Aggregation–Diffusion Equations

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North meets South, May 28th 2021

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This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 883363)

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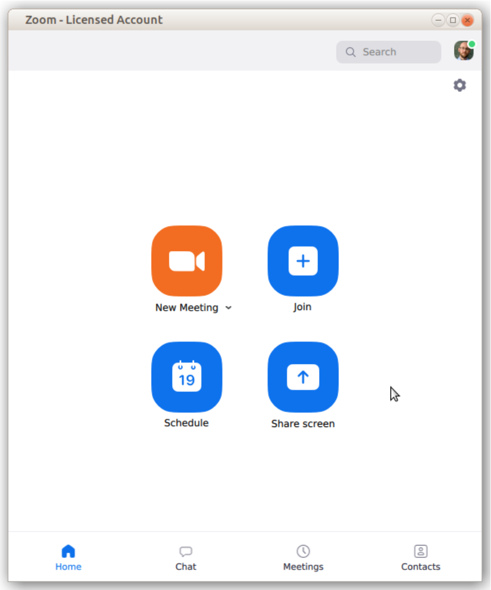
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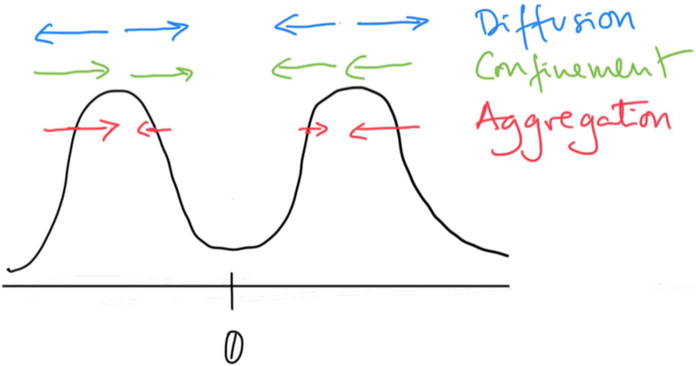

The aim of this talk is to explain the modeling and theory behind the following model for aggregation-diffusion phenomena:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \Big(\rho \nabla (\underbrace{U'(\rho)}_{\text{Diffusion}} + \underbrace{V}_{\text{Confinement}} + \underbrace{W * \rho}_{\text{Aggregation}}) \Big) \tag{ADE}$$

We will discuss the range of power-type aggregation and diffusion

$$U'(\rho) = \tfrac{m}{m-1} \rho^{m-1}, \qquad V(x) = \frac{|x|^\alpha}{\alpha}, \qquad \text{and} \qquad W(x) = \frac{|x|^\lambda}{\lambda}.$$

If V, W are bounded below, we can always assume $V, W \geq 0$.

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Conservation equation. Let ρ be a density and $\omega \subset \mathbb{R}^d$ any control volume, if j is the out-going flux

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \rho \,\mathrm{d}x = -\int_{\partial\omega} j \cdot n \,\mathrm{d}S = -\int_\omega \nabla \cdot j \,\mathrm{d}x$$

Linear Darcy's law: flux opposing the gradient $j = -\nabla \rho$ yields

$$\frac{\partial \rho}{\partial t} = \Delta \rho \tag{HE}$$

The confinement can be added as a drift $j = -\nabla \rho - \rho \nabla V$.

Non-linear Darcy's law: $j = -\nabla \varphi(\rho)$ for some non-decreasing $\varphi : \mathbb{R} \to \mathbb{R}$

$$\frac{\partial \rho}{\partial t} = \Delta \varphi(\rho). \tag{DE}$$

When $\varphi(\rho) = \rho^m$ for $m > 0$ this is called Porous Medium Equation [Vázquez 2006].
Notice $\nabla \varphi(\rho) = \nabla \cdot (\varphi'(u) \nabla \rho)$ so $\quad U''(\rho) = \frac{\varphi'(\rho)}{\rho}$.

