THE (CLASSICAL) RIEMANN–HILBERT CORRESPONDENCE

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Contents

Our goal in these notes is to present a detailed proof of the Riemann–Hilbert correspondence in its classical setting, which relates the solutions of a certain ordinary differential equation on a complex domain to representations of the fundamental group of that domain. In addition, we will discuss some generalizations of this correspondence to higher-dimensional complex manifolds. Our main reference will be [2].

1. Fundamental group, covering spaces and local systems

1.1. **Covering spaces.** Let us recall some basic topological notions. Given a topological space X, a *cover* (or *covering space*) of X is a topological space Y together with a surjective continuous map $p : Y \to X$ such that, for each $p \in X$, there exists an open neighbourhood $p \in U \subseteq X$ such that $p^{-1}(U) = \bigsqcup_{i\in I} V_i$ (i.e. $p^{-1}(U)$ is a disjoint union of open sets of Y) and $p_{|V_i} : V_i \to U$ is a homeomorphism for each $i \in I$.

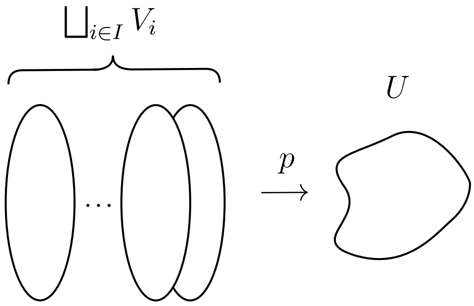

FIGURE 1. Cover of a topological space.

A *morphism* between two covers $Y_1 \overset{p_1}{\to} X$ and $Y_2 \overset{p_2}{\to} X$ is a continuous map $f : Y_1 \to Y_2$ such that $p_1 = p_2 \circ f$ (i.e. making the corresponding diagram commute). We denote by $\mathrm{Hom}_X(Y_1, Y_2)$ the set of such maps. These concepts allow us to consider the category of covers of X with morphisms of cover as morphisms. It will be denoted by $\mathbf{Cov}(X)$.

The following result will be useful in what follows.

Lemma 1.1. *Let $p : Y \to X$ be a cover, Z a connected space and $f, g : Z \to Y$ continuous maps such that $p \circ f = p \circ g$. If there exists a point $z_0 \in Z$ such that $f(z_0) = g(z_0)$, then $f = g$.*

Proof. Let us consider the subset $A = \{z \in Z : f(z) = g(z)\}$ of Z. Since there exists a point $z_0 \in Z$ such that $f(z_0) = g(z_0)$, the set A is non-empty. Let us show that A is open in Z. Let $z \in A$ and consider the point $y = f(z) = g(z) \in Y$. Given a connected open neighbourhood U of $p(y) \in X$, let $V_i \subseteq Y$ be the open component of $p^{-1}(U)$ homeomorphic to U such that $y \in V_i$. The continuity of f and g guarantees the existence of an open neighbourhood W^z of $z \in Z$ such that $f(W^z), g(W^z) \subseteq V_i$. Let us suppose that there exists a point $w \in W$ such that $f(w) \neq g(w)$. Hence, since p maps V_i homeomorphically onto U_i, we have that $p(f(w)) \neq p(g(w))$, which contradicts that $p \circ f = p \circ g$. Thus, $f_{|W^z} = g_{|W^z}$, so $z \in W^z \subseteq A$ and A is open in Z. A similar argument shows that A is closed in Z, so we conclude that $A = Z$ and $f = g$. □

As a consequence, any automorphism of a connected cover with a fixed point is trivial.

Given a discrete topological space $F \neq \varnothing$, the first projection $X \times F \to X$ is a cover called the *trivial cover*. A cover isomorphic to some trivial cover is called *trivial*.

Proposition 1.2. *Given a cover $p : Y \to X$, each point of X has an open neighbourhood U such that the restriction of p to $p^{-1}(U)$ is a trivial cover. Moreover, if X is connected, all the fibres $p^{-1}(x)$ are all homeomorphic to the same discrete space I.*

Proof. Let $x \in X$ be a point and $U \subseteq X$ an open neighbourhood of x. Thus, we can write $p^{-1}(U) = \bigsqcup_{i\in I} V_i$ for some open subsets $V_i \subseteq Y$ for each $i \in I$. Hence, the map

$f : p^{-1}(U) \to U \times I$ given by $f(v_i) = (p(v_i), i)$ for $v_i \in V_i$ maps homeomorphically $p^{-1}(U)$ onto $U \times I$, where I is equipped with the discrete topology. This map turns out to be an isomorphism of covers. □

Let us consider a group G (left-)acting continuously on a topological space Y. We say that the action of G on Y is *even* if each point $y \in Y$ has an open neighbourhood $y \in U \subseteq Y$ such that $gU \cap U \neq \varnothing$ for some $g \in G$ if and only if $g = e$. It can be proved that a continuous even action of a group G on a topological space Y yields to a cover $Y \to Y/G$ given by the canonical projection.

Given a connected and locally simply connected[1] topological space X and a cover $p : Y \to X$, let us consider the group $\mathrm{Aut}(Y|X)$ of automorphisms of the cover, i.e. the group of isomorphisms from $Y \xrightarrow{p} X$ to itself (this is usually called the *deck group* of the cover). We have a continuous left-action of $\mathrm{Aut}(Y|X)$ on Y. Moreover, it is easy to see that each element of $\mathrm{Aut}(Y|X)$ maps $p^{-1}(x)$ onto itself for each $x \in X$, hence we have a continuous left-action of $\mathrm{Aut}(Y|X)$ on $p^{-1}(x)$ for each $x \in X$.

If $p : Y \to X$ is a connected cover (i.e. Y is a connected space), the action of $\mathrm{Aut}(Y|X)$ of Y is even. Something like a reciprocal of this statement is that, if G is a group acting evenly on a connected space Y, the automorphism group of the cover $p : Y \to Y/G$ is isomorphic to G.

Given a connected cover $p : Y \to X$, since p is $\mathrm{Aut}(Y|X)$-invariant, the action of $\mathrm{Aut}(Y|X)$ on Y gives rise to a factorisation of p given by

$$p : Y \longrightarrow Y/\mathrm{Aut}(Y|X) \xrightarrow{\overline{p}} X,$$

where $Y \to Y/\mathrm{Aut}(Y|X)$ is the canonical projection. The cover $p : Y \to X$ is called a *Galois cover* (or *regular cover* or *normal cover*) if the map $\overline{p}$ is a homeomorphism. In particular, if a group G acts evenly on a connected topological space Y, the cover $Y \to Y/G$ is Galois.

Theorem 1.3. ***Galois Correspondence for Covering Spaces I***
Let $p : Y \to X$ be a Galois cover and let us consider the deck group $G = \mathrm{Aut}(Y|X)$. Given a subgroup $H \subseteq G$, the map p induces a cover $\overline{p}_H : Y/H \to X$. Reciprocally, given a connected cover $q : Y' \to X$ and a morphism $f \in \mathrm{Hom}_X(Y, Y')$, the map $f : Y \to Y'$ is a Galois cover and $Y' \cong Y/H$ for the subgroup $H = \mathrm{Aut}(Y|Y') \subseteq G$. Moreover, the assignments

$$\mathbf{Sub}(G) \longrightarrow \mathbf{Cov}(X) \;:\; H \mapsto Y/H \qquad \mathbf{Cov}(X) \longrightarrow \mathbf{Sub}(G) \;:\; Y' \mapsto \mathrm{Aut}(Y|Y')$$

induce a bijection between subgroups of G and covers $Y' \to X$. Moreover, $Y' \to X$ is Galois if and only if H is a normal subgroup of G, in which case $\mathrm{Aut}(Y'|X) \cong G/H$.

[1] Recall that a topological space is locally (simply, path-) connected if each point has a basis of (simply, path-) connected open neighbourhoods.

Proof. A proof can be found in [2, Theorem 2.2.10]. □

1.1.1. *The Universal Cover.* Let $p : Y \to X$ be a simply connected cover with X a path-connected and locally simply connected space. If $q : Y' \to X$ is another simply connected cover, it can be proved that there exists a unique homeomorphism $f : Y \to Y'$ such that $p = q \circ f$ (i.e. making the corresponding diagram commute). This fact can be rephrased by saying that two simply connected covers are isomorphic. For this reason, a simply connected cover $p : Y \to X$ is called a *universal cover*.

In order to give a more general construction, let us assume for now that X is connected and locally simply connected topological space. Let us recall some basic topological notions. A *path* in X is a continuous map $\gamma : [0, 1] \to X$. We say that γ is *closed* if $\gamma(0) = \gamma(1)$. Two paths $\gamma, \sigma : [0, 1] \to X$ are *homotopic* if $\gamma(0) = \sigma(0)$, $\gamma(1) = \sigma(1)$ and there exists a continuous map $H : [0, 1] \times [0, 1] \to X$ such that $H(0, t) = \gamma(t)$ and $H(1, t) = \sigma(t)$ for each $t \in [0, 1]$.

