A DECOMPOSITION FOR PLANE DOMAINS WITH THE QUASIHYPERBOLIC METRIC

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Abstract. It is known that complete Riemannian surfaces can be obtained by pasting three kinds of pieces. In this paper we prove an analogous result in the context of plane domains with their quasihyperbolic metrics. In order to do it, we prove several facts about quasihyperbolic closed geodesics of independent interest; for instance, we characterize the existence of closed geodesics, and we show that they have finite topology.

1. Introduction

A domain is an open connected set $\Omega \subsetneq \mathbb{R}^n$. Given a rectifiable curve $\gamma \subset \Omega$, its quasihyperbolic length is the length induced by the density $1/\delta_\Omega(x)$, with $\delta_\Omega(x) = d_{\mathbb{R}^n}(x, \partial\Omega) = d_{\mathbb{R}^n}(x, \Omega^c)$, i.e.,

$$L_\Omega(\gamma) = \int_\gamma \frac{ds}{\delta_\Omega(x)}\,,$$

where ds is the differential of the Euclidean arclength. The quasihyperbolic distance in Ω, denoted by d_Ω, is the distance induced by L_Ω, i.e.,

$$d_\Omega(x_1, x_2) = \inf \big\{ L_\Omega(\gamma) : \, \gamma \text{ is a curve joining } x_1 \text{ and } x_2 \text{ in } \Omega \big\}.$$

The quasihyperbolic metric of a domain in $\mathbb{R}^n$ was introduced by Gehring and Palka [6] in 1976, and it has turned out to be a useful tool, for example, in harmonic analysis and many subfields of geometric function theory, for instance, in the study of quasiconformal maps of $\mathbb{R}^n$ and of Banach spaces [14], analysis of metric spaces [9] and hyperbolic type metric [8]. Also, there is quite a strong relationship between uniform domains and the quasihyperbolic metric. Most of the basic results on the quasihyperbolic metric can be found in [6], [5] and [10].

However, there are just a few papers studying the geometric properties of this metric. It is known [5, Lemma 1] that a quasihyperbolic geodesic between given points always exists. G. Martin [10, Corollary 4.8] proved in 1985 that quasihyperbolic geodesics are C^1 smooth with Lipschitz continuous derivatives, Väisälä [13] showed that if $d_\Omega(x_1, x_2) < 2$ then the geodesic arc joining x_1 to x_2 is unique. The proofs required new ideas, since even in domains with smooth boundary the density $1/\delta_\Omega(x)$ need not be differentiable.

The celebrated Classification Theorem of compact surfaces says that every orientable compact topological surface is homeomorphic either to a sphere or to a sphere with handles attached, see e.g. [11]. In the Riemannian setting one has the following: *Every orientable compact Riemannian surface which is homeomorphic neither to the sphere nor to the torus can be obtained by gluing Y-pieces along their bounding geodesics.*

A *halfplane* is a simply connected open subset of the hyperbolic plane $\mathbb{H}$ whose boundary is a unique geodesic line. A *generalized Y-piece* is either a Y-piece or a Y-piece for which one or several boundary geodesics are replaced by punctures. In [2] the following result was obtained

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Theorem A. ([2, Theorem 1.2]) *Every complete orientable Riemannian surface with constant curvature $K = -1$, which is not the punctured disc, is the union (with pairwise disjoint interiors) of generalized Y-pieces, funnels and halfplanes.*

The fact that the boundaries of generalized Y-pieces are simple closed geodesics facilitates the applications of Theorem A, since cutting and pasting surfaces along such type of curves is easy.

The present paper deals with plane domains with their quasi-hyperbolic metric. Since that metric is only Lipschitz, the behavior of closed geodesics, their existence, and their uniqueness, are not trivial to study.

Theorem 4.3, the main result in this paper, is a decomposition for plane domains with their quasihyperbolic metric. It is analogous to Theorem A quoted above. This decomposition is made possible by other crucial results. In Section 3 we determine when a general kind of closed geodesics, called limit geodesics, do exist. In Section 5 we show that closed geodesics have finite topology; this is specially valuable, as Section 6 exhibits examples of quite unexpected behavior by quasihyperbolic geodesics: two distinct quasihyperbolic geodesics may be tangent at a point; further, a *sliding* occurs when such geodesics have proper segments (of positive length) that coincide for a while; *self-slidings* of a single quasihyperbolic geodesic also occur.

Just as half-planes are needed in Theorem A, simply connected open sets are a must in Theorem 4.3; this is made obvious by Examples 6.2 and 6.4.

2. Background

Let Ω be any domain in $\mathbb{C}$, endowed with its quasihyperbolic metric. For a closed set $A \subset \Omega$ and $r > 0$, we define $\overline{B}(A, r) := \{ z \in \Omega \,:\, d_\Omega(p, z) \le r \text{ for all } p \in A \}$, which is also closed. For closed $A, A' \subset \Omega$, the *Hausdorff distance* between them is

$$d_H(A, A') \;=\; \inf \{ r > 0 \,:\, A' \subseteq \overline{B}(A, r) \text{ and } A \subseteq \overline{B}(A', r) \} \,.$$

In the proof of Theorem 4.3 we use the following result, a consequence of the proof of [3, Theorem 4.2].

