_n) + o(t_n^2||v_n||^2) \end{aligned}$$

Consequently

$$\langle \zeta, h_n\rangle \leqslant \frac{2t_n d(y_0,z_0) D_1 d(y_0,z_0)(v_n) + \|\psi''(y_0)\| t_n^2 \|v_n\|^2 + o(t_n^2\|v_n\|^2)}{t_n(d(exp_{y_0}(t_n v_n), z_0) + d(y_0,z_0))} + \frac{1}{t_n} o(t_n\|h_n\|).$$

Now, by letting $t_n \downarrow 0$, $h_n \to h$ and $v_n \to v$, we have

$$\langle \zeta, h\rangle \leqslant \frac{2d(y_0,z_0)D_1 d(y_0,z_0)(v)}{2d(y_0,z_0)}.$$

Simplifying the expression above yields

$$\langle \zeta, h\rangle \leqslant D_1 d(y_0,z_0)(v) = \left\langle \frac{-exp_{y_0}^{-1}(z_0)}{d(y_0,z_0)}, v\right\rangle.$$

As for the second part it is enough to observe that when $h \neq 0$,

$$||\zeta||||h|| \geqslant \langle \zeta, -h\rangle \geqslant \left\langle \frac{exp_{y_0}(z_0)}{d(y_0, z_0)}, v\right\rangle$$

and when $h = 0$,

$$0 \leqslant \left\langle \frac{-exp_{y_0}(z_0)}{d(y_0, z_0)}, v\right\rangle. \qquad \square$$

The differentiability condition on $d^2(., z_0)$ is satisfied, for example, if i_M is infinite, or more general, if i_M is larger than the distance between z_0 and $G(x_0)$. This Lemma can be viewed as a sort of Chain Rule. In fact, when $g\colon M \to M$ is single-valued and differentiable, we have $D\varphi(x_0)(h) = \left\langle \frac{\partial d(z_0, g(x_0))}{\partial x}, Dg(x_0)(h)\right\rangle$. However, the equality may fail in general even when we deal with single-valued functions. Indeed, consider $g\colon \mathbb{R} \to \mathbb{R}$ such that $g(x) = |x| + 1$, $z_0 = 0$, (hence $\varphi = g$), $0 \in D^-\varphi(0)$, $h = 1$ and $v = 1$; in this case we have $\langle \zeta, h\rangle = 0 < 1 = \langle 1, v\rangle$.

Definition 3 Let M be a Riemannian manifold. We say that a set-valued mapping $H\colon M \rightrightarrows M$ has the Aubin-property with modulus L provided that for every $x_1, x_2 \in M$ and every $y_1 \in H(x_1)$, there is a $y_2 \in H(x_2)$ satisfying $d(y_1, y_2) \leqslant Ld(x_1, x_2)$.

It is easy to see that if H has the Aubin-property with modulus L, then the function $\varphi(x) = d(z_0, H(x))$ is L-Lipschitz.

Definition 4 Let M be a Riemannian manifold. We say that a set-valued mapping $G\colon M \rightrightarrows M$ satisfies the weak derivative condition, (WDC) for short, for a positive constant A, if for every $x_0, z_0 \in M$ such that $d(G(x_0), z_0) < i_{z_0}$, there are $h \in T_{x_0}M, ||h|| = 1$, and $v \in DG(x_0|y_0)(h)$ such that $\langle exp_{y_0}^{-1}(z_0), v\rangle \geqslant Ad(z_0, y_0)$, where $y_0 \in G(x_0)$ with $d(z_0, G(x_0)) = d(y_0, z_0)$.

In the following theorem we will require our Riemannian manifold M to have an abelian Lie group structure in order that the distance is translation invariant (see [6]).

Theorem 5 *Let M be a complete Riemannian manifold with an abelian Lie group structure, and $a \in M$. Let $G\colon B(a, R) \rightrightarrows M$ be a set-valued mapping satisfying (WDC) for $A > 0$. Let $H\colon M \rightrightarrows M$ be a set-valued mapping and have the Aubin-property with modulus $L < A$. Assume also that for every $x \in B(a, R)$, at least one of the sets $G(x)$, $H(x)$ is compact, and that $d(-H(x), G(x)) < i_M$. Then the equation $0_M \in F(x) = G(x) + H(x)$ has a solution in $B(a, R)$ provided that $F(a) \cap B(0, R(A-L)) \neq \emptyset$.*

Proof A solution of the equation is a zero of the function $f(x) = d(0, F(x)) = d(G(x), -H(x))$ (where we denote $-H(x) := \{-x \mid x \in H(x)\}$). Assume on the contrary that there is no solution. For any $x_0 \in B(a, R)$ and $\zeta \in \partial_P f(x_0)$ we have that $f(x_0) > 0$. Let $z_0 \in -H(x_0)$ and $y_0 \in G(x_0)$ be such that $f(x_0) = d(z_0, y_0)$. The proximal inequality tells us that there is a positive constant σ, such that for every x near x_0, we have:

$$d(G(x), -H(x)) = f(x) \geqslant d(z_0, y_0) + \left\langle \zeta, exp_{x_0}^{-1}(x)\right\rangle - \sigma d(x, x_0)^2,$$

that is

$$d(G(x), -H(x)) - \langle \zeta, exp_{x_0}^{-1}(x) \rangle + \sigma d(x, x_0)^2 \geqslant d(z_0, y_0) - \langle \zeta, exp_{x_0}^{-1}(x_0) \rangle,$$

hence

$$d(z_0, -H(x)) + d(z_0, G(x)) - \langle \zeta, exp_{x_0}^{-1}(x) \rangle + \sigma d(x, x_0)^2 \geqslant d(z_0, y_0) - \langle \zeta, exp_{x_0}^{-1}(x_0) \rangle.$$