Consider an stochastic particle jumping over the mesh $\{..., -h, 0, h, 2h, ...\}$ $(h > 0)$.
Let X_n be the position after n jumps. Assume the jump probabilities are

$$\mathbb{P}(X_{n+1} = jh \mid X_n = ih) = \begin{cases} \frac{1}{2} & \text{if } |i - j| = 1, \\ 0 & \text{otherwise} \end{cases}$$

Define $U_j^n = P(X_n = hj)$ and assume the initial distribution U_j^0 is given.
Then $U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n)$ or, for $\tau = h^2$

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \frac{1}{2} \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}.$$

This is a classical discretisation of the stochastic version of (HE): $\partial_t \rho = \frac{1}{2} \Delta \rho$.

The time continuous extension of X_n version is the Wiener process $X_t = W_t$.
This gives rise to the intuition (which has to be understood in terms of the Îto calculus)

$$dX_t = dW_t$$

Consider 1 particle. Using a similar arguments, for the stochastic equation

$$\mathrm{d}X_t = \underbrace{\mu(t, X_t)}_{\text{drift}}\,\mathrm{d}t + \underbrace{\sigma(t, X_t)\,\mathrm{d}W_t}_{\text{diffusion}}$$

its probability density ρ is the solution of the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t}(t, x) = -\nabla \cdot (\mu(t, x)\rho(t, x)) + \Delta(D(t, x)\rho(t, x))$$

where $D = \frac{\sigma^2}{2}$.

Consider N with positions X_i of masses a_i and the attracting/repulsive system[1]

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \underbrace{a_j \nabla W(X_i - X_j)}_{\text{Aggregation}} \underbrace{- a_i \nabla V(X_i)}_{\text{Confinement}}, \qquad i = 1, \cdots, N$$

We say that these are aggregation potentials when $\nabla W(x) \cdot x, \nabla V(x) \cdot x \geq 0$.

The typical example is $W(x) = \frac{|x|^\lambda}{\lambda}$ and $V(x) = \frac{|x|^\alpha}{\alpha}$.

The empirical distribution is defined as $\mu_t^N = \sum_{j=1}^{N} a_j \delta_{X_j(t)}$.

It is easy to see that, in the sense of distributions, μ^N solves the **Aggregation Equation**

$$\partial_t \mu = \nabla \cdot (\mu \nabla (W * \mu + V)) \tag{AE}$$

Often $a_i = 1/N$ and we simply play with the initial distribution of $X_1(0), \cdots, X_N(0)$.

[1] Assume $\nabla W(0) = 0$

Imagine now we have N stochastic particles at positions $X_1(t), \cdots, X_N(t)$. We assume they have equal mass.

Recall the empirical measure $\mu_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_j(t)}$

Assume the evolution of the particles is given by the system of SODEs

$$\mathrm{d}X_i = -\frac{1}{N}\sum_{j\neq i} \nabla W(X_i - X_j) - \frac{1}{N}\nabla V(X_i) + \sqrt{2D}\,\mathrm{d}W_t^i$$

The limit as $N \to \infty$ is the solution of

$$\partial_t \rho = \nabla \cdot (\rho \nabla (W * \rho + V)) + D\Delta\rho.$$

This corresponds to $U(\rho) = D\rho\log\rho$.

Mean-Field Approximation: As $N \to \infty$

$$\mu_0^N \to \rho_0 \text{ in the tight topology} \implies \mu_t^N \to \rho(t) \text{ in law for a.e. } t > 0.$$

For the details see, e.g., [Jabin and Wang 2017].

[1] Convergence in law: pointwise convergence of distribution functions at continuity points of the limit

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Joining the many particle approximation with the Porous Medium diffusion:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \Big(\rho \nabla (U'(\rho) + V + W * \rho)\Big) \tag{ADE}$$

Some famous examples

Model	U	V	W
Heat Equation	$\rho \log \rho$	0	0
Porous Medium Equation $m \neq 1$	$\frac{1}{m-1}\rho^m$	0	0
Fokker-Planck	$\rho \log \rho$	$\frac{1}{2}\lvert x\rvert^2$	0
Patlak-Keller-Segel	$\rho \log \rho$	0	$\chi \log \lvert x\rvert$
Swarming / Herding	0	0	$\frac{1}{a}\lvert x\rvert^a - \frac{1}{b}\lvert x\rvert^b$

Conservation?