Theorem B. *Let Ω be any domain in $\mathbb{C}$, endowed with its quasi-hyperbolic metric and E an end of Ω. Then E is a collared end if and only if there exists a sequence $\{\alpha_n\}$ of closed curves converging to E and representing a single non-trivial free homotopy class.*

Quasihyperbolic geodesics are usually defined by minimization ot length among all paths with the same pair of endpoints, thus forcing them to be arcs. Instead, it is useful for our purposes to require minimization of length within a given free homotopy class of *closed* curves.

Definition 2.1. *Let Ω be any domain in $\mathbb{C}$, endowed with its quasi-hyperbolic metric, and let $[\alpha_0]$ be a non-trivial free homotopy class in this domain. We say that a closed curve $\alpha \in [\alpha_0]$ is a* closed geodesic *in Ω, or that it is a* minimizer *for the class $[\alpha_0]$, if $L_\Omega(\alpha) = \inf \{ L_\Omega(\sigma) \,:\, \sigma \in [\alpha_0] \}$. We do not consider constant paths to be closed geodesics.*

G. Martin [10, Corollary 4.8] has shown that quasihyperbolic geodesics are $C^{1,1}$, i.e., they are C^1 and their first derivatives are Lipschitz. Same is true for closed geodesics.

3. Limit geodesics

In the quasihyperbolic metric, a Jordan curve may be homotopic to a minimizing geodesic that is not a Jordan curve. However, we are going to see that the minimizer can be chosen *without self-crossings*, and also that when two Jordan curves are disjoint their chosen minimizers *do not cross.* We define the absence of crossings and self-crossings by requiring that the configuration (one or two geodesics) can be turned into an embedded one by arbitrarily small perturbations. We restrict attention to quasihyperbolic geodesics because, as stated in Theorem 5.1, they make up configurations of finite complexity.

Definition 3.1. *Let Ω be any domain in $\mathbb{C}$ endowed with its quasi-hyperbolic metric. A* limit geodesic *is a closed quasihyperbolic geodesic that is a uniform limit of simple closed paths. We say that two limit geodesics* do not cross *if the configuration of the two is a uniform limit of pairs of disjoint simple closed paths.*

The *interior* of a Jordan curve $\alpha \subset \mathbb{C}$ is the bounded connected component $\mathrm{Int}(\alpha)$ of $\mathbb{C} \setminus \alpha$, while the *exterior of* α is the unbounded component. The meanings of the expressions 'lies interior to α', 'is surrounded by α', and 'lies exterior to α' are the obvious ones. A point p interior to α has rotation index $i(\alpha, p) = \pm 1$; an exterior point q has $i(\alpha, q) = 0$.

Theorem 3.2. *Let Ω be any domain in $\mathbb{C}$, endowed with its quasihyperbolic metric and let $[\alpha_0]$ be a non-trivial homotopy class represented by a Jordan curve $\alpha_0 \subset \Omega$. This class contains at least one quasihyperbolic minimizer, except in one special case: when Ω^c consists of more than one point and is all surrounded by α_0.*

It is possible to choose, for each such class not in the special case, a limit geodesic as minimizer, in such a way that whenever two classes are distinct, and represented by disjoint Jordan curves, the chosen limit geodesic minimizers do not cross.

Proof. We consider three mutually excluding cases.
Case 1: Ω^c has at least two points and is all surrounded by α_0. Choose two points $z_0, z_1 \in \Omega^c$, and let $\gamma(t) = z_0 + r(t)e^{i\theta(t)} \subset \Omega$ be any counterclockwise Jordan curve that surrounds Ω^c, then:

$$L_\Omega(\gamma) = \int_\gamma \frac{|dz|}{\delta_\Omega(z)} \geq \int_\gamma \frac{|dz|}{|z - z_0|} = \int_0^1 \left| \frac{r'(t)}{r(t)} + i\theta'(t) \right| dt \geq \int_0^1 \theta'(t)\, dt = 2\pi\ . \tag{3.1}$$

The value 2π is a lower bound for the quasihyperbolic length of all curves in this homotopy class. It is in fact the infimum, as we see by considering circles centered at z_0 with radius going to infinity. Of the two inequalities in formula (3.1), the second one is strict unless $r'(t) \equiv 0$. The curve γ thus has $L_\Omega(\gamma) > 2\pi$ if it is not a circle centered at z_0. Likewise it has $L_\Omega(\gamma) > 2\pi$ if it is not a circle centered at z_1, but a circle cannot be centered at both points simultaneously. In this case, thus, there is no minimizer.

Case 2: $\Omega^c \cap \mathrm{Int}(\alpha_0)$ consists of exactly one point p. Near the point p we have $\delta_\Omega(z) = |z - p| = r$. The quasihyperbolic metric of Ω is, in this neighborhood, same as in the product cylinder $[0, \infty) \times S^1$, where the coordinate along the $[0, \infty)$ factor is $\log r$ and the circle factor has length 2π. The circles $r = \text{constant}$ are the minimizers.

Case 3: Ω^c contains at least two points interior to α_0 and at least one point exterior to α_0. We can choose distinct points $p_1, p_2, q \in \Omega^c$, with p_1, p_2 lying interior to α_0 and q exterior to α_0, plus a path $\beta(t) : [0, 1] \to \mathbb{C}$ with $q = \beta(0)$, $p_1 = \beta(1)$, and $\beta\big((0, 1)\big) \subset \Omega$. Let D be a closed Euclidean disc centered at q and disjoint from $\{p_1, p_2\}$. Let D_1 be a closed Euclidean disc centered at p_1 and disjoint from $\{p_2\} \cup D$.