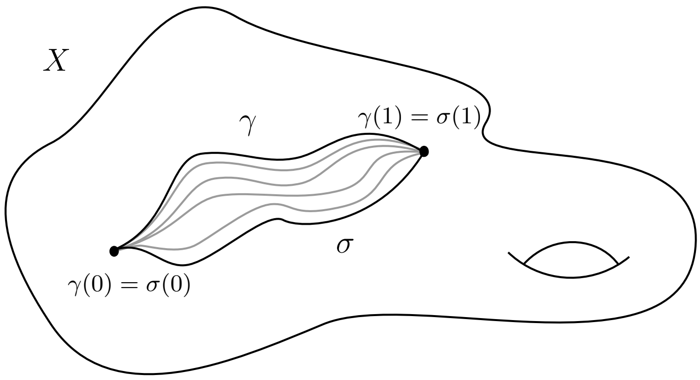

FIGURE 2. Homotopic paths.

Given two paths $\gamma, \sigma : [0, 1] \to X$ with $\gamma(0) = \sigma(1)$, we define its *juxtaposition*[2] as the continuous map $\gamma * \sigma : [0, 1] \to X$ given by

$$(\gamma * \sigma)(t) = \begin{cases} \sigma(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases} .$$

It is well-known that, given a point $x \in X$, the set $\pi_1(X, x)$ of homotopy classes of closed paths based on x together with the binary operation given by $[\alpha] * [\beta] = [\alpha * \beta]$, where $[\cdot]$ denotes the homotopy class, is a group called the *fundamental group* of X based at the point x.

[2]Note that the above convention for juxtaposition of paths differs from the convention of many textbooks.

We have all the ingredients to define our candidate for universal cover. Given a point $x \in X$, let us consider the set $\widetilde{X}_x$ of homotopy classes of paths $\sigma : [0,1] \to X$ with $\sigma(0) = x$. Since homotopic paths have the same endpoints, the map $p : \widetilde{X}_x \to X$ given by $p([\sigma]) = \sigma(1)$ is well-defined. Let us now consider a topology on $\widetilde{X}_x$ giving a basis of open neighbourhoods of a given $z_x = [\sigma] \in \widetilde{X}_x$. Let U^z be a simply connected open neighbourhood of $z = p(z_x) = \sigma(1) \in X$. We define $\widetilde{U}^{z_x}$ to be the set of homotopy classes of juxtapositions $\gamma * \sigma$, where $\gamma : [0,1] \to X$ is a path with $\gamma(0) = z$ and $\gamma([0,1]) \subseteq U^z$. Note that $\widetilde{U}^{z_x}$ is well-defined since U^z is simply connected.

Hence, if $\{U_i^z\}_{i \in I}$ is a basis of simply connected open neighbourhoods of z (whose existence is guaranteed since X is locally simply connected), $\left\{\widetilde{U}_i^{z_x}\right\}_{i \in I}$ is a basis of open neighbourhoods of z_x. Moreover, p is a continuous map with respect to this topology and a connected cover. Indeed, given a point $z \in X$ and a simply connected open neighbourhood U^z, we have that $p^{-1}(U^z) = \bigsqcup_{z_x \in p^{-1}(z)} \widetilde{U}^{z_x}$ and $p_{|\widetilde{U}^{z_x}}$ is a homeomorphism for each $z_x \in p^{-1}(z)$.

Proposition 1.4. *The cover* $\widetilde{X}_x \to X$ *is Galois.*

Let us now assume that X is path-connected and locally simply connected. In this case, given two distinct points $x, y \in X$, there exists a path $\sigma : [0,1] \to X$ joining them, i.e. such that $\sigma(0) = x$ and $\sigma(1) = y$. In this case, the map $\widetilde{X}_y \to \widetilde{X}_x$ given by $[\gamma] \mapsto [\gamma * \sigma]$ is an isomorphism of covers, hence the construction of the space $\widetilde{X}_x$ described above is unique (up to isomorphism) and does not depend on the choice of the base point $x \in X$, so we can drop the subindex of $\widetilde{X}_x$.

Proposition 1.5. *Let* X *be a path-connected and locally simply connected topological space. The cover* $\widetilde{X} \to X$ *is simply connected, therefore it is a universal cover.*

In particular, since simply connected covers are isomorphic, last result ensures that every universal cover is isomorphic to $\widetilde{X} \to X$, so this cover is usually called *the* universal cover.

Remark 1.6. Although we will be able to prove Propositions 1.4 and 1.5 as consequences of Theorem 1.12, we state these properties here to conclude the discussion on universal covers. Furthermore, they will no longer be used in what follows, so we omit their proofs.

Let us conclude the section by computing the automorphism group $\mathrm{Aut}(\widetilde{X}_x|X)$.

Proposition 1.7. *There is a group isomorphism* $\mathrm{Aut}(\widetilde{X}_x|X) \cong \pi_1(X, x)$.

Proof. As we already know, there is a continuous left-action of $\mathrm{Aut}(\widetilde{X}_x|X)$ on $\widetilde{X}_x$, thus we can regard the said action as a continuous right-action of $\mathrm{Aut}(\widetilde{X}_x|X)^{op}$, where $(\cdot)^{op}$

denotes the opposite group[3]. It is easy to check that the map

$$\widetilde{X}_x \times \pi_1(X,x) \longrightarrow \widetilde{X}_x \;:\; (z_x = [\sigma], [\alpha]) \mapsto z_x[\alpha] = [\sigma * \alpha]$$

is a right-action of $\pi_1(X,x)$. Moreover, given a fixed element $[\alpha] \in \pi_1(X,x)$, the map $\phi_{[\alpha]} : \widetilde{X}_x \to \widetilde{X}_x$ given by $\phi_{[\alpha]}(z_x) = z_x[\alpha]$ is a continuous map, so the right-action of $\pi_1(X,x)$ is continuous. Moreover, if $z_x = [\sigma]$, we have that

$$(p \circ \phi_{[\alpha]})(z_x) = p(\phi_{[\alpha]}(z_x)) = p(z_x[\alpha]) = p([\sigma * \alpha]) = (\sigma * \alpha)(1) = \sigma(1) = p([\sigma]) = p(z_x),$$

thus $\phi_{[\alpha]} : \widetilde{X}_x \to \widetilde{X}_x$ is an automorphism of covers, i.e. $\phi_{[\alpha]} \in \mathrm{Aut}(\widetilde{X}_x|X)$. Therefore, we have a group homomorphism $\pi_1(X,x) \to \mathrm{Aut}(\widetilde{X}_x|X)^{op}$ given by $[\alpha] \mapsto \phi_{[\alpha]}$ (we consider the homomorphism in this way so that it is compatible with the right-actions on $\widetilde{X}_x$ described above). Let us denote by $\mathsf{c}_x \in \widetilde{X}_x$ the homotopy class of the constant path based on x. Given a non-trivial element $[\alpha] \in \pi_1(X,x)$, $\phi_{[\alpha]}(\mathsf{c}_x) = \mathsf{c}_x[\alpha]$ cannot be equal to c_x since $[\alpha]$ is non-trivial, so the homomorphism is injective. Let $\phi \in \mathrm{Aut}(\widetilde{X}_x|X)$ be an automorphism and let $z_x = [\sigma_1] \in \widetilde{X}_x$ be a point. If we consider the point $\phi(z_x) = w_x = [\sigma_2] \in \widetilde{X}_x$, we have that $\sigma_1(1) = \sigma_2(1)$. Hence, $\sigma_1^{-1} * \sigma_2$ is a closed path based on x in X which verifies $\sigma_1 * (\sigma_1^{-1} * \sigma_2) = \sigma_2$. Hence, the automorphism $\phi_{[\sigma_1^{-1} * \sigma_2]} \circ \phi^{-1}$ of $\widetilde{X}_x$ satisfies that

$$(\phi_{[\sigma_1^{-1} * \sigma_2]} \circ \phi^{-1})(w_x) = \phi_{[\sigma_1^{-1} * \sigma_2]}(\phi^{-1}(w_x)) = \phi_{[\sigma_1^{-1} * \sigma_2]}(z_x) = z_x[\sigma_1^{-1} * \sigma_2] = [\sigma_2] = w_x,$$

i.e. $w_x \in \widetilde{X}_x$ is a fixed point. Therefore, since $\widetilde{X}_x \to X$ is connected, Lemma 1.1 assures that $\phi_{[\sigma_1^{-1} * \sigma_2]} = \phi$, so the group homomorphism $\pi_1(X,x) \to \mathrm{Aut}(\widetilde{X}_x|X)^{op}$ is surjective and hence an isomorphism.