In conservation laws, we expect $\int_{\mathbb{R}^d} \rho(t) = \int_{\mathbb{R}^d} \rho_0$
(i.e. $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$, we expect $\rho(t) \in \mathcal{P}(\mathbb{R}^d)$)

A direct computation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \rho \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \lim_{R \to \infty} \int_{B_R} \rho \, \mathrm{d}x = \lim_{R \to \infty} \int_{\partial B_R} j \frac{x}{|x|} \, \mathrm{d}S \xrightarrow{?} 0.$$

Sometimes mass is not conserved.
Let $\partial_t \rho = \Delta \rho^m$ with $d \geq 3$, $m < \frac{d-2}{d}$, and $\rho_0 \in L^q(\mathbb{R}^d)$ with $q = \frac{(1-m)d}{2}$ Details

$$\|\rho(t)\|_{L^q} \searrow 0, \qquad \text{as } t \nearrow T^* < \infty.$$

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Limitations of the L^1 framework

For (DE) **entropy solutions**:

$$\rho_0 \in L^1 \implies \exists! \rho \in C([0, +\infty); L^1(\mathbb{R}^d))$$

(see, e.g. [Kružkov 1970; Carrillo 1999])

Keller-Segel with $\chi > \chi^*$ (e.g. [Herrero and Velázquez 1996] for $d = 2$):

$$\rho(t) \longrightarrow M\delta_0 + f \qquad \text{as } t \nearrow T < \infty.$$

The so-called **chemotactic collapse**

The L^1 framework is not enough!

Total variation: $\|\mu\|_{TV} = |\mu|(\mathbb{R}^d)$.

However, if $a \neq b$ then $\|\delta_a - \delta_b\|_{TV} = 2$.

For the particle system $t \mapsto \mu_t^N$ is not $C\Big([0,T]; (\mathcal{M}(\mathbb{R}^d), \|\cdot\|_{TV})\Big)$.

We want to construct a distance between measures such that

$$d(\delta_a, \delta_b) = |a - b|.$$

Other people in measure distances. Optimal transport

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$T : X \to Y$ transport $\mu \in \mathcal{P}(X)$ into $\nu \in \mathcal{P}(Y)$ if $\nu(B) = \mu(T^{-1}(B))$, i.e. $\nu = T_{\#}\mu$.

Monge's transport problem:

$$\min_{T:\nu=T_{\#}\mu} \int_X c(x, T(x))\, \mathrm{d}\mu(x)$$

The limitation is that mass $x \mapsto T(x)$ so the mass of a Dirac cannot be split:
i.e. $\nu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1 \neq T_{\#}\delta_0$ for any T.

A generalisation is through transport plans between μ and ν:

$$\Pi(\mu,\nu) = \Big\{\pi \in \mathcal{P}(X \times Y) : \quad \pi(A \times Y) = \mu(A), \quad \pi(X \times B) = \nu(B)\Big\}.$$

Kantorovich's transport problem:

$$\min_{\pi \in \Pi(\mu,\nu)} \iint_{X \times Y} c(x,y)\, \mathrm{d}\pi(x,y)$$

Under some conditions, the problems are equivalent. See [Villani 2009; Carrillo 2021].

The Wasserstein distance between $\mu, \nu \in \mathcal{P}(X)$ with $c(x, y) = |x - y|^p$

$$d_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} |x - y|^p \, \mathrm{d}\pi(x, y) \right)^{\frac{1}{p}}$$

When there exists optimal T_0, we have the geodesic $\mu_t = ((1-t)\mathrm{id}_{\mathbb{R}^d} + tT_0)_{\#}\mu$.

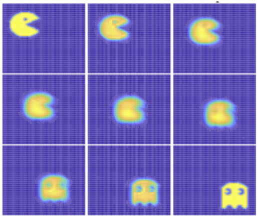

Figure: Computation of a interpolation measure by the Monge-Kantorovich problem with $p = 2$ between Pac-Man and the Ghost characteristic sets suitably normalized.
[Carrillo, Craig, Wang, and Wei 2019]

Wasserstein is good for (AE)

To compute $d_p(\delta_a, \delta_b)$ we first note that $\Pi(\delta_a, \delta_b) = \{\delta_{(a,b)}\}$.
Hence $d_p(\delta_a, \delta_b) = |a - b|$.