Let $\alpha \in [\alpha_0]$ be any curve with $L_\Omega(\alpha) \leq L_\Omega(\alpha_0) + 1$. Since $i(\alpha, q) = 0$ and $i(\alpha, p_1) = \pm 1$, the path β must intersect α. There is a point $p_\alpha \in \alpha$ outside $D \cup D_1$, otherwise we would have $\alpha \subset D$ or $\alpha \subset D_1$, and in both cases $i(\alpha, p_2) = 0$, which is false. Starting at p_α, we cannot go along α and get very close to p_1 or to q, because this would force $L_\Omega(\alpha) > L_\Omega(\alpha_0) + 1$. All this provides a compact arc $A_0 \subset \beta$ intersected by all this curves α which, in turn, are all contained in the compact set $A = \overline{B}\big(\, A_0\, ,\, L_\Omega(\alpha_0) + 1\, \big) \subset \Omega$.

Now we approximate the quasihyperbolic metric $\lambda_0\, |dz|^2$ by Riemannian metrics $\lambda\, |dz|^2$ near the compact set A. Each of these metrics satisfies $m_\lambda^2 \lambda_0 \leq \lambda \leq M_\lambda^2 \lambda_0$, with m_λ, M_λ as close to 1 as we please. Minimize Riemannian length among those closed paths $\alpha \in [\alpha_0]$ whose Riemannian length is at most that of α_0. The quasihyperbolic length of those paths satisfies the inequality $L_\Omega(\alpha) \leq (M_\lambda / m_\lambda) \cdot L_\Omega(\alpha_0)$ and, for m_λ, M_λ close enough to 1, we have $L_\Omega(\alpha) \leq 1 + L_\Omega(\alpha_0)$. Then all those paths are contained in the fixed compact set A and Arzelá-Ascoli provides, for each metric $\lambda |dz|^2$, a closed minimizer $\alpha_\lambda \in [\alpha_0]$. By the results in [4], the Riemannian geodesics α_λ are smooth Jordan curves.

Among the Riemannian metrics $\lambda |dz|^2$ select a sequence $\{\lambda_n |dz|^2\}$ converging uniformly on A to the quasihyperbolic metric. Get a corresponding sequence $\{\alpha_n\}$ of geodesics all contained in A and each minimizing length in $[\alpha_0]$ for the respective metric $\lambda_n |dz|^2$. A second use of Arzelá-Ascoli yields a subsequence $\{\alpha_k\}$ converging uniformly to a quasihyperbolic minimizer α_∞ for the class $[\alpha_0]$. Now α_∞ is a limit geodesic, because it is approximated by the Jordan curves α_k. This completes the analysis of the three cases.

For each limit geodesic α_∞, constructed as the uniform limit of a sequence of Jordan curves α_k, we define the *limit interior* as the limit (in Hausdorff distance) of the compact sets $\overline{\mathrm{Int}(\alpha_k)}$. This set has finite complexity by Theorem 5.1.

Suppose now that $\alpha, \beta \subset \Omega$ are two disjoint Jordan curves, with non-trivial homotopy classes $[\alpha] \neq [\beta]$ neither of which is in Case 1. If α surrounds β, then the limit interior of the limit geodesic chosen for $[\alpha]$

contains the limit geodesic chosen for $[\beta]$, hence they do not cross. If α and β have disjoint interiors; then the chosen limit geodesics for $[\alpha]$ and $[\beta]$ have non-overlapping limit interiors and, again, do not cross. □

4. Structure theorem

Definition 4.1. *Given any domain Ω in $\mathbb{C}$, endowed with its quasi-hyperbolic metric, we define:*

A funnel *is the closed subset of Ω placed between a limit geodesic and a connected component of $\overline{\mathbb{C}} \setminus \Omega = \Omega^c \cup \{\infty\}$ with more than one point.*

A puncture *in Ω is the closed subset of Ω lying between a simple closed geodesic with length 2π and an isolated point in Ω^c.*

The puncture at infinity of Ω *is (if it exists) the collared end in Ω defined by the inclusion $\mathbb{C} \setminus D \subset \Omega$, where D is a suitable closed disc.*

A geodesic domain G *is a closed subset of Ω, neither simply nor doubly connected, limited by finitely many non-crossing limit geodesics. It may contain the puncture at infinity but neither funnels nor punctures.*

Generalizing the classical definition with negative curvature to our context, we call Y-piece *a compact geodesic domain limited by three limit geodesics. Likewise we call* exterior Y-piece *a non-compact geodesic domain that contains the puncture at infinity and is limited by two limit geodesics.*

The domain Ω has a (unique) puncture at infinity if and only if Ω^c is compact. According to Theorem 3.2, this type of collared end is the only one with no closed geodesic in its homotopy class.

An exterior Y-piece can be cut by a Jordan curve into a "topological Y-piece" and a puncture at infinity, but the cutting curve can never be a closed geodesic.