Note that, although we have proved the existence of an isomorphism between $\pi_1(X,x)$ and $\mathrm{Aut}(\widetilde{X}_x|X)^{op}$, the natural isomorphism between a group and its opposite gives us the stated isomorphism. □

This result allows us to deduce a particular version of the Galois correspondence.

Corollary 1.8. ***Galois Correspondence for Covering Spaces II***
Consider the cover $p : \widetilde{X}_x \to X$. Given a subgroup $H \subseteq \pi_1(X,x)$, the map p induces a cover $\overline{p}_H : \widetilde{X}_x/H \to X$. Reciprocally, given a connected cover $p : Y \to X$ and a morphism $f \in \mathrm{Hom}_X(\widetilde{X}_x, Y)$, the map $f : \widetilde{X}_x \to Y$ is a Galois cover and $Y \cong \widetilde{X}_x/H$ for the subgroup $H = \mathrm{Aut}(\widetilde{X}_x|Y) \subseteq G$. Moreover, the assignments

$$\mathbf{Sub}(\pi_1(X,x)) \longrightarrow \mathbf{Cov}(X) \;:\; H \mapsto \widetilde{X}_x/H \qquad \mathbf{Cov}(X) \longrightarrow \mathbf{Sub}(\pi_1(X,x)) \;:\; Y \mapsto \mathrm{Aut}(\widetilde{X}_x|Y)$$

induce a bijection between subgroups of $\pi_1(X,x)$ and covers $Y \to X$. Moreover, $Y \to X$ is Galois if and only if H is a normal subgroup of $\pi_1(X,x)$, in which case $\mathrm{Aut}(Y|X) \cong \pi_1(X,x)/H$.

[3] Given a group G, its opposite group G^{op} is the group which is setwise identical to G and whose binary operation is given by $(g,h) \mapsto h \cdot g$, where $\cdot$ is the binary operation of G. Of course, there is a natural isomorphism $G \cong G^{op}$.

1.2. **The monodromy action.** We now want to show that given a cover $p : Y \to X$, the fibre $p^{-1}(x)$ carries an action of the fundamental group $\pi_1(X, x)$ for each $x \in X$. Let us assume for the rest of the section that X is a connected and locally simply connected space, although some results can be proved with weaker conditions.

Lemma 1.9. ***Path Lifting Property***
Let $p : Y \to X$ be a cover, $y \in Y$ and $x = p(y)$:

(i) *given a path $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, there exists a unique path $\widetilde{\gamma} : [0, 1] \to Y$ with $\widetilde{\gamma}(0) = y$ and $p \circ \widetilde{\gamma} = \gamma$;*
(ii) *moreover, let $\sigma : [0, 1] \to X$ be a path homotopic to γ. Then $\widetilde{\gamma}$ and $\widetilde{\sigma}$ are homotopic.*

Proof. A proof can be found in [2, Lemma 2.3.2]. □

We are now ready to define the left-action of $\pi_1(X, x)$ on $p^{-1}(x)$. Given $y \in p^{-1}(x)$ and $[\alpha] \in \pi_1(X, x)$, let us define the map

$$\pi_1(X, x) \times p^{-1}(x) \longrightarrow p^{-1}(x) \quad : \quad ([\alpha], y) \mapsto [\alpha]y = \widetilde{\alpha}(1),$$

where $\widetilde{\alpha}$ is the unique lifting of $\alpha : [0, 1] \to X$ with $\widetilde{\alpha}(0) = y$ and $p \circ \widetilde{\alpha} = \alpha$. Note that this map is well-defined due to Lemma 1.9. Moreover, since $([\alpha] * [\beta])\, y = [\alpha]\,([\beta]y)$, the map described above is therefore a left-action of $\pi_1(X, x)$ on $p^{-1}(x)$ called the *monodromy action* of the cover.

The main purpose of this section is to show that the correspondence between covers and its monodromy actions gives us an equivalence of categories. Given a fixed point $x \in X$, let us consider the category $\pi_1(X, x)$-**Set** whose objects are sets equipped with a left-action of $\pi_1(X, x)$ and whose morphisms are $\pi_1(X, x)$-maps, i.e. set-theoretical maps $f : A \to B$ such that $f([\alpha]r) = [\alpha]f(r)$ for each $r \in A$ and $[\alpha] \in \pi_1(X, x)$. Hence, we can assign each cover $p : X \to Y$ to the fibre $p^{-1}(x)$, which carries the monodromy action. Moreover, let $f : Y_1 \to Y_2$ be a morphism of covers. By definition, f restricts to a map $p_1^{-1}(x) \to p_2^{-1}(x)$ which we shall denote by f_x. Given $[\alpha] \in \pi_1(X, x)$ and a point $y \in p_1^{-1}(x)$, the uniqueness of the lifting ensures that

$$f_x([\alpha]y) = f_x\left(\widetilde{\alpha}(1)\right) = \left(f_x \circ \widetilde{\alpha}\right)(1) = \widehat{\alpha}(1) = [\alpha]f_x(y),$$

where $\widetilde{\alpha}$ is the unique lifting of α to Y_1 with $\widetilde{\alpha}(0) = y$ and $\widehat{\alpha}$ is the unique lifting of α to Y_2 with $\widehat{\alpha}(0) = f(y)$. Thus, we can conclude the following.

Proposition 1.10. *Given a point $x \in X$, the assignment* $\mathrm{Fib}_x : \mathbf{Cov}(X) \to \pi_1(X, x)$-**Set** *given by*

$$\begin{array}{ccc} Y \xrightarrow{p} X & \mapsto & \mathrm{Fib}_x(Y) = p^{-1}(x) \\ f : Y_1 \longrightarrow Y_2 & \mapsto & \mathrm{Fib}_x(f) = f_x : p_1^{-1}(x) \to p_2^{-1}(x) \end{array}$$

is a (covariant) functor.

This functor will gives us the equivalence of categories we are looking for. However, we need to do some preliminary work. Let us start by recalling some categorical facts about equivalence of categories.

Recall that a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is *fully faithful* if each map $F_{AB} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is bijective for each pair of $\mathcal{C}_1$-objects A and B. We say that F is *essentially surjective* if each $\mathcal{C}_2$-object is isomorphic to some object of the form $F(A)$ for some $\mathcal{C}_1$-object A. It turns out that these conditions are sufficient for F to be an equivalence of categories.

Lemma 1.11. *A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. A proof can be found in [2, Lemma 1.4.9]. □

Let us check the relation between the cover $\widetilde{X}_x \to X$ and the functor Fib_x. Note that, since X is not assumed to be path-connected, $\widetilde{X}_x$ may depend on the base point $x \in X$.

Theorem 1.12. *The functor* Fib_x *is representable by the cover $\widetilde{X}_x \to X$.*

Proof. We divide the proof into three statements.

(**I**) *For each cover $q : Y \to X$, there is a bijection between the sets* $\mathrm{Fib}_x(Y)$ *and* $\mathrm{Hom}_X(\widetilde{X}_x, Y)$.

Given a cover $q : Y \to X$, we have to show that each point $q^{-1}(x) = \mathrm{Fib}_x(Y)$ corresponds to a unique morphism of covers $\widetilde{X}_x \to Y$.

Given a point $y \in q^{-1}(x)$, let us define a morphism of covers $\pi_y : \widetilde{X}_x \to Y$. Given a point $z_x = [\sigma] \in \widetilde{X}_x$, we define $\pi_y(z_x) = \widetilde{\sigma}(1)$, where $\widetilde{\sigma} : [0, 1] \to Y$ is the unique lifting of the path $\sigma : [0, 1] \to X$ with $\widetilde{\sigma}(0) = y$. It can be proved that π_y is a continuous map. Moreover, since π_y satisfies the chain of equalities

$$(q \circ \pi_y)(z_x) = q\left(\widetilde{\sigma}(1)\right) = \left(q \circ \widetilde{\sigma}\right)(1) = \sigma(1) = p([\sigma]) = p(z_x),$$

π_y is a morphism of covers. Hence, we have an assignment $q^{-1}(x) \to \mathrm{Hom}_X(\widetilde{X}_x, Y)$ given by $y \mapsto \pi_y$. Conversely, given a morphism of covers $\phi : \widetilde{X}_x \to Y$, let us consider the point $\phi(\mathsf{c}_x) \in Y$, where $\mathsf{c}_x \in \widetilde{X}_x$ is the homotopy class of the constant path. Since $p = q \circ \phi$, we have that

$$q(\phi(\mathsf{c}_x)) = (q \circ \phi)(\mathsf{c}_x) = p(\mathsf{c}_x) = x,$$

so $\phi(\mathsf{c}_x) \in q^{-1}(x)$. Thus, we also have an assignment $\mathrm{Hom}_X(\widetilde{X}_x, Y) \to q^{-1}(x)$ given by $\phi \mapsto \phi(\mathsf{c}_x)$, which is precisely the inverse of $y \mapsto \pi_y$.