If we define $\psi(x) = d(z_0, -H(x)) + d(z_0, G(x)) - \langle \zeta, exp_{x_0}^{-1}(x) \rangle + \sigma d(x, x_0)^2$, we have that ψ has a local minimum at x_0, hence

$$0 \in \partial_P \psi(x_0) = -\zeta + \partial_P(h + \varphi)(x_0)$$

where $h(x) = d(z_0, -H(x))$ and $\varphi(x) = d(z_0, G(x))$ (see [5] for the sum rule, and keep in mind that the function $x \mapsto \langle -\zeta, exp_{x_0}^{-1}(x) \rangle + \sigma d(x, x_0)^2$ is C^2 near x_0, with derivative $-\zeta$, because $\partial_P \langle \zeta, exp_{x_0}^{-1}(\dot{}) \rangle(x_0) = \langle \zeta, dexp_{x_0}^{-1}(x_0) \rangle = \langle \zeta, Id_{TM_{x_0}} \rangle = \zeta$).

Therefore, for this $\zeta \in \partial_P(h + \varphi)(x_0)$, according to the fuzzy sum rule (see [3, Theorem 3.8]), for every $\varepsilon > 0$, there exist $x_1, x_2 \in M$, $\zeta_2 \in \partial_P h(x_2)$ and $\zeta_1 \in \partial_P \varphi(x_1)$ such that

$$d(x_i, x_0) < \varepsilon, |h(x_2) - h(x_0)| < \varepsilon, |\varphi(x_1) - \varphi(x_0)| < \varepsilon$$

and

$$||\zeta - (L_{x_1 x_0}(\zeta_1) + L_{x_2 x_0}(\zeta_2))||_{x_0} < \varepsilon,$$

because h is L-Lipschitz. Hence we have

$$||\zeta|| \geqslant ||L_{x_1 x_0}(\zeta_1) + L_{x_2 x_0}(\zeta_2)||_{x_0} - \varepsilon \geqslant ||L_{x_1 x_0}(\zeta_1)||_{x_0} - ||L_{x_2 x_0}(\zeta_2)||_{x_0} - \varepsilon$$

with $\zeta_1 \in \partial_P \varphi(x_1)$ and $||L_{x_2 x_0}(\zeta_2)||_{x_0} \leqslant L$ (because $\|\zeta_2\|_{x_2} \leqslant L$).

Now, applying the (WDC) condition for G, x_1 and z_0, there exist $h_1 \in TM_{x_1}$, with $||h_1|| = 1$, and $v_1 \in DG(x_1|y_1)(h_1)$ such that $\langle exp_{y_1}^{-1}(z_0), v_1 \rangle \geqslant Ad(z_0, y_1)$. Then we can apply Lemma 2 (indeed, $d(-H(x), G(x)) < i_M$ implies that $d^2(., z_0)$ is C^2 at y_0, hence at y_1 as well because this point can be taken as close to y_0 as required) to deduce that

$$||\zeta_1|| \geqslant \left\langle \frac{exp_{y_1}^{-1}(z_0)}{d(y_1, z_0)}, v_1 \right\rangle \geqslant A.$$

Hence, $||\zeta|| \geqslant ||L_{x_1 x_0}(\zeta_1)||_{x_0} - ||L_{x_2 x_0}(\zeta_2)||_{x_0} - \varepsilon \geqslant A - L - \varepsilon$, and letting ε go to 0, we get $||\zeta|| \geqslant A - L$. Now we can finally apply the Decrease Principle (see [2, Theorem 16]) to obtain a contradiction: $0 \leqslant \inf\{f(x) : x \in B(a, R)\} \leqslant f(a) - R(A - L) < 0$ (recall that $f(a) = d(0, F(a)) < R(A - L)$).

Therefore there necessarily exists $x_0 \in M$ such that $f(x_0) = 0$, and consequently $0 \in F(x_0)$. □

Remark 6 It is worth noting that in the statement 5 we can take any other point z_0 instead of 0_M, and the result holds with an analogous proof.

Remark 7 We can use Lemma 2 with the proximal subdifferential because

$$\partial_p f(x_0) \subset D^- f(x_0).$$

In view of the preceding theorem, we can think about a similar problem

$$0_M \in F(x) = G(x) + P(y) + H(x),$$

where $P\colon X \to M$ is a K-Lipschitz function defined on X, a metric space of parameters. In this situation it can be very useful to know something about the continuity properties of the solution mapping $S\colon X \rightrightarrows M$ defined as

$$S(y) := \{x : 0_M \in G(x) + P(y) + H(x)\};$$

in this respect we obtain the following.