Proposition 4.2. *Let Ω be any domain in $\mathbb{C}$, endowed with its quasi-hyperbolic metric. Every geodesic domain $G \subset \Omega$ is a finite union (with pairwise disjoint interiors) of Y-pieces and, at most, an exterior Y-piece. Furthermore, the exterior Y-piece appears in this union if and only if Ω^c is a compact set and the geodesic domain contains a neighborhood of infinity in Ω.*

Proof. We denote by $\gamma_1, \gamma_2, \dots, \gamma_k$ the limit geodesics in ∂G. We can choose pairwise disjoint Jordan curves $g_1, g_2, \dots, g_k$ in Ω such that $g_j \in [\gamma_j]$ for each j. Let G' be the closed region with boundary $g_1 \cup \cdots \cup g_k$. Topologically G' is a disc with $k-1$ holes (or the complex plane with k holes) and we can cut it into finitely many topological Y-pieces $Y_1, \dots, Y_s$ (and maybe also an exterior Y-piece Y_0). We consider the set $\{g_1, \dots, g_k, \eta_1, \dots, \eta_h\}$ of pairwise disjoint Jordan curves in $\cup_n \partial Y_n$ and modify it in the following way. For $i = 1, \dots, k$ replace g_i with γ_i. For $j = 1, \dots, h$ choose a limit geodesic $\gamma_{k+j} \in [\eta_j]$; this geodesic exists because η_j separates two pieces none of which is a puncture at infinity. By Theorem 3.2, the limit geodesics $\gamma_1, \dots, \gamma_{k+h}$ do not cross, therefore $\gamma_{k+1}, \dots, \gamma_{k+h}$ lie inside G and in fact split it into the required finite union of Y-pieces and, perhaps, one exterior Y-piece in addition. □

Theorem 4.3. *For each domain Ω in $\mathbb{C}$, endowed with its quasi-hyperbolic metric, which is neither simply nor doubly connected, there exists a set $H \subseteq \Omega$ made of Y-pieces, funnels, punctures and, at most, an exterior Y-piece, all glued together by sharing boundary geodesics, in such a way that Ω is the union of the closure of H and simply connected open sets. Furthermore, the exterior Y-piece appears in this decomposition if and only if Ω^c is a compact set.*

Assume first that the fundamental group of Ω is finitely generated. Thus, $\overline{\mathbb{C}} \setminus \Omega$ has just a finite number of connected components $C_0, C_1, \dots, C_k$, with $\infty \in C_0$. Notice that $k \geq 1$, because $\pi_1(\Omega)$ has at least two generators. For each $1 \leq j \leq k$, let F_j be a funnel or puncture in Ω such that C_j is contained in the interior of the limit geodesic ∂F_j. If Ω^c is not compact, i.e. $C_0 \neq \{\infty\}$, let F_0 be the funnel in Ω between C_0 and a limit geodesic ∂F_0 that separates $C_1, \dots, C_k$ from C_0. Then the closure of $\Omega \setminus \cup_{j=1}^n F_j$ (if Ω^c is compact) or $\Omega \setminus \cup_{j=0}^n F_j$ (if Ω^c is not compact) is a geodesic domain, and Proposition 4.2 gives the result in this case.

Assume now that Ω has infinitely generated fundamental group. The proof in this case will take up the rest of this Section, including proofs of some lemmas and propositions. Fix a point $p_0 \in \Omega$. Since Ω^c has more than one point, we can consider the Poincaré metric ρ in Ω, a complete Riemannian metric with constant curvature -1. As ρ is real analytic, the boundary of the Poincaré ball $B_\rho(r)$ centered at p_0 is a finite union of pairwise disjoint Jordan curves (see [7, Theorem 1.2]) except for $r \in \mathcal{R}$ where $\mathcal{R}$ is some countable set of numbers. Start with $r_1 \notin \mathcal{R}$ such that the fundamental group of the ball $B_\rho(r_1)$ induces a subgroup of $\pi_1(\Omega)$ with at least two generators. Inductively, once r_{n-1} has been chosen we take $r_n \notin \mathcal{R}$ with $r_n > \max\{r_{n-1}, n\}$. Each $B_n = B_\rho(r_n)$ induces a non-cyclic subgroup of $\pi_1(\Omega)$ and has boundary made of finitely many pairwise disjoint Jordan curves.

Call a boundary component of ∂B_n *inessential* if it is contractible in Ω, and *essential* if it is not contractible in Ω. Let $\widehat{B}_n \subset \Omega$ be the union of B_n with the closures of the interiors of its inessential boundary components.

Now $\partial\widehat{B}_n$ is made of the essential boundary components $\{\eta_i^n\}_{i\in I_n^0}$ of ∂B_n. In particular, write η_0^n for the outer component; which surrounds all other boundary components. Replace the family of curves $\{\eta_i^n\}_{i\in I_n^0}$ by a family of geodesics $\{\gamma_i^n\}_{i\in I_n}$, using the following inductive rules. If the class $[\eta_i^n]$ is not in Case 1 of Theorem 3.2, and it does not appear among the classes $[\eta_j^{n-1}]$ (in particular, if $n = 1$), then let γ_i^n be a limit geodesic in this class and include the index i in I_n. If $n > 1$ and there is $\gamma_j^{n-1} \in [\eta_i^n]$, then choose $\gamma_i^n = \gamma_j^{n-1}$ and include the index i in I_n. If the outer component η_0^n is in Case 1 of Theorem 3.2, forget this curve and exclude the index 0 from I_n. When done, either $I_n = I_n^0$ or $I_n = I_n^0 \setminus \{0\}$.