(**II**) *For each cover $q : Y \to X$, the Hom-set* $\mathrm{Hom}_X(\widetilde{X}_x, Y)$ *is equipped with a left-action of $\pi_1(X, x)$.*

We have a right-action of $\mathrm{Aut}(\widetilde{X}_x|X)$ on $\mathrm{Hom}_X(\widetilde{X}_x, Y)$. Indeed, consider the map

$$\mathrm{Hom}_X(\widetilde{X}_x, Y) \times \mathrm{Aut}(\widetilde{X}_x|X) \longrightarrow \mathrm{Hom}_X(\widetilde{X}_x, Y) \ : \ (f, \phi) \mapsto f\phi = f \circ \phi.$$

Hence, we obtain a left-action of $\mathrm{Aut}(\widetilde{X}_x|X)^{op}$ on $\mathrm{Hom}_X(\widetilde{X}_x, Y)$. In particular, the bijection given in (**I**) gives us a left-action of $\mathrm{Aut}(\widetilde{X}_x|X)^{op}$ on $\mathrm{Fib}_x(Y) = q^{-1}(x)$. Moreover, this action coincides with the monodromy action of $\pi_1(X, x)$ on $q^{-1}(x)$, so again, (**I**) assures us that the left-action of $\mathrm{Aut}(\widetilde{X}_x|X)^{op}$ on $\mathrm{Hom}_X(\widetilde{X}_x, Y)$ coincides with the given action of $\pi_1(X, x)$, so $\mathrm{Hom}_X(\widetilde{X}_x, -)$ can be regarded as a functor $\mathbf{Cov}(X) \to \pi_1(X, x)$-**Set**.

(**III**) *There exists a natural isomorphism* $\tau : \mathrm{Fib}_x \to \mathrm{Hom}_X(\widetilde{X}_x, -)$.

Parts (**I**) and (**II**) of the proof allow us to conclude that, for each cover $q : Y \to X$, we have an isomorphism of $\pi_1(X, x)$-sets $\tau_Y : q^{-1}(x) \to \mathrm{Hom}_X(\widetilde{X}_x, Y)$ given by $\tau_Y(y) = \pi_y$ and whose inverse is given by $\tau_Y^{-1}(\phi) = \phi(\mathsf{c}_x)$.

Given two covers $Y \xrightarrow{q} X$, $Y' \xrightarrow{q'} X$ and a morphism of covers $f : Y \to Y'$, let us consider the points $y \in Y$ and $y' = f(y) \in Y'$. We have to check if the diagram

$$\begin{array}{ccc} q_1^{-1}(x) & \xrightarrow{\tau_Y} & \mathrm{Hom}_X(\widetilde{X}_x, Y) \\ {\scriptstyle \mathrm{Fib}_x(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(\widetilde{X}_x, f)} \\ q_2^{-1}(x) & \xrightarrow{\tau_{Y'}} & \mathrm{Hom}_X(\widetilde{X}_x, Y') \end{array}$$

is commutative. On the one hand, we have that

$$\mathrm{Hom}_X(\widetilde{X}_x, f)(\tau_Y(y)) = \mathrm{Hom}_X(\widetilde{X}_x, f)(\pi_y) = f \circ \pi_y.$$

On the other hand, it holds that

$$\tau_{Y'}(\mathrm{Fib}_x(f)(y)) = \tau_{Y'}(f_x(y)) = \tau_{Y'}(y') = \pi'_{y'}.$$

Therefore, the uniqueness of the lifting ensures that $f \circ \pi_y = \pi'_{y'}$, so the commutativity of the diagram holds and hence τ is a natural isomorphism. □

We are now ready to prove the main theorem of the section.

Theorem 1.13. ***Correspondence Between Covers and*** $\pi_1(X, x)$***-Sets***
Let X *be a connected and locally simply connected space. Given a base point* $x \in X$*, the functor* $\mathrm{Fib}_x : \mathbf{Cov}(X) \to \pi_1(X, x)$-**Set** *induces an equivalence of categories.*

Proof. Due to Lemma 1.11, we have to prove that Fib_x is both fully faithful and essentially surjective.

(**I**) Fib_x *is fully faithful.*

Given two covers $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ and a morphism $\phi : \mathrm{Fib}_x(Y_1) \to \mathrm{Fib}_x(Y_2)$, we have to show that there exists a unique morphism of covers $f : Y_1 \to Y_2$ such that $\phi = \mathrm{Fib}_x(f)$. It is not restrictive to assume that both Y_1 and Y_2 are connected (otherwise, the argument could be repeated on pairs of connected components). Due to the fact that $\mathrm{Hom}_X(\widetilde{X}_x, -)$ is naturally isomorphic to Fib_x, we can regard ϕ as a map $\phi : \mathrm{Hom}_X(\widetilde{X}_x, Y_1) \to \mathrm{Hom}_X(\widetilde{X}_x, Y_2)$ which sends the map π_y to the map $\pi'_{\phi(y)}$ (see the proof of Theorem 1.12). Due to the Galois correspondence for covers, we know that $\pi_y : \widetilde{X}_x \to Y_1$ is a Galois cover and it induces an isomorphism of covers $\overline{\pi}_y : \widetilde{X}_x/\mathrm{Aut}(\widetilde{X}_x|Y) \to Y_1$. Moreover, since $H_y = \mathrm{Aut}(\widetilde{X}_x|Y_1)$ can be seen as the stabiliser of π_y by the right-action of $\mathrm{Aut}(\widetilde{X}_x|X)$ on $\mathrm{Hom}_X(\widetilde{X}_x, Y_1)$, we know that H_y embeds into $H_{\phi(y)} = \mathrm{Aut}(\widetilde{X}_x|Y_2)$, i.e. the stabiliser of $\pi'_{\phi(y)}$ by the right-action of $\mathrm{Aut}(\widetilde{X}_x|X)$ on $\mathrm{Hom}_X(\widetilde{X}_x, Y_2)$. Hence, the map $\pi'_{\phi(y)} : \widetilde{X}_x \to Y_2$ is a Galois cover and induces a morphism of covers $\widehat{\pi'}_{\phi(y)} : \widetilde{X}_x/H_y \to Y_2$ (which may fail to be an isomorphism). Therefore, the composition

$$f : Y_1 \xrightarrow{(\overline{\pi})_y^{-1}} \widetilde{X}_x/H_y \xrightarrow{\widehat{\pi'}_{\phi(y)}} Y_2$$

is the our morphism of covers.

(II) Fib_x *is essentially surjective.*

Let A be a left $\pi_1(X,x)$-set. It is sufficient to prove that A is isomorphic to the fibre of some cover when A contains a single $\pi_1(X,x)$-orbit. Otherwise, we can decompose A into its $\pi_1(X,x)$-orbits. Given a point $a \in A$, let us consider the stabiliser $H = \mathrm{Stab}_{\pi_1(X,x)}(a) \subseteq \pi_1(X,x)$. Hence, the Galois correspondence ensures the existence of the cover $p_H : \widetilde{X}_x/H \to X$ we are looking for. □

1.3. **Locally constant sheaves.** In a categorical language, a *section* is a right-inverse of some morphism. Particularly, in the category **Top** of topological spaces, given a continuous map $p : Y \to X$, a section of p over U for some open subset $U \subseteq X$ is a continuous map $s : U \to Y$ such that $p \circ s = \mathrm{id}_Y$.

In this context, for each open subset $U \subseteq X$, we define the set $\mathcal{F}_Y(U)$ to be the set of sections $s : U \to Y$. Moreover, given another open subset $V \subseteq U$, we define the map $\mathcal{F}_Y(U) \to \mathcal{F}_Y(V)$ by restricting sections over U to V. This construction yields to a sheaf

$$\mathcal{F}_Y : \mathbf{Open}(X) \longrightarrow \mathbf{Set}$$

called the *sheaf of sections* of $p : Y \to X$.

Remark 1.14. Some of the most relevant examples of sheaves arise as the sheaf of sections of some morphism. Apart from the above example, we could consider the sheaf of sections of a vector bundle on an algebraic variety, or more generally of a fibre bundle. In a more technical sense, any sheaf is the sheaf of sections over its étale space (see below for the formal definition). This is why the elements of a sheaf are called the sections of the sheaf.

As before, let us assume for the rest of the section that X is a fixed connected and locally simply connected topological space.

