On maxitive monetary risk measures

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(Based on joint work with Michael Kupper).

Abstract

We provide a structural theory of monetary risk measures with the crucial property of being maxitive. In particular, we show that the basic theory of large deviations is covered by this general theory. We prove for this type of monetary risk measures the Varadhan-Bryc equivalence between *large deviation principle* (LDP) and *Laplace principle* (LP), and extended versions of Bryc's theorem, and Cramer's theorem. As an application, we provide a comparison result for the asymptotic behavior of two random walks.

1 Motivation

- The theory of risk measures is one of the main directions of recent developments in stochastic optimization. It has found multitude of applications.
- Risk measures are connected to nonlinear expectations, model uncertainty/model-free finance, and robust optimization.
- Typical examples include: expectations, coherent/convex risk measures, nonlinear expectations...

Throughout, we fix a completely regular topological space S endowed with the Borel σ-algebra $\mathcal{B}(S)$.

Definition 1. *A **monetary risk measure** is a function $\phi\colon L^\infty(S) \to \mathbb{R}$ that satisfies the following axioms:*

1. Normalization*: $\phi(0) = 0$,*
2. Translation invariance*: $\phi(f + c) = \phi(f) + c$ for every constant c,*
3. Monotonicity*: $\phi(f) \le \phi(g)$ if $f \le g$.*

*A monetary risk measure ϕ is **maxitive** if $\phi(f \vee g) \le \phi(f) \vee \phi(g)$ for all $f, g \in L^\infty(S)$.*

In the following we focus on maxitive monetary risk measures. In contrast to an expectation, where losses are averaged over different states, a maxitive monetary risk measure does not compensate losses by large profits in other states.

In probability theory, the theory of large deviations concerns the asymptotic behaviour of sequences of probability distributions, see [2].

Example 1.1. *Suppose that $X_1, X_2, X_3, \cdots$ are S-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The **asymptotic entropic risk measure***

$$\phi\colon L^\infty(S) \to \mathbb{R}, \quad \phi(f) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[\exp(nf(X_n))]$$

*is a maxitive monetary risk measure (for simplicity, we assume that the limit exists). The sequence (X_n) is said to satisfy the **Large deviation principle** (LDP) with rate function $I\colon S \to [0,\infty]$ if*

$$-\inf_{x\in \mathrm{int}(A)} I(x) \le \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \le -\inf_{x\in \mathrm{cl}(A)} I(x),$$

for all $A \in \mathcal{B}(S)$.

The LDP has the following equivalent formulation in terms of the asymptotic entropic risk measure

$$\phi(f) = \sup_{x\in S}\{f(x) - I(x)\} \quad \text{for all } f \in C_b(S).$$

*This form is called **Laplace principle** (LP). An important example is the case when $X_n = \frac{1}{n}(\xi_1 + \cdots + \xi_n)$ is the sample mean of i.i.d. real-valued random variables $\xi_1, \xi_2, \cdots$. In that case, the LDP is satisfied and the rate function is the convex conjugate of the logarithmic moment generating function $\Lambda(t) = \log \mathbb{E}_{\mathbb{P}}[e^{t\xi_1}]$.*

2 Large deviation principle and Laplace principle

Suppose that $\phi\colon L^\infty(S) \to \mathbb{R}$ is a monetary risk measure. For every $A \in \mathcal{B}(S)$, we define its *concentration* as

$$J_A := \lim_{M\to-\infty} \phi(M 1_{A^c}).$$

Inspired by Example 1.1, we define.

Definition 2. *Suppose that $\phi\colon L^\infty(S) \to \mathbb{R}$ is a monetary risk measure and $I\colon S \to [0,\infty]$ is a rate function.*

- *We say that ϕ satisfies the **Large deviation principle** (LDP) with rate function $I(\cdot)$ if*

$$-\inf_{x\in \mathrm{int}(A)} I(x) \le J_A \le -\inf_{x\in \mathrm{cl}(A)} I(x)$$

- *We say that ϕ satisfies the **Laplace principle** (LP) with rate function $I(\cdot)$ if*

$$\phi(f) = \sup_{x\in S}\{f(x) - I(x)\} \quad \text{for all } f \in C_b(S).$$

Both principles are equivalent and uniquely determine the rate function. More precisely, we have the following.

Theorem 1. *Suppose that $\phi\colon L^\infty(S) \to \mathbb{R}$ is a maxitive monetary risk measure and $I\colon S \to [0,\infty]$ a rate function. Then, the following are equivalent:*

1. *ϕ satisfies the LDP with rate function $I(\cdot)$,*
2. *ϕ satisfies the LP with rate function $I(\cdot)$.*

In that case, $I(x) = \sup_{f\in C_b(S)}\{f(x) - \phi(f)\}$.

3 Large deviation results for maxitive monetary risk measures

We prove several analogues of large deviation results for maxitive monetary risk measures. For instance, we have the following version of Bryc's theorem.

