On the dimension of harmonic/elliptic measures

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Elliptic partial differential equations

- Let $A = (a_{ij})_{1\le i,j\le n+1}$ be an $(n+1)\times(n+1)$ matrix whose entries $a_{ij} : \mathbb{R}^{n+1} \to \mathbb{R}$ are measurable functions in $L^\infty(\mathbb{R}^{n+1})$.
- Suppose the matrix is **elliptic**, i.e., there exists $\lambda \ge 1$ such that for all $\xi \in \mathbb{R}^{n+1}$ and a.e. $x \in \mathbb{R}^{n+1}$,

$$\lambda^{-1}|\xi|^2 \le \langle A(x)\xi, \xi\rangle, \quad \langle A(x)\xi, \eta\rangle \le \lambda|\xi||\eta|.$$

- We study the divergence form PDE $L_A u = -\operatorname{div}(A\nabla u)$.
- A function u is said to be L_A-harmonic in an open set U if

$$\int A\nabla u \nabla\varphi = 0 \quad \text{for every } \varphi \in C_0^\infty(U).$$

- Suppose also that the coefficients of A are **Lipschitz**, i.e.,

$$|a_{ij}(x) - a_{ij}(y)| \le C_L|x-y|$$

- When $A = Id$ the operator is the Laplacian, $L_{Id} = -\Delta$, and it is said harmonic instead of L_{Id}-harmonic.

Definition of the elliptic measure

- Let Ω be a regular domain for the L_A-Dirichlet problem, also called Wiener regular.
- Given a function $g \in C(\partial\Omega)$, let u_g be the L_A-harmonic extension of g on Ω.
- Fix a point $p \in \Omega$. The operator $T : C(\partial\Omega) \to \mathbb{R}$, defined as $T(g) = u_g(p)$, is linear, bounded and positive.
- By the Riesz representation theorem there exists a unique Radon measure ω_Ω^p with total mass 1 such that

$$u_g(p) = \int_{\partial\Omega} g(\xi)\, d\omega_\Omega^p(\xi) \quad \text{for every } g \in C(\partial\Omega).$$

The dimension of the harmonic measure

Some known results on the dimension of the harmonic measure, i.e., on

$$\dim_{\mathcal{H}} \omega_\Omega^p := \inf\{\dim_{\mathcal{H}} F : \omega_\Omega^p(F^c) = 0\}.$$

In the plane, i.e., when $\Omega \subset \mathbb{R}^2$:

- [JW88] There exists a subset $F \subset \partial\Omega$ with $\omega_\Omega^p(F) = 1$ and $\dim_{\mathcal{H}} F \le 1$ for every $p \in \Omega$.
- [Wol93] There exists a subset $F \subset \partial\Omega$ with $\omega_\Omega^p(F) = 1$ and with σ-finite $\mathcal{H}^1$ measure for every $p \in \Omega$.

In higher dimensions, $\Omega \subset \mathbb{R}^{n+1}$ with $n \ge 2$, the behaviour of the harmonic measure is different:

- [Bou87] There exists $0 < b_n << 1$ such that $\dim_{\mathcal{H}} \omega_\Omega^p \le n + 1 - b_n$, with b_n depending only on dimension of the space, i.e., independent on the set Ω.
  - The optimal value of b_n is only known when $n = 1$ with $b_1 = 1$.
- [Wol95] There exists an open set $\Omega_n \subset \mathbb{R}^{n+1}$ such that $\dim_{\mathcal{H}} \omega_{\Omega_n}^p > n$.
  - It is not longer true in the plane by the results in [JW88] and [Wol93].

Counterexample to general planar elliptic measures

If we do not requiere any regularity on the coefficients of the matrix A, then the analogous results of [JW88] and [Wol93] in the plane are no longer true:

- [Swe92] For any $\varepsilon > 0$ one can construct a set and an elliptic operator in divergence form whose associated measure has support of Hausdorff dimension $2 - \varepsilon$.
  - From this we conclude that in this setting there is no optimal value b_1 in the plane for the Bourgain's result [Bou87].

Analogous results for p-harmonic measures

- A function u is said to be p-harmonic if it is solution to the operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$.
- Similar results about the dimension of the analogous p-harmonic measure are known **only** (as far as I know) for Reifenberg flat sets with small constant, or simply connected domains.

Reifenberg flat sets

- Let $\Omega \subset \mathbb{R}^{n+1}$ $(n+1 \ge 1)$ be an open set, and let $0 < \delta < 1/2$, $r_0 > 0$. We say that Ω is a (δ, r_0)–**Reifenberg flat** domain if:

(a) For every $x \in \partial\Omega$ and every $0 < r \le r_0$, there exists a hyperplane $\mathcal{P}(x,r)$ containing x such that

$$\operatorname{dist}_{\mathcal{H}}(\partial\Omega \cap B(x,r), \mathcal{P}(x,r) \cap B(x,r)) \le \delta r,$$

(b) and for every $x \in \partial\Omega$ and every $0 < r \le r_0$, one of the connected components of

$$B(x,r) \cap \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, \mathcal{P}(x,r)) \ge 2\delta r\}$$

is contained in Ω and the other is contained in $\mathbb{R}^{n+1} \setminus \Omega$.