The correct space to work with this distance is

$$\mathcal{P}_p(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \, \mathrm{d}\mu(x) \right\}$$

We endow $\mathcal{P}_p(\mathbb{R}^d)$ with the distance d_p.

Theorem 1 [Carrillo, DiFrancesco, Figalli, Laurent, and Slepčev 2011]

$W \in C(\mathbb{R}^d) \cap C^1(\mathbb{R}^d \setminus \{0\})$, even, λ-convex ($\lambda \leq 0$), $W(x) \leq C(1 + |x|^2)$, and $V = 0$ then there exists a unique solution[1] of (AE) in $C([0, +\infty); \mathcal{P}_2(\mathbb{R}^d))$.

[1] solution in the sense of *curve of maximal slope* of the energy functional

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Gradient flow in $\mathbb{R}^d$

Let $F : \mathbb{R}^d \to \mathbb{R}$. Imagine we look for $\operatorname{argmin} F$.

We call **gradient flow** of F the flow of the ODE $$\frac{\mathrm{d}X}{\mathrm{d}t} = -\nabla F(X(t))$$

If F is strictly convex, for any $X(0)$ we have $X(t) \to X_\infty = \operatorname{argmin} F$.

If $D^2 F \geq \lambda I$ then $|X(t) - X_\infty| \leq e^{-\lambda t} |X_0 - X_\infty|$.

The Heat Equation as a gradient flow in H^1

Let $\mathcal{F} : H^1(\mathbb{R}^d) \to \mathbb{R}$ be defined as $\mathcal{F}[\rho] = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla \rho|^2$

We compute the first variation (i.e. Gateaux derivative). Taking $\varphi \in H^1(\mathbb{R}^d)$

$$\left\langle \frac{\delta \mathcal{F}}{\delta \rho}[\rho_0], \varphi \right\rangle = \lim_{\varepsilon \to 0} \frac{\mathcal{F}[\rho_0 + \varepsilon\varphi] - \mathcal{F}[\rho_0]}{\varepsilon}$$
$$= \int_{\mathbb{R}^d} \nabla \rho_0 \nabla \varphi + \lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \int_{\mathbb{R}^d} |\nabla \varphi|^2$$

We define the gradient

$$\nabla_{H^1} \mathcal{F}[\rho_0] = \frac{\delta \mathcal{F}}{\delta \rho}[\rho_0] = -\Delta \rho_0 \qquad \text{in } H^{-1}$$

Remark

We can rewrite the Heat Equation

$$\frac{\partial \rho}{\partial t} = -\nabla_{H^1} \mathcal{F}[\rho(t)], \quad \text{where } \mathcal{F}[\rho] = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla \rho|^2 \qquad \text{(HE)}$$

$\mathcal{F}$ is strictly convex in $H^1(\mathbb{R}^d)$. Naturally, $\rho(t) \to 0$ which is the minimiser of $\mathcal{F}$.
In general, the $\nabla_{H^1}\mathcal{F}$ is given by the Euler-Lagrange equations ▸ Details

For $\mathcal{F} : L^1 \cap \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ formally speaking Details

$$\nabla_{d_2} \mathcal{F}[\rho] = -\nabla \cdot \left(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho} \right)$$

Remark

If $W(x) = W(-x)$, we can formally rewrite the Aggregation-Diffusion problem as[2]

$$\frac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)], \quad \text{where } \mathcal{F}[\rho] = \int_{\mathbb{R}^d} \left(U(\rho) + V\rho + \tfrac{1}{2}\rho(W * \rho) \right). \qquad \text{(ADE)}$$

Formally, $\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^d} \rho \left| \nabla \frac{\delta \mathcal{F}}{\delta \rho}[\rho] \right|^2$. This is called *energy dissipation* estimate.