Corollary 8 *Let M be a complete Riemannian manifold with an abelian Lie group structure. Let H and G be set-valued mappings which satisfy the hypothesis in Theorem 5 for $R = \infty$. We assume that $d(-H(x), G(x) + P(y)) \leqslant i_M$ for all $x \in M$, $y \in X$. Then, for any given $y \in X$, the equation $0_M \in F(x) = G(x) + P(y) + H(x)$ has a solution, and the set-valued solution mapping $S\colon X \rightrightarrows M$, $y \to S(y) := \{x : 0_M \in G(x) + P(y) + H(x)\}$ has the Aubin-property with modulus $\frac{K}{A-L}$.*

Proof For a given point $y \in X$, the mapping $G(x) + P(y)$ satisfies the (WDC)-condition because so does G. From Theorem 5 we know that, given a y_1 we have an $x_1 \in S(y_1)$. Now we take another point y_2, and we apply Theorem 5 to the mappings $H(.)$ and $G(.) + P(y_2)$ with $R = \frac{d(y_1,y_2)K}{A-L}$ so as to find a $x_2 \in S(y_2) \cap B(x_1, R)$. In order to do so, we need that

$$G(x_1) + H(x_1) + P(y_2) \cap B(0_M, (A - L)R) \neq \emptyset, \tag{1}$$

which (taking into account that $0_M \in G(x_1) + H(x_1) + P(y_1)$) is implied by

$$P(y_2) - P(y_1) \in B(0_M, (A - L)R), \tag{2}$$

which in turn is true because we have taken $R = \frac{d(y_1,y_2)K}{A-L}$, so

$$d(P(y_2), P(y_1)) \leqslant Kd(y_2, y_1) = (A - L)R.$$

Therefore

$$d(x_2, x_1) \leqslant R = \frac{K}{A-L} d(y_1, y_2).$$

We have thus proved that given an $x_1 \in S(y_1)$ we can find an $x_2 \in S(y_2)$ satisfying the previous inequality, and this means that S has the Aubin-property with modulus $\frac{K}{A-L}$. □

3 Applications

The equations considered in [10, Section 5] are similar to those of Theorem 5. The problem in [10] is how to find solutions of

$$0 \in f(x, y) + Q(x, y),$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuously differentiable function, and $Q: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a set-valued mapping which is defined as follows:

$$Q(x, y) = \begin{cases} \partial\varphi(g(x, y)) & \text{if } g(x, y) \in \operatorname{dom}\varphi \\ \emptyset & \text{otherwise} \end{cases}$$

where $\varphi: \mathbb{R}^m \to \overline{\mathbb{R}}$ is an extended-real-valued proper function, $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable at the points in question (that is dom(φ)), and $\partial\varphi(x_0)$ is the basic subdifferential (with the notation in [10]), and when it coincides with the Frechet subdifferential the function is called subdifferentially regular (that is the case of the convex functions, the smooth functions, "max functions", etc; see [9]) as in the following example.

Our problem is, on the one hand, less general, because we consider values x and parameters y as separate values. But, on the other hand, our assumptions in Theorem 5 are less restrictive than those of [10]. For instance, we don't ask for the existence of solutions as the Theorem 5.1 of [10]. We don't need that the function f is C^1 smooth, and neither that the function Q is a subdifferential. Moreover, there are cases where we can apply either method to get the result. For example, we can apply Corollary 8 to the equation which appears in Section 5 of [10], which corresponds to a contact problem with nonmonotone friction and can be described as follows:

$$0 \in Ay + p(x) + \partial\phi(Dy) \tag{β}$$

where $x \in \mathbb{R}^n$, A is an $m \times m$ positively definite "stiffness" matrix, $p: \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable vector function related to external forces and D is an $m \times m$ nonsingular matrix defined by a quadrature formula. The function ϕ is of the form

$$\phi(z) = \sum_{i=1}^{m} \varphi_i(z_i) \quad \text{with } z \in \mathbb{R}^m,$$

and the functions φ_i that are used in [10] are as follows:

$$\varphi_i(z_i) = \begin{cases} (-k_1 + k_2 z_0) z_i + \frac{k_2}{2}(z_0)^2 & \text{if } z_i < -z_0 \\ -k_1 z_i - \frac{k_2}{2}(z_i)^2 & \text{if } z_i \in [-z_0, 0), \\ k_1 z_i - \frac{k_2}{2}(z_i)^2 & \text{if } z_i \in [0, z_0), \\ (k_1 - k_2 z_0) z_i + \frac{k_2}{2}(z_0)^2 & \text{if } z_i \geqslant z_0, \end{cases}$$

where $z_0 > 0$, $k_1 > 0$, and $k_2 > 0$ are given parameters. It is not difficult to prove that Ay has the (WDC)-condition for the constant $\frac{1}{\|A^{-1}\|}$, the mapping $\partial\phi(Dy)$ has the Aubin-property for $L = \sqrt{n}\|D\| Max\{2k_1, k_2\}$. Then, when we have that

$$\sqrt{n}\|D\| Max\{2k_1, k_2\} < \frac{1}{\|A^{-1}\|}$$

and the function $p(x)$ is Lipschitz, we can assure that the equation (β) has solution, and the solution has the Aubin property for the constant

$$\frac{K}{\frac{1}{\|A^{-1}\|} - \sqrt{n}\|D\| Max\{2k_1, k_2\}}$$

where K is the Lipschitz constant of $p(x)$.