Fix a ball B_n. Since $\{\eta_i^n\}_{i\in I_n}$ are pairwise disjoint Jordan curves, Theorem 3.2 says that the limit geodesics $\{\gamma_i^n\}_{i\in I_n}$ do not cross, and there should be a geodesic domain G_n limited by them. We proceed to the precise construction of G_n. There are a constant L_n and a compact set $A_n \subset \Omega$ such that for all $i \in I_n$ the loops in $[\eta_i^n]$ with quasihyperbolic length not greater than L_n are contained in A_n. Approximate the quasihyperbolic metric near A_n by a sequence $\left\{\lambda_\nu\,|dz|^2\right\}_{\nu=1}^\infty$ of Riemannian metrics. For each positive integer ν get a set of closed Riemannian minimizers for the classes $[\eta_i^n]$ with $i \in I_n$. By [4], these Riemannian geodesics are simple and pairwise disjoint, hence bound a closed domain $G(\nu) \subset \mathbb{C}$.

Lemma 4.4. *For all ν we have $G(\nu) \subset \Omega$.*

Proof. Suppose that $0 \in I_n$. In this case $\partial G(\nu)$ contains a Riemannian geodesic $\gamma_0(\nu)$ homotopic to the outer component η_0^n of $\partial\widehat{B}_n$. Each point $z \in G(\nu)$ lies interior to $\gamma_0(\nu)$ and exterior to the other geodesics in $\partial G(\nu)$; if z did not belong to Ω, then it would also lie interior to η_0^n and exterior to all other η_i^n, hence it woud be $z \in \widehat{B} \subset \Omega$, a contradiction. A similar argument applies to the case $0 \notin I_n$, only without $\gamma_0(\nu)$. □

For a fixed ball B_n, the domains $G(\nu)$ are either all compact or all non-compact. The second possibility only happens when the outer component of $\partial\widehat{B}_n$ is homotopic to the puncture at infinity of Ω, and then all domains $G(\nu)$ contain the puncture at infinity.

After passing to a subsequence, the boundaries $\{\partial G(\nu)\}_{\nu=1}^\infty$ converge uniformly and $\left\{G(\nu)\right\}_{\nu=1}^\infty$ is a Cauchy sequence in Hausdorff distance. We define the closed set $G_n \subset \Omega$ as the Hausdorff limit $G_n := \lim_{\nu\to\infty} G(\nu)$.

We define the *boundary* of the geodesic domain G_n as $\partial G_n = \cup_{i\in I_n}\gamma_i^n = \lim_{\nu\to\infty}\partial G(\nu)$. This set contains the topological frontier: $\operatorname{Fr} G_n \subseteq \partial G_n$, but this inclusion could be strict.

Proposition 4.5. *The sets G_n are path connected.*

Proof. Suffices to prove that any two points in ∂G_n are joined by a path in G_n.

Fix two geodesics $\alpha_1, \beta_1 \subset \partial G(1)$. For each $\nu > 1$ let $\alpha_\nu, \beta_\nu \subset \partial G(\nu)$ be the geodesics homotopic to α_1 and β_1, respectively. Let M be an apriori bound for the quasihyperbolic distance between α_ν and β_ν, and choose paths $\xi_\nu \subset \Omega$ that join α_ν to β_ν and have $L_\Omega(\xi_\nu) \le M$. Every time ξ_ν exits $G(\nu)$ through a geodesic $\gamma \subset \partial G(\nu)$, it must re-enter through the same γ and we can replace the part of ξ_ν between those two events with an arc of γ. Doing this as many times as necessary, we get a new path $\overline{\xi}_\nu \subset G(\nu)$ joining α_ν to β_ν.

If $\overline{\xi}_\nu$ visits a geodesic $\gamma \subset \partial G(\nu)$ more than once, we can replace the part between the first and last visit by a single arc of γ. This procedure leads to yet another path $\widetilde{\xi}_\nu \subset G(\nu)$, still joining α_ν to β_ν and having $L_\Omega(\widetilde{\xi}_\nu) \le M + L$, where L is an apriori bound for $L_\Omega\big(\partial G(\nu)\big)$.

The sequence $\{\widetilde{\xi}_\nu\}_{\nu=1}^\infty$ is contained in a compact subset of Ω and has length uniformly bounded by $M+L$. We can reparametrize $\widetilde{\xi}_\nu(t) : [0, M+L] \to \Omega$ so that the sequence becomes uniformly Lipschitz. By Arzelá-Ascoli there is a subsequence converging uniformly to a Lipschitz path $\xi_\infty \subset G_n$ that joins the two boundary geodesics $\alpha_\infty = \lim_{\nu\to\infty}\alpha_\nu$ and $\beta_\infty = \lim_{\nu\to\infty}\beta_\nu$. □

Lemma 4.6. *Let γ be a limit geodesic for which there is a natural number N with $\gamma \subset \partial G_n$ for every $n \geq N$. Then γ is the border of a funnel or a puncture in Ω.*

Proof. It is well-known that $\operatorname{dist}_\rho \leq 2d_\Omega$ (see, e.g., [1, Theorem 1-11]). For $n \geq N$, let us consider the Jordan curve $\eta_n \subset \partial B_\rho(r_n)$ which is freely homotopic to γ. Since $2\liminf_{n\to\infty} d_\Omega(p_0, \eta_n) \geq \lim_{n\to\infty} \operatorname{dist}_\rho(p_0, \eta_n) = \lim_{n\to\infty} r_n = \infty$, and η_n belongs to a single non-trivial free homotopy class for every $n \geq N$, Theorem B gives that $\{\eta_n\}$ converges to a collared end F. Since γ is a limit geodesic and $\eta_n \in [\gamma]$ for every $n \geq N$, the collared end F must be a funnel or a puncture. $\square$

Let us continue now with the proof of Theorem 4.3. By construction we have $G_n \subseteq G_{n+1}$. We can take a subsequence of radii $\{r_h\}$ such that $G_h \subsetneq G_{h+1}$, and besides, if $\partial G_h \cap \partial G_{h+1}$ contains some limit geodesic γ, then γ is also in ∂G_N for all $N > h$ (such γ is, by Lemma 4.6, the border of a funnel or a puncture). This subsequence can be constructed because, once we have arrived at the geodesic domain G_h, we only need to examine the long-term behavior of a finite number of boundary components, namely, those of G_h.

