Unique continuation at the boundary for elliptic PDEs in Lipschitz domains with small constant

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Fang-Hua Lin's question

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Let u be a harmonic function in Ω, continuous up to the boundary, and that vanishes in a relatively open subset Σ of the boundary $\partial\Omega$. Assume the singular set $S(u) \cap \Sigma = \{x \in \Sigma \mid \nabla u = 0\}$ has positive surface measure.

Is it true that $u \equiv 0$ in Ω? Still open!

Partial results: It is known to be true for convex Lipschitz domains and for Lipschitz domains with small Lipschitz constant [3]. That is, domains such that its boundary locally coincides with a Lipschitz graph with Lipschitz constant small enough depending on the ambient dimension d.

Main theorem

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, and Σ be a relatively open subset of $\partial\Omega$ which coincides with a Lipschitz graph with Lipschitz constant $\tau < \tau_0$ (small). Let $u \not\equiv 0$ be a harmonic function in Ω, continuous in $\overline{\Omega}$ that vanishes in Σ. Then there exists some small constant $\epsilon_1(d) > 0$ and a family of open balls $(B_i)_i$, $i \in \mathbb{N}$ centered on Σ such that

- $u|_{B_i \cap \Omega}$ is either strictly positive or negative, for all $i \in \mathbb{N}$,
- $K \setminus \bigcup_i B_i$ has Minkowski dimension at most $d - 1 - \epsilon_1$ for any compact $K \subset \Sigma$.

Remarks:

- The theorem is also true for solutions of divergence form elliptic PDEs with Lipschitz coefficients $\operatorname{div}(A(x)\nabla u) = 0$! No hope for matrices with Hölder coefficients in general...
- A C^1 domain is locally a Lipschitz domain with small Lipschitz constant.

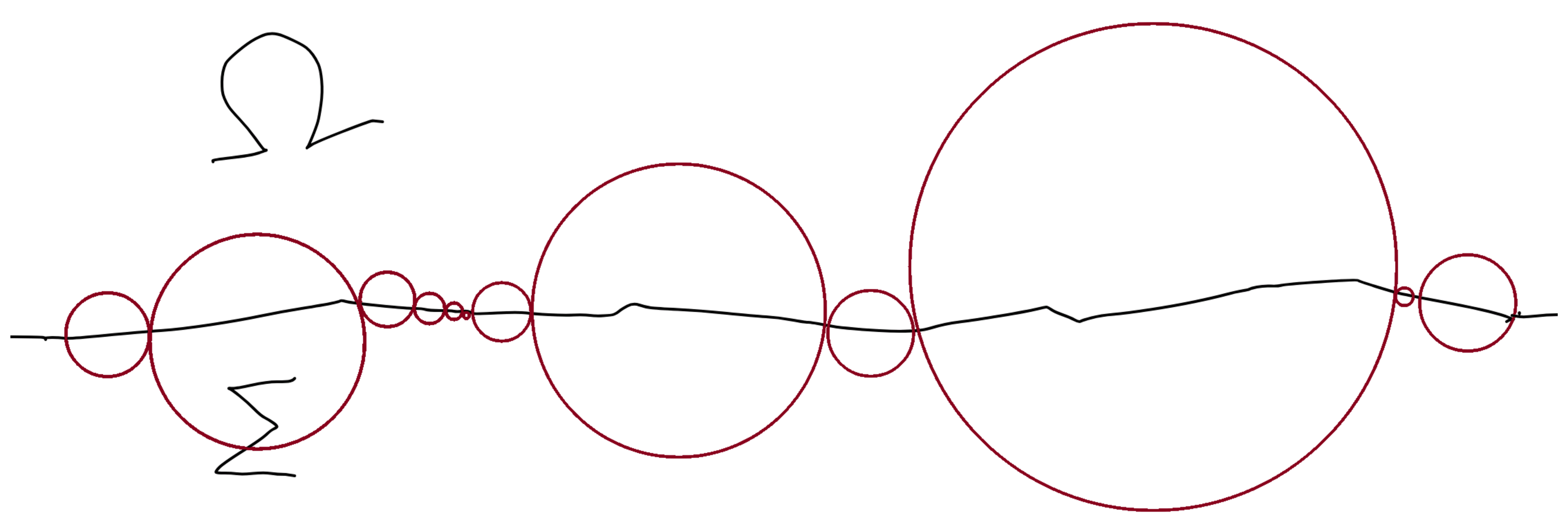


Figure 1. Example of decomposition given by the main theorem.

Partial answer to F.-H. Lin's question in the small constant case

Using the previous result, we can recover a partial answer to the previous question, also for solutions of $\operatorname{div}(A\nabla u) = 0$! For harmonic functions, this result was proved last year in [3] with a different proof.