Proposition 1.15. *Given a cover $p : Y \to X$, the sheaf $\mathcal{F}_Y$ is locally constant. As a consequence, the sheaf $\mathcal{F}_Y$ is constant if and only if the cover is trivial.*

Proof. Let $x \in X$ be a point and $U \subseteq X$ an open connected neighbourhood of x such that the restriction $p : p^{-1}(U) \to U$ is trivial, i.e. isomorphic to some trivial cover $U \times I \to U$. Given a section $s : U \to Y$ (i.e. an element $s \in \mathcal{F}_Y(U)$), $s(U)$ is an open connected subset of Y mapped isomorphically onto U by p, so it is a connected component of $p^{-1}(U)$ which is in a bijective correspondence with I, so sections $U \to Y$ correspond in a bijective way to points of I. Thus, the restriction sheaf $(\mathcal{F}_Y)_{|U}$ is isomorphic to the constant sheaf defined by the set I, so $\mathcal{F}_Y$ is a locally constant sheaf. □

The following result is an immediate consequence of the above proposition.

Corollary 1.16. *Given a cover $p : Y \to X$ and a point $x \in X$, the stalk $(\mathcal{F}_Y)_x$ is in a bijective correspondence with $p^{-1}(x)$.*

Proof. This is just a consequence of the second part of Proposition 1.2. □

Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ two covers and $f : Y_1 \to Y_2$ a morphism of covers. Let us consider the family of morphisms (in **Set**)

$$\Sigma(f) = \{\Sigma(f)_U : \mathcal{F}_{Y_1}(U) \longrightarrow \mathcal{F}_{Y_2}(U)\}_{U \in \mathbf{Open}(X)},$$

where $\Sigma(f)_U(s) = f \circ s$. Indeed, this is a well-defined map since

$$p_2 \circ \Sigma(f)_U(s) = p_2 \circ (f \circ s) = (p_2 \circ f) \circ s = p_1 \circ s = \mathrm{id}_U$$

so $\Sigma(f)_U(s) \in \mathcal{F}_{Y_2}(U)$. Thus, we can conclude the following.

Proposition 1.17. *The assignment $\Sigma : \mathbf{Cov}(X) \to \mathbf{LCSh}(X)$ given by*

$$\begin{array}{ccc} Y \xrightarrow{p} X & \mapsto & \Sigma(Y) = \mathcal{F}_Y \\ f : Y_1 \longrightarrow Y_2 & \mapsto & \Sigma(f) : \mathcal{F}_{Y_1} \to \mathcal{F}_{Y_2} \end{array}$$

is a (covariant) functor.

Given a presheaf $\mathcal{F}$ on X, let us define the set $X_{\mathcal{F}} = \bigsqcup_{x \in X} \mathcal{F}_x$ and consider the map $p_{\mathcal{F}} : X_{\mathcal{F}} \to X$ given by $p_{\mathcal{F}}(v) = x$ for each $v \in \mathcal{F}_x$ and each point $x \in X$. Let us now define a topology on $X_{\mathcal{F}}$. Given an open subset $U \subseteq X$ and a section $s \in \mathcal{F}(U)$, let us consider the map $i_s : U \to X_{\mathcal{F}}$ given by $i_s(u) = [s(u)]$, where $[\cdot]$ denotes the equivalence class of the relation described in the definition of stalk at a point. We define the topology on $X_{\mathcal{F}}$ to be the coarsest (i.e. smallest) topology in which the sets $i_s(U)$ are open. Hence, $p_{\mathcal{F}} : X_{\mathcal{F}} \to X$ and the maps i_s are continuous maps with respect to this topology. The topological space $X_{\mathcal{F}}$ is called the *étale space* of the presheaf $\mathcal{F}$.

Proposition 1.18. *If $\mathcal{F}$ is a locally constant sheaf on X, its étale space $p_{\mathcal{F}} : X_{\mathcal{F}} \to X$ is a cover.*

Proof. Let $U \subseteq X$ be an open connected subset such that $\mathcal{F}_{|U}$ is a constant sheaf. Hence, it is easy to check that $\mathcal{F}_x = I$ for each $x \in X$ and some set I. Hence, we have that

$$p_{\mathcal{F}}^{-1}(U) = \bigsqcup_{x \in U} \mathcal{F}_x = \bigsqcup_{x \in U} I = U \times I.$$

Therefore, the cover $p_{\mathcal{F}} : p_{\mathcal{F}}^{-1}(U) \to U$ is isomorphic to the trivial cover $U \times I \to U$, where I is equipped with the discrete topology. Hence, $p_{\mathcal{F}} : X_{\mathcal{F}} \to X$ is a cover. □

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. As we know, the morphism φ induces a family of morphisms (in **Set**) $\{\varphi_x : \mathcal{F}_x \to \mathcal{G}_x\}_{x \in X}$. Hence, the map

$$\text{Étale}(\varphi) : X_{\mathcal{F}} \longrightarrow X_{\mathcal{G}}$$

given by $\text{Étale}(\varphi)_{|\mathcal{F}_x} = \varphi_x$ is a well-defined continuous map. Moreover, if we restrict ourselves to the full subcategory of locally constant sheaves on X, the continuous map $\text{Étale}(\varphi)$ turns out to be a morphism of covers. Thus, we can conclude the following.

Proposition 1.19. *The assignment* $\text{Étale} : \mathbf{LCSh}(X) \to \mathbf{Cov}(X)$ *given by*

$$\begin{array}{ccc} \mathcal{F} & \mapsto & \text{Étale}(\mathcal{F}) = p_{\mathcal{F}} : X_{\mathcal{F}} \longrightarrow X \\ \varphi : \mathcal{F} \longrightarrow \mathcal{G} & \mapsto & \text{Étale}(\varphi) : X_{\mathcal{F}} \to X_{\mathcal{G}} \end{array}$$

is a (covariant) functor.

We are now ready to prove the main theorem of the section.

Theorem 1.20. ***Covers and Locally Constant Sheaves***
Let X be a connected and locally simply connected space. The functors $\Sigma : \mathbf{Cov}(X) \to \mathbf{LCSh}(X)$ and $\text{Étale} : \mathbf{LCSh}(X) \to \mathbf{Cov}(X)$ are naturally isomorphic, i.e. they induce an equivalence of categories.

Proof. We have to check that, given a locally constant sheaf $\mathcal{F}$ on X and a cover $p : Y \to X$, there are functorial isomorphisms $\mathcal{F} \cong \mathcal{F}_{X_{\mathcal{F}}}$ and $Y \cong X_{\mathcal{F}_Y}$. On the one hand, consider the morphism of sheaves $\mathcal{F} \to \mathcal{F}_{X_{\mathcal{F}}}$ given by the family $\{\mathcal{F}(U) \to \mathcal{F}_{X_{\mathcal{F}}}(U) \ : \ s \mapsto i_s\}_{U \in \mathbf{Open}(X)}$. Indeed, these maps are well-defined since

$$(p_{\mathcal{F}} \circ i_s)(x) = p_{\mathcal{F}}(i_s(x)) = p_{\mathcal{F}}([s(x)]) = x,$$

so $i_s \in \mathcal{F}_{X_{\mathcal{F}}}(U)$. On the other hand, since $X_{\mathcal{F}_Y} = \bigsqcup_{x \in X} (\mathcal{F}_Y)_x \cong \bigsqcup_{x \in X} p^{-1}(x)$, we have a map $Y \to X_{\mathcal{F}_Y}$ that sends each point $y \in Y$ with $p(y) = x$ to its corresponding point in the stalk $p^{-1}(x) \cong (\mathcal{F}_Y)_x$.

Let us check that these maps are isomorphisms by restricting ourselves to a distinguished open covering. Indeed, let $\{U_i\}_{i\in I}$ be an open covering of X such that $\mathcal{F}_{|U_i}$ is constant for each $i \in I$. Hence, if we replace X by some U_{i_0}, we may assume that $\mathcal{F}$ is a constant sheaf on X with constant value a fixed set I. Therefore, since there is an isomorphism of covers $X_{\mathcal{F}} \cong X \times I$, where I is equipped with the discrete topology, and there is an isomorphism of sheaves between the constant sheaf $\mathcal{F}$ and the sheaf of sections of the trivial cover $\mathcal{F}_{X\times I}$, the said isomorphisms $\mathcal{F} \cong \mathcal{F}_{X_{\mathcal{F}}}$ and $Y \cong X_{\mathcal{F}_Y}$ hold. Moreover, the corresponding commutative diagrams are satisfied, so $\Sigma \circ \text{Étale} \cong \text{Id}_{\mathbf{LCSh}(X)}$ and $\text{Étale} \circ \Sigma \cong \text{Id}_{\mathbf{Cov}(X)}$, as we wanted to show. □

We may combine this theorem with Theorem 1.13 to obtain the following result.

Theorem 1.21. ***Locally Constant Sheaves and $\pi_1(X,x)$-Sets***
Let X be a connected and locally simply connected space, and let $x \in X$ be a base point. The functor $\text{Stalk}_x : \mathbf{LCSh}(X) \to \pi_1(X,x)\text{-}\mathbf{Set}$ induces an equivalence of categories.