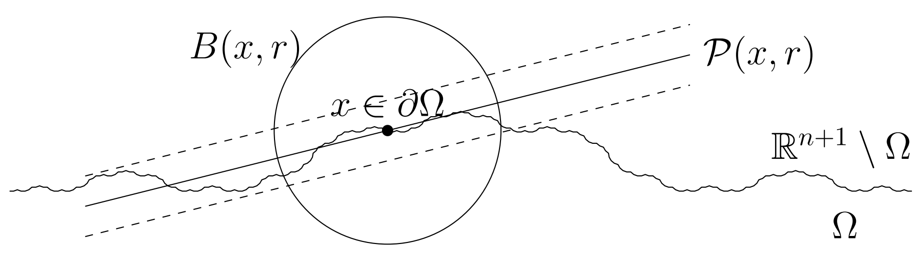

Figure 1. Reifeberg flat domain example.

Theorem (Work in progress)

Let A be a real elliptic Lipschitz (not necessarily symmetric) matrix. Let $\Omega \subset \mathbb{C}^*$ be (δ, r_0)–Reifenberg flat (not necessarily bounded) with bounded boundary $\partial\Omega$, and $p \in \Omega$.

Then there exists $\delta_0 = \delta_0(\lambda)$ such that if $0 < \delta \le \delta_0$ then there is a set $F \subset \partial\Omega$ satisfying $\omega_\Omega^p(F) = 1$ and with σ–finite one–dimensional Hausdorff measure. In particular $\dim_{\mathcal{H}} F \le 1$, and hence $\dim_{\mathcal{H}} \omega_\Omega^p \le 1$.

- It is the analogue of [Wol93] for the elliptic case, whenever the matrix is Lipschitz and the set is Reifenberg flat with small constant.

– A similar proof in [Wol93] is used to obtain Theorem from the following Main Lemma.

Main Lemma (Work in progress)

Let A be a real elliptic Lipschitz (not necessarily symmetric) matrix, and set $0 < r \le 1$ small enough such that $r \cdot C_L \|A\|_{L^\infty(\mathbb{R}^2)} \le 1$. Let $\Omega \subset \mathbb{C}^*$ be (δ, r_0)–Reifenberg flat (not necessarily bounded) with bounded boundary $\partial\Omega$, and let $p \in \Omega$ such that $\operatorname{dist}(p, \partial\Omega) > r_0$. Then there exists $\delta_0 = \delta_0(\lambda)$ such that if $0 < \delta \le \delta_0$, then we have the following:

For any $0 < \tau < 1$, sufficiently large M (big enough) and ρ such that $0 < \rho/r < 1/M$ there is a set $F \subset \partial\Omega$ such that $\omega_\Omega^p(F) \ge C^{-1}\tau$ and with a covering $F \subset \bigcup_i B(z_i, r_i)$ where

$$\sum_i r_i \le CM^\tau, \quad \sum_{i: r_i > \rho} r_i \le CM^{-1},$$

with universal constant $C \ge 1$.

Preliminar reductions (Main Lemma)

1. Rescaling it suffices to prove for $C_L \le 1$.
2. By means of a diagonal linear deformation we can reduce to the case that the symmetric part $A_0 := \frac{A+A^T}{2}$ is of the form $A_0 = RBR$, where R is a Lipschitz rotation, and B is a Lipschitz diagonal matrix.

Sketch of the proof [1/2] (Main Lemma)

It is based on the proof in [Wol93] for the capacity density condition (CDC) case.

1. Select balls centered on $\partial\Omega$ with 'high density' scaling in a ε–Lipschitz way.
2. Modify the domain Ω by removing these balls and obtain a new domain $\widetilde{\Omega}$, and its elliptic measure $\widetilde{\omega}$.
3. The new domain is smooth except at finitely many points.
4. The new domain $\widetilde{\Omega}$ is Reifenberg flat with small constant, provided the initial domain has enough small Reifenberg flat constant.

