UNIVERSALITY OF $C_\phi - \lambda$ ON WEIGHTED HARDY SPACES

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16.1.2014

Motivation

Denote the separable ∞-dimensional Hilbert space by $\mathcal{H}$ and the set of all bounded linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$.

We say that a closed subspace $\mathcal{M} \subseteq \mathcal{H}$ is *invariant* under an operator $T \in \mathcal{L}(\mathcal{H})$ (or shortly, T-*invariant*) if $T(\mathcal{M}) \subseteq \mathcal{M}$. Clearly, every operator has the *trivial* invariant subspaces $\{0\}$ and $\mathcal{H}$.

A famous open problem (*Invariant subspace problem*, ISP) in the field of operator theory asks if *every* bounded linear operator on $\mathcal{H}$ has a *non-trivial* invariant subspace.

ISP and universal operators

The connection between *universal operators* and ISP:

Theorem

Every operator $T \in \mathcal{L}(\mathcal{H})$ has a non-trivial invariant subspace if and only if all *minimal* invariant subspaces of a *universal operator* are 1-dimensional.

Definition

We say that an invariant subspace $\mathcal{M} \subset \mathcal{H}$ is minimal if the only invariant subspace contained in $\mathcal{M}$ is $\{0\}$.

Definition

An operator $U \in \mathcal{L}(\mathcal{H})$ is universal if for *every* operator $T \in \mathcal{L}(\mathcal{H})$ there exists $\alpha \in \mathbb{C} \setminus \{0\}$ and an U-invariant subspace $\mathcal{M} \subseteq \mathcal{H}$ such that the operators $\alpha T : \mathcal{H} \longrightarrow \mathcal{H}$ and $U_{|\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ are *similar* i.e. $\alpha T = \psi^{-1} U_{|\mathcal{M}} \psi$ for some isomorphism $\psi : \mathcal{H} \longrightarrow \mathcal{M}$.

Universality of the operator $C_\phi - \lambda : H^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D})$

Denote $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Consider the composition operator C_ϕ, $C_\phi f = f \circ \phi$, induced by a hyperbolic Möbius mapping $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ on the classical Hardy space $H^2(\mathbb{D})$. In their article *Invertible composition operators on H^p* (1987) E. Nordgren, P. Rosenthal and F.S. Wintrobe proved that when λ belongs to the point spectrum of C_ϕ, the operator

$$C_\phi - \lambda : H^2(\mathbb{D}) \longrightarrow H^2(\mathbb{D}) \text{ is universal.}$$

The proof relies on a beautiful theorem of S. Caradus (*Universal operators and invariant subspaces*, 1969) :

Theorem

If an operator $U \in \mathcal{L}(\mathcal{H})$

(i) is surjection $\mathcal{H} \longrightarrow \mathcal{H}$

(ii) $\dim Ker(U) = \infty$

then U is a universal operator.

The Weighted Hardy Spaces $\mathcal{H}^\gamma$

We denote by $\mathcal{H}^\gamma$, $\gamma \in [0, 2]$, the Hilbert space consisting of analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ for which

$$\|f\|_\gamma^2 := \sum_{n=0}^{\infty} |c_n|^2 (n+1)^{1-\gamma} < \infty, \; f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

When

$\gamma = 0,$ we have the Dirichlet space $\mathcal{D}_2$

$\gamma = 1,$ we have the classical Hardy space $H^2(\mathbb{D})$

$\gamma = 2,$ we have the Bergman space $A^2(\mathbb{D})$.

Note: $\mathcal{H}^{\gamma_1} \subset \mathcal{H}^{\gamma_2}$ whenever $\gamma_1 < \gamma_2$.

- *My main research question concerns the operator $C_\phi - \lambda$ and its universality (or non-universality) on the whole scale of weighted Hardy spaces $\mathcal{H}^\gamma$, $\gamma \in [0, 2]$.*

Inner functions

As usual, let $H^\infty(\mathbb{D})$ denote the bounded analytic functions in $\mathbb{D}$.