By Proposition 4.2, each connected component of the closure of $G_{h+1} \setminus G_h$ is a finite union (with pairwise disjoint interiors) of Y-pieces and, at most, an exterior Y-piece.

For each h, let us define H_h as the closed subset of Ω obtained as the union of G_h and the funnels and punctures whose boundaries are contained in ∂G_h. Define also H as the union $H := \cup_h H_h$.

By construction, any two limit geodesics $\gamma_h \subset \partial H_h$ and $\gamma_{h+1} \subset \partial H_{h+1}$ are non-homotopic in Ω.

If $\Omega = H$ there is nothing else to prove, but $\Omega \setminus \overline{H}$ can be a non-empty set, see Examples 6.2 and 6.4. In any case H "captures all the homotopy of Ω"; let us see that it captures even more.

Lemma 4.7. *Every Jordan curve $\alpha_0 \subset \Omega$ with non-trivial homotopy class intersects the set H.*

Proof. Choose a radius r_h, so that the ball $B = B_\rho(r_h)$ contains α_0. Let G_h be the geodesic domain that corresponds to B.

Part 1. Let us see that α_0 intersects G_h or is homotopic to an essential boundary component of B.

For each connected component η_i of ∂B we have either $\eta_i \subset \operatorname{Int}(\alpha_0)$ (possible only for the inner components of ∂B) or $\eta_i \cap \operatorname{Int}(\alpha_0) = \emptyset$. If no essential component η_i lied interior to α_0, then we would have $\operatorname{Int}(\alpha_0) \subset \widehat{B} \subset \Omega$, and $[\alpha_0]$ would be trivial, contrary to our hypotheses. If only one essential component η_{i_0} lies interior to α_0, then $\alpha_0 \cup \eta_{i_0}$ is the boundary of an annulus contained in $\widehat{B}$ and α_0 is homotopic to η_{i_0}, as claimed. If all essential inner components of ∂B lie interior to α_0, then α_0 is homotopic to the outer component of ∂B, again proving our claim.

The remaining possibility is that there are three essential inner components of ∂B, say η_1, η_2, η_3, the first two lying interior to α_0 and the third one exterior to α_0. It is possible to choose, for $j = 1, 2, 3$, a point $z_j \in \Omega^c \cap \operatorname{Int}(\eta_j)$.

For $j = 2, 3$, let $\gamma_j \subset \partial G_h$ be the limit geodesic chosen in $[\eta_j]$. If α_0 intersects γ_2 or γ_3, the claim is true. Assume that α_0 is disjoint from $\gamma_2 \cup \gamma_3$. If γ_3 lied interior to α_0, then we would have $i(\gamma_3, z_3) = 0 \neq i(\eta_3, z_3)$, impossible, hence γ_3 lies exterior to α_0. If γ_2 lied exterior to α_0, then a close enough Jordan curve $\widetilde{\gamma}_2$ would also lie exterior to α_0 and either $\operatorname{Int}(\alpha_0) \subset \operatorname{Int}(\widetilde{\gamma}_2)$ or $\operatorname{Int}(\alpha_0) \cap \operatorname{Int}(\widetilde{\gamma}_2) = \emptyset$; in the first case we would have $i(\widetilde{\gamma}_2, z_1) = \pm 1 \neq i(\gamma_2, z_1)$, impossible; in the second case we would have $i(\widetilde{\gamma}_2, z_2) = 0 \neq i(\gamma_2, z_2)$, again impossible; therefore γ_2 lies interior to α_0.

Now γ_2 lies interior to α_0 while γ_3 lies exterior to α_0. The set G_h thus visits the interior and the exterior of α_0 and, since by Proposition 4.5 it is path connected, it must intersect α_0.

Part 2: suppose α_0 is homotopic to an essential boundary component η_i of B, hence homotopic to a limit geodesic γ_i in ∂G_h or perhaps to the puncture at infinity.

Assume first that α_0 is homotopic to the puncture at infinity, which in turn is contained in an exterior Y-piece P. It is impossible to have $\alpha_0 \cap P = \emptyset$, because then α_0 could not be homotopic to η_i. Hence, α_0 intersects H in this case.

Assume now that α_0 is homotopic to a limit geodesic γ_i in ∂G_h. If γ_i is not in ∂G_{h+1}, then α_0 is not homotopic to any essential boundary component of $B_\rho(r_{h+1})$ and, by Part 1, it intersects G_{h+1}. By Lemma 4.6, the only alternative option for γ_i is to be the boundary of a funnel or puncture F. If some non-empty part $\xi \subset \alpha_0$ lies on γ_i or is on the side of γ_i where F is, then α intersects F.

