Operator algebras and modular entropy in holography

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Operators algebras and modular entropy in holography.

- The main goal will be to motivate the use of operator algebras in physics.
- In particular, we will focus on its application to quantum field theory (AQFT) and how it allows us to better understand certain phenomena such as entanglement.
- **It is not just a mathematical trick**.
- We will focus on its recent application to holography in particular in calculations of Rényi entropy and Ryu-Takayanagi as a quantum error correction code (QECC).

Quantum mechanics postulates and entanglement

- A system A in quantum mechanics is described by a Hilbert space $\mathcal{H}_A$.
- States given by positive hermitian matrices with unit trace, which we call density matrices: $\rho \in M_{n\times n}(\mathbb{C})$.
- An observable $\mathcal{O}$ will be given by a self-adjoint matrix acting in $\mathcal{H}$ and the expectation value of said observable is given by:

$$\langle\mathcal{O}\rangle = \mathrm{Tr}[\rho\mathcal{O}] \tag{1}$$

- Time evolution of state is given by an unitary matrix, where H is a hermitian matrix called the Hamiltonian.

$$\rho(t) = e^{iHt}\rho e^{-iHt} \tag{2}$$

Quantum mechanics postulates and entanglement

- We can then create from states in A and B a joint state in $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Given two states in A and B:

$$\rho_{AB} = \rho_A \otimes \rho_B \tag{3}$$

- A state in a bipartite system is said to be separable if $\rho \in \mathcal{H}_{AB}$ can be written as:

$$\rho = \sum_{j=1}^{n} p_j \rho_{A,j} \otimes \rho_{B,j} \tag{4}$$

- States which are not separable are said to be entangled.

Quantum mechanics postulates and entanglement

- We will say that a state is pure if there exists $v \in \mathcal{H}$ such that $\rho = v^\dagger v = |v\rangle \langle v|$. A state that is not pure is called mixed.
- Given a state ρ_{AB} we can construct a state in one of the subsystems by what it is called: **partial trace**:

$$\mathrm{Tr}_B[\rho_{AB}] := (I_A \otimes \mathrm{Tr}_B)(\rho_{AB}) \tag{5}$$

Measures of entanglement

Definition (Shannon entropy)

Let X be a discrete random variable on a finite set $\{x_i\}_{i=1}^{n}$ with a probability distribution function $p(x) = \Pr(X = x)$ then we can define the Shannon entropy as:

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x) \tag{6}$$

Definition (von Neumann entropy)

Let $\rho \in \mathcal{L}(\mathcal{H})$ be a state where $\dim \mathcal{H} < \infty$ then we can define the von Neumann entropy as:

$$S = -\mathrm{Tr}[\rho \log \rho] = \langle - \log \rho \rangle_\rho \tag{7}$$

We call the following expression **relative entropy**:

$$S(\rho || \sigma) = -S(\rho) - \mathrm{Tr}[\rho \log \sigma] \tag{8}$$

Measures of entanglement

Theorem (Strong subadditivity)

Let $\rho_{ABC} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ and ρ_i the corresponding reduced matrices to $\mathcal{H}_i$ then the following inequality holds:

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}) \tag{9}$$

- Similarly one can define mutual information as:

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \tag{10}$$

Definition

Rényi entropies: Let $\alpha \in [0, \infty]$ then:

$$S_\alpha = \frac{1}{1-\alpha} \ln Tr[\rho^\alpha] \text{ for } \alpha \in (1, \infty)$$

$$S_0 = \ln \text{rank}(\rho)$$

$$S_\infty = -\ln ||\rho||_\infty$$

Measures of entanglement

- Thus, entanglement tells us that there are "hidden" correlations between the two subsystems.

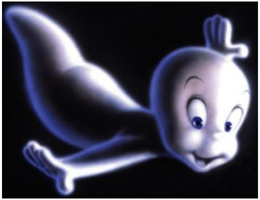


Figure: Spooky action at a distance

- Suppose we have a state ρ_A in a certain system $\mathcal{H}_{\mathcal{A}}$. We can purify it: find a bigger Hilbert space $\mathcal{H}_B$ and a pure state ρ such that $\rho_A = \mathrm{Tr}_B[\rho]$.
- Through purification one can find that a pure state in a bipartite system $\mathcal{H}_{AB}$ is separable if and only if the reduced states (partial trace) are also pure.