Remark 9 Let us observe that when $H \equiv 0$, the assumptions in Theorem 5 are clearly relaxed. First, we only need that $\varphi(x) = d(0_M, G(x))$ be lower semicontinuous, which is a very weak condition. For example, the single-valued function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = sign(x)$, shows that continuity can be dispensed with. Secondly, (WDC) reads as follows: *for every $x_0 \in M$, there are $h \in TM_{x_0}$, $||h|| = 1$, and $v \in DG(x_0|y_0)(h)$ such that $\left\langle \frac{exp_{y_0}^{-1}(0_M)}{d(0_M, y_0)}, v \right\rangle \geqslant A$, where $y_0 \in G(x_0)$ satisfies that $d(0_M, G(x_0)) = d(0_M, y_0)$.*

Moreover, when $H \equiv 0$, it is not necessary for M to have a Lie group structure, and we can replace 0_M with a given point p_0.

This observation allows us to deduce the following result, whose original proof (in the case when $M = X$, a Hilbert space) is due to Ekeland (see [1] and [7]).

Theorem 10 *Let M be a complete Riemannian manifold and $F\colon M \rightrightarrows M$ an upper semicontinuous map with compact values. Let us fix a point z_0 and assume that there is a positive constant C such that for every $x \in M$ there are a $y \in F(x)$, with $d(z_0, y) = d(z_0, F(x))$, and an $h \in TM_x$, $C||h|| \leqslant d(z_0, y)$ such that $exp_y^{-1}(z_0) \in DF(x|y)(h)$. Assume also that $d(z_0, F(x)) < i_{z_0}$. Then there exists a solution x_0 of $z_0 \in F(x)$.*

Proof If $h = 0$ for some x, we deduce that $\left\langle \frac{-exp_y^{-1}(z_0)}{d(z_0, y)}, exp_y^{-1}(z_0) \right\rangle \geqslant 0$, as a consequence of Lemma 2, hence $exp_y^{-1}(z_0) = 0 \Rightarrow y = z_0 \in F(x)$. Otherwise (if $y \neq z_0$) we have $v = \frac{exp_y^{-1}(z_0)}{||h||} \in DF(x|y)\left(\frac{h}{||h||}\right)$ and

$$\left\langle \frac{exp_y^{-1}(z_0)}{d(z_0, y)}, v \right\rangle = \frac{d(z_0, y)^2}{||h|| d(z_0, y)} = -\frac{d(z_0, y)}{||h||} \geqslant C,$$

so we may apply Theorem 5 with $H = 0$. □

We now introduce a condition which is slightly stronger than (WDC), but much easier to verify in practice.

Definition 11 We say that a set-valued mapping $G\colon M \rightrightarrows M$ satisfies the derivative condition, (DC) for short, for a positive constant A at $x_0 \in M$, if $d(x_0, G(x_0)) < i_{x_0}$ and for every $e \in TM_{y_0}$ with $||e|| = 1$, $y_0 \in G(x_0)$, $d(x_0, G(x_0)) = d(x_0, y_0)$, there exist $h \in TM_{x_0}$, $||h|| = 1$, and $v \in DG(x_0|y_0)(h)$ such that $\langle e, v\rangle \geqslant A$.

Observe that (DC) is symmetric in the sense that it is equivalent to the fact that for every $e \in TM_{y_0}$, $||e|| = 1$, where $y_0 \in G(x_0)$, such that $d(x_0, G(x_0)) < i_{x_0}$, there are $h \in TM_{x_0}$, $||h|| = 1$, and $v \in DG(x_0|y_0)(h)$ such that $\langle e, v\rangle \leqslant -A$. We will denote this condition by $(-DC)$.

From Theorem 5, we deduce an analogous result with the (DC) (or the $(-DC)$ condition) instead of the WDC condition.

Corollary 12 *Let M be a complete Riemannian manifold with an abelian Lie group structure, and $a \in M$. Let $G\colon B(a, R) \subset M \rightrightarrows M$ be a set-valued mapping, and $A > 0$. Assume that G satisfies (DC), or equivalently $(-DC)$, for A at every point $x_0 \in B(a, R) \subset M$. Let $H\colon M \rightrightarrows M$ have the Aubin-property with modulus $L < A$. Assume also that at least one of the sets $G(x)$, $H(x)$ is compact and that $d(-H(x)$,*

$G(x)) < i_M$ for each x. Then the equation $0_M \in F(x) = G(x) + H(x)$ has a solution in $B(a, R)$ provided that $F(a) \cap B(0, R(A - L)) \neq \emptyset$.

Now we introduce a fixed point result, whose prof is similar to that of Theorem 5.