Finally, α_0 could be a Jordan curve homotopic to the boundary γ_i of F but disjoint from F. But on the side of γ_i opposite to F we must have another piece P of the decomposition, and if α_0 intersects P, then it intersects H and we are done. Now P cannot be a funnel or a puncture, because then we would have $\Omega = F \cup P$, a domain with cyclic fundamental group. Thus P is either a Y-piece or an exterior Y-piece. It is impossible that α_0 be disjoint from $F \cup P$, because then α_0 could not be homotopic to γ_i. Hence, α_0 intersects H and the proof is finished. □

The following result completes the proof of Theorem 4.3.

Proposition 4.8. *Each connected component J of $\Omega \setminus \overline{H}$ is simply connected.*

Proof. Let $\gamma_0 \subset J$ be any loop. Slightly perturb γ_0 into a closed path $\gamma \subset J$ in general position. This does not change the homotopy class, but now γ has a finite number of transverse self-intersections and $\mathbb{C} \setminus \gamma$ has finitely many bounded components, each contractible. Let U be any of those components.

The Jordan curve $\partial U \subset \gamma \subset J$ is disjoint from $\overline{H}$ and, by Lemma 4.7, it is contractible in Ω. Tus $U \subset \Omega$.

Since the connected set $\overline{H}$ is disjoint from ∂U, either $\overline{H} \subset U$ or $\overline{H} \cap U = \emptyset$. But if we had $\overline{H} \subset U$ then H would induce the trivial subgroup in $\pi_1(\Omega)$, which is false, hence $\overline{H}$ and U are disjoint. Then U is a connected open subset of $\Omega \setminus \overline{H}$ and it intersects J, therefore $U \subset J$.

Since all bounded components of $\mathbb{C} \setminus \gamma$ are contained in J, the path γ is contractible in J and so is γ_0. □

5. Finite topology of closed geodesics

Here we give a counting result for tangencies and slidings of *maximal* length: while the two segments remain together we count the sliding as a single one.

Theorem 5.1. *Let Ω be any domain in $\mathbb{C}$, endowed with its quasi-hyperbolic metric. Given closed geodesics α and β, and*

$$N := \left\lfloor \frac{L_\Omega(\alpha)}{2} \right\rfloor + 1, \qquad N' := \left\lfloor \frac{L_\Omega(\beta)}{2} \right\rfloor + 1,$$

the configuration $\{\alpha, \beta\}$ has a finite topological complexity with upper bounds:
at most $N(N-1)/2$ self-intersections/self-slidings for α,
at most $N'(N'-1)/2$ self-intersections/self-slidings for β,
and at most NN' intersections/slidings between α and β.

Proof. We start with a closed geodesic γ with $L_\Omega(\gamma) = \ell$. Choose a positive integer N with $\ell/N < 2$ and divide γ into segments $\gamma_1, \dots, \gamma_N$ each with $L_\Omega(\gamma_i) < 2$. By a result of Väisälä [13], the γ_i are Jordan arcs.

Let $1 \le i \ne j \le N$ and consider α_i and α_j. Parametrize α_i and α_j by arclength, with resulting parametrizations $f_i : [0, \ell_i] \to \Omega$ and $f_j : [0, \ell_j] \to \Omega$. Define $a, b \in [0, \ell_i]$ as the first and last values of t such that $f_i(t) \in \alpha_i \cap \alpha_j$. For any point $f_i(t) \in \alpha_i \cap \alpha_j$ we must have $t \in [a, b]$, in other words $\alpha_i \cap \alpha_j \subseteq \beta_i := \alpha_i\big([a, b]\big)$.

If $a = b$ then $\alpha_i \cap \alpha_j = \{\alpha_i(a)\}$ consists of a singe point.

Suppose now that $a < b$ and define β_j as the segment of α_j whose endpoints are $\alpha_i(a), \alpha_i(b)$. The segments β_i, β_j are geodesics, they have the same endpoints, and the same length which is less than 2. By [13], we must have $\beta_i = \beta_j$. Hence, $\alpha_i \cap \alpha_j = \beta_i = \beta_j$ and so, α_i and α_j have a single sliding: a geodesic segment whose endpoints are $\alpha_i(a), \alpha_i(b)$. They have no other intersection.

The conclusion is that two segments, with length less than 2, have at most a single intersection point (tangential or transverse) or a single sliding, and never both. This counts as one intersection. As the number of pairs $i < j$ is $N(N-1)/2$, a closed geodesic with length less than $2N$ has at most $N(N-1)/2$ self-intersections/self-slidings.

By the same argument, if α, β are closed geodesics with $L_\Omega(\alpha) < 2N$ and $L_\Omega(\beta) < 2N'$, then they have at most NN' intersections/slidings. □

6. Examples

We begin by showing that infinitely many quasihyperbolic geodesics can have the same initial point and same initial tangent direction.

Example 6.1. *Consider $\Omega = \mathbb{C} \setminus \{-1,1\}$ with its quasihyperbolic metric. For $0 < r \le 1$, the following curves are closed geodesics in Ω, each a minimizer for its homotopy class:*

$$a_r = \{\, z \,:\, |z+1| = r \,\} \quad , \quad b_r = \{\, z \,:\, |z-1| = r \,\}\,,$$

and we see that for $r=1$ they are tangent at the point $z=0$. Let D, D' be the closed discs bounded by a_1 and b_1, respectively. One can check that a geodesic joining 0 to any point $z \notin D \cup D'$ is exterior to both a_1 and b_1, which forces it to be tangent at 0 to both a_1 and b_1.