Gibbs states

- In QM Gibbs states are an example of equilibrium states:

$$\rho = \frac{e^{-\beta H}}{Z} \tag{11}$$

  Where $Z = \mathrm{Tr}[e^{-\beta H}]$ and H is a self-adjoint operator acting in H.

- Their von Neumann entropy reproduces the second law of thermodynamics:

$$S = -\mathrm{Tr}[\rho \log \rho] = \beta \mathrm{Tr}[\rho H] - \beta \log Z \tag{12}$$

$$S = \beta E - \beta F \tag{13}$$

C^* algebras

Definition (C^* algebras)

We will say that a Banach $*$-algebra is a C^* algebra if it posses the following property:

$$||A^*A|| = ||A||^2 \tag{14}$$

Theorem (Characterization of C^* algebras)

Let $\mathcal{A}$ be a C^* algebra, then $\mathcal{A}$ is isomorphic to a normed closed self-adjoint algebra of bounded operators on a Hilbert space

Theorem (Characterization of commutative C^* algebras)

Let $\mathcal{A}$ be a commutative C^* algebra, then it it is isometrically isomorphic to the algebra of continuous functions that vanish at infinity $C_0(X)$ where X is a locally compact Haussdorff space.

Representation of C*-algebras

Definition (Representation of C*-algebras)

Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{A}$ a C*-algebra and $\pi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H})$ a $*$-morphism. We say that the pair $(\pi, \mathcal{H})$ is a representation of $\mathcal{A}$.

- Cyclic representations $(\pi, \mathcal{H}, x)$ i.e $\{\pi(A)x : A \in \mathcal{A}\}$ is dense in $\mathcal{H}$.
- Irreducible representations are those where the only invariant subspaces under $\pi(\mathcal{A})$ are $\{\mathcal{H}, 0\}$.

States

Definition (States)

Let $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ be a linear functional over a C^*-algebra. We will say that it is a state if it verifies:

1. $\omega(AA^*) \geq 0$
2. $||\omega|| = 1$

- The set of states $\mathcal{E}_{\mathcal{A}}$ is a convex subset of $\mathcal{A}^*$.

Definition (Pure states)

We will say that ω is a pure state if the only functionals bounded by it are of the form: $\lambda\omega$ with $0 \leq \lambda \leq 1$. We denote the set of pure states as: $\mathcal{P}_{\mathcal{A}}$.

- If $\mathcal{A}$ is a unital algebra then $\mathcal{E}_{\mathcal{A}}$ is $*$-weakly compact, convex and its extreme points are pure states.

GNS construction

Theorem (GNS construction (1943))

Let ω be a positive functional of $\mathcal{A}$ a C^* algebra st. $||\omega|| = 1$. Then there exists a cyclic representation: $(\mathcal{H}_\omega, \pi_\omega, x_\omega)$ such that:

$$\omega(A) = (x_\omega, \pi_\omega(A)x_\omega). \tag{15}$$

For all $A \in \mathcal{A}$, with $||x_\omega|| = ||\omega|| = 1$. This representation is unique up to unitary equivalence.

- The GNS representations associated to pure states will be irreducible. Moreover for each element $A \in \mathcal{A}$ we can associate a pure state such that: $\omega(A^*A) = ||A||^2$. This will allow us to construct a Hilbert space for the first characterization:

$$\mathcal{H} = \bigoplus_{\omega \in \mathcal{E}_\mathcal{A}} \mathcal{H}_\omega \quad \pi = \bigoplus_{\omega \in \mathcal{E}_\mathcal{A}} \pi_\omega \tag{16}$$

- For the second characterization the locally compact Haussdorff space of the theorem will be the set of the pure states, which will be related

GNS and the postulates of quantum mechanics:

- We can translate the postulates of QM to this new formalism:
- We will begin with a C^* algebra of observables. States are now positive linear functionals.
- This allows us to introduce locality more naturally.
- In the case of QFT the vacuum state corresponds to the cyclic and separating vector (Reeh-Schlieder).