Theorem 2. *Let $\phi\colon L^\infty(S) \to \mathbb{R}$ be a maxitive monetary risk measure. Suppose that for every $N \in \mathbb{N}$ there exists $K \subset S$ compact such that*

$$J_{K^c} \le -N.$$

Then ϕ satisfies the LDP and LP with rate function $I(x) = \sup_{x\in S}\{f(x) - \phi(f)\}$.

In the case that S is a topological vector space, we have the following general version of Cramer's theorem.

Theorem 3. *Let $\phi\colon L^\infty(E) \to \mathbb{R}$ be a maxitive monetary risk measure, where E is a topological vector space. Suppose that ϕ satisfies the LDP with a convex rate function $I\colon E \to [0,\infty]$. Then*

$$I(x) = \Lambda^*(x) = \sup_{\mu\in E^*}\{\mu(x) - \Lambda(\mu)\},$$

where $\Lambda(\mu) := \lim_{N\to\infty} \lim_{M\to\infty} \phi((\mu \vee -M) \wedge N)$, and E^* is the dual space of E.

4 Maxitive integral representation

One of the ideas behind the latter results is the maxitive integration representation of a monetary risk measure. A function $J\colon \mathcal{B}(S) \to [\infty, 0]$ is called a **concentration** if

1. $J_\emptyset = -\infty$, $J_S = 0$;
2. $J_A \le J_B$ if $A \subset B$.

The **maxitive integral** of $f \in L^0(S)$ with respect to J is defined by

$$\int^\vee f dJ := \sup_{c\in\mathbb{R}}\{c + J_{\{f\ge c\}}\},$$

see [1]. Denote by $LSC(E)$ the set of all lower semincontinuous real-valued functionals, and by $USC(E)$ all upper semincontinuous real-valued functionals. We provide the following duality bounds.

Theorem 4. *Let $I\colon S \to [0,\infty]$ a function. Then*

$$\int^\vee f dJ \ge \sup_{x\in S}\{f(x) - I(x)\}\, \forall f \in LSC(E) \quad \Longleftrightarrow \quad J_O \ge -\inf_{x\in O} I(x)\, \forall O \subset S \text{ open.} \tag{1}$$

$$\int^\vee f dJ \sup_{x\in S}\{f(x) - I(x)\}\, \forall f \in USC(E) \quad \Longleftrightarrow \quad J_C \le -\inf_{x\in C} I(x)\, \forall C \subset S \text{ closed.} \tag{2}$$

In Theorem 4, we do not assume maxitivity. However, the duality bounds (1) and (2) together imply that ϕ_J is maxitive in the following weak sense.

Definition 3. *We say that a monetary risk measure $\phi\colon L^\infty(S) \to \mathbb{R}$ is **weakly maxitive** if*

$$\phi(f) \le \vee_{i=1}^N \phi(g_i) \quad \text{whenever } f \in USC(S),\ g_i \in LSC(S), \text{ and } f \le \vee_{i=1}^N g_i.$$

The maxitive integral is a monetary risk measure which is maxitive when J is maxitive. In the converse direction, Cattaneo [1] proved that every maxitive risk measure is represented by a maxitive integral. We have the same representation on continuous bounded functionals if we assume only weak maxitivity.

Theorem 5. *If $\phi\colon L^\infty(S) \to \mathbb{R}$ is a weakly maxitive monetary risk measure and $J_A := \lim_{M\to-\infty} \phi(M 1_{A^c})$, then*

$$\phi(f) = \int^\vee f dJ \quad \text{for all } f \in C_b(S).$$

5 Application: Asymptotic comparison of random walks

Next, we apply the theory above to asymptotically compare two random walks. Fritz [3] derived a formula to analyze how the tail probabilities of one random walk decay relative to those of another random walk. By application of the representation theory of maxitive monetary risk measures, the Fritz's formula extends to the following general LDP. Next, we state the result in the real-valued case.

Theorem 6. *Suppose that $X_n = \frac{1}{n}\sum_{i=1}^n \xi_i$ and $Y_n = \frac{1}{n}\sum_{i=1}^n \eta_i$ for some i.i.d. sequences (ξ_i), (η_i) of real-valued random variables with bounded support. Then, for every $a \in \mathbb{R}$, it holds*

$$-\inf_{x>a}\{I_\xi(x) - I_\eta(x)\} \le \sup_{\varepsilon>0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sup_{c\in\mathbb{R}} \frac{\mathbb{P}(X_n \ge a \vee c)}{\mathbb{P}(Y_n \ge c - \varepsilon)} \le -\inf_{x\ge a}\{I_\xi(x) - I_\eta(x)\},$$

where

$$I_\xi(x) = \sup_{y\ge 0}\{xy - \log \mathbb{E}[e^{y\xi}]\}, \quad I_\eta(x) = \sup_{y\ge 0}\{xy - \log \mathbb{E}[e^{y\eta}]\}.$$

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