Proof. Note that the composition functor given by

$$\mathbf{LCSh}(X) \xrightarrow{\text{Étale}} \mathbf{Cov}(X) \xrightarrow{\text{Fib}_x} \pi_1(X,x)\text{-}\mathbf{Set}$$

satisfies $(\text{Fib}_x \circ \text{Étale})(\mathcal{F}) = \text{Fib}_x(\text{Étale}(\mathcal{F})) = \text{Fib}_x(X_{\mathcal{F}}) = p_{\mathcal{F}}^{-1}(x) = \mathcal{F}_x = \text{Stalk}_x(\mathcal{F})$. □

The following result establishes a version of the above correspondence in which the locally constant sheaves are sheaves of R-modules, where R is a commutative ring with unit.

Theorem 1.22. ***Locally Constant Sheaves of R-Modules and $R[\pi_1(X,x)]$-Modules***
Let X be a connected and locally simply connected space, and let $x \in X$ be a base point. The category of locally constant sheaves of R-modules is equivalent to the category of left $R[\pi_1(X,x)]$-modules.

Proof. We know that the stalk $\mathcal{F}_x$ is an R-module by definition and it has a left $\pi_1(X,x)$-action equipped as a set. Moreover, since the action of $\pi_1(X,x)$ is compatible with the R-module structure, the stalk $\mathcal{F}_x$ is precisely a left $R[\pi_1(X,x)]$-module. □

1.3.1. *Complex Local Systems.* A locally constant sheaf of finite-dimensional $\mathbb{C}$-vector spaces on X is called a *complex local system* on X. This definition can be analogously stated for an arbitrary field K, but for our purpose, we restrict ourselves to the field $K = \mathbb{C}$.

Moreover, the dimension of the stalks of a local system is constant on each connected component, so since X is connected, all stalks have the same dimension, which we call the *dimension* of the local system. We denote the category of complex local systems on X by $\mathbf{Loc}_{\mathbb{C}}(X)$.

As a consequence of Theorem 1.22 and the well-known equivalence between the category of finite-dimensional representations of a group and finitely generated left-modules over the group algebra, we have the following result.

Corollary 1.23. *Local Systems and Representations of* $\pi_1(X,x)$
Let X be a connected and locally simply connected space, and let $x \in X$ be a base point. The category of complex local systems on X is equivalent to the category of finite-dimensional complex representations of $\pi_1(X,x)$.

Therefore, if we fix an integer $n \in \mathbb{Z}_{\geq 1}$, the last theorem states that an n-dimensional complex local system on X is essentially the same thing as a group homomorphism $\pi_1(X,x) \to \mathrm{GL}_n(\mathbb{C})$. This representation of $\pi_1(X,x)$ is called the *monodromy representation* of the local system.

2. Monodromy representations of ordinary differential equations

Let $D \subseteq \mathbb{C}$ be a domain, i.e. a connected open subset. We devote this section to present the relation between complex local systems on D and solutions of certain homogeneous ODEs.

Consider the n-th order linear differential equation over D given by

$$y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_{n-1}(z)y' + a_n(z)y = 0,$$

where $a_i \in \mathcal{O}(D)$ for each $i = 1, \ldots, n$. Given an open subset $U \subseteq D$, let us denote by $\mathrm{Sol}(U)$ the set of holomorphic solutions to the previous differential equations over U. Hence, we know that $\mathrm{Sol}(U)$ is closed under $\mathbb{C}$-linear combinations, so $\mathrm{Sol}(U)$ is a complex vector space for each open set $U \subseteq D$. Moreover, given a point $z_0 \in D$, the Cauchy–Kovalevskaya theorem implies the existence of a connected open neighbourhood $U \subseteq D$ containing z_0 such that $\mathrm{Sol}(U)$ has a finite $\mathbb{C}$-basis $\{y_1, \ldots, y_n\}$, so $\mathrm{Sol}(U) \cong \mathbb{C}^n$. Moreover, since the restrictions of the basics solutions $y_1, \ldots, y_n$ to smaller open sets still form a basis for the solutions, we deduce that Sol is a locally constant subsheaf of $\mathcal{O}^n$ of complex vector spaces. We summarise these information in the following result.

Proposition 2.1. *Let $D \subseteq \mathbb{C}$ be a domain. Consider the n-th order linear differential equation over D given by*

$$y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_{n-1}(z)y' + a_n(z)y = 0,$$

where $a_i \in \mathcal{O}(D)$ for each $i = 1, \ldots, n$. The sheaf of solutions Sol *is a complex local system of dimension n over D.*

Due to Theorem 1.23 and the previous result, for a fixed point $z \in D$, we can assign to the complex local system Sol a representation $\pi_1(D, z) \to \mathrm{GL}_n(\mathbb{C})$. Let us now try to describe this representation explicitly.

Given a closed path $\sigma : [0,1] \to D$ based at a point $z \in D$, consider its homotopy class $\gamma = [\sigma] \in \pi_1(D, z)$. Given an element $s_z \in \mathrm{Sol}_z$, we can understand s_z as the germ of a solution of our ODE in a neighbourhood of $z \in D$[4]. Consider the cover $p_{\mathrm{Sol}} : D_{\mathrm{Sol}} \to D$ associated to the complex local system Sol. By definition of the monodromy action, the element γ acts on $s_z \in p_{\mathrm{Sol}}^{-1}(z) = \mathrm{Sol}_z$ as $\gamma s_z = \widetilde{\sigma}(1) \in p_{\mathrm{Sol}}^{-1}(z) = \mathrm{Sol}_z$, where $\widetilde{\sigma}$ is the unique lift of σ to D_{Sol}. Due to the compactness of the unit interval, there exist some open sets $U_1, \ldots, U_k \subseteq D$ such that $\sigma^{-1}(U_1), \ldots, \sigma^{-1}(U_k)$ cover $[0,1]$ and such that $\mathrm{Sol}_{|U_i}$ is constant for each $i = 1, \ldots, k$. Moreover, there are solutions $y_i \in \mathrm{Sol}(U_i)$ for each $i = 1, \ldots, k$ such that y_i and y_{i+1} agree on $U_i \cap U_{i+1}$ for each $i = 1, \ldots, k-1$. In this case, the germ of y_1 at z is s_z, and the germ of y_k at z is γs_z. That is why γs_z is called the *analytic continuation of the germ* s_z. Note that, the assignment

$$\gamma \in \pi_1(D, z) \mapsto (s_z \in \mathrm{Sol}_z \cong \mathbb{C}^n \mapsto \gamma s_z \in \mathrm{Sol}_z \cong \mathbb{C}^n)$$

provides a group homomorphism $\rho : \pi_1(D, z) \to \mathrm{Aut}(\mathrm{Sol}_z) \cong \mathrm{GL}_n(\mathbb{C})$, which is precisely the monodromy representation of the local system Sol.

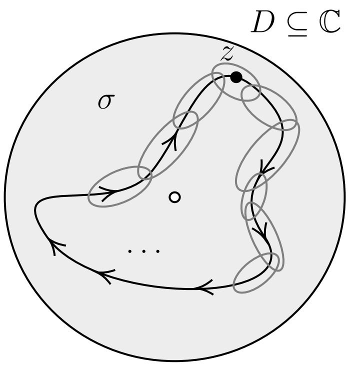

FIGURE 3. Trivialization of the closed path.

The existence (and uniqueness) of the analytic continuation of the germ s_z along the path σ is guaranteed by the fact that Sol is a locally constant sheaf. In contrast, for an arbitrary germ of the sheaf $\mathcal{O}^n$, analytic continuation along a path may not yield a globally well-defined extension.

Remark 2.2. A similar reasoning can be made when one considers solutions of a system of n linear differential equations in n variables, that is, a system of the form $Y' = A(z)Y$, where $A(z)$ is an $n \times n$ matrix of holomorphic functions.

[4]Indeed, the elements of Sol_z are germs of local holomorphic solutions, which equivalently correspond to initial condition data $(y(z), y'(z), \ldots, y^{(n-1)}(z)) \in \mathbb{C}^n$.

So far, we have associated the so-called monodromy representation to a certain homogeneous ordinary differential equation. But what about the converse? Well, we have the following. Take $D = \mathbb{C} - \{z_0, \dots, z_m\}$ and consider the n-th order linear differential equation over D given by

$$y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_{n-1}(z)y' + a_n(z)y = 0,$$

where $a_i \in \mathcal{O}(D)$ for each $i = 1, \dots, n$. We say that the point $z_j \in \mathbb{C}$ is a *regular singularity* if $(z - z_j)^i a_i(z)$ is holomorphic for each $i = 1, \dots, n$. A linear differential equation whose singular points are all regular is called a *fuchsian* equation. In this context, we have the following.

Problem 2.3. ***Hilbert's 21st problem***
Given a representation $\rho : \pi_1(D, z) \to \mathrm{GL}_n(\mathbb{C})$, does there exist a fuchsian system of linear differential equations whose monodromy representation equals ρ?