Sketch of the proof [2/2] (Main Lemma)

5. Relation between the final and inital elliptic measure. *We want to transfer information from the final measure and domain, to the initial setting.*
6. Every point has not excessive density for the new elliptic measure with respect to lenght.
7. *The most delicate step*: If the set is Reifenberg flat with **small enough constant**,

$$-\infty < C < \int_{\partial\widetilde{\Omega}} \frac{d\widetilde{\omega}}{d\sigma}(\xi) \log \frac{d\widetilde{\omega}}{d\sigma}(\xi)\, d\sigma(\xi).$$

8. From the last two steps we can find a collection of balls satisfying the conditions in Main Lemma.

Reifenberg flat with dimension larger than 1

- Despite we are requiring in Main Lemma and Theorem to work with **Reifenberg flat domains with small enough constant**, one can construct such sets with Hausdorff dimension strictly largen than 1.
- **Example:** Construct the analogue of the Koch snowflake with small angle, as small as desired.

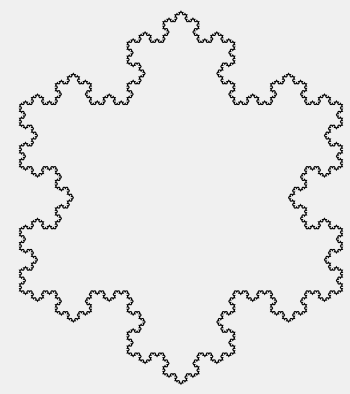

Figure 2. $\frac{\pi}{3}$–Koch snowflake.

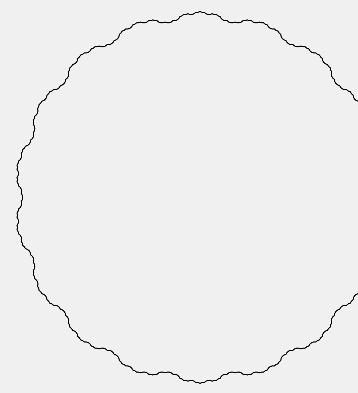

Figure 3. $\frac{\pi}{9}$–Koch snowflake.

- The smaller the angle, the smaller the δ–Reifenberg flat constant. See Figure 1.

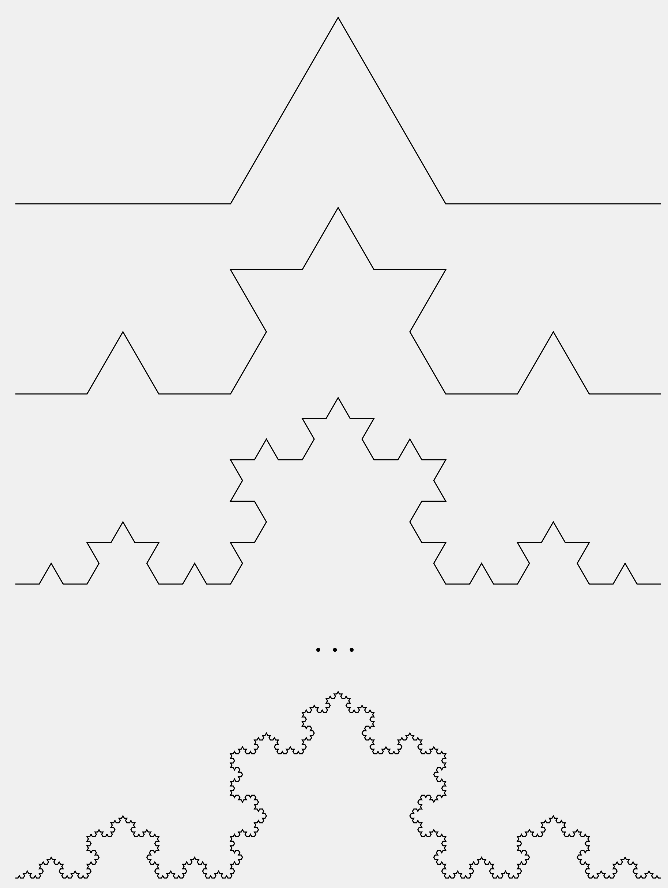

Figure 4. Side construction of the $\frac{\pi}{3}$–Koch snowflake.

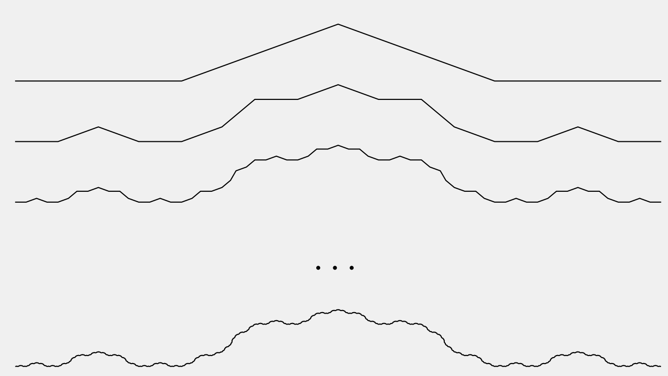

Figure 5. Side construction of the $\frac{\pi}{9}$–Koch snowflake.

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