The precise definition of solution is the notion of *curves of maximal slope*. Details

[2] Due to the convolution, $\mathcal{F}$ is non-local and $\mathcal{F}[\rho] \neq \int_{\mathbb{R}^d} F(x, \rho(x))\, \mathrm{d}x$. $\frac{\delta F}{\delta \rho}$ can be computed directly

The extension of convexity in $\mathbb{R}^d$ that is suitable in $\mathcal{P}_2$ is **displacement convexity** (see [McCann 1997]).

There is a suitable theory for gradient flow of $\mathcal{F}$ in d_2 (see [Ambrosio, Gigli, and Savare 2005])

In fact, as $t \to \infty$ we have

$$\mathcal{F}[\rho(t)] \searrow \inf_{\rho \in \mathcal{P}_2 \cap L^1} \mathcal{F}.$$

Under stronger hypothesis

$$d_p(\rho(t), \operatorname{argmin} \mathcal{F}) \searrow 0.$$

Due to the estimate above, at an minimiser ρ_∞, we have

$$\frac{\delta \mathcal{F}}{\delta \rho}[\rho_\infty] = C.$$

In some cases, the free energy $\mathcal{F}$ for (ADE) is displacement convex (see [Carrillo and Slepčev 2009]).

This does not hold for $\partial_t \rho = \Delta \rho^m$ with $m < \frac{d-2}{d}$ (where solutions leave $\mathcal{P}$).

When $\inf \mathcal{F} = -\infty$, then we do not expect an asymptotic equilibrium.
There might be intermediate asymptotics, recovered by rescaling.

Actually, we need to consider the extension of $\mathcal{F}$ to $\mathcal{M}(\mathbb{R}^d)$, which we denote $\widetilde{\mathcal{F}}$ (see [Demengel and Temam 1986]) Details

If $\mu_\infty \in \operatorname{argmin} \widetilde{\mathcal{F}}$, we expect it to be a local attractor but there is no guarantee.

The first variation:

$$\frac{\delta \mathcal{F}}{\delta \rho} = U'(\rho) + V + W * \rho.$$

Asymptotic behaviour of some examples

$\rho_0 \in \mathcal{P} \cap L_c^\infty(\mathbb{R}^d)$

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(HE)	$\rho(t) \sim (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{\lvert x\rvert^2}{4t}\right) \to 0$
Fokker-Planck	$\rho(t) \to (4\pi)^{-\frac{d}{2}} \exp\left(-\frac{\lvert x\rvert^2}{4}\right)$
PME $m \in \left(\frac{d-2}{d}, 1\right) \cup (1, +\infty)$ (see [Vázquez 2006])	$\rho(t) \sim t^{-\alpha}\left(C_1 - C_2\lvert x\rvert^2 t^{\frac{2\alpha}{d}}\right)_+^{\frac{1}{m-1}} \to 0$ where $\alpha = \frac{d}{d(m-1)-2}$
PME $m < \frac{d-2}{d}$	$\rho(t) \to 0$ as $t \to T$
Keller-Segel (see [Herrero and Velázquez 1996])	$\rho(t) \to M\delta_0 + f$ as $t \to T$ (where $M > 0$ if $\chi > \chi^*$)

Scaling of $\mathcal{F}$

Case $U = \frac{1}{m-1}\rho^m$, $V = c|x|^\alpha$ and $W = d|x|^\lambda$ when $\alpha, \lambda > 0$

Let $0 \le \rho_1 \in C_c^\infty(\mathbb{R}^d)$ with $\rho_1 \ge 0$ and $\int_{\mathbb{R}^d} \rho_1 = 1$.

Define $\rho_k(x) = k^d \rho(kx)$. $\quad \rho_k \to \begin{cases} \delta_0 & k \to \infty, \\ 0 & k \to 0. \end{cases}$

Scaling the integrals

$$\mathcal{F}[\rho_k] = k^{d(m-1)} \int_{\mathbb{R}^d} U(\rho_1)\,\mathrm{d}x + k^{-\alpha} \int_{\mathbb{R}^d} V\rho_1 + \frac{k^{-\lambda}}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho_1(x)\rho_1(y)\,\mathrm{d}y.$$

When $m > 1$, $\mathcal{F}[\rho] > 0$ for all ρ.