Theorem 13 *Let M be a complete Riemannian manifold with an abelian Lie group structure, and $a \in M$. Let $G\colon B(a, R) \subset M \rightrightarrows M$ be a set-valued mapping, and $A > 1$. Assume that G satisfies the (WDC) condition for A. Let $H\colon M \rightrightarrows M$ be a set-valued mapping with the Aubin-property with modulus $L < A - 1$. Assume also that for each $x \in B(a, R)$ at least one of the sets $G(x)$, $H(x)$ is compact, and that $d(x - H(x), G(x)) < i_M$. Then the mapping $F(x) = G(x) + H(x)$ has a fixed point in $B(a, R)$ provided that $F(a) \cap B(a, R(A - L - 1)) \neq \emptyset$.*

Proof A solution of $x \in F(x) = G(x) + H(x)$ is a zero of $f(x) = d(x, F(x)) = d(G(x), x - H(x))$. Assume on the contrary that there is no solution. For any $x_0 \in B(a, R)$ and $\zeta \in \partial_P f(x_0)$ we have that $f(x_0) > 0$. Let $z_0 \in x_0 - H(x_0)$ and $y_0 \in G(x_0)$ be such that $f(x_0) = d(G(x_0), x_0 - H(x_0)) = d(y_0, z_0)$. The Proximal inequality gives us

$$d(G(x), x - H(x)) = f(x) \geqslant f(x_0) + \left\langle \zeta, exp_{x_0}^{-1}(x) \right\rangle - \sigma(d(x, x_0)^2),$$

hence

$$d(G(x), x - H(x)) - \left\langle \zeta, exp_{x_0}^{-1}(x) \right\rangle \geqslant d(z_0, y_0) - \left\langle \zeta, exp_{x_0}^{-1}(x_0) \right\rangle - \sigma(d(x, x_0)^2)$$

If we define $\psi(x) = d(z_0, x - H(x)) + d(z_0, G(x)) - \langle \zeta, x \rangle + \sigma(d(x, x_0)^2)$, we conclude that ψ has a local minimum at x_0, therefore

$$0 \in \partial_P \psi(x_0) = -\zeta + \partial(h + \varphi)(x_0)$$

because $h(x) = d(z_0, x - H(x))$, $\varphi(x) = d(z_0, G(x))$ (see [5] for the sum rule), and the function $x \to \left\langle -\zeta, exp_{x_0}^{-1}(x) \right\rangle + \sigma d(x, x_0)^2$ is C^2 with derivative $-\zeta$. Then, using the Fuzzy Sum Rule (and taking into account that h is $(L + 1)$-Lipschitz), for every $\varepsilon > 0$, there are x_1, x_2 and $\zeta_1 \in \partial_P \varphi(x_1)$, $\zeta_2 \in \partial_P h(x_2)$ such that

(1) $d(x_i, x_0) < \varepsilon$, $d(h(x_2), h(x_0)) < \varepsilon$ and $d(\varphi(x_1), \varphi(x_0)) < \varepsilon$
(2) $||\zeta - (L_{x_1 x_0}(\zeta_1) + L_{x_2 x_0}(\zeta_2)||_{x_0} < \varepsilon$.

Hence

$$||L_{x_1 x_0}(\zeta_1) + L_{x_2 x_0}(\zeta_2)||_{x_0} \leqslant \varepsilon + ||\zeta||_{x_0}.$$

Given x_1 and z_0, there are $h_1 \in T_{x_1} M$, $||h_1|| = 1$ and $v_1 \in DG(x_1 \mid y_1)(h_1)$ such that

$$\left\langle exp_{y_1}^{-1}(z_0), v_1 \right\rangle \geqslant A d(z_0, y_1)$$

since G enjoys the (WDC) condition.

On the other hand, we can apply Lemma 2 thanks to the fact that $d(x - H(x), G(x)) < i_M$, so we can deduce that

$$||\zeta_1||_{x_1} \geqslant \left\langle \frac{exp_{y_1}^{-1}(z_0)}{d(y_1, z_0)}, \frac{v_1}{||h_1||} \right\rangle = \left\langle \frac{exp_{y_1}^{-1}(z_0)}{d(y_1, z_0)}, v_1 \right\rangle \geqslant A,$$

and therefore

$$||\zeta|| \geqslant ||L_{x_1x_0}(\zeta_1)||_{x_0} - ||L_{x_2x_0}(\zeta_2)||_{x_0} - \varepsilon \geqslant A - ||L_{x_2x_0}(\zeta_2)|| - \varepsilon \geqslant A - L - 1 - \varepsilon$$

since $||\zeta_1||_{x_1} = ||L_{x_1x_0}(\zeta_1)||_{x_0}$ and $||L_{x_2x_0}(\zeta_2)||_{x_0} = ||\zeta_2||_{x_2} \leqslant L + 1$.

Finally, from the Decrease Principle, we obtain the following contradiction

$$0 \leqslant inf\{f(x) : x \in B(a, R)\} \leqslant f(a) - R(A - L - 1) < 0,$$

since $f(a) = d(a, F(a)) < R(A - L - 1)$. □

This theorem improves the result of [2, Theorem 38], as it holds for set-valued mappings and the hypotheses are less restrictive. Theorem 38 of [2] guarantees a solution of

$$x \in G(x) + H(x)$$

under the following assumptions:

(1) $G\colon M \to M$ is a single-valued smooth function, C-Lipschitz in $B(x_0, R)$,
(2) $H\colon M \to M$ is a single-valued L-Lipschitz function,
(3) $G(x) + H(x) \notin sing(x) \cup sing(G(x))$ for every $x \in B(x_0, R)$,
(4) $\langle L_{x,H(x)+G(x)}h, L_{G(x),H(x)+G(x)}dG(x)(h)\rangle_{H(x)+G(x)} \leqslant K < 1$ for all $x \in B(x_0, R)$ and $h \in TM_x$ with $\|h\|_x = 1$ and that
(5) $L < 1 - K$ and $d(x_0, x_0 + H(x_0)) < R(1 - K - L)$.