Von Neumann algebras and Tomita-Takesaki theory

Definition (von Neumann algebra)

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a $*$-sub algebra of $\mathcal{L}(\mathcal{H})$. We will say that $\mathcal{M}$ is a von Neumann algebra if it satisfies:

$$\mathcal{M} = \mathcal{M}'' \tag{17}$$

- The von Neumann Bicommutant theorem (17) translates this algebraic property into a topological one:

Theorem (Bicommutant theorem)

Let $\mathcal{M}$ be a von Neumann algebra the following properties are equivalent:

1. $\mathcal{M} = \mathcal{M}''$
2. $\mathcal{M}$ *is* (σ)*-weakly closed*
3. $\mathcal{M}$ *is* (σ)*-strongly closed*
4. $\mathcal{M}$ *is* (σ)*-strongly** *closed*

- In order to classify von Neumann algebras, Tomita-Takesaki theory

von Neumann algebras and Tomita-Takesaki

Definition (C* dynamical system)

Let G be a locally compact group, $\mathcal{A}$ a C* algebra and $\tau : G \to \mathrm{Aut}(\mathcal{A})$ a strongly continuous representation of G in the automorphism group of $\mathcal{A}$. Then we call the triple $\{\mathcal{A}, G, \tau\}$ a C* dynamical system. We take $G = \mathbb{R}$.

Theorem

Let ω be a state of a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ then the following conditions are equivalent:

1. *ω is normal (i.e σ-weakly continuous)*
2. *There exists a positive trace class operator ρ with $\mathrm{Tr}[\rho] = 1$ such that:*

$$\omega(A) = \mathrm{Tr}[\rho A] \tag{18}$$

Definition (Faithful states)

A state ω in a von Neumann algebra $\mathcal{M}$ is faithful if for all $A \in \mathcal{M}$:

von Neumann algebras and Tomita-Takesaki

- We begin by studying the relation between $\mathcal{M}$ and its commutant $\mathcal{M}'$.

Theorem

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{R} \subset \mathcal{H}$ a subset. Then the two following conditions are equivalent:

1. *$\mathcal{R}$ is cyclic for $\mathcal{M}$.*
2. *$\mathcal{R}$ is separating for $\mathcal{M}'$*

- Said relationship is even stronger for σ-finite algebras. We have the following theorem:

Theorem

Let $\mathcal{M}$ be a von Neumann algebra, then the following conditions are equivalent:

1. *$\mathcal{M}$ is σ-finite.*
2. *There exists a faithful and normal state.*
3. *$\mathcal{M}$ is isomorphic to a von Neumann algebra $\pi(\mathcal{M})$ that admits a*

von Neumann algebras and Tomita-Takesaki

- Suppose there exists a cyclic and separating vector $x \in \pi(\mathcal{M})$. This allows us to define anti-linear operators in $\mathcal{M}$ and $\mathcal{M}'$:

$$S_0 A x = A^* x \tag{20}$$

$$F_0 A' x = A'^* x \tag{21}$$

- These operators are closable and through the polar decomposition we can find a anti-unitary operator called: conjugation operator J and an positive self-adjoint operator Δ such that:

$$S = \overline{S_0} = J\Delta^{1/2} \tag{22}$$

- Several relations hold for these operators:

$$\Delta = FS \tag{23}$$

$$\Delta^{-1} = SF \tag{24}$$

$$\Delta^{-\frac{1}{2}} = J\Delta^{\frac{1}{2}}\Delta \tag{25}$$

Tomita-Takesaki theorem:

Theorem (Tomita-Takesaki theorem (1967))

Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector x, let Δ be the associated modular operator and J the modular conjugation. It follows that:

$$J\mathcal{M}J = \mathcal{M}' \tag{26}$$

$$\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \tag{27}$$

- The modular operator gives rise to a uniparametric group of automorphisms of the algebra called the **modular group**.

$$\sigma_t^{\omega}(A) = \pi_{\omega}^{-1}(\Delta^{it}\pi_{\omega}(A)\Delta^{-it}) \tag{28}$$

- The modular group satifies the following property:

$$\omega(\sigma_{\frac{i}{2}}(A)\sigma_{-\frac{i}{2}}(B))) = \omega(BA) \tag{29}$$