- In $\mathcal{F}[\rho_k]$ the exponents are $d(m-1) > 0$ whereas $-\alpha, -\lambda < 0$, and the constants ≥ 0.
- If $V = W = 0$, then we minimise $k \to 0$ and we get full diffusion $\rho_k \to 0$.
- If $V > 0$ or $W > 0$, then we have balancing effects. There can be minimisers $\rho_\infty \in L^1$.

When $m \in (0,1)$, $\int_{\mathbb{R}^d} U(\rho_1) < 0$ and $d(m-1) < 0$.

- If $d(1-m) < \max\{\alpha, \lambda\}$ then the diffusion is dominant $k \to 0$ gives $\inf \mathcal{F} = -\infty$.
- If $d(1-m) > \max\{\alpha, \lambda\}$ aggregation is dominant. There can be minimisers.
- We call fair competition range to $m = \frac{d - \max\{\alpha,\lambda\}}{d}$.

Minimization and (ADE) when $V = 0$

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Minimisation for $U = \frac{m}{m-1}\rho^m$, $V = 0$, and $W(x) = |x|^\lambda/\lambda$:

- [Carrillo, Hittmeir, Volzone, and Yao 2019]:
  Any minimiser is $\mu_\infty = \rho_\infty + M\delta_0$ with ρ_∞ radially symmetric
- [Carrillo, Delgadino, Dolbeault, Frank, and Hoffmann 2019]:
  $\lambda > 0$ and $m \in (0, 1)$
  - $\mathcal{F}$ is bounded below iff $m \in (\frac{d}{d+\lambda}, 1)$
  - If $m > \frac{d}{d+\lambda}$ there exists a minimiser of the form $\mu_\infty = \rho_\infty + M\delta_0$
  - If $\lambda \in [2, 4]$ or $\lambda \geq 1$ and $m \geq 1 - \frac{1}{d}$, then the global minimiser is unique (up to translation).
  - $M = 0$ if $\lambda \in (0, 2 + \frac{4}{(N-2)_+})$ and $m \in (\frac{d}{d+\lambda}, 1)$
- [Carrillo, Delgadino, Frank, and Lewin 2020]:
  - If $\lambda = 4$ and $d \leq 5$ then $M = 0$.
  - If $\lambda = 4$ and $d \geq 6$ then $M = 0$ if and only if $m \geq \frac{d-2}{d+4}\left(1 + \frac{4}{3d}\right)$.
  - Numerical results for $\lambda = 2k$

Asymptotic behaviour as $t \to \infty$

- [Cañizo, Carrillo, and Schonbek 2012]
- [Carrillo, G-C, Yao, and Zeng 2021]:
  $W \in \mathcal{W}^{1,\infty}$, $\nabla W \in L^{n-\varepsilon}$, $\Delta W \in L^{\frac{n}{2}-\varepsilon}$ then ρ behaves like (HE) .

Minimization and (ADE) when $W = 0$

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When $W = 0$, $\frac{\delta \mathcal{F}}{\delta \rho} = 0$ leads to

$$\rho_{V+h} = \left(\tfrac{1-m}{m}(V+h)\right)^{\frac{-1}{1-m}}, \qquad a_{V+h} = \int_{\mathbb{R}^d} \rho_{V+h}, \qquad h \geq 0$$

[Carrillo, G-C, and Vázquez 2021]: $m \in (0,1)$ and $V \in W^{2,\infty}_{loc}(\mathbb{R}^d)$ $(\alpha \geq 2)$.