So this theorem demands that both of the functions are Lipschitz, and in Theorem 13 we only need that H have the Aubin-property and we don't need that G is smooth. Moreover the condition in hypothesis (4) is stronger than the (WDC)-condition.

Now we will study the special case of single-valued functions g. We assume that g is continuous. As we noted above, $Dg(x_0|g(x_0))(h)$, now denoted by $Dg(x_0)(h)$, is the set of limits, $\lim \frac{exp^{-1}_{g(x_0)}(g(exp_{x_0}(t_nh_n)))}{t_n}$, for sequences $h_n \to h$ and $t_n \downarrow 0$. Let us give a proof of this fact. Indeed,

$$v \in Dg(x \mid g(x_0))(h) \Leftrightarrow \exists h_n \to h, \quad h_n \in T_{x_0}M, \quad \exists v_n \to v, \quad v_n \in T_{g(x_0)}M$$

$$\text{and } \exists t_n \downarrow 0 \text{ such that, } \lim_n (h_n, v_n) = (h, v) \text{ and } exp_{g(x_0)}(t_nv_n) \in g(exp_{x_0}(t_nh_n)),$$

hence

$$exp_{g(x_0)}(t_nv_n) \equiv g(exp_{x_0}(t_nh_n))$$

since $g(exp_{x_0}(t_nh_n))$ is single-valued. Therefore

$$t_nv_n = exp^{-1}_{g(x_0)}(g(exp_{x_0}(t_nh_n))) \Leftrightarrow v_n = \frac{exp^{-1}_{g(x_0)}(g(exp_{x_0}(t_nh_n)))}{t_n}$$

and then, for all $v \in Dg(x_0 \mid g(x_0))(h) \equiv Dg(x_0)(h)$ there exist sequences t_n, h_n such that

$$v = \lim_n v_n = \lim_n \frac{exp^{-1}_{g(x_0)}(g(exp_{x_0}(t_nh_n)))}{t_n}.$$

Consequently the (DC) condition at x_0 is equivalent to the following one: for every $e \in T_{g(x_0)}M$, $||e|| = 1$, assuming that $d(x_0, g(x_0)) < i_{x_0}$, there exist sequences $h_n \to h$ and $t_n \downarrow 0$, $||h|| = 1$, such that

$$\left\langle e, exp_{g(x_0)}^{-1}(g(exp_{x_0}(t_n h_n)))\right\rangle \geqslant t_n A \quad (sDC)$$

Similarly $(-DC)$ at x_0 means that for every $e \in TM_{g(x_0)}$, $||e|| = 1$, assuming that $d(x_0, g(x_0)) < i_{x_0}$, there exist sequences $h_n \to h$ and $t_n \downarrow 0$, $||h|| = 1$, such that

$$\left\langle e, exp_{g(x_0)}^{-1}(g(exp_{x_0}(t_n h_n)))\right\rangle \leqslant -t_n A \quad (-sDC)$$

Therefore, for single-valued functions g, Theorems 5 and 13 can be rewritten using the (sDC) or the $(-sDC)$ condition instead of the (WDC) condition. In this case we do not need to assume that $G(x)$ and $H(x)$ are compact, because we know $g(x)$ that is a single point. The rest of the conditions are the same.

Remark 14 The graphical derivative of a single-valued function $g: M \to M$, always contains the one-side directional derivatives

$$g'(x_0, h) = \lim_{t \downarrow 0} \frac{exp_{g(x_0)}^{-1}(g(exp_{x_0}(th)))}{t}$$

provided that they exist. That is the case for Gateaux differentiable functions for instance.

Therefore we can use the one-side directional derivatives in order to simplify the conditions of Theorem 5. So instead of the WDC condition, we may require that for every $x_0 \in B(a, R)$ either

$$\inf_{e \in T_{g(x_0)}M, ||e||=1} \left(\sup_{h \in T_{x_0}M, ||h||=1} \langle e, g'(x_0, h)\rangle \right) \geqslant A$$

or

$$\sup_{e \in T_{g(x_0)}M, ||e||=1} \left(\inf_{h \in T_{x_0}M, ||h||=1} \langle e, g'(x_0, h)\rangle \right) \leqslant -A,$$

and leave the rest of the statement untouched.

As for fixed point results, and concerning one-side directional derivatives, we have the following corollary.

Corollary 15 *Let M be a complete Riemannian manifold with an abelian Lie group structure, and $a \in M$. Let $g: B(a, R) \to M$ be a continuous function with one-side directional derivatives at every point. Assume that for a positive constant $A > 1$, we have that for every $x_0 \in B(a, R)$ either*

$$\inf_{e \in T_{g(x_0)}M, ||e||=1} \left(\sup_{h \in T_{x_0}M, ||h||=1} \langle e, g'(x_0, h)\rangle \right) \geqslant A$$

or

$$\sup_{e \in T_{g(x_0)}M, ||e||=1} \left(\inf_{h \in T_{x_0}M, ||h||=1} \langle e, g'(x_0, h)\rangle \right) \leqslant -A$$

Let $H: M \rightrightarrows M$ have the Aubin-property with modulus $L < A - 1$. We assume than $d(x - H(x), g(x)) < i_M$. Then $F = g + H$ has a fixed point, provided that $F(a) \cap B(a, R(A - L - 1)) \neq \emptyset$.