We say that $\varphi \in H^\infty(\mathbb{D})$ is an *inner function* if $\lim_{r \longrightarrow 1^-} |\varphi(re^{i\theta})| = 1$ for almost every θ. (Note that $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$ so that every $\varphi \in H^\infty(\mathbb{D})$ has boundary values a.e. on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.) From the maximal principle it follows that $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$.

As a special case of inner functions we have the *Möbius mappings* of the unit disc $\mathbb{D}$ that are of the form

$$\phi(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}, \quad |a| < 1,\ \theta \in \mathbb{R}.$$

Inner functions: Möbius mappings

Möbius mappings $\mathbb{D} \longrightarrow \mathbb{D}$ can be divided into classes by their *fixed points* : A Möbius mapping $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ is

- *parabolic* if it has one fixed point on the circle $\mathbb{T}$
- *elliptic* if it has two fixed points, one of them in $\mathbb{D}$ and the other in $\overline{\mathbb{C}} \setminus \mathbb{D}$
- *hyperbolic* if it has two fixed points (other being *attractive* and another *repulsive*) , both of them in the circle $\mathbb{T}$.

If $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ is a hyperbolic Möbius mapping we can find a Möbius mapping $h : \Pi_+ \longrightarrow \mathbb{D}$ such that

$$h^{-1} \circ \phi \circ h = f, \; f(z) = \lambda_\phi z.$$

The *characteristic constant* $\lambda_\phi \in \mathbb{R}_+ \setminus \{0, 1\}$ does not depend on the conjugate function h and is unique up to its reciprocal $1/\lambda_\phi$. Hence we can assume that $\lambda_\phi \in (0, 1)$.

The hyperbolic Möbius mapping ϕ is said to be *normalised* if it fixes points -1 and 1.

The sequence $\left(\phi_j(0)\right)_{j\in\mathbb{Z}}$

A normalised hyperbolic Möbius mapping ϕ can be written in the form

$$\phi(z) = \frac{z+r}{1+rz}, \text{ where } r = \frac{1-\lambda_\phi}{1+\lambda_\phi} \in (0,1).$$

We denote the iterates of ϕ by $(j = 1, 2, \dots)$

$$\phi_j := \phi \circ \cdots \circ \phi, \quad \phi_{-j} := \phi^{-1} \circ \cdots \circ \phi^{-1} \quad (j \text{ times}) \text{ and } \phi_0 = id.$$

Write $z_j := \phi_j(0)$ for all $j \in \mathbb{Z}$. Now $z_j \in \mathbb{R}$ and $-z_j = z_{-j}$ for all $j \in \mathbb{Z}$.

- The sequence $(z_j)_{j\in\mathbb{Z}}$ is an *interpolating sequence*!

Definition

We say that a sequence $(z_j)_{j\in\mathbb{Z}} \subset \mathbb{D}$ is interpolating if for every bounded sequence $(c_j)_{j\in\mathbb{Z}} \subset \mathbb{C}$ there exists a function $f \in H^\infty(\mathbb{D})$, $f \not\equiv 0$, such that $f(z_j) = c_j$ for all $j \in \mathbb{Z}$.

The Blaschke product B_ϕ

The interpolating sequence $(z_j)_{j\in\mathbb{Z}}$ satisfy the Blaschke condition $\sum_{j\in\mathbb{Z}}(1-|z_j|) < \infty$!

Denote the Blaschke product associated to the sequence $(z_j)_{j\in\mathbb{Z}}$ by

$$B_\phi(z) := z \prod_{j\in\mathbb{Z}\setminus\{0\}} \frac{|z_j|}{z_j} \frac{z_j - z}{1-\bar{z}_j z}$$

- B_ϕ is an eigenfunction with eigenvalue -1 of the composition operator C_ϕ!