First, we replace $\mathbb{R}^d$ by B_R, and add no-flux condition:

- For $\rho_0 \in L^1$, there is a global solution with mass conservation
- $\mathcal{F}$ always minimises in $\mathcal{P}_2(B_R)$
- The minimiser is
$$\mu_\infty = \begin{cases} \rho_{V+h} & \text{if there exists } h \text{ s.t. } a_{V+h} = 1, \\ \rho_V + (1 - a_V)\delta_0 & \text{if } a_0 < 1. \end{cases}$$
- If $a_V < 1$, $\exists \rho_0$ such that $\forall r$, $\displaystyle\int_{B_r} \rho(t)\,\mathrm{d}x \nearrow 1 - a_0 + \int_{B_r} \rho_0$ as $t \nearrow \infty$

We study the equation satisfies by the mass function $M(t,r) = \int_{B_r} \rho(t)\,\mathrm{d}x$
We prove similar results in $\mathbb{R}^d$, with V that ensure minimisation and compactness of $\mathcal{F}$.

Thank you for your attention!

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PME: Finite time extinction

Let $\partial_t \rho = \Delta \rho^m$ with $m < \frac{d-2}{d}$ and $d \geq 3$ and $\rho_0 \in L^q(\mathbb{R}^d)$ with $q = \frac{(1-m)d}{2}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{q} \int_{\mathbb{R}^d} \rho^q \overset{\text{PDE}}{=} -C \int_{\mathbb{R}^d} |\nabla \rho^{\frac{m+q}{2}}|^2 \overset{\text{Sobolev}}{\leq} -C \left(\int_{\mathbb{R}^d} \rho^{\frac{m+q}{2} 2^*} \right)^{\frac{1}{2^*}}$$

where $2^* = \frac{1}{2} - \frac{1}{d}$.
The equation $\frac{\mathrm{d}}{\mathrm{d}t} X = -C X^\alpha$ where $\alpha < 1$ has finite time extinction.

$\mathcal{P}_2$ is not a vector space, so there we are not using the intrinsic notion of Fréchet gradient.
The correct notion is **Fréchet subdifferentials** (we will not define it here).
Also, we can see $\mathcal{P}_2$ inside the space of measures.
Fix $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Then, the tangent space is given by

$$\mathrm{Tan}_{\rho_0}\mathcal{P}_2(\mathbb{R}^d) = \left\{ \xi : \exists \zeta_n \in C_c(\mathbb{R}^d, \mathbb{R}) \text{ s.t. } \int_{\mathbb{R}^d} |\xi - \nabla\zeta_n|^2 \, \mathrm{d}\rho_0 \to 0 \right\}$$

Take $\xi = \nabla\zeta$ with $\zeta \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Then, by [Ambrosio, Gigli, and Savare 2005, Lemma 5.5.3]

$$\rho_\varepsilon = (1_{\mathbb{R}^d} + \varepsilon\xi)_\# \rho_0 = \frac{\rho_0}{\det(1_{\mathbb{R}^d} + \varepsilon\nabla\xi)} \circ (1_{\mathbb{R}^d} + \varepsilon\xi)^{-1}$$

The map $(x, \varepsilon) \mapsto \rho_\varepsilon(x)$ is C^2 and

$$\lim_{\varepsilon\to 0} \rho_\varepsilon = \rho_0, \qquad \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \rho_\varepsilon = -\nabla\cdot(\rho\xi).$$

For ε small enough $1_{\mathbb{R}^d} + \varepsilon\nabla\zeta$ is an optimal transport map, so ρ_ε is a constant-speed geodesic.
Hence, using standard variation formulae (see [Giaquinta and Hildebrandt 1996])

$$\lim_{\varepsilon\to 0} \frac{\mathcal{F}[\rho_\varepsilon] - \mathcal{F}[\rho_0]}{\varepsilon} = -\int_{\mathbb{R}^d} \frac{\delta F}{\delta\rho}[\rho_0]\nabla\cdot(\rho\xi) = \int_{\mathbb{R}^d} \nabla\zeta\nabla\frac{\delta F}{\delta\rho}[\rho_0]\,\mathrm{d}\rho$$

This characterises $\nabla_{d_2}\mathcal{F} = -\nabla\cdot(\rho\nabla\frac{\delta F}{\delta\rho})$ in a broad distributional sense.

Convex functions of a measure

Following [Demengel and Temam 1986]

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Given

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} f(\rho)\,\mathrm{d}x$$

where $f : \mathbb{R} \to \mathbb{R}$.