Example: Bipartite system

- Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite system and consider the von Neumann algebra: $\mathcal{A} = \mathcal{L}(\mathcal{H}) \otimes I$ y $\mathcal{A}' = I \otimes \mathcal{L}(\mathcal{H})$. Then we can write our cyclic vector as:

$$x_\omega = \sum_{k=1}^{n} \lambda_k x_k \otimes y_k \ ; \ \lambda_k > 0 \tag{30}$$

- We can calculate the S and F operators:

$$S(x_j \otimes y_i) = \frac{\lambda_j}{\lambda_i} x_i \otimes y_j \tag{31}$$

$$S^*(x_j \otimes y_i) = \frac{\lambda_i}{\lambda_j} x_i \otimes y_j \tag{32}$$

$$\Delta(x_j \otimes y_i) = S^* S(x_j \otimes y_i) = \frac{|\lambda_j|^2}{|\lambda_i|^2} x_j \otimes y_i \tag{33}$$

Example: Bipartite system

- Let $x_i \otimes y_j \to |i\rangle \otimes |j\rangle'$
- If we introduce the following notation the density matrix given by the cyclic and separating state: $\rho = |x_\omega\rangle \langle x_\omega|$ and the reduced matrices:

$$\rho_A = \sum_i |\lambda_i|^2 |i\rangle \langle i| \tag{34}$$

$$\rho_B = \sum_i |\lambda_i|^2 |i\rangle' \langle i|' \tag{35}$$

- One finds that the modular operator it's just:

$$\Delta = \rho_A \otimes \rho_B^{-1} \tag{36}$$

- And the **modular group** is given by:

$$\Delta^{it}(A_{ij} \otimes I)\Delta^{-it} = \rho_A^{it} A_{ij} \rho_A^{-it} \otimes I \tag{37}$$

Applications of Tomita-Takesaki modular theory

- Similar to how we developed the modular operator we can define a relative modular operator via two cyclic and separating vectors ψ and ϕ. We can define $S_{0_{\phi,\psi}} A\phi = A^*\psi$ and from its closure:

$$\Delta_{\phi,\psi} = F_{\phi,\psi} S_{\phi,\psi} \tag{38}$$

Definition (Relative modular entropy)

Let ω_1 and ω_2 be two faithful normal positive linear functionals over a von Neumann algebra $\mathcal{M}$. Let ψ and ϕ be the two vector representatives in the natural positive cone $\mathcal{P}$. Then we define the relative entropy as:

$$S(\omega_1|\omega_2) = \int_0^\infty d(\psi, E(\lambda)\psi) \log \lambda \tag{39}$$

Where E is the spectral family of the relative modular operator:

$$\Delta_{\phi,\psi} = \int_0^\infty dE(\lambda) d\lambda \tag{40}$$

Applications of Tomita-Takesaki modular theory

- The modular group will allow us to generalize the concept of equilibrium with thermal entropy (Gibbs state).
- First note that for the Gibbs state:

$$\langle A \rangle_\rho = \frac{\mathrm{Tr}[e^{-\beta H} A]}{\mathrm{Tr}[e^{-\beta H}]} \tag{41}$$

$$\omega(A\tau_t(B))|_{t=i\beta} = \omega(BA) \tag{42}$$

$$\tau_t(A) = e^{itH} A^{-itH} \tag{43}$$

- This motivates the following definition:

Definition (KMS-state)

Let $(\mathcal{A}, \tau)$ be a C* dynamical system. The state ω over $\mathcal{A}$ is said to e a (τ, β)-KMS state if:

$$\omega(A\tau_{i\beta}(B)) = \omega(BA) \tag{44}$$

Modular group and KMS condition

Theorem (Takesaki)

Let $\mathcal{M}$ is a von Neumann algebra and ω a normal state on $\mathcal{M}$. The following conditions are equivalent:

1. *ω is faithful as a state on $\pi_\omega(\mathcal{M})$*
2. *There exists a σ-weakly continuous one-prameter group τ of automorhpisms of $\mathcal{M}$ such that ω is a τ-KMS state.*

Example: Causal Development of a Sphere

- Consider a sphere of dimension n and radius R on a Cauchy slice of Minkowski space at $t = 0$. Its causal development:

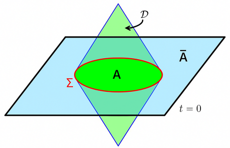


Figure: Causal development of a sphere of radius R and dimension n

$$ds^2 = \Omega^2 \left(-d\tau^2 + du^2 + \sinh^2 u \, d\Omega_{d-2}^2\right)$$

- Ignoring the factor Ω^2, the space corresponds to a **hyperbolic cylinder**.