The answer is, in general, negative (see [3, Chapter 2]). However, as we will illustrate in the next example, the answer is affirmative for 1-dimensional representations of the punctured plane.

Example 2.4. Take $D = \mathbb{C} - \{0\}$ and consider a point $z \in D - \{r \in \mathbb{R} : r < 0\}$. Given a complex number $m \in \mathbb{C}^*$, consider the 1-dimensional representation $\rho_m : \pi_1(D, z) \to \mathbb{C}^*$ defined by $\rho(\gamma) = m$, where $\gamma = [\sigma]$ is the generator of $\pi_1(D, z) \cong \mathbb{Z}$ and $\sigma(t) = ze^{2\pi i t}$. Now, consider the differential equation

$$y' = \frac{\mu}{z} y,$$

where $\mu \in \mathbb{C}$ verifies $m = e^{2\pi i \mu}$. Let us prove that the monodromy representation of this equation is precisely ρ_m. At the point z, a local solution to this equation is given by $y(z) = e^{\mu \operatorname{Log}(z)}$, where $\operatorname{Log}(\cdot)$ denotes the principal branch of the complex logarithm. As we analytically continue the germ $s_z = [y] \in \mathrm{Sol}_z$ along γ, the branch of the logarithm jumps by $2\pi i$, since

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z) \mapsto \operatorname{Log}(z) + 2\pi i = \ln|z| + i(\operatorname{Arg}(z) + 2\pi).$$

Thus, the analytic continuation of (the germ of) the solution $y(z) = e^{\mu \operatorname{Log}(z)}$ is given by

$$y(z) \mapsto e^{\mu\,(\operatorname{Log}(z)+2\pi i)} = e^{2\pi i \mu} y(z) = m y(z).$$

Therefore, the monodromy representation of this differential equation is precisely ρ_m.

3. Connections and the Riemann–Hilbert correspondence

In the previous section, given a complex domain $D \subseteq \mathbb{C}$, we associated to each homogeneous ODE a representation of the fundamental group of D. We also verified that 1-dimensional representations of the punctured complex plane occur as monodromy representations of certain ODEs. To establish a well-founded correspondence, however, it is necessary to work within a category that captures the holomorphic data inherent to these ODEs. For our purposes, this will be the category of holomorphic connections.

In what follows, $D \subseteq \mathbb{C}$ denotes a complex domain. Let us recall some notions about sheaf theory. As before, let $\mathcal{O}$ be the **Ring**-sheaf of holomorphic functions on our domain $D \subseteq \mathbb{C}$. Recall that a *sheaf of $\mathcal{O}$-modules* is an **Ab**-sheaf $\mathcal{F}$ on D such that for each open set $U \subseteq D$, the abelian group $\mathcal{F}(U)$ carries the structure of a $\mathcal{O}(U)$-module, and such that for every inclusion $V \hookrightarrow U$ of open subsets of D, the square

$$\begin{array}{ccc} \mathcal{O}(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

is commutative. A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of sheaves of abelian groups that is compatible with the $\mathcal{O}$-module structure. We say that the sheaf of $\mathcal{O}$-modules $\mathcal{F}$ is *locally free* if for every point $z \in D$, there exists an open set $U \subseteq D$ containing z such that $\mathcal{F}_{|U} \cong \mathcal{O}^n_{|U}$ for some $n \in \mathbb{Z}_{\geq 1}$. The integer n is called the *rank* of the locally free sheaf. We recall the following fundamental result.

Theorem 3.1. ***Locally free sheaves and holomorphic vector bundles***
There is an equivalence between the category of locally free sheaves on D and the category of holomorphic vector bundles over D.

Recall that a *holomorphic 1-form* on D is a complex differential 1-form on D that can be written, locally, in the form $\omega = f dz$, where f is a holomorphic function. Thus, we can consider the *sheaf of holomorphic 1-forms* Ω^1_D, which is a sheaf of $\mathcal{O}$-modules on D.

A *holomorphic connection* on D is a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a locally free sheaf on D and $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}} \Omega^1_D$ is a morphism of sheaves of $\mathbb{C}$-vector spaces[5] that satisfies the so-called *Leibniz rule*:

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for all open subsets $U \subseteq D$, $f \in \mathcal{O}(U)$ and $s \in \mathcal{E}(U)$. A *morphism* $\phi : (\mathcal{E}, \nabla) \to (\mathcal{E}', \nabla')$ between two connections is a morphism of $\mathcal{O}$-modules $\phi : \mathcal{E} \to \mathcal{E}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\nabla} & \mathcal{E} \otimes_{\mathcal{O}} \Omega^1_D \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle \phi \otimes id_{\Omega^1_D}} \\ \mathcal{E}' & \xrightarrow{\nabla'} & \mathcal{E}' \otimes_{\mathcal{O}} \Omega^1_D \end{array}$$

is commutative.

As we said, the category $\mathbf{Conn}_{\text{hol}}(D)$ of holomorphic connections on D will be the appropriate setting for the Riemann–Hilbert correspondence. A natural question arises: what is the relation between holomorphic connections and ODEs?

[5]That is, locally, a connection is $\mathbb{C}$-linear.

Example 3.2. Consider the sheaf $\mathcal{O}^n$ on D (which is, of course, locally free[6]) together with the connection map

$$\nabla_0 : \mathcal{O}^n \longrightarrow \mathcal{O}^n \otimes_{\mathcal{O}} \Omega^1_D = (\Omega^1_D)^n$$

defined by $\nabla_0(f_1, \dots, f_n) = (df_1, \dots, df_n)$. Let ∇ be another connection map on $\mathcal{O}^n$. Then, since

$$(\nabla - \nabla_0)(fs) = f(\nabla - \nabla_0)(s),$$

we deduce that $\nabla - \nabla_0$ is an $\mathcal{O}$-linear map (i.e. tensorial), so $\nabla - \nabla_0 \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^n, (\Omega^1_D)^n)$. Consequently, $\nabla - \nabla_0$ is given by an $n \times n$ matrix Λ of 1-forms (i.e. $\Lambda_{ij} = f_{ij} dz$), so

$$\nabla(f) = \nabla_0(f) + \Lambda f.$$

Therefore, a section $f \in \mathcal{O}^n(U)$ on some open set $U \subseteq D$ verifies $\nabla(f) = 0$ if, and only if, it is a solution of the system of ODEs given by $Y' = A(z)Y$, where $A(z)_{ij} = -f_{ij}(z)$.

The previous example shows that, given a holomorphic connection $(\mathcal{E}, \nabla)$ and an open set $U \subseteq D$, the sections $s \in \mathcal{E}(U)$ which verify $\nabla(s) = 0$ are the analogues of solutions of a system of ODEs. These sections are called *horizontal*, and form a (sub)sheaf $\mathcal{E}^\nabla \subseteq \mathcal{E}$ of $\mathbb{C}$-vector spaces. Using the previous example and the fact that over a non-compact Riemann surface every locally free sheaf is free, we deduce the following result.

Proposition 3.3. *Given a holomorphic connection $(\mathcal{E}, \nabla)$, the sheaf of horizontal sections $\mathcal{E}^\nabla$ is a complex local system.*

Now, we are ready to prove the classical version of the Riemann–Hilbert correspondence on a complex domain $D \subseteq \mathbb{C}$.

Theorem 3.4. *Riemann–Hilbert correspondence on* D
The assignment $\mathfrak{H} : \mathbf{Conn}_{\mathrm{hol}}(D) \to \mathbf{Loc}_{\mathbb{C}}(D)$ given by

$$\begin{array}{rcl} \mathcal{E} & \mapsto & \mathfrak{H}((\mathcal{E}, \nabla)) = \mathcal{E}^\nabla \\ \phi : (\mathcal{E}, \nabla) \longrightarrow (\mathcal{E}', \nabla') & \mapsto & \mathfrak{H}(\phi) = \phi_{|\mathcal{E}^\nabla} \end{array}$$

is a (covariant) functor which induces an equivalence of categories.

Proof. Let us construct a functor in the opposite direction. Given a complex local system $\mathcal{L}$ on D, consider the sheaf $\mathcal{E}_{\mathcal{L}}$ on D defined by

$$\mathcal{E}_{\mathcal{L}}(U) = \mathcal{L}(U) \otimes_{\mathbb{C}} \mathcal{O}(U)$$

for any open set $U \subseteq D$. Given a point $z \in D$, there exists some neighbourhood $U \subseteq D$ containing z such that $\mathcal{L}_{|U} \cong \mathbb{C}^n$ (as sheaves) for some $n \in \mathbb{Z}_{\geq 1}$. Therefore, we have that

$$\mathcal{E}_{\mathcal{L}}(U) = \mathcal{L}(U) \otimes_{\mathbb{C}} \mathcal{O}(U) \cong \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}(U) \cong \mathcal{O}(U)^n,$$

so $\mathcal{E}_{\mathcal{L}}$ is locally free.