The questions is what is the natural lower semicontinuous extension of $\mathcal{F}$ to $\mathcal{M}(\mathbb{R}^d)$ with the weak-$\star$ topology.

Given a measure μ and mollifiers η_ε we define $\rho_\varepsilon = \mu * \eta_\varepsilon$.
For $|f(\xi)| \le C(1 + |\xi|)$ define

$$f_\infty(\xi) = \lim_{t\to\infty} \frac{f(t\xi)}{t}.$$

Since we can use the Lebesgue decomposition theorem $\mu = \rho\,\mathrm{d}x + \mu^s$, where ρ is the Radon-Nikodym derivative of μ. Then

$$\widetilde{F}[\mu] = \int_{\mathbb{R}^d} f(\rho)\,\mathrm{d}x + f_\infty(\mu^s).$$

The notion of $f_\infty(\mu^s)$ is tricky (but possible) to define.
If $f(s) = s^m$ with $m < 1$, then $f_\infty = 0$.

Curves of maximal slope

(see [Ambrosio, Gigli, and Savare 2005])

Typically, $\frac{\partial \rho}{\partial t} = -\nabla_X \mathcal{F}[\rho(t)]$ for $X = L^2, H^1$ is satisfied in the dual sense.

The way in which $\frac{\partial \rho}{\partial t} = -\nabla_{d_2} \mathcal{F}[\rho(t)]$ in rather tricky since $\mathcal{P}_2$ is not linear a space.

The main idea is the equivalence for $u : [0,T] \to \mathbb{R}^d$ that

$$u'(t) = -\nabla \mathcal{F}(u), \qquad \Longleftrightarrow \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F} \circ u) = -|\nabla F(u)||u'| & \text{orientation} \\ |u'| = |\nabla \mathcal{F}(u)| & \text{norm} \end{cases}$$

We define the metric slopes

$$|\mu'|(t) = \limsup_{h \to 0} \frac{d_2(\mu(t+h), \mu(t))}{h}, \qquad |\partial \mathcal{F}|[\mu] = \limsup_{\nu \to \mu} \frac{(\mathcal{F}[\mu] - \mathcal{F}[\nu])_+}{d_2(\mu, \nu)}$$

Definition 2 Maximal slope curve

A locally abs. cont. curve $t \mapsto \mu(t) \in \mathcal{P}_2(\mathbb{R}^d)$ such that $t \mapsto \mathcal{F}[\mu(t)]$ is abs. cont. and

$$\frac{1}{2}\int_s^t |\mu'|^2(r)\,\mathrm{d}r + \frac{1}{2}\int_s^t |\partial \mathcal{F}|^2[\mu(r)]\,\mathrm{d}r \le \mathcal{F}[\mu(s)] - \mathcal{F}[\mu(t)] \qquad \forall 0 \le s < t \le T$$

Let

$$\mathcal{F}[\rho] = \int_{\mathbb{R}^d} F(x, \rho(x), \nabla\rho(x))\, \mathrm{d}x.$$

Expanding $F(x, s, \xi)$ in Taylor expansion yields

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\mathcal{F}[\rho_0 + \varepsilon\varphi] - \mathcal{F}[\rho_0]}{\varepsilon} &= \int_{\mathbb{R}^d} \left(\tfrac{\partial F}{\partial s}(x, \rho_0, \nabla\rho_0)\varphi + \tfrac{\partial F}{\partial \xi}(x, \rho_0, \nabla\rho_0) \cdot \nabla\varphi \right) \\ &= \int_{\mathbb{R}^d} \left(\tfrac{\partial F}{\partial s}(x, \rho_0, \nabla\rho_0) - \nabla \cdot \left[\tfrac{\partial F}{\partial \xi}(x, \rho_0, \nabla\rho_0) \right] \right) \varphi\end{aligned}$$

Thus

$$\nabla_{H_1} \mathcal{F}[\rho_0] = \frac{\delta \mathcal{F}}{\delta \rho}[\rho_0] = \frac{\partial F}{\partial s}[\rho_0] - \nabla \cdot \left(\frac{\partial F}{\partial \xi}[\rho_0] \right).$$

This is the Euler-Lagrange equation!