The following example shows that $\Omega \setminus \overline{H}$ may be non-empty.

Example 6.2. *Let us define*

$$\mathbb{D} = \big\{ z \in \mathbb{C} :\, |z| < 1 \big\}\,, \quad \Omega = \mathbb{D} \setminus \cup_{n=1}^{\infty} \big\{ -1 + 1/n \big\}\,, \quad W = \big\{ x+iy \in \mathbb{D} :\, x > 1/2 \big\}\,.$$

The set H corresponding to this domain contains no outer funnel and no exterior Y-piece. It consists only of geodesic domains G_h and the punctures around the points $-1+1/n$. We claim that every limit geodesic in Ω is disjoint from W, hence the G_h are disjoint from W. Since the punctures are also disjoint from W, at least one simply connected piece is needed in the decomposition of Ω.

Let γ be a limit geodesic in Ω and suppose that there exists $z_0 = x_0 + iy_0 \in \gamma$ with $x_0 > 1/2$. The connected component γ_0 of $\gamma \cap \overline{W}$ that contains z_0 is an arc joining two points $1/2 + iy_1, 1/2 + iy_2 \in \mathbb{D}$. For each $x+iy \in \gamma_0$ we have $d_{\mathbb{C}}(x+iy, \partial\Omega) \le d_{\mathbb{C}}(1/2+iy, \partial\Omega)$, with strict inequality for $x_0 + iy_0$. Therefore, if g is the Euclidean segment joining $1/2+iy_1$ and $1/2+iy_2$, we have $L_\Omega(g) < L_\Omega(\gamma_0)$ and, since g and γ_0 are homotopic in Ω rel endpoints, we would deduce that γ is not minimizing in its homotopy class.

We shall see now that a narrow straight corridor in the domain Ω forces geodesics to have (self)-slidings.

Lemma 6.3. *Let Ω be a domain that for some $a > 2$ contains the open rectangle $U_a = (-1,1) + i(-a,a)$, while the long sides $L_a^{\pm} = \pm 1 + i[-a,a]$ of that rectangle lie entirely in Ω^c.*

If γ is a geodesic in Ω joining a point on the short side $\{x - ia \,:\, x \in (-1,1)\} \cap \Omega$ with a point on the other short side $\{x + ia \,:\, x \in (-1,1)\} \cap \Omega$, and contained in $\overline{U}_a \cap \Omega$, then the segment $L_4 = i[-a+2, a-2]$, four units shorter than $[-a,a]$, is part of γ.

Proof. The segment $L_2 = i[-a+1, a-1]$, two units shorter than $[-a,a]$, separates $Q_a^- := (-1,0] + L_2 \subset U_a$ from $Q_a^+ := [0,1) + L_2 \subset U_a$. In the left rectangle Q_a^- we have $\delta_\Omega(x+iy) = 1+x$, hence the restriction to Q_a^- of the quasihyperbolic metric of Ω coincides with the restriction to Q_a^- of the Poincaré metric in the half-plane $\{x+iy \in \mathbb{C} :\, x > -1\}$. Thus, the geodesics of Ω in Q_a^- consist of straight segments orthogonal to L_a^-, subarcs of half-circles orthogonal to L_a^-, and segments contained in L_2. A symmetric result holds for the right rectangle Q_a^+. All these parts must be put together so that the result is a C^1 curve. One checks, by inspection, that if γ goes all the way from $\{y = 1-a\}$ to $\{y = a-1\}$, then it is the union of a straight segment that contains L_4 and at most two subarcs of circles centered at $L_a^+ \cup L_a^-$ and with radius 1. $\square$

A first application of Lemma 6.3 is the existence of a Y-piece with the shape shown in Figure 1. The domain Ω is the complex plane minus two pairs of parallel straight segments, indicated by thick lines in the figure, which delimit narrow rectangular corridors in Ω. The Y-piece consists of the shaded area plus the thin segments (one vertical, one horizontal). One of the boundary geodesics has two self-slidings, because it goes twice along each corridor, while the other two are Jordan curves. The three boundary geodesics have a common sliding along the thin vertical segment.

Example 6.4. *Let Ω be the plane domain*

$$\Omega = \big\{ x+iy \,:\, x > 0\,,\; |y| < 1 \,\big\} \setminus \cup_{n=1}^{\infty} \{5n\}\,.$$

Lemma 6.3 implies that the decomposition is $\Omega = H \cup J$, as shown in Figure 2. The boundary ∂J is not an embedded geodesic.

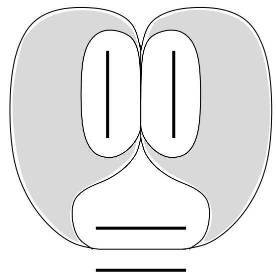

FIGURE 1. A Y-piece.

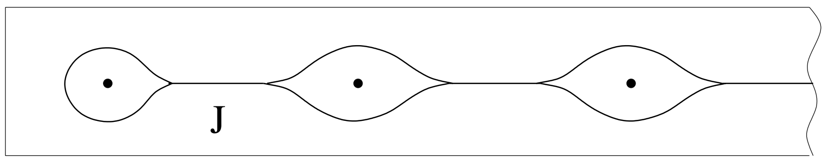

FIGURE 2

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