Proof Taking $\varepsilon > 0$ small enough so that $F(a) \cap B(a, R(A - L - 1 - \varepsilon)) \neq \emptyset$, we have that for all $e \in T_{g(x_0)}M$, $||e|| = 1$ there exists $h \in T_{x_0}M$ such that $\langle e, g'(x_0, h)\rangle \geqslant A - \varepsilon$ (or $\langle e, g'(x_0, h)\rangle < -A + \varepsilon$), and we can now apply the Theorem 13 with the (sDC) or the $(-sDC)$ instead of the (WDC)-condition. □

The above result yields a differentiable test to ensure that a given function is onto.

Corollary 16 *Let M be a complete Riemannian manifold with an abelian Lie group structure and with an infinite injectivity index i_M. Let $g: M \to M$ be a continuous function with one-side directional derivatives at every point. Assume that for a positive constant A, we have that for every $x_0 \in M$ either*

$$\inf_{e \in T_{g(x_0)}M, ||e||=1} \left(\sup_{h \in T_{x_0}M, ||h||=1} \langle e, g'(x_0, h)\rangle \right) \geqslant A,$$

or

$$\sup_{e \in T_{g(x_0)}M, ||e||=1} \left(\inf_{h \in T_{x_0}M, ||h||=1} \langle e, g'(x_0, h)\rangle \right) \leqslant -A.$$

Let H be a set-valued mapping with the Aubin-property with modulus L, and $L < A$. Then $g + H$, and in particular g too, are onto.

Proof Given a $y_0 \in M$, from Theorem 5, with the condition of Remark 14 instead of the (WDC)-condition, applied to the function $g(x) - y_0$, with $R = +\infty$, we deduce the existence of a $x_0 \in M$ such that $0_M \in g(x_0) - y_0 + H(x_0)$. □

For instance, if we consider a polynomial $P \in \mathcal{P}(^2X)$ (where X is a Hilbert space) such that $P(h) \geqslant A$ for every $h \in S_X$, and an L-Lipschitz function $f: X \to X$, with $L < A$, then the function $G(x) = DP(x) + f(x)$ is onto.

The case H constant, that is 0-Lipschitz, gives us an interesting result on discrete dynamical systems.

Corollary 17 *Let M be a complete Riemannian manifold with an abelian Lie group structure and $a \in M$. Let $g: B(a, R) \subset M \to M$ be a continuous function, $A > 0$. Assume that it has one-side directional derivatives at every point, $g'(x_0, h)$, and moreover that for every $x_0 \in B(a, R)$ either*

$$\inf_{e \in T_{g(x_0)}M, ||e||=1} \left(\sup_{h \in T_{x_0}M, ||h||=1} \langle e, g'(x_0, h)\rangle \right) \geqslant A,$$

or

$$\sup_{e \in T_{g(x_0)}M, ||e||=1} \left(\inf_{h \in T_{x_0}M, ||h||=1} \langle e, g'(x_0, h)\rangle \right) \leqslant -A.$$

Let S be a closed subset, such that $d(x, S)$ is attained for every $x \in M$. We assume that $d(g(x), 0_M) < i_0$ where i_0 is the injectivity radius of M at 0_M. Then $g(B(a, R)) \cap S \neq \emptyset$ provided that $d(g(a), S) \leqslant RA$.

Proof The result follows from Theorem 5, with the condition of Remark 14 instead of the (WDC)-condition, applied to the mappings $-g$ and $H(x) = S$ constantly. □

In [11] a Banach Fixed Point Theorem for set valued mappings is proved. With the tools we have developed we can now easily deduce a similar result for Riemannian manifolds.

Theorem 18 *Let M be a complete Riemannian manifold with an abelian Lie group structure. Assume that $H\colon M \rightrightarrows M$ is an L-pseudo-contractive (with the Aubin-property with modulus $L < 1$) set valued mapping, and that $d(-H(x), -x) < i_M$. Then H has a fixed point. Moreover, $B(a, R)$ contains a fixed point provided that $H(a) \cap B(a, R(1 - L)) \neq \emptyset$.*

Proof It is enough to apply Theorem 5, with the condition of Remark 14 instead of the (WDC)-condition, to $g(x) = -x$, H, and $A = 1$. Then $F(x) = -x + H(x)$ satisfies $F(a) \cap B(0_M, R(1 - L)) \neq \emptyset$ since $H(a) \cap B(a, R(1 - L)) \neq \emptyset$, and we can deduce that there is a $x_0 \in B(a, R)$ such that $0 \in F(x_0)$, or equivalently $x_0 \in H(x_0)$. □

Another interesting application of Theorem 5 is the existence of solutions for families of equations. To this end we first need to establish the following viability Theorem.