Conformal Transformations

- A **conformal transformation** is a diffeomorphism that relates two metrics by a factor Ω^2, called the *conformal factor*:

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x).$$

- These transformations form the **conformal group**, which includes the Poincaré group along with dilations and special conformal transformations.

$$X^\mu \to \frac{X^\mu - b^\mu (X \cdot X)}{1 - 2b \cdot X + b^2 (X \cdot X)}$$

- A **conformal field theory** (CFT) is a theory whose action is invariant under this group. For a CFT, we can map $\mathcal{D}(A)$ to $\mathbb{R} \times \mathbb{H}^n$:

$$ds^2 = \Omega^2 \left(-d\tau^2 + du^2 + \sinh^2 u \, d\Omega_{d-2}^2 \right)$$

AdS/CFT Correspondence

- Originally developed by Maldacena and Witten in the context of string theory, the AdS/CFT correspondence establishes a dictionary between:
  - A gravity theory in an asymptotically $\mathrm{AdS} \times M$ space, where M is a compact manifold.
  - A conformal field theory (CFT), which is a gauge theory with a large number of degrees of freedom and a broad spectrum at low energies.
- Formally, the correspondence equates the partition functions:

$$Z_{CFT}[g, J] = Z_{gravity}[g, J]$$

- There's a relationship between the fields in the AdS and the operators in the CFT.

$$\lim_{r \to \infty} r^{\Delta} \phi_i(r, t, \Omega) = \mathcal{O}_i(t, \Omega) \tag{45}$$

Ryu–Takayanagi Formula

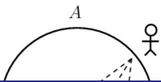
- The Bekenstein–Hawking formula relates the entropy of a black hole to the area of its event horizon.

$$S = \frac{A}{4G} \tag{46}$$

- The entanglement entropy of a subregion of the CFT can be computed as:

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_n^{d+2}}$$

- The entropy is related to the area of a minimal surface whose boundary coincides with that of region A. In the case of AdS_3, this corresponds to the area enclosed by a geodesic whose endpoints match those of A.



Properties of Conformal Transformations and Thermality

- The **Rindler wedge**, or causal development of the right half-plane, is given by:

$$W : \left\{ X^{\pm} \geq 0 \quad ; \quad X^{\pm} \equiv X^1 \pm X^0 \right\}$$
$$X^1 = e^u \cosh(\tau) \qquad ; \qquad X^0 = e^u \sinh(\tau)$$

Theorem

Bisognano-Wichmann *states that the Minkowski vacuum $|\Omega\rangle$, when restricted to the Rindler wedge, is a thermal state with respect to ∂_τ.*

Conformal Transformation between $\mathcal{D}(A)$ and the Rindler Wedge

- **Special conformal transformations** the Rindler wedge to the causal development $\mathcal{D}(A)$.
- Myers, Casini and Huertas showed that under this map the thermal state with respect to Rindler time translations H_τ is mapped to a thermal state in $\mathcal{D}(A)$, with $\beta = 2\pi R$.

Conformal Mapping of CFT Operators

- Primary operators in a CFT transform locally under conformal maps as:

$$\phi(x) = \Omega(X)^{\Delta}\, U_0\, \phi(X)\, U_0^{-1},$$

where Δ is the scaling dimension of the field.