Sketch of proof

1. Decomposition of Σ in balls given by the main theorem.
2. Boundary Harnack inequality on each ball gives comparability between the normal derivative of u on Σ and the density of harmonic measure a.e.
3. Dahlberg's theorem (density of harmonic measure is an $\mathcal{A}_\infty$ weight with respect to surface measure) implies that $\partial_\nu u \neq 0$ a.e.

Discussion of the main theorem

Define $S'(u) \cap \Sigma := \{x \in \Sigma \mid u \text{ changes sign in every neighborhood } N_x \cap \Omega \text{ of } x\}$. Then, the main theorem shows $\dim_{\overline{\mathcal{H}}}(S'(u) \cap \Sigma) \le d - 1 - \epsilon_1$.

Motivation to consider $S'(u) \cap \Sigma$ instead of $S(u) \cap \Sigma$

1. In the case the domain is $C^{1,\alpha}$ smooth (or $C^{1,\mathrm{Dini}}$), the singular set coincides with the set where u changes sign nearby:
$$S(u) \cap \Sigma = S'(u) \cap \Sigma.$$
2. In the Lipschitz case with small constant, we can upper bound the dimension of $S'(u) \cap \Sigma$ by $d - 1 - \epsilon_1$ whereas we cannot improve the bound $\dim_{\overline{\mathcal{M}}}(S(u) \cap \Sigma) \le d - 1$.

Example of harmonic function with large singular set

First, note that the outer unit normal $\nu(x)$ and the gradient $\nabla u(x)$ need not exist at every point $x \in \Sigma$. So we consider another singular set at the boundary

$$\tilde{S}(u) \cap \Sigma = \left\{ x \in \Sigma \mid \limsup_{\substack{y \in \Omega \\ y \to x}} \left| \frac{u(y) - u(x)}{y - x} \right| = 0 \right\}.$$

Let $\mathcal{C}_{2k+1}$ be the Cantor set such that at each step we divide each segment into $2k+1$ segments and take out the middle one, for $k \ge 1$. Note that, $\dim_{\mathcal{H}}(\mathcal{C}_{2k+1}) = \frac{\log 2k}{\log 2k+1} \to_{k \to \infty} 1$.

Define $E \subset [0,1] \subset \mathbb{R}$ as a concatenation of (compressed) Cantor sets

$$E = \bigcup_{k \ge 1} \left(1 - 2^{-k-1}\right) \frac{\mathcal{C}_{2k+1}}{2^k}$$

Let $\tilde{C}_\alpha(x)$ be a vertical open cone with aperture α and vertex at x and

$$\Omega := B(0,2) \cap \bigcup_{x \in E} \tilde{C}_\alpha(x).$$

Figure 2. Union of cones $\tilde{C}_\alpha(x)$ for x in the third iteration of the $1/3$-Cantor set.

Let u be the Green function of Ω with pole at $(1/2, 1)$. Then $\tilde{S}(u)$ contains most points in E, and

$$\dim_{\mathcal{H}} \tilde{S}(u) \cap \partial\Omega = 1 \quad \text{whereas} \quad S'(u) \cap \partial\Omega = \emptyset.$$

The proof of this statement uses Ahlfors' distortion theorem.

Main references

[1] A. Logunov. *Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure.* Ann. of Math. 187, no. 1 (2018): 221-239.

[2] A. Logunov, E. Malinnikova, N. Nadirashvili, and F. Nazarov. *The sharp upper bound for the area of the nodal sets of Dirichlet Laplace eigenfunctions.* Geom. Funct. Anal. 31, 1219–1244 (2021).

[3] X. Tolsa. *Unique continuation at the boundary for harmonic functions in C^1 domains and Lipschitz domains with small constants.* Preprint arXiv:2004.10721 (2021). To appear in Comm. Pure Appl. Math.