[6] A sheaf of $\mathcal{O}$-modules that is isomorphic to $\mathcal{O}^n$ for some n is called a *free sheaf of rank n*.

As before, consider an open subset $U \subseteq D$ such that $\mathcal{L}|_U \cong \mathbb{C}^n$ and fix a $\mathbb{C}$-basis $\{s_1, \dots, s_n\}$ of $\mathcal{L}(U)$. Then, each section of $\mathcal{E}_{\mathcal{L}}(U)$ can be uniquely written as a sum $\sum_{i=1}^n s_i \otimes f_i$, where $f_i \in \mathcal{O}(U)$. Now, define $\nabla_{\mathcal{L}}|_U$ by

$$\nabla_{\mathcal{L}}\Big(\sum_{i=1}^n s_i \otimes f_i\Big) := \sum_{i=1}^n s_i \otimes df_i.$$

Since two bases of $\mathcal{L}(U)$ differ by a matrix whose entries lie in $\mathbb{C}$, this definition does not depend on the choice of basis. Therefore, the $\nabla_{\mathcal{L}}|_U$ defined over the various U patch together to give a map $\nabla_{\mathcal{L}}$ defined over the whole of D. □

It is worth noting that the Riemann–Hilbert correspondence holds for arbitrary Riemann surfaces. Moreover, it can be generalised to higher-dimensional complex manifolds, provided that an additional condition is imposed on the holomorphic connections, namely, that they are *flat*, i.e. that they have zero curvature. This condition is automatically satisfied for one-dimensional complex manifolds.

Appendix A. Sheaves

Sheaf theory allows us to study some algebraic data attached to the open sets of a topological space and defined locally with respect to them. In particular, it plays a crucial role in the theory of algebraic geometry. Throughout this dissertation, we will assume basic concepts and results from category theory. Our main reference for the categorical machinery will be [1].

For the sake of simplicity, in what follows **C** denotes a category from among the category **Set** of sets, the category **Ab** of abelian groups or the category **Ring** of commutative unitary rings.

Given a topological space X, let us consider the category **Open**(X), whose objects are the open subsets of X and whose Hom-sets are given by

$$\mathrm{Hom}(V, U) = \begin{cases} \varnothing & \text{if} \quad V \nsubseteq U \\ \{V \hookrightarrow U\} & \text{if} \quad V \subseteq U \end{cases}.$$

A **C**-*presheaf* $\mathcal{F}$ on X is a contravariant functor $\mathcal{F} : \mathbf{Open}(X) \to \mathbf{C}$. Let us establish some related notations. Given $U \in \mathbf{Open}(X)$, the elements of $\mathcal{F}(U)$ are usually called *sections* for further discussion) over U. If $V \subseteq U$ is an open set, the morphism

$$\mathcal{F}(V \hookrightarrow U) : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

is denoted by ρ_{UV}. Given a section $s \in \mathcal{F}(U)$, we denote by $s_{|V}$ the induced section $\rho_{UV}(s) \in \mathcal{F}(V)$.

A **C**-presheaf $\mathcal{F}$ on a topological space X is a **C**-*sheaf* if it satisfies the following conditions:

(i) **Locality**. If $U \subseteq X$ is an open set, $\{V_i\}_{i\in I}$ is an open covering of U and $s, t \in \mathcal{F}(U)$ are sections such that $s_{|V_i} = t_{|V_i}$ for each $i \in I$, then $s = t$.

(ii) **Gluing**. Let $U \subseteq X$ be an open set and $\{V_i\}_{i\in I}$ an open covering of U. Given sections $s_i \in \mathcal{F}(U_i)$ such that $s_{i|V_i\cap V_j} = s_{j|V_i\cap V_j}$ for each $i, j \in I$, there is a (unique) section $s \in \mathcal{F}(U)$ such that $s_{|V_i} = s_i$ for each $i \in I$.

Example A.1. Sheaves of functions. Given two topological spaces X and Y, we can assign to each open subset $U \subseteq X$ the ring of continuous functions $\mathcal{C}(U, Y)$. Thus, we can consider the *sheaf of continuous functions* $\mathcal{C}(-, Y)$. In this case, given an open set $V \subseteq U$ and a continuous function $f : U \to Y$, $f_{|V}$ is just the restriction of f to V. As a special case, if X were a complex manifold, we could consider the *sheaf of holomorphic functions* $\mathcal{O}(-, \mathbb{C})$, which will be simply denoted as $\mathcal{O}$.

Example A.2. Constant sheaves. Given a topological space X and a discrete space I (i.e. a topological space with the discrete topology), consider the sheaf $\mathcal{F}_I$ defined by $\mathcal{F}_I = \mathcal{C}(-, I)$. Note that, if $U \subseteq X$ is a connected open subset, a continuous map $U \to I$ must be constant, so $\mathcal{F}_I(U) = I$. The sheaf $\mathcal{F}_I$ is called the *constant sheaf* on X with value I. More generally, a sheaf is called *constant* if it is isomorphic (see below for the definition of morphism of sheaves) to a constant sheaf $\mathcal{F}_I$ for some discrete space I.

Given a presheaf $\mathcal{F}$ on a topological space X and a fixed open subset $U \subseteq X$, we define its *restriction* to U as the presheaf $\mathcal{F}_{|U} : \mathbf{Open}(U) \to \mathbf{C}$ given by $\mathcal{F}_{|U}(V) = \mathcal{F}(V)$. This concept allows us to define the notion of a locally constant sheaf, which will appear later in this dissertation.

Example A.3. Locally constant sheaves. Given a topological space X, we say that a $\mathbf{C}$-sheaf $\mathcal{F}$ on X is *locally constant* if each point $p \in X$ has an open neighbourhood $U \subseteq X$ such that $\mathcal{F}_{|U}$ is a constant $\mathbf{C}$-sheaf.

Given a $\mathbf{C}$-presheaf on a topological space X and a point $p \in X$, the *stalk* of $\mathcal{F}$ at the point p is the quotient object given by

$$\mathcal{F}_x = \left(\sqcup_{x\in U\in\mathbf{Open}(X)}\mathcal{F}(U)\right) / \sim,$$

where $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ satisfy $s \sim t$ if and only if there exists an open set $x \in W \subseteq U \cap V$ such that $s_{|W} = t_{|W}$. In a categorical language, the stalk of $\mathcal{F}$ at $x \in X$ can be defined as the direct limit

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U)$$

over the open sets $U \in \mathbf{Open}(X)$ with $x \in U$ ordered by inclusion. Note that, given a point $x \in X$, the assignment $\mathcal{F} \mapsto \mathcal{F}_x$ gives rise to the *stalk functor* $\mathrm{Stalk}_x : \mathbf{PSh}(X) \to \mathbf{C}$, where $\mathbf{PSh}(X)$ denotes the category of $\mathbf{C}$-presheaves.

Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathbf{C}$-presheaves on a topological space X. A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is just a natural equivalence of functors, i.e. a family of morphisms $\{\varphi(U) :$

$\mathcal{F}(U) \to \mathcal{G}(U)\}_{U \in \mathbf{Open}(X)}$ such that, given an inclusion arrow $V \hookrightarrow U$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle \rho^{\mathcal{F}}_{UV}} \downarrow & & \downarrow {\scriptstyle \rho^{\mathcal{G}}_{UV}} \\ \mathcal{F}(U) & \xrightarrow{\varphi(V)} & \mathcal{G}(U) \end{array}$$

In particular, an isomorphism is a morphism which has a two-sided inverse.

Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X, a morphism of sheaves (i.e. a morphism of its underlying presheaves) $\varphi : \mathcal{F} \to \mathcal{G}$ induces a family of morphisms on the stalks $\{\varphi_x : \mathcal{F}_x \to \mathcal{G}_x\}_{x \in X}$. Moreover, φ is an isomorphism if and only if φ_x is an isomorphism for every $x \in X$. The opposite, however, is not true: a collection of isomorphisms $\{\mathcal{F}_x \to \mathcal{G}_x\}_{x \in X}$ does not guarantee the existence of an isomorphism of sheaves $\mathcal{F} \to \mathcal{G}$.

Given a continuous map between topological spaces $f : X \to Y$ and a $\mathbf{C}$-sheaf $\mathcal{F}$ on X, we define the *direct image* of $\mathcal{F}$ as the $\mathbf{C}$-sheaf $f_*\mathcal{F}$ on Y given by $f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$ for each $Y \in \mathbf{Open}(Y)$. This definition allows us to state the following result.

Proposition A.4. *Let $f : X \to Y$ be a continuous map between topological spaces. f_* defines a (covariant) functor $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$, where $\mathbf{Sh}(\cdot)$ denotes the category of $\mathbf{C}$-sheaves over the corresponding topological space.*

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