Theorem 19 *Let M be a complete Riemannian manifold with an abelian Lie group structure. Let $g\colon B(a, R) \subset M \to M$ be a continuous function, $A > 0$, P a compact topological space of parameters, $h\colon B(a, R) \times P \to M$. We define $f(x, \alpha) = g(x) + h(x, \alpha)$. Assume that*

(1) *$d(-h(x, \alpha), g(x)) < i_M$ for all $\alpha \in P$.*
(2) *g satisfies either the (sDC) or the $(-sDC)$ at every $x \in B(a, R)$.*
(3) *$h_x\colon P \to M$ is continuous for every $x \in B(a, R)$.*
(4) *For every $x_1, x_2 \in B(a, R)$ and $\alpha_1 \in P$, there is a $\alpha_2 \in P$, such that $d(h(x_1, \alpha_1), h(x_2, \alpha_2)) \leqslant Ld(x_1, x_2)$, with $L < A$.*
(5) *There is $\alpha_0 \in P$ such that $d(0_M, f(a, \alpha_0)) \leqslant R(A - L)$.*

Then the problem $f(x, \alpha) = 0$, $x \in B(a, R)$, $\alpha \in P$ has a solution.

Proof Consider the set valued mapping $C\colon B(a, R) \subset M \rightrightarrows M$ defined by $C(x) = \{f(x, \alpha)\}_{\alpha \in P} = g(x) + \{h(x, \alpha)\}_{\alpha \in P} = g(x) + H(x)$. The set $H(x)$ is compact by (3), H is L-Lipschitz by (4), and $C(a) \cap B(0, R(A - L)) \neq \emptyset$ by (5). Hence, by Theorem 5 with the (sDC) or the $(-sDC)$ condition instead of the (WDC)-condition, there is a $x_0 \in B(a, R)$ such that $0 \in C(x_0)$, or equivalently $f(x_0, \alpha_0) = 0$ for an $\alpha_0 \in P$. □

Remark 20 Condition (3) is trivially met if we consider a finite parameter set. On the other hand, condition (4) is weaker than requiring that all the h_α are L-Lipschitz. As a matter of fact it is not necessary to consider continuous functions, as the following example shows: $P = \{1, 2\}$, $E \subset \mathbb{R}$, $h_1 = \chi_E$, $h_2 = \chi_{E^c}$.

A different approach would be to consider small perturbations of functions with zeros or with fixed points. For the sake of simplicity we will consider only the case $X = \mathbb{R}^n$.

Theorem 21 *Let $g\colon X \to X$ be a continuous function, with continuous one-side directional derivatives, $g'(.,h)$. Let $a \in X$ such that $g(a) = 0$. Assume that either $\inf_{e\in S_X}(\sup_{h\in S_X}\langle e, g'(a,h)\rangle) > 0$ or $\sup_{e\in S_X}(\inf_{h\in S_X}\langle e, g'(a,h)\rangle) < 0$. Let $H\colon X \rightrightarrows X$ be a set-valued mapping with the Aubin-property with modulus $L < A$. Then $F = g + \alpha H$ has a zero, provided that α is small enough.*

Proof Assume that $\inf_{e\in S_X}(\sup_{h\in S_X}\langle e, g'(a,h)\rangle) > 0$. For every $e \in S_X$, let $h_e \in S_X$ and $C_e > 0$ be such that $\langle e, g'(a,h_e)\rangle = C_e$. Let U_e be a neighborhood of e, and $r_e > 0$ such that $\langle \tilde{e}, g'(x,h_e)\rangle > \frac{C_e}{2}$ whenever $\tilde{e} \in U_e$ and $x \in B(a, r_e)$. We may cover S_X with a finite number of these neighborhoods, $U_{e_1}, \dots, U_{e_n}$, by compactness. If $R = \min(r_{e_1}, \dots, r_{e_n})$, we have that for every $x_0 \in B(a, R)$ and every $e \in S_X$ there is an $h \in S_X$ ($h = h_{e_1}, \dots$ or h_{e_n}) such that $\langle e, g'(x_0,h)\rangle > A$ for $A = \min\left(\frac{C_{e_1}}{2}, \dots, \frac{C_{e_n}}{2}\right)$.

Finally, if α is small enough so that the Lipschitz constant of αH is smaller than A and $\alpha||H(a)|| < R(A - L)$, we may apply Theorem 5 with the condition of Remark 14 instead of the (WDC)-condition, and conclude that $g + \alpha H$ has a zero in $B(a, R)$. □

Remark 22 In order to establish the above result for Riemannian manifolds, we would have to assume both $g'(.,h)$ and $g'(a,.)$ continuous, since we would be working on different tangent spaces depending on the point.

Similarly we have the following.

Theorem 23 *Let $g\colon B(a,R) \to X$ be a continuous function, with continuous one-side directional derivatives, $g'(.,h)$. Let a be a fixed point for g. Assume that either $\inf_{e\in S_X}(\sup_{h\in S_X}\langle e, g'(a,h) - h\rangle) > 0$ or $\sup_{e\in S_X}(\inf_{h\in S_X}\langle e, g'(a,h) - h\rangle) < 0$. Let H: $X \rightrightarrows X$ be a Lipschitz set valued mapping. Then $F = G + \alpha H$ has a fixed point, provided that α is small enough.*

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