Main tool: *Almgren's frequency function*

Let

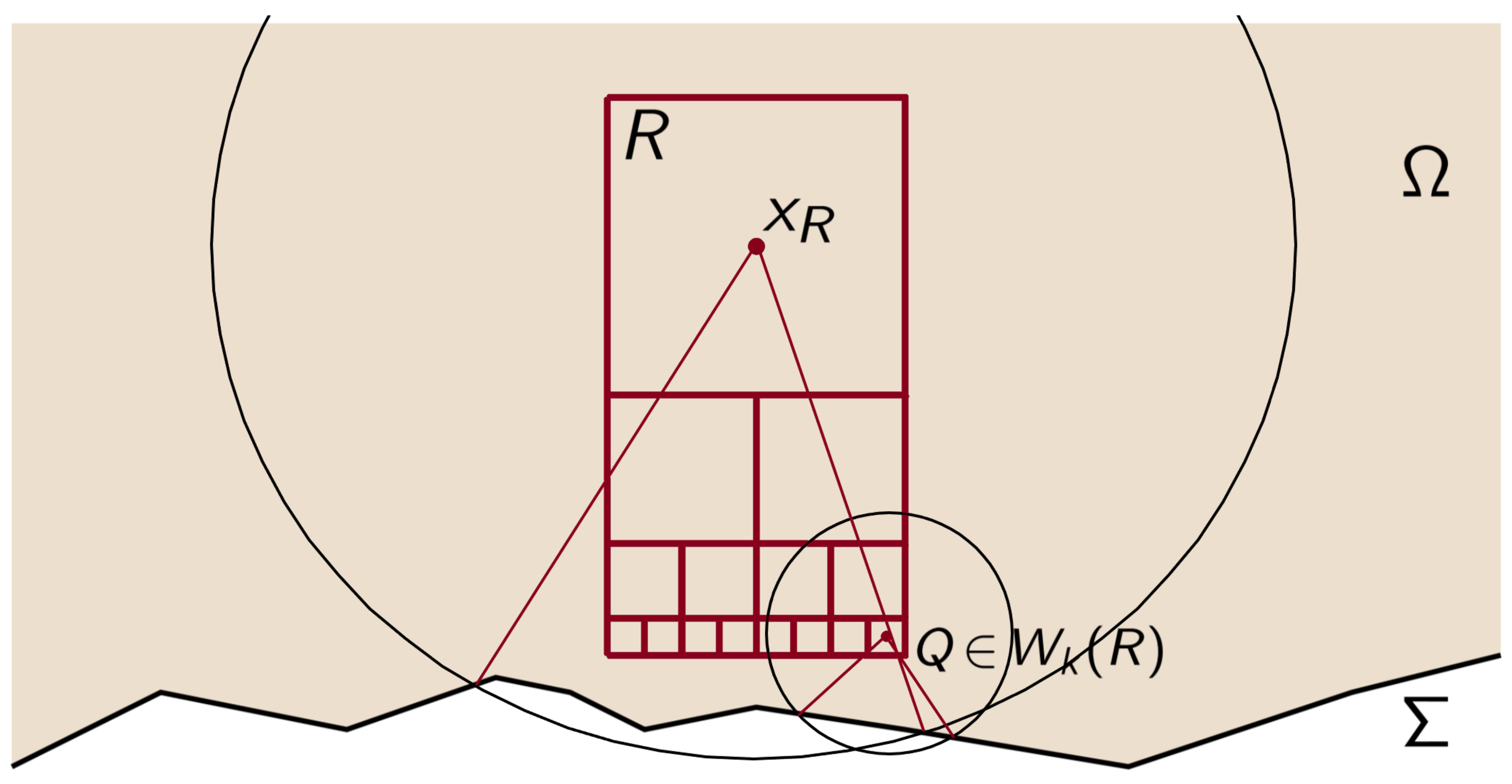
$$H(x,r) := r^{1-d} \int_{\partial B(x,r) \cap \Omega} u^2 \, d\sigma(y) \quad \text{and} \quad N(x,r) := r \partial_r \log H(x,r).$$

Main properties:

- If u is a harmonic homogeneous polynomial of degree n and $\Omega = \mathbb{R}^d$, $N(0,r) \equiv 2n$ for all $r > 0$.
- $N(x,r)$ is monotone non-decreasing in r as long as $B(x,r) \subset \Omega$.
- N controls the doubling of H
$$2^{N(x,r)} \le \frac{H(x,2r)}{H(x,r)} \le 2^{N(x,2r)}.$$
- Additivity properties (hyperplane lemma in [1]).
- $N(x,r)$ is also monotone non-decreasing in r if $B(x,r) \cap \partial\Omega \subset \Sigma$ and $B(x,r) \cap \Omega$ is star-shaped with respect to x.

Why do we need *small Lipschitz constant?*

In the proof of the main theorem we consider a Whitney cube decomposition $\mathcal{W}$ of the domain Ω and a family of balls $(B(x_Q, A\ell(Q)))_{Q \in \mathcal{W}}$ where x_Q is the center of the cube Q, $\ell(Q)$ is the side length of Q and A is a large constant depending on the dimension d. To preserve the monotonicity of the frequency function N inside these balls, we need $B_Q \cap \Omega$ to be star-shaped with respect to x_Q, which can only be ensured if the Lipschitz constant of the boundary is small enough.



Open questions

- Full answer for F.-H. Lin's question.
- Similar results for domains with lower regularity (chord-arc domains with small constant?).
- Good dimension estimates for the set where u changes sign nearby.
- Can we find good bounds of the dimension of the set of points with *order of vanishing* ∞, that is the set of points where
$$\lim_{r \to 0} r^{-n} \int_{B(x,r) \cap \Omega} |u(x)| \, dx = 0$$
for all $n \ge 0$?

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