- Practically, vacuum correlators on region $\mathcal{R}$ are mapped to vacuum correlators on $\mathcal{D}$:

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \Omega(X_1)^{\Delta_1} \cdots \Omega(X_n)^{\Delta_n} \langle \phi_1(X_1) \cdots \phi_n(X_n) \rangle.$$

- The KMS condition is interpreted as the following periodicity:

$$\mathcal{O}(s) = U(s)\mathcal{O}U(-s)$$
$$U(t) = \rho^{it} := \Delta^{it}$$
$$\langle \mathcal{O}_1(i)\mathcal{O}_2 \rangle = \langle \mathcal{O}_2\mathcal{O}_1 \rangle$$

- The thermal character is carried from the Rindler wedge to $\mathcal{D}(A)$ and to $\mathbb{R} \times \mathbb{H}^{d-1}$

Error Correction Codes

- We encode Alice's message $|s_1 s_2 ... s_k\rangle$ into the code subspace $\mathcal{H} = \mathcal{H}_{code} \otimes \mathcal{H}_B$ as $|s_1 s_2 ... s_k \; 00...0\rangle$.
- Errors $\mathcal{E}$ are described by quantum channels (TPCP maps) that act nontrivially on l qubits of the message.
- An error is said to be correctable if there exists another quantum channel $\mathcal{R}$ along with an auxiliary space $\mathcal{H}_A$ such that: $\mathcal{R} \circ \mathcal{E} = I_k \otimes M_{BA}$.
- There are several characterizations; in particular, we say a code $|\tilde{i}\rangle$ is correctable if its density matrix $\rho = |\tilde{i}\rangle \langle \tilde{i}|$ factorizes as:

$$\rho_{\mathcal{H}} = \rho_{\mathcal{H}_{code}} \otimes \rho_{\mathcal{H}_B} \tag{47}$$

- This characterization establishes an equivalence between having an expression analogous to RT and the ability to correct errors.

Qutrit Code

- Suppose Alice wants to send a message $|\psi\rangle \in \mathbb{C}^3$ to Bob. $|\psi\rangle = \sum_{i=1}^{3} C_i |i\rangle$. We can encode it through the following isometries:

$$|0\rangle \longrightarrow |\tilde{0}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$$

$$|1\rangle \longrightarrow |\tilde{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle)$$

$$|2\rangle \longrightarrow |\tilde{2}\rangle = \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle)$$

- We can recover it using unitaries U_{ij} that act nontrivially on two qutrits.

$$(U_{12} \otimes I_3)|\tilde{i}\rangle = |i\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \tag{48}$$

- Considering locality, the behavior of the qutrit code is analogous to the reconstruction of bulk fields ϕ_i from operators in the CFT.

Quantum Error Correction and Holographic Reconstruction

In the AdS/CFT context, quantum error correction explains how bulk information is encoded in the boundary theory.

- The **code subspace** $\mathcal{H}_{\text{code}}$ is generated by the action of CFT operators $\mathcal{O}_i$ on the vacuum:

$$\mathcal{H}_{\text{code}} = \text{span}\{\mathcal{O}_i \left|\Omega\right\rangle\}.$$

- Bulk fields $\phi_i(x)$ are represented on the boundary by operators $\mathcal{O}_i$, encoding the bulk degrees of freedom.
- Although the CFT has fewer spatial dimensions, it contains more degrees of freedom and acts as the complete physical system.

Operator Reconstruction and Error Correction

Consider a partition of the boundary into A and $\overline{A}$: $\mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$.

- If errors on $\overline{A}$ can be corrected, then bulk operators can be represented using only region A:

$$\tilde{\mathcal{O}}_A \left|\psi\right\rangle = \mathcal{O} \left|\psi\right\rangle , \quad \forall \left|\psi\right\rangle \in \mathcal{H}_{\text{code}}.$$

- This property reflects the **local holographic reconstruction**: bulk operators in the causal wedge of A have equivalent representations in the CFT restricted to A.
- The holographic code ensures that bulk information is protected against the loss of part of the boundary.

Harlow's Theorem: Algebraic Structure of RT

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$ and a code subspace $\mathcal{H}_{\text{code}}$ with von Neumann algebra M. Then, the following statements are equivalent:

1. **Algebraic Ryu–Takayanagi formula:**

$$S(\tilde{\rho}_A) = \mathrm{Tr}(\tilde{\rho}\mathcal{L}_A) + S(\tilde{\rho}, M),$$

   where $\mathcal{L}_A$ acts as the "area operator".

2. **Subregion duality/Error correction** Every bulk observable has a representation in both A and $\overline{A}$.
3. **Equality of relative entropies (JLMS formula):**

$$S(\tilde{\rho}_A \| \tilde{\sigma}_A) = S(\tilde{\rho}, \tilde{\sigma}, M).$$