The composition operator C_ϕ

Assume that $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is analytic. *A composition operator* C_φ induced by the function φ is defined by setting

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

- The operator C_φ is bounded in each of the spaces $\mathcal{H}^\gamma$, $\gamma \in [0, 2]$, at least when $\varphi :: \mathbb{D} \longrightarrow \mathbb{D}$ is a Möbius mapping.

Hyperbolic composition operator

Assume $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ is a hyperbolic Möbius mapping. We say that C_ϕ is a *hyperbolic composition operator*.

If ϕ and $\tilde{\phi}$ and are, respectively, a normalised and an arbitrary hyperbolic Möbius mappings in the same conjugacy class then the composition operators C_ϕ and $C_{\tilde{\phi}}$ are similar. It suffices to consider the operator C_ϕ where ϕ fixes points -1 and 1.

The spectrum of C_ϕ on $\mathcal{H}^\gamma$ when

$\gamma = 0$ is the unit cirle $\mathbb{T}$

$0 < \gamma < 1$ is not yet known

$1 \leq \gamma \leq 2$ is

$$\sigma^\gamma(C_\phi) = \{\alpha \in \mathbb{C} : \lambda_\phi^{\gamma/2} \leq |\alpha| \leq \lambda_\phi^{-\gamma/2}\}$$

and the point spectrum is

$$\sigma_p^\gamma(C_\phi) = \{\alpha \in \mathbb{C} : \lambda_\phi^{\gamma/2} < |\alpha| < \lambda_\phi^{-\gamma/2}\}.$$

The role of the spectrum

Consider the operator $C_\phi - \lambda$, where $\lambda \in \sigma(C_\phi)$.

- $C_\phi - \lambda$ can **not** be universal if $\lambda \in \partial\sigma(C_\phi)$.
- $C_\phi - \lambda$ on $\mathcal{H}^0 = \mathcal{D}_2$ can **not** be universal for any $\lambda \in \mathbb{C}$!
- Every point in the point spectrum (i.e. eigenvalue) of C_ϕ on $\mathcal{H}^\gamma$, $\gamma \in [1, 2]$, are of ∞-multiplicity..
- i.e., $\dim \operatorname{Ker}(C_\phi - \lambda) = \infty$ for all $\lambda \in \sigma_p^\gamma(C_\phi)$!

Difficulties

The techniques in proving that $C_\phi - \lambda$ is *onto* on $H^2(\mathbb{D})$ whenever $\lambda \in \sigma_p(C_\phi)$ rely heavily on the properties that are characteristic for $H^2(\mathbb{D})$, e.g.

- Multiplying by *inner functions* is *isometry* on $H^2(\mathbb{D})$..

..since functions in the Hardy space $H^2(\mathbb{D})$ satisfy

$$\|f\|_{H^2}^2 = \lim_{r \longrightarrow 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty$$

and $\lim_{r \longrightarrow 1^-} |\varphi(re^{i\theta})| = 1$ for almost every θ when φ is an inner function!

- $H^2(\mathbb{D}) = \oplus_{n=0}^{\infty} B_\phi^{2n} \left(B_\phi^2 H^2(\mathbb{D})\right)^\perp$.

Consequences

Notice that the invariant subspaces of $C_\phi - \lambda$ are exactly those that are invariant under C_ϕ. Therefore the positive answer to ISP follows if it can be proved that all the minimal invariant subspaces of the operator C_ϕ on $H^2(\mathbb{D})$ are of dimension 1.

After the article of Nordgren et al. several people (for example V. Matache, V. Chkliar, J. H. Shapiro, E.A. Gallardo-GutiÃ©rrez and P. Gorkin) have searched and found some conditions for a function $f \in H^2(\mathbb{D})$ for which the C_ϕ-invariant subspace

$$K_f := \overline{span}\left\{ C_\phi^n f : n \in \mathbb{Z} \right\}$$

is minimal or non-minimal.

Gracias!